MUTLIPLE-SUBARC APPROACH FOR SOLVING MINIMAX PROBLEMS OF OPTIMAL CONTROL

by

PANCHAPAKESAN VENKATARAMAN

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APPROVED, THESIS COMMITTEE:

Angelo Miele, Professor
Mechanical Engineering Department
Chairman

Frederic A. Wierum, Professor
Mechanical Engineering Department

Yildiz Bayazitoglu, Associate Professor
Mechanical Engineering Department

Houston, Texas

May 1981
Numerical solutions of minimax problems of optimal control are obtained through a multiple-subarc approach, used as a sequel to a single-subarc approach. The problems are solved by means of the sequential gradient-restoration algorithm.

First, a transformation technique is employed in order to convert minimax problems of optimal control into the Mayer-Bolza problem of the calculus of variations. The transformation requires the proper augmentation of the state vector $x(t)$, the control vector $u(t)$, and the parameter vector $\pi$. As a result of the transformation, the unknown minimax value of the performance index becomes a component of the vector parameter $\pi$ being optimized. The transformation technique is then employed in conjunction with the sequential gradient-restoration algorithm for solving optimal control problems on a digital computer.

The algorithm developed in the thesis belongs to the class of sequential gradient-restoration algorithms. The sequential gradient-restoration algorithm is made up of a sequence of two-phase cycles, each cycle consisting of a gradient phase and a restoration phase. The principal property of this algorithm is that it produces a sequence of feasible suboptimal solutions. Each feasible
solution is characterized by a lower value of the minimax performance index than any previous feasible solution. To facilitate numerical implementation, the intervals of integration are normalized to unit length.

Several numerical examples are presented to illustrate the present approach. For comparison purposes, the analytical solutions, the single-subarc solutions, and the multiple-subarc solutions are presented.

Key Words. Minimax problems, minimax optimal control, numerical methods, continuous approach, single-subarc approach, multiple-subarc approach, transformation techniques, sequential gradient-restoration algorithms.
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DEDICATION

The author wishes to dedicate this work to his wife Jayanti, who has always been a constant source of encouragement and a very helpful factor in his desire to return to academic work.
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1. **INTRODUCTION**

In recent years, considerable research has been done on the problem of optimizing a trajectory from the standpoint of an integral performance index. This problem can be formulated in the form of the classical problem of Bolza, which is described as follows: Minimize the functional

\[
J = \int_0^\tau f(x,u,\pi,\theta)\,d\theta + \left[h(x,\rho)\right]_0 + \left[g(x,\pi)\right]_\tau,
\]

subject to the differential constraints

\[
\frac{dx}{d\theta} = \phi(x,u,\pi,\theta), \quad 0 \leq \theta \leq \tau,
\]

the initial conditions

\[
[\omega(x,\pi)]_0 = 0,
\]

and the final conditions

\[
[\psi(x,\pi)]_\tau = 0.
\]

Here, \(x(\theta)\) is the state vector, \(n\)-dimensional; \(u(\theta)\) is the control vector, \(m\)-dimensional; \(\pi\) is the parameter vector, \(p\)-dimensional; \(\theta\) is the independent variable, \(0 \leq \theta \leq \tau\); \(\theta=0\) is the initial time, and \(\theta=\tau\) is the final time. The functions \(f, h, g\) are scalar; the function \(\phi\) is an \(n\)-vector; the function \(\omega\) is a \(c\)-vector; and the function \(\psi\) is a \(q\)-vector.
An important extension of the above problem is the inclusion of a set of nondifferential constraints, described by

$$S(x,u,\pi,\Theta) = 0, \quad 0 \leq \Theta \leq \varpi,$$

where the function $S$ is a $k$-vector. For easy identification, the problem represented by Eqs. (1)-(4) is called Problem (P1); the problem represented by Eqs. (1)-(5) is called Problem (P2).

In the terminology of the calculus of variations, the above problem is called the Bolza problem; it includes as particular cases the Lagrange problem and the Mayer problem. The former occurs when $h=0$, $g=0$, and the latter occurs when $f=0$. For the above Bolza problem, the necessary conditions for an extremum can be found, for example, in Refs. 1-3. Computer algorithms of the first-order type can be found, for instance, in Refs. 4-5.

The formulation (1)-(5) omits an important class of problems. These problems occur when the minimization of the integral performance index (1) is replaced with the minimization of a local performance index having one of the following forms:

$$I = \max_\Theta F(x,\pi,\Theta), \quad 0 \leq \Theta \leq \varpi,$$

$$I = \max_\Theta F(x,u,\pi,\Theta), \quad 0 \leq \Theta \leq \varpi.$$

These problems are called Chebyshev problems or minimax problems of optimal control. The function $F$ is called the minimax function. For easy identification,
the problem represented by Eq. (6) and Eqs. (2)-(4) or (2)-(5) is called Problem (Q1); the problem represented by Eq. (7) and Eqs. (2)-(4) or (2)-(5) is called Problem (Q2).

It must be noted that Chebyshev-type problems occur frequently in various branches of engineering. In aerospace engineering, the following Chebyshev problems are of interest for the reentry of a variable-geometry ballistic missile and the reentry of a space glider: (R1) minimization of the peak deceleration; (R2) minimization of the peak dynamic pressure; (R3) minimization of the peak heating rate at a particular point; and (R4) minimization of the peak surface-integrated heating rate. In civil engineering, the following Chebyshev problem is of interest for an arch having variable cross section: (R5) minimization of the peak deflection for a given load distribution. In environmental engineering, the following Chebyshev problem is of interest for an ecological system (lake) subject to pollutant injection: (R6) minimization of the peak algae concentration during the year. Starting from the analytical formulation of Problems (R1) through (R6), we recognize that they are particular cases of either Problem (Q1) or Problem (Q2).

Previous research on the analytical and/or numerical solution of Problem (Q1) can be found in Refs. 6-9. In Ref. 6, Johnson observed that the problem of minimizing the local performance index (6), subject to (2)-(4), can be replaced with the problem of minimizing the integral performance index

\[ K = \left\{ \int_0^1 \left[ F(x, \pi, \theta) \right]^q d\theta \right\}^{1/q}, \] (8)
for $q \rightarrow \infty$. When (6) is replaced with (8), Problem (Q1) reduces to Problem (P1), an idea exploited by Michael in Ref. 7. The drawback of this approach is that one must solve a large number of Bolza problems for increasing values of the exponent $q$. Hence, algorithms based on the equivalence between the performance indexes (6) and (8) might be expensive from the CPU time viewpoint.

Warga (Ref. 8) noted the analogy of Problem (Q1) with bounded-state problems. Then, Powers (Ref. 9) exploited this analogy in connection with a multiple subarc approach; he applied gradient algorithms to the numerical solution of these problems.

Previous research on the analytical solution of Problem (Q2) can be found in the work of Holmaker (Refs. 10-11). In particular, Ref. 10 deals with nonautonomous systems, and Ref. 11 deals with autonomous systems.

In this thesis, we treat Chebyshev problems of both Type (Q1) and Type (Q2). A Chebyshev problem of Type (Q1) can be converted into a Bolza problem of Type (P2) by exploiting its analogy with a bounded-state problem in combination with a transformation of the Jacobson type (Ref. 12). Generally speaking, this transformation results in an increase in the dimension of the state vector. However, in some cases, the increase in the dimension of the state vector can be limited or prevented by using a transformation of the Miele-Wu-Liu type (Ref. 13). On the other hand, a Chebyshev problem of Type (Q2) can be converted into a Bolza problem of Type (P2) by exploiting its analogy with a bounded-control problem in combination with a transformation of the Valentine type (Ref. 14). In this case, the dimension of the state vector remains unchanged.
After a Chebyshev problem has been converted into the Bolza problem, some of the existing first-order algorithms can be employed in order to find numerical solutions on a digital computer, for instance, the sequential gradient-restoration algorithm. In this connection, one has two choices: the single-subarc approach (Refs. 4-5) and the multiple-subarc approach (Refs. 15-16).

In the single-subarc approach, the extremal arc is viewed as a single subarc, even though a portion of it may lie extremely close to the boundary. Along this single subarc, the control vector is regarded to be a continuous function of the time.

In the multiple subarc approach, the extremal arc is viewed as being composed of several subarcs, some internal to the boundary and some lying on the boundary. At the points of junction of different subarcs, discontinuities in the control are allowed.

It is clear that, if the exact solution of an optimization problem is characterized by discontinuities in the control, the multiple-subarc approach is superior to the single-subarc approach. Computationally speaking, however, it is important to make an intelligent guess on the number and types of subarcs composing the extremal arc.

\[1\] Here, the word "boundary" is employed to denote the state boundary for a minimax problem of Type (Q1) and the control boundary for a minimax problem of Type (Q2).
In this thesis, we employ the single-subarc approach in order to determine a first approximation to the extremal solution. Then, we employ the multiple-subarc approach as a sequel to the single-subarc approach in order to determine a second approximation to the extremal solution. We illustrate this sequential approach through several numerical examples. For comparison purposes, we show the analytical solutions, the single-subarc solutions, and the multiple-subarc solutions.
2. \textbf{TRANSFORMATION TECHNIQUES}

By means of transformation techniques akin to those employed in Refs. 12-14, the Chebyshev problems of either Type (Q1) or Type (Q2) can be converted into the Bolza problem.

Problem (Q1). In this section, we consider Problem (Q1) represented by Eqs. (6) and (2)-(4): Minimize the functional

\[
I = \max_\theta F(x,\pi,\theta), \quad 0 \leq \theta \leq \tau, \tag{9}
\]

with respect to the state \(x(\theta)\), the control \(u(\theta)\), and the parameter \(\pi\) which satisfy the constraints

\[
dx/d\theta = \Phi(x,u,\pi,\theta), \quad 0 \leq \theta \leq \tau, \tag{10}
\]

\[
[\omega(x,\pi)]_0 = 0, \tag{11}
\]

\[
[\psi(x,\pi)]_\tau = 0. \tag{12}
\]

Transformation of Problem (Q1). For any admissible choice of the state \(x(\theta)\), the control \(u(\theta)\), and the parameter \(\pi\), let \(F_*\) denote the maximum value (or peak value) achieved by the function \(F(x,\pi,\theta)\) along the interval of integration. With this understanding, the functional (9) can be rewritten as

\[
I = F_*, \tag{13}
\]

\[
F_* - F(x,\pi,\theta) \geq 0, \quad 0 \leq \theta \leq \tau. \tag{14}
\]

As a consequence, problem (9)-(12) is now replaced with problem (10)-(14).
This is a Bolza problem complicated by the fact that the state inequality constraint (14) must be satisfied everywhere along the trajectory.

The conversion of problem (10)-(14) to problem (1)-(5) requires the proper augmentation of the state vector, the control vector, and the parameter vector, as well as the proper modification of the constraining relations. In this connection, an important element is the order of the state inequality constraint (14), that is, the order of the minimax function \( F(x,\pi,\theta) \).

A minimax function \( F(x,\pi,\theta) \) is defined to be of order \( k \) if the \( k \)th total time derivative of \( F(x,\pi,\theta) \) is the first to contain the control explicitly. As an example, if the minimax function is of order \( k=1 \), we have

\[
F = F(x,\pi,\theta), \quad \frac{dF}{d\theta} = G(x,u,\pi,\theta). \tag{15}
\]

As another example, if the minimax function is of order \( k=2 \), we have

\[
F = F(x,\pi,\theta), \quad \frac{dF}{d\theta} = G(x,\pi,\theta), \quad \frac{d^2F}{d\theta^2} = H(x,u,\pi,\theta). \tag{16}
\]

**Case \( k=1 \).** Introduce the auxiliary state variable \( y(\theta) \) and the auxiliary control variable \( w(\theta) \) defined by

\[
F - F(x,\pi,\theta) = y^2, \tag{17a}
\]

\[
\frac{dy}{d\theta} = w. \tag{17b}
\]
Observe that the first time derivative of (17a) has the form

\[ G(x,u,\pi,\theta) + 2yw = 0 . \]  \hspace{1cm} (18)

Then, we replace the inequality constrained problem (10)-(14) with the following equality constrained problem:

\[ I = F_*, \]  \hspace{1cm} (19)

\[ \frac{dx}{d\theta} = \phi(x,u,\pi,\theta), \quad 0 \leq \theta \leq \tau, \]  \hspace{1cm} (20a)

\[ \frac{dy}{d\theta} = w, \quad 0 \leq \theta \leq \tau, \]  \hspace{1cm} (20b)

\[ G(x,u,\pi,\theta) + 2yw = 0, \quad 0 \leq \theta \leq \tau, \]  \hspace{1cm} (21)

\[ [\omega(x,\pi)]_0 = 0, \]  \hspace{1cm} (22a)

\[ [F_* - F(x,\pi,\theta) - y^2]_0 = 0, \]  \hspace{1cm} (22b)

\[ [\psi(x,\pi)]_\zeta = 0 . \]  \hspace{1cm} (23)

Upon augmenting the state, the control, and the parameter as follows:

\[^2\text{The superscript } T \text{ denotes transposition of vector or matrix.}\]
we see that problem (19)-(23) is identical with problem (1)-(5).

Case k=2. Introduce the auxiliary state variables \( y(\theta), z(\theta) \) and the auxiliary control variable \( w(\theta) \) defined by

\[
F_* - F(x,\pi,\theta) = y^2 ,
\]

\[
dy/d\theta = z ,
\]

\[
dz/d\theta = w .
\]

Observe that the first and second time derivatives of (25a) have the form

\[
G(x,\pi,\theta) + 2yz = 0 ,
\]

\[
H(x,u,\pi,\theta) + 2z^2 + 2yw = 0 .
\]
Then, we replace the inequality constrained problem (10)-(14) with the following equality constrained problem:

\[
\mathcal{I} = F^*,
\]

\[
\frac{dx}{d\theta} = \Phi(x,u,\pi,\theta), \quad 0 \leq \theta \leq \zeta, \quad (27)
\]

\[
\frac{dy}{d\theta} = z, \quad 0 \leq \theta \leq \zeta, \quad (28a)
\]

\[
\frac{dz}{d\theta} = w, \quad 0 \leq \theta \leq \zeta, \quad (28c)
\]

\[
H(x,u,\pi,\theta) + 2xz^2 + 2yw = 0, \quad 0 \leq \theta \leq \zeta, \quad (29)
\]

\[
\left[ \omega(x,\pi) \right]_0 = 0, \quad (30a)
\]

\[
\left[ f^* - f(x,\pi,\theta) - y^2 \right]_0 = 0, \quad (30b)
\]

\[
\left[ g(x,\pi,\theta) + 2yz \right]_0 = 0, \quad (30c)
\]

\[
\left[ \psi(x,\pi) \right]_\zeta = 0. \quad (31)
\]

Upon augmenting the state, the control, and the parameter as follows:
\[ \tilde{X} = [x^T, y, z]^T, \quad (32a) \]

\[ \tilde{u} = [u^T, w]^T, \quad (32b) \]

\[ \tilde{\pi} = [\pi^T, F^*]^T, \quad (32c) \]

we see that problem (27)-(31) is identical with problem (1)-(5).

**Problem (Q2).** In this section, we consider Problem (Q2) represented by Eqs. (7) and (2)-(4): Minimize the functional

\[ I = \max_{\theta} F(x, u, \pi, \theta), \quad 0 \leq \theta \leq \gamma, \quad (33) \]

with respect to the state \( x(\theta) \), the control \( u(\theta) \), and the parameter \( \pi \) which satisfy the constraints

\[ \frac{dx}{d\theta} = \phi(x, u, \pi, \theta), \quad 0 \leq \theta \leq \gamma, \quad (34) \]

\[ [\omega(x, \pi)]_o = 0, \quad (35) \]

\[ [\psi(x, \pi)]_\gamma = 0. \quad (36) \]
Transformation of Problem (Q2). For any admissible choice of the state $x(\theta)$, the control $u(\theta)$, and the parameter $\pi$, let $F_*$ denote the maximum value (or peak value) achieved by the function $F(x,\pi,u,\theta)$ along the interval of integration. With this understanding, the functional (33) can be rewritten as

$$I = F_*.$$  \hfill (37)

$$F_* - F(x,u,\pi,\theta) \geq 0, \quad 0 \leq \theta \leq \tau. \hfill (38)$$

As a consequence, problem (33)-(36) is now replaced with problem (34)-(38). This is a Bolza problem, complicated by the fact that the control inequality constraint (38) must be satisfied everywhere along the trajectory.

The conversion of problem (34)-(38) to problem (1)-(5) requires the proper augmentation of the control vector and the parameter vector, as well as the proper modification of the constraining relations. In this connection, we introduce an auxiliary control variable $w(\theta)$ defined by

$$F_* - F(x,u,\pi,\theta) = w^2. \hfill (39)$$

Then, we replace the inequality constrained problem (34)-(38) with the following equality constrained problem:

$$I = F_*.$$

$$I = F_*.$$  \hfill (40)
\[
d\bar{x}_1 d\theta = \phi(x, u, \pi, \theta), \quad 0 \leq \theta \leq \tau, \quad (41)
\]

\[
F_\theta - F(x, u, \pi, \theta) - w^2 = 0, \quad 0 \leq \theta \leq \tau, \quad (42)
\]

\[
[\omega(x, \pi)]_0 = 0, \quad (43)
\]

\[
[\psi(x, \pi)]_{\tau} = 0. \quad (44)
\]

Upon augmenting the state, the control, and the parameter as follows:

\[
\tilde{x} = x, \quad (45a)
\]

\[
\tilde{u} = [u^T, w]^T, \quad (45b)
\]

\[
\tilde{\pi} = [\pi^T, F_\theta]^T, \quad (45c)
\]

we see that problem (40)-(44) is identical with problem (1)-(5).
3. **PROBLEM FORMULATION**

After a Chebyshev problem has been converted into the Bolza problem, one has two choices: the single-subarc approach and the multiple-subarc approach. Regardless of the approach employed, it is convenient to normalize the time in such a way that each subarc composing the extremal arc has a normalized time length equal to one.

In this thesis, we illustrate the multiple-subarc approach by considering the particular case of an extremal arc including three subarcs. However, it must be noted that the method presented here is general and can be extended to any number of subarcs.

**Single-Subarc Approach.** In this approach, the time transformation is the following:

\[ \theta = \gamma t, \quad 0 \leq t \leq 1. \]  

(46)

The time \( t=0 \) denotes the initial point; and the time \( t=1 \) denotes the final point.

Upon redefining the parameter vector as follows:

\[ \tilde{\pi} = \pi, \quad \text{if } \gamma \text{ is fixed}, \]  

(47a)

\[ \tilde{\pi} = [\pi^T, \gamma]^T, \quad \text{if } \gamma \text{ is free}, \]  

(47b)
and upon dropping the tilde, the Bolza problem of Type (P2) governed by Eqs. (1)-(5) is reformulated as follows: Minimize the functional

$$I = \int_0^1 L(x,u,v,t)\,dt + \left[ h(x,v)\right]_0 + \left[ g(x,v)\right]_1,$$

with respect to the state $x(t)$, the control $u(t)$, and the parameter $\pi$ which satisfy the constraints

$$\dot{x} = \Phi(x,u,v,t), \quad 0 \leq t \leq 1,$$  
$$S(x,u,v,t) = 0, \quad 0 \leq t \leq 1,$$  
$$\left[ \omega(x,v)\right]_0 = 0,$$  
$$\left[ \psi(x,v)\right]_1 = 0.$$

**Multiple-Subarc Approach.** In this approach, the time transformation is the following:

$$\theta = \Theta_1 t, \quad 0 \leq t \leq 1,$$  
$$\theta = \Theta_1 + (\Theta_2 - \Theta_1)(t-1), \quad 1 \leq t \leq 2,$$  
$$\theta = \Theta_2 + (\zeta - \Theta_2)(t-2), \quad 2 \leq t \leq 3.$$
The time \( t=0 \), denotes the initial point; the time \( t=1 \) denotes the endpoint of the first subarc; the time \( t=2 \) denotes the endpoint of the second subarc; and time \( t=3 \) denotes the final point.

Upon redefining the parameter vector as follows:

\[
\tilde{\pi} = [\pi^T, \theta_1, \theta_2]^T, \quad \text{if } \tau \text{ is fixed},
\]

\[
\tilde{\pi} = [\pi^T, \theta_1, \theta_2, \tau]^T, \quad \text{if } \tau \text{ is free},
\]

and upon dropping the tilde, the Bolza problem of Type (P2) governed by Eqs. (1)-(5) is reformulated as follows: Minimize the functional

\[
I = \int_0^3 f(x,u,\pi,t) \, dt + [h(x,\pi)]_0 + [q(x,\pi)]_3,
\]

with respect to the state \( x(t) \), the control \( u(t) \), and the parameter \( \pi \) which satisfy the constraints

\[
\dot{x} = \phi(x,u,\pi,t), \quad 0 \leq t \leq 3,
\]

\[
S(x,u,\pi,t) = 0, \quad 0 \leq t \leq 3,
\]

\[
[\omega(x,\pi)]_0 = 0,
\]

\[
[\psi(x,\pi)]_3 = 0.
\]
Remark 3.1. We note that, in the formulation (55)-(59), the functions $f, \phi, S$ might have different specifications in each of the subarcs composing the extremal arc, more specifically,\(^3\)

\[
\begin{align*}
  f &= f_I(x, u, \pi, t), & 0 \leq t \leq 1, \quad (60a) \\
  f &= f_{II}(x, u, \pi, t), & 1 \leq t \leq 2, \quad (60b) \\
  f &= f_{III}(x, u, \pi, t), & 2 \leq t \leq 3; \quad (60c) \\
  \phi &= \phi_I(x, u, \pi, t), & 0 \leq t \leq 1, \quad (61a) \\
  \phi &= \phi_{II}(x, u, \pi, t), & 1 \leq t \leq 2, \quad (61b) \\
  \phi &= \phi_{III}(x, u, \pi, t), & 2 \leq t \leq 3; \quad (61c) \\
  S &= S_I(x, u, \pi, t), & 0 \leq t \leq 1, \quad (62a) \\
  S &= S_{II}(x, u, \pi, t), & 1 \leq t \leq 2, \quad (62b) \\
  S &= S_{III}(x, u, \pi, t), & 2 \leq t \leq 3. \quad (62c)
\end{align*}
\]

\(^3\)In Eqs. (60)-(62), the subscripts I, II, III refer to the first, second, and third subarc, respectively.
We shall take advantage of this flexibility, when attempting to solve minimax problems of optimal control by means of the multiple-subarc approach, employed as a sequel to the single-subarc approach.

**Remark 3.2.** For the sake of generality, we shall derive (i) the first-order conditions for a minimum and (ii) the sequential gradient-restoration algorithm by imbedding problem (55)-(59) in a more general problem, one in which the system of boundary conditions (58)-(59) is supplemented by the possible presence of entrance conditions and exit conditions imposed on the middle subarc:

\[
\left[ M(x,\pi) \right]_1 = 0 , \quad (63)
\]

\[
\left[ N(x,\pi) \right]_2 = 0 . \quad (64)
\]

Here, M is a d-vector and N is an e-vector.
4. **FIRST-ORDER CONDITIONS**

From calculus of variations, it is known that problem (55)-(64) can be recast as that of minimizing the augmented functional $\mathcal{J}$, subject to (56)-(64). Here,

$$\mathcal{J} = I + L,$$

(65)

and the Lagrangian $L$ is given by

$$L = \int_0^3 \lambda^T(\dot{x} - \Phi) \, dt + \int_0^3 \rho^T S \, dt$$

$$+ (\sigma^T \omega)_0 + (\xi^T M)_1 + (\eta^T N)_2 + (\mu^T \psi)_3 .$$

(66)

In (66), $\lambda(t)$ is an $n$-vector variable Lagrange multiplier; $\rho(t)$ is a $k$-vector variable Lagrange multiplier; $\sigma$ is a $c$-vector constant Lagrange multiplier; $\xi$ is a $d$-vector constant Lagrange multiplier; $\eta$ is an $e$-vector constant Lagrange multiplier; and $\mu$ is a $q$-vector constant Lagrange multiplier.

After performing the customary integration by parts, the Lagrangian $L$ can be rewritten as$^4$

$$L = \int_0^3 (-\dot{x}^T x - \dot{x}^T \Phi + \dot{\rho}^T S) \, dt + (\lambda^T x + \sigma^T \omega)_0 + \left[ (\lambda^T x)_{\ominus} - (\lambda^T x)_{\oplus} + \xi^T M \right]$$

$^4$The subscript $\ominus$ denotes conditions preceding a corner point, and the subscript $\oplus$ denotes conditions following a corner point.
The optimal solution \( x(t), u(t), \pi \) and the associated multipliers \( \lambda(t), \rho(t), \sigma, \xi, \eta, \mu \) must satisfy the feasibility equations (56)-(64) and the following first-order optimality conditions:

\[
\dot{\lambda} = f_x - \phi_x \lambda + S_x \rho, \quad 0 \leq t \leq 3,
\]

\[
f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 3,
\]

\[
\int_0^3 (f_{\pi} - \phi_{\pi} \lambda + S_{\pi} \rho) \, dt + (h_{\pi} + \omega_{\pi} \sigma)\,0 + (M_{\pi} \xi)_1 + (N_{\pi} \eta)_2 + (q_{\pi} + \psi_{\pi} \nu)_3 = 0,
\]

\[
(-\lambda + h_x + \omega_x \sigma)_0 = 0,
\]

\[
(\lambda_\Theta - \lambda_\Theta - M_x \xi)_1 = 0,
\]

\[
(\lambda_\Theta - \lambda_\Theta - N_x \eta)_2 = 0,
\]

\[
(\lambda + q_x + \psi_x \mu)_3 = 0.
\]
In (68)-(74), the subscripts represent partial derivatives. It should be noted that the partial derivatives of \( f, h, g \) with respect to \( x, u, \pi \) yield vectors; on the other hand, the partial derivatives of \( \phi, S, \omega, \)
\( M, N, \psi \) with respect to \( x, u, \pi \) yield matrices.

**Approximate Methods.** In general, the differential system (56)-(64) and (68)-(74) is nonlinear. Consequently, approximate methods must be used to seek a solution iteratively. In this connection, define the norm squared of a vector \( \nu \) to be

\[
Z(\nu) = \nu^T \nu.
\] (75)

Then, the functionals

\[
P = \int_0^3 Z(\dot{x} - \phi) dt + \int_0^3 Z(s) dt + Z(\omega) + Z(M) + Z(N) + Z(\psi)
\]

and

\[
Q = \int_0^3 Z(\dot{\lambda} - f_x + \phi_x + S_x \rho) dt + \int_0^3 Z(f_u - \phi_u + \phi + S_u \rho) dt
\]

\[
+ Z \left[ \int_0^3 (f_n - \phi_n + S_n \rho) dt + (h_n + \omega_n \rho) + (M_n \xi) + (N_n \eta) \right] + Z(\lambda + h_x + \omega_x \rho) + Z(\lambda - \lambda - M_x \xi)
\]
\[ + \frac{L}{\lambda + q_x + \psi x \mu} \]

measure the errors in the feasibility conditions and the optimality conditions, respectively.

For the exact optimal solution, one must have

\[ P = 0, \quad Q = 0. \]

(78)

For an approximation to the optimal solution, one must have

\[ P \leq \epsilon_1, \quad Q \leq \epsilon_2, \]

(79)

where \( \epsilon_1, \epsilon_2 \) are small, preselected numbers.
5. **SEQUENTIAL GRADIENT-RESTORATION ALGORITHM**

The sequential gradient-restoration algorithm (SGRA) is an iterative technique which includes a sequence of two-phase cycles, each composed of a gradient phase and a restoration phase (Refs. 4-5). This technique is designed to achieve a decrease in the functional $I$ and/or the augmented functional $J$ between the endpoints of each cycle, while the constraints are satisfied to a predetermined accuracy. The two phases of a cycle are called the gradient phase and the restoration phase.

The gradient phase is started when the inequality (79-1) is satisfied. It involves a single iteration. In each gradient iteration, the objective is to reduce the functional $I$ and/or the augmented functional $J$, while the constraints are satisfied to the first order.

The restoration phase is started when the inequality (79-1) is violated. It involves one or more iterations. In each restorative iteration, the objective is to reduce the functional $P$, while the constraints are satisfied to first order and the norm squared of the variations of the control, the parameter, and the initial state vector is minimized. The restoration phase is terminated whenever the inequality (79-1) is satisfied.

The algorithm is stopped when Ineqs. (79) are both satisfied.

The sequential gradient-restoration algorithm for a general Bolza problem has been developed by Miele et al for both the continuous version (Ref. 4-5) and the three-subarc version (Refs. 15-16). With reference to the three-subarc version, a particular set of initial conditions was assumed in Refs. 15-16 (state vector given at the initial point). Here, a three-subarc version of SGRA is developed for a general set of boundary conditions.
Notation. Let $x(t), u(t), \pi$ denote the nominal functions; let $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ denote the varied functions; and let $\Delta x(t), \Delta u(t), \Delta \pi$ denote the perturbations of $x(t), u(t), \pi$ about the nominal values. Assume that the perturbations $\Delta x(t), \Delta u(t), \Delta \pi$ are linear in the stepsize $\alpha$, where $\alpha > 0$; and let $A(t), B(t), C$ denote the perturbations per unit stepsize. Then, the following relations hold:

\begin{align*}
\tilde{x}(t) &= x(t) + \Delta x(t) = x(t) + \alpha A(t), \\
\tilde{u}(t) &= u(t) + \Delta u(t) = u(t) + \alpha B(t), \\
\tilde{\pi} &= \pi + \Delta \pi = \pi + \alpha C.
\end{align*}

With regard to the functionals, let $I, J, P$ denote values associated with the nominal functions; let $\tilde{I}, \tilde{J}, \tilde{P}$ denote values associated with the varied functions; and let $\Delta I, \Delta J, \Delta P$ denote the total variations of these functionals caused by the perturbations $\Delta x(t), \Delta u(t), \Delta \pi$.

The perturbations $\Delta x(t), \Delta u(t), \Delta \pi$ must be determined so as to achieve at least one of the following descent properties:

$\Delta I < 0$, and/or $\Delta J < 0$, and/or $\Delta P < 0$.  \hfill (81)

Inequalities (81) can be enforced by proper selection of the stepsize, if we can choose functions $A(t), B(t), C$ such that the following first-
variation properties are satisfied:

\[ \delta I < 0, \quad \text{and/or} \quad \delta J < 0, \quad \text{and/or} \quad \delta P < 0. \]  

(82)

**First Variations.** After simple manipulations, the first variations of the functionals \( I, J, P \) take the following form:

\[ \delta I = \int_0^3 (f_x^T \Delta x + f_u^T \Delta u + f_{\pi}^T \Delta \pi) \, dt \]

\[ + (h_x^T \Delta x + h_{\pi}^T \Delta \pi)_o + (q_x^T \Delta x + q_{\pi}^T \Delta \pi)_o, \]  

(83)

\[ \delta J = \int_0^3 (\dot{\lambda} + f_x^T \phi \lambda + S_x^T \phi \lambda + \dot{\lambda} + S_x^T \phi \lambda + \dot{\lambda} + S_x^T \phi \lambda + \dot{\lambda} + S_x^T \phi \lambda + \dot{\lambda} + S_x^T \phi \lambda) \Delta x \, dt \]

\[ + \left[ \left[ (f_{\pi}^T \phi \lambda + S_{\pi}^T \phi \lambda) \Delta u \right] + (h_{\pi}^T \phi \lambda + S_{\pi}^T \phi \lambda) \right] \Delta \pi \]

\[ + \left[ (\dot{\lambda} + h_x^T \Delta x + \omega_x \sigma)^T \Delta x \right] \Delta \pi \]  

(84)

\[ \delta P = 2 \int_0^3 (\dot{\phi}^T \phi \lambda + S_x^T \phi \lambda + \dot{\phi}^T \phi \lambda + S_x^T \phi \lambda + \dot{\phi}^T \phi \lambda + S_x^T \phi \lambda + \dot{\phi}^T \phi \lambda + S_x^T \phi \lambda) \, dt \]

\[ + 2 \int_0^3 s^T (S_x^T \Delta x + S_u^T \Delta u + S_{\pi}^T \Delta \pi) \, dt \]

\[ + 2 \left[ M^T (M_x^T \Delta x + M_{\pi}^T \Delta \pi) \right] \Delta \pi \]  

(85)
These equations must be completed by the relation

\[ K = \int_0^T \Delta u^T \Delta u \, dt + \Delta \pi^T \Delta \pi + \Delta x^T(0) \Delta x(0). \]  

(86)

**Gradient Phase.** Let \( x(t), u(t), \pi \) denote nominal functions satisfying (56)-(64). Let \( \bar{x}(t), \bar{u}(t), \bar{\pi} \) denote varied functions also satisfying (56)-(64). To first order, the perturbations \( \Delta x(t), \Delta u(t), \Delta \pi \) must satisfy the linearized constraints

\[ \Delta \dot{x} = \Phi_x^T \Delta x + \Phi_u^T \Delta u + \Phi_\pi^T \Delta \pi, \quad 0 \leq t \leq 3, \]  

(87)

\[ s_x^T \Delta x + s_u^T \Delta u + s_\pi^T \Delta \pi = 0, \quad 0 \leq t \leq 3, \]  

(88)

\[ (\omega_x^T \Delta x + \omega_\pi^T \Delta \pi)_0 = 0, \]  

(89)

\[ (M_x^T \Delta x + M_\pi^T \Delta \pi)_1 = 0, \]  

(90)

\[ (N_x^T \Delta x + N_\pi^T \Delta \pi)_2 = 0, \]  

(91)

\[ (\psi_x^T \Delta x + \psi_\pi^T \Delta \pi)_3 = 0. \]  

(92)

In order to satisfy the first-variation property (82-2), we choose the following special variations of the control, the parameter, and the
initial state vector:

$$\Delta u = -\alpha (f_u - \Phi_u \lambda + S_u \rho), \quad 0 \leq t \leq 3,$$

$$\Delta \pi = -\alpha \left[ \int_0^3 (f_\pi - \Phi_\pi \lambda + S_\pi \rho) \, dt + (h_\pi + \omega_\pi \sigma) \right]_0,$$

$$\Delta x(0) = -\alpha (-\lambda + h_x + \omega_x \sigma)_0,$$

where $\alpha$ denotes the gradient stepsize. The multipliers $\lambda(t), \rho(t), \sigma, \xi, \eta, \mu$ appearing in (93)-(95) must be consistent with the relations

$$\dot{\lambda} = f_x - \Phi_x \lambda + S_x \rho, \quad 0 \leq t \leq 3,$$

$$(\lambda - \lambda_0 - M_x \xi)_1 = 0,$$

$$(\lambda - \lambda_0 - N_x \eta)_2 = 0,$$

$$(\lambda + g_x + \psi_x \mu)_3 = 0.$$

When the above special variations are introduced into Eq.(84), the first variation of the augmented functional reduces to

$$\delta J = -\alpha Q,$$
where Q is the error in the optimality conditions (77), which reduces to

\[
Q = \int_0^3 Z \left( \int_0^3 (f_u - \phi_u \lambda + S_u \rho) \, dt + Z \left[ \int_0^3 (f_\pi - \phi_\pi \lambda + S_\pi \rho) \, dt + (h_\pi + \omega_\pi \sigma) \right] \right) + (M_1 \xi + (N_1 \eta)_1 \xi + (g_1 + \psi_1 \mu)_1 \xi) + Z (-\lambda + h_x + \omega_x \sigma) \, dt. \tag{101}
\]

Since \( Q > 0 \) and \( \alpha > 0 \), we have that \( \delta J < 0 \). Hence, for \( \alpha \) sufficiently small, the decrease of the augmented functional is guaranteed.

Introducing the perturbations per unit stepsize defined in (80), the linearized constraints (87)-(92) and the special variations (93)-(99) then become

\[
\dot{A} = \phi_x^T A + \phi_u^T B + \phi_\pi^T C, \quad 0 \leq t \leq 3, \tag{102}
\]

\[
S_x^T A + S_u^T B + S_\pi^T C = 0, \quad 0 \leq t \leq 3, \tag{103}
\]

\[
(\omega_x^T A + \omega_\pi^T C) = 0, \tag{104}
\]

\[
(M_x^T A + M_\pi^T C)_1 = 0, \tag{105}
\]

\[
(N_x^T A + N_\pi^T C)_2 = 0, \tag{106}
\]

\[
(\psi_x^T A + \psi_\pi^T C)_3 = 0, \tag{107}
\]
and

\[
B = -(f_u - \phi_u \lambda + S u \rho), \quad 0 \leq t \leq 3, \tag{108}
\]

\[
C = - \left[ \int_0^3 (f_\pi - \phi_\pi \lambda + S_\pi \rho) \, dt + (h_\pi + \omega_\pi \sigma)_0 + (M_\pi \xi)_1 + (N_\pi \eta)_2 + (g_\pi + \psi_\pi \mu)_3 \right], \tag{109}
\]

\[
A(0) = -(-\lambda + h_x + \omega_x \sigma)_0, \tag{110}
\]

\[
\dot{\lambda} = f_x - \phi_x \lambda + S_x \rho, \quad 0 \leq t \leq 3, \tag{111}
\]

\[
(\lambda_\phi - \lambda_\phi - M_x \xi)_1 = 0, \tag{112}
\]

\[
(\lambda_\phi - \lambda_\phi - N_x \eta)_2 = 0, \tag{113}
\]

\[
(\lambda + q_x + \psi_x \mu)_3 = 0 \tag{114}
\]

The error in the optimality conditions \( Q \) can be written as

\[
\Theta = \int_0^3 B^T B \, dt + C^T C + A^T(0) A(0). \tag{115}
\]
The differential system (102)-(114) is linear and nonhomogenous in the functions $A(t)$, $B(t)$, $C$ and the multipliers $\lambda(t)$, $\rho(t)$, $\sigma, \xi, \eta, \mu$. It can be solved without assigning a value to the gradient stepsize $\alpha$. The technique employed to solve this system is the method of particular solutions (Refs. 17-18). After the linear, multipoint boundary value problem is solved, the value of $\alpha$ is selected so to enforce the descent requirement (81-2).

Restoration Phase. Let $x(t), u(t), \pi$ denote nominal functions violating at least one of Eqs. (56)-(64). Let $\bar{x}(t), \bar{u}(t), \bar{\pi}$ denote varied functions satisfying (56)-(64). To first order, the perturbations $\Delta x(t), \Delta u(t), \Delta \pi$ must satisfy the linearized constraints

\[
\Delta \dot{x} = \Phi_x^T \Delta x + \Phi_u^T \Delta u + \Phi_{\pi}^T \Delta \pi - \alpha (\dot{x} - \Phi), \quad 0 \leq t \leq 3, \quad (116)
\]

\[
S_x^T \Delta x + S_u^T \Delta u + S_{\pi}^T \Delta \pi + \alpha S = 0, \quad 0 \leq t \leq 3, \quad (117)
\]

\[
(\omega_x^T \Delta x + \omega_{\pi}^T \Delta \pi + \alpha \omega) = 0, \quad (118)
\]

\[
(M_x^T \Delta x + M_{\pi}^T \Delta \pi + \alpha M) = 0, \quad (119)
\]

\[
(N_x^T \Delta x + N_{\pi}^T \Delta \pi + \alpha N) = 0, \quad (120)
\]

\[
(\psi_x^T \Delta x + \psi_{\pi}^T \Delta \pi + \alpha \psi) = 0, \quad (121)
\]
where \( \alpha \) denotes the restoration stepsize, a scaling factor in the range 
\( 0 \leq \alpha \leq 1. \)

When the variations defined by (116)-(121) are employed, the first 
variation of the constraint error (85) becomes

\[
\delta P = -2\alpha P. \tag{122}
\]

Since \( P > 0 \) and \( \alpha > 0 \), we have that \( \delta P < 0 \). Hence, for \( \alpha \) sufficiently small, 
the decrease of the constraint error is guaranteed.

Since Eqs. (116)-(121) admit an infinite number of solutions, an 
additional requirement must be introduced to uniquely define the restoration 
algorithm. This additional requirement is that the restoration be accomplished 
with the least-square change of the control, the parameter, and the initial 
state vector. Hence, we minimize the quadratic functional

\[
k = \left(\frac{1}{2\alpha}\right) \left[ \int_0^3 \Delta u^T \Delta u \, dt + \Delta \pi^T \Delta \pi + \Delta x(0) \Delta x(0) \right], \tag{123}
\]

with respect to the perturbations \( \Delta x(t), \Delta u(t), \Delta \pi \) which satisfy the linearized 
constraints (116)-(121).

The special variations of the control, the parameter, and the initial 
state vector solving the above auxiliary minimization problem are given 
by

\[
\Delta u = \alpha \left( \phi_u \lambda - s_u \rho \right), \quad 0 \leq t \leq 3, \tag{124}
\]

\[
\Delta \pi = \alpha \left[ \int_0^3 (\phi_\pi \lambda - s_\pi \rho) \, dt - (\omega_\pi \sigma) \right]_0
\]
\[-(M_\pi \xi)_1 - (N_\pi \eta)_2 - (\psi_\pi \mu)_3, \tag{125}\]

\[\Delta x(0) = \alpha (\lambda - \omega_x \sigma)_o, \tag{126}\]

where \(\alpha\) denotes the restoration stepsize. The multipliers \(\lambda(t), \rho(t), \sigma, \xi, \eta, \mu\) appearing in (124)-(126) must be consistent with the relations

\[\dot{\lambda} = -\phi_x \lambda + S_x \rho, \tag{127}\]

\[(\lambda_0 - \lambda_0 - M_x \xi)_A = 0, \tag{128}\]

\[(\lambda_0 - \lambda_0 - N_x \eta)_2 = 0, \tag{129}\]

\[(\lambda + \psi_x \mu)_3 = 0. \tag{130}\]

Introducing the perturbations per unit stepsize defined in (80), the linearized constraints (116)-(121) and the special variations (124)-(130) then become

\[\dot{A} = \Phi_x^T A + \Phi_u^T B + \Phi_\pi^T C - (\dot{x} - \phi), \quad 0 \leq t \leq 3, \tag{131}\]

\[S_x^T A + S_u^T B + S_\pi^T C + S = 0, \quad 0 \leq t \leq 3, \tag{132}\]

\[(\omega_x^T A + \omega_\pi^T C + \omega)_o = 0, \tag{133}\]

\[(M_x^T A + M_\pi^T C + M)_c = 0, \tag{134}\]
\[ \left( N_x^T A + N_\pi^T C + N \right)_2 = 0, \quad (135) \]
\[ \left( \psi_x^T A + \psi_\pi^T C + \psi \right)_3 = 0, \quad (136) \]

and

\[ B = \left( \phi_u \lambda - s_u \rho \right), \quad 0 \leq t \leq 3, \quad (137) \]
\[ C = \int_0^3 \left( \phi_\pi \lambda - s_\pi \rho \right) dt - (\omega_\pi \sigma)_o - \left( M_\pi \xi \right)_1 - \left( N_\pi \eta \right)_2 - (\psi_\pi \mu)_3, \quad (138) \]
\[ A(0) = (\lambda - \omega_x \sigma)_o, \quad (139) \]
\[ \dot{\lambda} = -\phi_x \lambda + s_x \rho, \quad 0 \leq t \leq 3, \quad (140) \]
\[ (\lambda_\Theta - \lambda_\Theta - M_x \xi)_1 = 0, \quad (141) \]
\[ (\lambda_\Theta - \lambda_\Theta - N_x \eta)_2 = 0, \quad (142) \]
\[ (\lambda + \psi_x \mu)_3 = 0. \quad (143) \]

The differential system (131)-(143) is linear and nonhomogenous in the functions $A(t), B(t), C$ and the multipliers $\lambda(t), \rho(t), \sigma, \xi, \eta, \mu$. 
It can be solved without assigning a value to the restoration stepsize $\alpha$. The technique employed to solve this system is the method of particular solutions (Refs. 17-18). After the linear, multipoint boundary value problem is solved, the value of $\alpha$ is selected so to enforce the descent requirement (81-3).

**Remark 5.1.** The solution of the linear, multipoint boundary-value problem characterizing the gradient phase and the restoration phase is subordinated to the following disequations:

\[
\det \left[ \omega^T \omega \right]_0 \neq 0, \quad (144)
\]

\[
\det \left[ S_u^T S_u \right] \neq 0, \quad 0 \leq t \leq 3. \quad (145)
\]
6. EXPERIMENTAL CONDITIONS AND COMPUTATIONAL DETAILS

To substantiate the approach of this thesis, four examples were solved (see Refs. 9-11 and Refs. 19-21). For comparison purposes, the analytical solutions, the single-subarc solutions, and the multiple-subarc solutions are given.

The sequential gradient-restoration algorithm (SGRA) was programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic. Computations were performed at Rice University using an ITEL AS/6 computer.

For the single-subarc version of SGRA, the interval of integration was divided into 100 steps. For the multiple-subarc version, the interval of integration of each subarc was divided into 50 steps. Two-subarc, three-subarc, and four-subarc versions of SGRA were developed, even though the analytical treatment of this thesis refers only to the three-subarc version.

The differential systems were integrated using Hamming's modified predictor-corrector method, with a special Runge-Kutta starting procedure. The definite integrals I, J, P, Q were computed using a modified Simpson's rule. Linear algebraic systems were solved using a standard Gaussian elimination routine. For both the gradient phase and the restoration phase, the linear, multipoint boundary-value problem was solved using the method of particular solutions.

Convergence Conditions. If P denotes the error in the constraints and Q denotes the error in the optimality conditions, the following stopping conditions were employed for the algorithm as a whole:
Gradient Stepsize. The gradient stepsize was determined with a two-step procedure involving (i) the determination of a reference stepsize and (ii) a bisection process.

First, the reference stepsize $\alpha_0$ was determined by means of a cubic interpolation process, which was stopped whenever the following inequality was satisfied:

$$\left| \tilde{f}(\alpha)/\tilde{f}(0) \right| \leq E-03.$$  \hspace{1cm} (147)

The number of Hermitian search steps required to satisfy Ineq. (147) was subject to the upper bound

$$N_3 \leq 5.$$ \hspace{1cm} (148)

Then, the gradient stepsize was determined by means of a bisection process, starting from $\alpha=\alpha_0$, which was stopped whenever the following inequalities were satisfied:

$$\tilde{f}(\alpha) < \tilde{f}(0),$$ \hspace{1cm} (149a)

$$\tilde{P}(\alpha) \leq 10.$$ \hspace{1cm} (149b)
Restoration Stepsize. The restoration stepsize was determined by means of a bisection process, starting from $\alpha=1$, which was stopped whenever the following inequality was satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0).$$  \hspace{1cm} (150)

Cycle Condition. For a complete gradient-restoration cycle, denote by $I_1$ the value of the functional (55) at the beginning of the cycle and by $I_2$ the value of the functional (55) at the end of the cycle. These values were required to be consistent with the inequality.

$$I_2 < I_1.$$ \hspace{1cm} (151)

If satisfaction occurs, the next cycle starts. If violation occurs, the gradient stepsize of the previous cycle is bisected as many times as needed until, after restoration, Ineq. (151) is satisfied.

Nonconvergence Conditions. The sequential gradient-restoration algorithm was programmed to stop whenever violation of any of the following inequalities occurred:

$$N \leq 50,$$ \hspace{1cm} (152a)

$$N_c \leq 30,$$ \hspace{1cm} (152b)

$$N_r \leq 15.$$ \hspace{1cm} (152c)
\[ N_{bg} \leq 10, \quad (152d) \]
\[ N_{br} \leq 10, \quad (152e) \]
\[ N_{bc} \leq 10. \quad (152f) \]

Here, \( N \) is the total number of iterations; \( N_C \) is the number of cycles; \( N_r \) is the number of restorative iterations for cycle; \( N_{bg} \) is the number of bisections of the gradient stepsize required to satisfy Ineqs. (149); \( N_{br} \) is the number of bisections of the restoration stepsize required to satisfy Ineq. (150); and \( N_{bc} \) is the number of bisections of the gradient stepsize required to satisfy Ineq. (151).
7. **NUMERICAL EXAMPLES**

In this section, four examples are described employing scalar notation. In particular, the symbols $x_i(t), i=1,2,...,n$, denote the components of the augmented state vector; the symbols $u_i(t), i=1,2,...,m$, denote the components of the augmented control vector; and the symbols $\pi_i, i=1,2,...,p$, denote the components of the augmented parameter vector.

**Example 7.1.** This example involves a minimax function of order $k=1$. Problem (Q1) is as follows: Minimize the functional

$$I = \max_\theta F, \quad F = x_2 \theta, \quad 0 \leq \theta \leq 1,$$

(153)

subject to the constraints

$$dx_1/d\theta = 5x_2, \quad 0 \leq \theta \leq 1,$$

(154a)

$$dx_2/d\theta = 5\sin u_1, \quad 0 \leq \theta \leq 1,$$

(154b)

$$x_1(0) = -4, \quad x_2(0) = 0,$$

(155)

$$x_1(1) = 0, \quad x_2(1) = 0.$$

(156)

For this example, the analytical solution is given in Ref. 19 and Table 1. The minimum value of the functional (153) is

$$I = 0.50231.$$

(157)
Single-Subarc Approach. We introduce the auxiliary state variable \( x_3(\Theta) \), the auxiliary control variable \( u_2(\Theta) \), and the auxiliary parameter \( \pi_1 \) defined by

\[
\pi_1 - x_2 \Theta = x_3^2, \tag{158a}
\]

\[
dx_3/d\Theta = 5u_2, \tag{158b}
\]

together with the time transformation

\[
\Theta = \tau, \quad 0 \leq \tau \leq 1. \tag{159}
\]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (153)-(156) as follows:

\[
I = \pi_1, \tag{160}
\]

\[
\dot{x}_1 = 5x_2, \quad 0 \leq \tau \leq 1, \tag{161a}
\]

\[
\dot{x}_2 = 5\sin u_1, \quad 0 \leq \tau \leq 1, \tag{161b}
\]

\[
\dot{x}_3 = 5u_2, \quad 0 \leq \tau \leq 1, \tag{161c}
\]

\[
x_2 + 5\tau \sin u_1 + 10x_3u_2 = 0, \quad 0 \leq \tau \leq 1, \tag{162}
\]

\[
x_1(0) = -4, \quad x_2(0) = 0, \quad x_3(0) = \pi_1, \tag{163}
\]
\[ x_1(t) = 0, \quad x_2(t) = 0. \]  

(164)

The numerical results are given in Table 2. Convergence to the desired stopping conditions (146) was achieved in N=35 iterations, which include 22 restorative iterations and 13 gradient iterations. The value achieved for the functional (160) is

\[ I = 0.50318, \]  

(165)

which is within 2/1000 of the analytically predicted minimum value (157).

**Multiple-Subarc Approach.** Inspection of Table 2 shows that the extremal arc exhibits the following characteristics: (i) the initial portion is on the control upper boundary, \( u_1 = \pi/2 \); (ii) the central portion is on the state boundary, \( u_2 = 0 \) and \( x_3 = 0 \); and (iii) the final portion is on the control lower boundary, \( u_3 = -\pi/2 \). Starting from the above observations, we attempt a three-subarc approach, in which we exploit (i), (ii), (iii) in the formulation of the problem and/or the choice of the nominal functions.

We introduce the auxiliary state variable \( x_3(\theta) \), the auxiliary control variable \( u_2(\theta) \), and the auxiliary parameter \( \pi_1 \) defined by (158), together with the time transformation

\[ \Theta = \pi_2 t, \quad 0 \leq t \leq 1, \]  

(166a)

Note that \( \pi_2 = \theta_1 \) and \( \pi_3 = \theta_2 \).
\[ \theta = \pi_2 + (\pi_3 - \pi_2)(t-1), \quad 1 \leq t \leq 2, \quad (166b) \]

\[ \theta = \pi_3 + (1-\pi_3)(t-2), \quad 2 \leq t \leq 3. \quad (166c) \]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (153)-(156) as follows:

\[ I = \pi_1, \quad (167) \]

\[ \dot{x}_1 = 5 \pi_2 x_2, \quad 0 \leq t \leq 1, \quad (168a) \]

\[ \dot{x}_1 = 5 (\pi_3 - \pi_2) x_2, \quad 1 \leq t \leq 2, \quad (168b) \]

\[ \dot{x}_1 = 5 (1-\pi_3) x_2, \quad 2 \leq t \leq 3, \quad (168c) \]

\[ \dot{x}_2 = 5 \pi_2 \sin u_1, \quad 0 \leq t \leq 1, \quad (169a) \]

\[ \dot{x}_2 = 5 (\pi_3 - \pi_2) \sin u_1, \quad 1 \leq t \leq 2, \quad (169b) \]

\[ \dot{x}_2 = 5 (1-\pi_3) \sin u_1, \quad 2 \leq t \leq 3, \quad (169c) \]

\[ \dot{x}_3 = 5 \pi_2 u_2, \quad 0 \leq t \leq 1, \quad (170a) \]

\[ \dot{x}_3 = 0, \quad 1 \leq t \leq 2, \quad (170b) \]

\[ \dot{x}_3 = 5 (1-\pi_3) u_2, \quad 2 \leq t \leq 3, \quad (170c) \]
\[ x_2 + 5 \pi_2 t \sin \theta_1 + 10x_3 u_2 = 0, \quad 0 \leq t \leq 1, \quad (171a) \]

\[ x_2 + 5 \left[ \pi_2 + (\pi_3 - \pi_2)(t-1) \right] \sin \theta_1 = 0, \quad 1 \leq t \leq 2, \quad (171b) \]

\[ x_2 + 5 \left[ \pi_3 + \left( 1 - \pi_3 \right)(t-2) \right] \sin \theta_1 + 10x_3 u_2 = 0, \quad 2 \leq t \leq 3, \quad (171c) \]

\[ x_1(0) = -4, \quad x_2(0) = 0, \quad x_3^2(0) = \pi_1, \quad (172) \]

\[ \pi_1 - x_2(1) \pi_2 - \varepsilon^2 = 0, \quad (173) \]

\[ x_1(3) = 0, \quad x_2(3) = 0. \quad (174) \]

Remark 7.1. Equation (173) constitutes the entrance condition into the middle subarc. It arises from the application of Eqs. (158a) and (166) at \( t=1 \), after an approximation is employed: the replacement of the exact condition \( x_3(1)=0 \) with the approximate condition \( x_3(1)=\varepsilon \), where \( \varepsilon \) is a small, positive number. This is done in order to avoid dissatisfaction of disequation (145) at \( t=1 \). As a consequence of this approximation, the value of the functional \( I \) is overestimated by the amount \( \varepsilon^2 \). The correct minimum value can be recovered a posteriori using the relation

\[ I' = I - \varepsilon^2. \quad (175) \]

Nominal Functions. The following nominal functions were assumed for Example 7.1:
\[ x_1 = -4 + 4t/3, \quad 0 \leq t \leq 1, \quad (176a) \]

\[ x_1 = -4 + 4t/3, \quad 1 \leq t \leq 2, \quad (176b) \]

\[ x_1 = -4 + 4t/3, \quad 2 \leq t \leq 3, \quad (176c) \]

\[ x_2 = 1.6t, \quad 0 \leq t \leq 1, \quad (177a) \]

\[ x_2 = 1.6, \quad 1 \leq t \leq 2, \quad (177b) \]

\[ x_2 = 1.6(3-t), \quad 2 \leq t \leq 3, \quad (177c) \]

\[ x_3 = \sqrt{1+e^2} + [\sqrt{1+e^2} - e]t, \quad 0 \leq t \leq 1, \quad (178a) \]

\[ x_3 = 0, \quad 1 \leq t \leq 2, \quad (178b) \]

\[ x_3 = \sqrt{1+e^2} - e]t + 3e - 2\sqrt{1+e^2}, \quad 2 \leq t \leq 3, \quad (178c) \]

\[ U_1 = \pi/2, \quad 0 \leq t \leq 1, \quad (179a) \]

\[ U_1 = 0, \quad 1 \leq t \leq 2, \quad (179b) \]

\[ U_1 = -\pi/2, \quad 2 \leq t \leq 3, \quad (179c) \]

\[ U_2 = -0.5t, \quad 0 \leq t \leq 1, \quad (180a) \]
\[ U_2 = 0, \quad 1 \leq t \leq 2, \quad (180b) \]

\[ U_2 = 0.5 (3-t), \quad 2 \leq t \leq 3, \quad (180c) \]

\[ \pi_1 = 0.6 + \epsilon^2, \quad \pi_2 = 0.4, \quad \pi_3 = 0.9, \quad (181) \]

with

\[ \epsilon = 0.1. \quad (182) \]

The numerical results are given in Table 3. Convergence to the desired stopping conditions (146) was achieved in \( N = 7 \) iterations, all of the restorative type. The value achieved for the functional (175) is

\[ I' = 0.50231, \quad (183) \]

which is identical with the analytically predicted minimum value (157).

**Example 7.2.** This example involves a minimax function of order \( k = 1 \). Problem (Q1) is as follows: Minimize the functional

\[ I = \max_{\Theta} F, \quad F = x_2, \quad 0 \leq \Theta \leq 1, \quad (184) \]

subject to the constraints

\[ \frac{dx_1}{d\Theta} = 5x_2, \quad 0 \leq \Theta \leq 1, \quad (185a) \]

\[ \frac{dx_2}{d\Theta} = 5\sin u_1, \quad 0 \leq \Theta \leq 1, \quad (185b) \]
\[ x_1(0) = -4, \quad x_2(0) = 0, \quad (186) \]
\[ x_1(t) = 0, \quad x_2(t) = 0. \quad (187) \]

For this example, the analytical solution is given in Ref. 19 and Table 4.

The minimum value of the functional (184) is

\[ I = 1.00000. \quad (188) \]

**Single-Subarc Approach.** We introduce the auxiliary state variable \( x_3(\theta) \), the auxiliary control variable \( u_2(\theta) \), and the auxiliary parameter \( \pi_1 \) defined by

\[ \pi_1 - x_2 = x_3^2, \quad (189a) \]
\[ dx_3/d\theta = 5u_2, \quad (189b) \]

together with the time transformation

\[ \theta = t, \quad 0 \leq t \leq 1. \quad (190) \]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (184)-(187) as follows:
\[ I = \pi_1 , \]  \hspace{1cm} (191)

\[ x_1' = 5x_2 , \quad 0 \leq t \leq 1, \]  \hspace{1cm} (192a)

\[ x_2' = 5 \sin u_1 , \quad 0 \leq t \leq 1, \]  \hspace{1cm} (192b)

\[ x_3' = 5u_2 , \quad 0 \leq t \leq 1, \]  \hspace{1cm} (192c)

\[ \sin u_1 + 2x_3u_2 = 0 , \quad 0 \leq t \leq 1, \]  \hspace{1cm} (193)

\[ x_1(0) = -4 , \quad x_2(0) = 0 , \quad x_3(0) = \pi_1 , \]  \hspace{1cm} (194)

\[ x_1(1) = 0 , \quad x_2(1) = 0 . \]  \hspace{1cm} (195)

The numerical results are given in Table 5. The algorithm was unable to achieve the desired stopping conditions (146) in a number of iterations consistent with Ineq. (152a). However, it was able to achieve the relaxed stopping conditions \( P \leq 10^{-8} \) and \( Q \leq 10^{-3} \) in \( N=17 \) iterations, which include 11 restorative iterations and 6 gradient iterations. The value achieved for the functional (191) is

\[ I = 1.00306 , \]  \hspace{1cm} (196)

which is within \( 3/1000 \) of the analytically predicted minimum value (188).
Multiple-Subarc Approach. Inspection of Table 5 shows that the extremal arc exhibits the following characteristics: (i) the initial portion is on the control upper boundary, \( u_1 = \pi/2 \); (ii) the central portion is on the state boundary, \( u_2 = 0 \) and \( x_3 = 0 \); and (iii) the final portion is on the control lower boundary, \( u_1 = -\pi/2 \). Starting from the above observations, we attempt a three-subarc approach, in which we exploit (i), (ii), (iii) in the formulation of the problem and/or the choice of the nominal functions.

We introduce the auxiliary state variable \( x_3(\theta) \), the auxiliary control variable \( u_2(\theta) \), and the auxiliary parameter \( \pi_1 \) defined by (189), together with the time transformation\(^6\)

\[
\begin{align*}
\theta &= \pi_2 t, & 0 \leq t \leq 1, \\
\theta &= \pi_2 + (\pi_3 - \pi_2)(t-1), & 1 \leq t \leq 2, \\
\theta &= \pi_3 + (1 - \pi_3)(t-2), & 2 \leq t \leq 3.
\end{align*}
\]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (184)-(187) as follows:

\[
I = \pi_1 ,
\]

\(^6\)Note that \( \pi_2 = \theta_1 \) and \( \pi_3 = \theta_2 \).
\[ \dot{x}_1 = 5 \pi_2 x_2 , \quad 0 \leq t \leq 1, \quad (199a) \]

\[ \dot{x}_1 = 5 (\pi_3 - \pi_2) x_2 , \quad 1 \leq t \leq 2, \quad (199b) \]

\[ \dot{x}_1 = 5 (1 - \pi_3) x_2 , \quad 2 \leq t \leq 3, \quad (199c) \]

\[ \dot{x}_2 = 5 \pi_2 \sin u_1 , \quad 0 \leq t \leq 1, \quad (200a) \]

\[ \dot{x}_2 = 5 (\pi_3 - \pi_2) \sin u_1 , \quad 1 \leq t \leq 2, \quad (200b) \]

\[ \dot{x}_2 = 5 (1 - \pi_3) \sin u_1 , \quad 2 \leq t \leq 3, \quad (200c) \]

\[ \dot{x}_3 = 5 \pi_2 u_2 , \quad 0 \leq t \leq 1, \quad (201a) \]

\[ \dot{x}_3 = 0 , \quad 1 \leq t \leq 2, \quad (201b) \]

\[ \dot{x}_3 = 5 (1 - \pi_3) u_2 , \quad 2 \leq t \leq 3, \quad (201c) \]

\[ \sin u_1 + 2x_3 u_2 = 0 , \quad 0 \leq t \leq 1, \quad (202a) \]

\[ \sin u_1 = 0 , \quad 1 \leq t \leq 2, \quad (202b) \]

\[ \sin u_1 + 2x_3 u_2 = 0 , \quad 2 \leq t \leq 3, \quad (202c) \]

\[ x_1(0) = -4 , \ x_2(0) = 0 , \ x_3^2(0) = \pi_1 , \quad (203) \]
\[ \Pi_1 - x_2 (1) - \varepsilon^2 = 0, \quad (204) \]
\[ x_1(3) = 0, \quad x_2(3) = 0. \quad (205) \]

**Remark 7.2.** Equation (204) constitutes the entrance condition into the middle subarc. It arises from the application of Eqs. (189a) and (197) at \( t=1 \), after an approximation is employed: the replacement of the exact condition \( x_3(1)=0 \) with the approximate condition \( x_3(1)=\varepsilon \), where \( \varepsilon \) is a small, positive number. This is done in order to avoid dissatisfaction of disequation (145) at \( t=1 \). As a consequence of this approximation, the value of the functional \( I \) is overestimated by the amount \( \varepsilon^2 \). The correct minimum value can be recovered a posteriori using the relation

\[ I' = I - \varepsilon^2. \quad (206) \]

**Nominal Functions.** The following nominal functions were assumed for Example 7.2:

\[ x_1 = -4 + 4 \frac{t}{3}, \quad 0 \leq t \leq 1, \quad (207a) \]
\[ x_1 = -4 + 4 \frac{t}{3}, \quad 1 \leq t \leq 2, \quad (207b) \]
\[ x_1 = -4 + 4 \frac{t}{3}, \quad 2 \leq t \leq 3, \quad (207c) \]
\[ x_2 = t, \quad 0 \leq t \leq 1, \quad (208a) \]

\[ x_2 = 1, \quad 1 \leq t \leq 2, \quad (208b) \]

\[ x_2 = 3 - t, \quad 2 \leq t \leq 3, \quad (208c) \]

\[ x_3 = \sqrt{1 + \varepsilon^2} + \left[ \varepsilon - \sqrt{1 + \varepsilon^2} \right] t, \quad 0 \leq t \leq 1, \quad (209a) \]

\[ x_3 = \varepsilon, \quad 1 \leq t \leq 2, \quad (209b) \]

\[ x_3 = \left[ \sqrt{1 + \varepsilon^2} - \varepsilon \right] t + 3 \varepsilon - 2 \sqrt{1 + \varepsilon^2}, \quad 2 \leq t \leq 3, \quad (209c) \]

\[ u_1 = \pi / 2, \quad 0 \leq t \leq 1, \quad (210a) \]

\[ u_1 = 0, \quad 1 \leq t \leq 2, \quad (210b) \]

\[ u_1 = -\pi / 2, \quad 2 \leq t \leq 3, \quad (210c) \]

\[ u_2 = -1, \quad 0 \leq t \leq 1, \quad (211a) \]

\[ u_2 = 0, \quad 1 \leq t \leq 2, \quad (211b) \]

\[ u_2 = 1, \quad 2 \leq t \leq 3, \quad (211c) \]

\[ \pi_1 = 1 + \varepsilon^2, \quad \pi_2 = 0.3, \quad \pi_3 = 0.7, \quad (212) \]

with
The numerical results are given in Table 6. Convergence to the desired stopping conditions (146) was achieved in N=9 iterations, all of the restorative type. The value achieved for the functional (206) is

\[ I' = 1.00000, \quad (214) \]

which is identical with the analytically predicted minimum value (188).

**Example 7.3.** This example involves a minimax function of order \( k=2 \). Problem (Q1) is as follows: Minimize the functional

\[ I = \max_\theta F, \quad F = x_1, \quad 0 \leq \theta \leq 1, \quad (215) \]

subject to the constraints

\[ \frac{dx_1}{d\theta} = 5x_2, \quad 0 \leq \theta \leq 1, \quad (216a) \]

\[ \frac{dx_2}{d\theta} = 5\sin u_1, \quad 0 \leq \theta \leq 1, \quad (216b) \]

\[ x_1(0) = 0, \quad x_2(0) = 1, \quad (217) \]

\[ x_1(1) = 0, \quad x_2(1) = -1. \quad (218) \]
For this example, the analytical solution is given in Ref. 19 and Table 7. The minimum value of the functional (215) is

\[ I = 0.50000. \quad (219) \]

**Single-Subarc Approach.** We introduce the auxiliary state variables \( x_3(\theta), x_4(\theta) \), the auxiliary control variable \( u_2(\theta) \), and the auxiliary parameter \( \pi_1 \) defined by

\[ \pi_1 - x_1 = x_3^2, \quad (220a) \]
\[ \frac{dx_3}{d\theta} = 5x_4, \quad (220b) \]
\[ \frac{dx_4}{d\theta} = 5u_2, \quad (220c) \]

together with the time transformation

\[ \Theta = t, \quad 0 \leq t \leq 1. \quad (221) \]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (215)-(218) as follows:

\[ I = \pi_1, \quad (222) \]
\[ x_1 = 5x_2, \quad 0 \leq t \leq 1, \quad (223a) \]
\[
x_2' = 5 \sin u_1, \\
x_3' = 5 x_4, \\
x_4' = 5 u_2, \\
\sin u_1 + 2 x_4^2 + 2 x_3 u_2 = 0,
\]

\[0 \leq t \leq 1, \quad (223b)\]
\[0 \leq t \leq 1, \quad (223c)\]
\[0 \leq t \leq 1, \quad (223d)\]
\[0 \leq t \leq 1, \quad (224)\]
\[x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = \pi_1, \quad 2 x_3(0) x_4(0) = -1, \quad (225)\]
\[x_1(1) = 0, \quad x_2(1) = -1. \quad (226)\]

If the Miele-Wu-Liu transformation is used (Ref. 13), the dimension of the state vector can be reduced from n=4 to n=2. This is achieved by reformulating problem (222)-(226) as follows:

\[
I = \pi_1, \\
x_3' = 5 x_4, \\
x_4' = 5 u_2, \\
\sin u_1 + 2 x_4^2 + 2 x_3 u_2 = 0, \\
\]

\[0 \leq t \leq 1, \quad (228a)\]
\[0 \leq t \leq 1, \quad (228b)\]
\[0 \leq t \leq 1, \quad (229)\]
\[x_3(0) = \pi_1, \quad 2 x_3(0) x_4(0) = -1, \quad (230)\]
Once problem (227)-(231) is solved, the eliminated state variables \( x_1(t), x_2(t) \) can be computed a posteriori using the relations

\[
\begin{align*}
  x_1 &= \Pi_1 - x_2^2, \\
  x_2 &= -2x_3x_4.
\end{align*}
\]  

The numerical results for problem (227)-(231) are given in Table 8. Convergence to the desired stopping conditions (146) was achieved in \( N=31 \) iterations, which include 19 restorative iterations and 12 gradient iterations. The value achieved for the functional (227) is

\[
I = 0.50040,
\]  

which is within \( 8/10,000 \) of the analytically predicted minimum value (219).

**Multiple-Subarc Approach.** Inspection of Table 8 shows that the extremal arc exhibits the following characteristics: (i) the initial portion is on the control lower boundary, \( u_1 = -\pi/2 \); (ii) the central portion is on the state boundary, \( u_2 = 0, x_3 = 0, x_4 = 0 \); and (iii) the final portion is on the control lower boundary, \( u_1 = -\pi/2 \). Starting from the above observations, we attempt a three-subarc approach in which we exploit
(i), (ii), (iii) in the formulation of the problem and/or the choice of
the nominal functions.

We introduce the auxiliary state variables $x_3(\theta), x_4(\theta)$, the auxiliary
control variable $u_2(\theta)$, and the auxiliary parameter $\pi_1$ defined by (220),
together with the time transformation\(^7\)

$$\Theta = \pi_2 t, \quad 0 \leq t \leq 1, \quad (234a)$$

$$\Theta = \pi_2 + (\pi_3 - \pi_2)(t-1), \quad 1 \leq t \leq 2, \quad (234b)$$

$$\Theta = \pi_3 + (1-\pi_3)(t-2), \quad 2 \leq t \leq 3. \quad (234c)$$

Then, we proceed in accordance with Sections 2-3, and we reformulate
problem (227)-(231) as follows:

$$I = \pi_1, \quad (235)$$

$$\dot{x}_3 = 5\pi_2 x_4, \quad 0 \leq t \leq 1, \quad (236a)$$

$$\dot{x}_3 = 5(\pi_3 - \pi_2) x_4, \quad 1 \leq t \leq 2, \quad (236b)$$

$$\dot{x}_3 = 5(1-\pi_3) x_4, \quad 2 \leq t \leq 3, \quad (236c)$$

\(^7\)Note that $\pi_2 = \Theta_1$ and $\pi_3 = \Theta_2$. 
\[
\begin{align*}
\dot{x}_4 &= 5\pi_2 u_2, & 0 \leq t \leq 1, & (237a) \\
\dot{x}_4 &= 0, & 1 \leq t \leq 2, & (237b) \\
\dot{x}_4 &= 5(1-\pi_3) u_2, & 2 \leq t \leq 3, & (237c) \\
\sin u_1 + 2x_4^2 + 2x_3 u_2 &= 0, & 0 \leq t \leq 1, & (238a) \\
\sin u_1 + 2x_4^2 &= 0, & 1 \leq t \leq 2, & (238b) \\
\sin u_1 + 2x_4^2 + 2x_3 u_2 &= 0, & 2 \leq t \leq 3, & (238c) \\
X_3^2(0) &= \pi_1, & 2X_3(0)x_4(0) &= -1, & (239) \\
X_3(1) - \varepsilon &= 0, & (240) \\
X_3(3) &= \pi_1, & 2X_3(3)x_4(3) &= 1. & (241)
\end{align*}
\]

Once problem (235)-(241) is solved, the eliminated state variables \(x_1(t), x_2(t)\) are computed a posteriori using (232).

**Remark 7.3.** Equation (240) constitutes the entrance condition into the middle subarc. It arises from the replacement of the exact condition \(x_3(1)=0\) with the approximate condition \(x_3(1)=\varepsilon\), where \(\varepsilon\) is a small, positive number. This is done in order to avoid dissatisfaction of dis-
equation (145) at \( t=1 \). As a consequence of this approximation, the value of the functional \( I \) is overestimated by the amount \( \epsilon^2 \). The correct minimum value can be recovered a posteriori using the relation

\[
I' = I - \epsilon^2. \tag{242}
\]

**Nominal Functions.** The following nominal functions were assumed for Example 7.3:

\[
x_3 = (\epsilon - \sqrt{\pi_1}) t + \sqrt{\pi_1}, \quad 0 \leq t \leq 1, \tag{243a}
\]

\[
x_3 = \epsilon, \quad 1 \leq t \leq 2, \tag{243b}
\]

\[
x_3 = (\sqrt{\pi_1} - \epsilon) t + 3\epsilon - 2\sqrt{\pi_1}, \quad 2 \leq t \leq 3, \tag{243c}
\]

\[
x_4 = -\frac{1}{2} t^2, \quad 0 \leq t \leq 1, \tag{244a}
\]

\[
x_4 = 0, \quad 1 \leq t \leq 2, \tag{244b}
\]

\[
x_4 = -\frac{1}{2} t^2, \quad 2 \leq t \leq 3, \tag{244c}
\]

\[
u_1 = \frac{-\pi}{2}, \quad 0 \leq t \leq 1, \tag{245a}
\]

\[
u_1 = 0, \quad 1 \leq t \leq 2, \tag{245b}
\]
\begin{align*}
U_1 &= -\pi/2, \\
&\quad 2 \leq t \leq 3, \quad \text{(245c)} \\
U_2 &= -1, \\
&\quad 0 \leq t \leq 1, \quad \text{(246a)} \\
U_2 &= 0, \\
&\quad 1 \leq t \leq 2, \quad \text{(246b)} \\
U_2 &= 1, \\
&\quad 2 \leq t \leq 3, \quad \text{(246c)} \\
\pi_1 &= 1.60, \quad \pi_2 = 0.25, \quad \pi_3 = 0.90, \quad \text{(247)}
\end{align*}

with

\[ \epsilon = 1. \quad \text{(248)} \]

The numerical results for problem (235)-(241) are given in Table 9. Convergence to the desired stopping conditions (146) was achieved in \( N = 5 \) iterations, all of the restorative type. The value achieved for the functional (242) is

\[ I' = 0.50000, \quad \text{(249)} \]

which is identical with the analytically predicted minimum value (219).

Example 7.4. This example involves a minimax function of order \( k = 0 \). Problem (Q2) is as follows: Minimize the functional
\[ I = \max_\theta F, \quad F = x_2^2 + u_1^2, \quad 0 \leq \theta \leq 1, \quad (250) \]

subject to the constraints

\[ \frac{dx_1}{d\theta} = x_2, \quad 0 \leq \theta \leq 1, \quad (251a) \]

\[ \frac{dx_2}{d\theta} = u_1, \quad 0 \leq \theta \leq 1, \quad (251b) \]

\[ x_1(0) = 0, \quad x_2(0) = 0, \quad (252) \]

\[ x_1(1) = 1, \quad x_2(1) = 0 \quad (253) \]

For this example, the analytical solution is given in Ref. 19 and Table 10. The minimum value of the functional (250) is

\[ I = 16.6822. \quad (254) \]

**Single-Subarc Approach.** We introduce the auxiliary control variable \( u_2(\theta) \) and the auxiliary parameter \( \pi_1 \) defined by

\[ \pi_1 - x_2^2 - u_1^2 = u_2^2, \quad (255) \]

together with the time transformation

\[ \theta = t, \quad 0 \leq t \leq 1. \quad (256) \]
Then, we proceed in accordance with Sections 2-3, and we reformulate problem \((250)-(253)\) as follows:

\[
\begin{align*}
I &= \pi_1, \tag{257} \\
\dot{x}_1 &= x_2, \quad 0 \leq t \leq 1, \tag{258a} \\
\dot{x}_2 &= u_1, \quad 0 \leq t \leq 1, \tag{258b} \\
\pi_1 - x_2^2 - u_1^2 - u_2^2 &= 0, \quad 0 \leq t \leq 1, \tag{259} \\
x_1(0) &= 0, \quad x_2(0) = 0, \tag{260} \\
x_1(1) &= 1, \quad x_2(1) = 0. \tag{261}
\end{align*}
\]

The numerical results are given in Table 11. The algorithm was unable to achieve the desired stopping conditions \((146)\) in a number of iterations consistent with Ineq. \((152a)\). However, it was able to achieve the relaxed stopping conditions \(P \leq E^{-08}\) and \(Q \leq E^{-02}\) in \(N=23\) iterations, which include 15 restorative iterations and 8 gradient iterations. The value achieved for the functional \((257)\) is

\[
I = 16.7007. \tag{262}
\]

which is within 7/1000 of the analytically predicted minimum value \((254)\).
Multiple-Subarc Approach. Inspection of Table 11 shows that the extremal arc exhibits the following characteristics: (i) the initial portion is on the control boundary, \( u_2 = 0 \); and (ii) the final portion is on the control boundary, \( u_2 = 0 \). Starting from these observations, we attempt a two-subarc approach, in which we exploit (i) and (ii) in the formulation of the problem and/or the choice of the nominal functions.

We introduce the time transformation\(^8\)

\[
\Theta = \pi_2 t, \quad 0 \leq t \leq 1, \quad (263a)
\]

\[
\Theta = \pi_2 + (1-\pi_2)(t-1), \quad 1 \leq t \leq 2. \quad (263b)
\]

Then, we proceed in accordance with Sections 2-3, and we reformulate problem (250)-(253) as follows:

\[
i = \pi_1, \quad (264)
\]

\[
x_1' = \pi_2 x_2, \quad 0 \leq t \leq 1, \quad (265a)
\]

\[
x_1' = (1-\pi_2) x_2, \quad 1 \leq t \leq 2, \quad (265b)
\]

\[
x_2' = \pi_2 u_1, \quad 0 \leq t \leq 1, \quad (266a)
\]

\(^8\)Note that \( \pi_2 = \theta_1 \).
\[ X_2 = (1 - \pi_2) u_1 , \quad 1 \leq t \leq 2, \quad (266b) \]

\[ \pi_1 - x_2^2 - u_1^2 = 0 , \quad 0 \leq t \leq 1, \quad (267a) \]

\[ \pi_1 - x_2^2 - u_1^2 = 0 , \quad 1 \leq t \leq 2, \quad (267b) \]

\[ x_1(0) = 0 , \quad x_2(0) = 0 , \quad (268) \]

\[ x_1(2) = 1 , \quad x_2(2) = 0 . \quad (269) \]

**Nominal Functions.** The following nominal functions were assumed for Example 7.4:

\[ x_1 = 0.5 t , \quad 0 \leq t \leq 1, \quad (270a) \]

\[ x_1 = 0.5 t , \quad 1 \leq t \leq 2, \quad (270b) \]

\[ x_2 = 2 t , \quad 0 \leq t \leq 1, \quad (271a) \]

\[ x_2 = -2 t + 4 , \quad 1 \leq t \leq 2, \quad (271b) \]

\[ u_1 = 4 , \quad 0 \leq t \leq 1, \quad (272a) \]

\[ u_1 = -4 , \quad 1 \leq t \leq 2, \quad (272b) \]

\[ \pi_1 = 16 , \quad \pi_2 = 0.4 . \quad (273) \]
The numerical results are given in Table 12. Convergence to the desired stopping conditions (146) was achieved in \( N = 4 \) iterations, all of the restorative type. The value achieved for the functional (264) is

\[
I = 16.6822, \tag{274}
\]

which is identical with the analytically predicted minimum value (254).
8. DISCUSSION AND CONCLUSIONS

Numerical solutions of minimax problems of optimal control are obtained through a multiple-subarc approach, used as a sequel to a single-subarc approach. The problems are solved by means of the sequential gradient-restoration algorithm.

First, a transformation technique is employed in order to convert minimax problems of optimal control into the Mayer-Bolza problem of the calculus of variations. The transformation requires the proper augmentation of the state vector $x(t)$, the control vector $u(t)$, and the parameter vector $\pi$. As a result of the transformation, the unknown minimax value of the performance index becomes a component of the vector parameter $\pi$ being optimized. The transformation technique is then employed in conjunction with the sequential gradient-restoration algorithm for solving optimal control problems on a digital computer.

The algorithm developed in this thesis belongs to the class of sequential gradient-restoration algorithms. The sequential gradient-restoration algorithm is made up of a sequence of two-phase cycles, each cycle consisting of a gradient phase and a restoration phase. The principal property of this algorithm is that it produces a sequence of feasible suboptimal solutions. Each feasible solution is characterized by a lower value of the minimax performance index than any previous feasible solution. To facilitate numerical implementation, the intervals of integration are normalized to unit length.
Four numerical examples are presented to illustrate the present approach. For comparison purposes, the analytical solutions, the single-subarc solutions, and the multiple-subarc solutions are presented. These multiple-subarc solutions include three subarcs for Examples 7.1, 7.2, 7.3 and two-subarcs for Example 7.4.

It is found that the combination of transformation technique and sequential gradient-restoration algorithm yields numerical solutions which are quite close to the analytical solutions. The relative differences between the numerical values of the minimax performance index and the analytical values are of order $10^{-3}$ if the single-subarc approach is employed and of order $10^{-5}$ or better if the multiple-subarc approach is employed. Thus, it appears that, by the combined use of the transformation technique and the sequential gradient-restoration algorithm, some rather complex problems have been solved in a relatively simple way.
Table 1. Analytical solution for Example 7.1.

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<th>( u_1 )</th>
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Table 3. Three-subarc solution for Example 7.1, $\varepsilon=0.1$.

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Table 4. Analytical solution for Example 7.2.

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Table 5. Single-subarc solution for Example 7.2.

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Table 6. Three-subarc solution for Example 7.2, $\varepsilon = 0.1$.

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Table 7. Analytical solution for Example 7.3.

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Table 8. Single-subarc solution for Example 7.3.

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<th>x₂</th>
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π₁ = 0.50040, I = 0.50040
Table 9. Three-subarc solution for Example 7.3, \(\epsilon = 1\).

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\(\pi_1 = 1.50000, \quad \pi_2 = 0.20001, \quad \pi_3 = 0.80003, \quad I = 1.50000, \quad I' = 0.50000\)
Table 10. Analytical solution for Example 7.4.

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Table 11. Single-subarc solution for Example 7.4.

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<tr>
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<td>4.0662</td>
<td>0.0015</td>
<td>0.0204</td>
<td>0.4079</td>
<td>16.7007</td>
</tr>
<tr>
<td>0.2</td>
<td>4.0052</td>
<td>0.0000</td>
<td>0.0814</td>
<td>0.8118</td>
<td>16.7007</td>
</tr>
<tr>
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<td>3.9041</td>
<td>0.0000</td>
<td>0.1825</td>
<td>1.2076</td>
<td>16.7007</td>
</tr>
<tr>
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<td>3.7640</td>
<td>-0.0052</td>
<td>0.3225</td>
<td>1.5914</td>
<td>16.7007</td>
</tr>
<tr>
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<td>3.6046</td>
<td>0.4999</td>
<td>1.9254</td>
<td>3.7073</td>
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<tr>
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<td>-3.7640</td>
<td>-0.0052</td>
<td>0.6774</td>
<td>1.5914</td>
<td>16.7007</td>
</tr>
<tr>
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<td>-3.9041</td>
<td>0.0000</td>
<td>0.8174</td>
<td>1.2076</td>
<td>16.7007</td>
</tr>
<tr>
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<td>-4.0052</td>
<td>0.0000</td>
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<td>16.7007</td>
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<tr>
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<td>0.0015</td>
<td>0.9795</td>
<td>0.4079</td>
<td>16.7007</td>
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<tr>
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<td>-4.0856</td>
<td>0.0900</td>
<td>1.0000</td>
<td>0.0000</td>
<td>16.6926</td>
</tr>
</tbody>
</table>

$\pi_1 = 16.7007, \quad I = 16.7007$
Table 12. Two-subarc solution for Example 7.4.

<table>
<thead>
<tr>
<th>t</th>
<th>θ</th>
<th>u₁</th>
<th>x₁</th>
<th>x₂</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>4.0843</td>
<td>0.0000</td>
<td>0.0000</td>
<td>16.6822</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1000</td>
<td>4.0639</td>
<td>0.0204</td>
<td>0.4077</td>
<td>16.6822</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2000</td>
<td>4.0029</td>
<td>0.0814</td>
<td>0.8114</td>
<td>16.6822</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3000</td>
<td>3.9019</td>
<td>0.1824</td>
<td>1.2070</td>
<td>16.6822</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4000</td>
<td>3.7619</td>
<td>0.3224</td>
<td>1.5905</td>
<td>16.6822</td>
</tr>
<tr>
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<td>3.5843</td>
<td>0.5000</td>
<td>1.9581</td>
<td>16.6822</td>
</tr>
</tbody>
</table>

| 1.0 | 0.5000 | -3.5843 | 0.5000 | 1.9581 | 16.6822 |
| 1.2 | 0.6000 | -3.7619 | 0.6775 | 1.5905 | 16.6822 |
| 1.4 | 0.7000 | -3.9019 | 0.8175 | 1.2070 | 16.6822 |
| 1.6 | 0.8000 | -4.0029 | 0.9185 | 0.8114 | 16.6822 |
| 1.8 | 0.9000 | -4.0639 | 0.9795 | 0.4077 | 16.6822 |
| 2.0 | 1.0000 | -4.0843 | 1.0000 | 0.0000 | 16.6822 |

π₁ = 16.6822,  π₂ = 0.50000,  I = 16.6822
Table 13. Summary of results, single-subarc approach.

<table>
<thead>
<tr>
<th>Example</th>
<th>$N_c$</th>
<th>$N_r$</th>
<th>$N_g$</th>
<th>$N$</th>
<th>$P$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>13</td>
<td>35</td>
<td>0.16E-13</td>
<td>0.96E-04</td>
</tr>
<tr>
<td>7.2</td>
<td>7</td>
<td>11</td>
<td>6</td>
<td>17</td>
<td>0.91E-12</td>
<td>0.68E-03</td>
</tr>
<tr>
<td>7.3</td>
<td>13</td>
<td>19</td>
<td>12</td>
<td>31</td>
<td>0.30E-14</td>
<td>0.99E-04</td>
</tr>
<tr>
<td>7.4</td>
<td>9</td>
<td>15</td>
<td>8</td>
<td>23</td>
<td>0.51E-11</td>
<td>0.85E-02</td>
</tr>
</tbody>
</table>

Table 14. Summary of results, multiple-subarc approach.

<table>
<thead>
<tr>
<th>Example</th>
<th>$N_c$</th>
<th>$N_r$</th>
<th>$N_g$</th>
<th>$N$</th>
<th>$P$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>7</td>
<td>0</td>
<td>7</td>
<td>0.18E-08</td>
<td>0.21E-20</td>
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<tr>
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<td>9</td>
<td>0.37E-13</td>
<td>0.45E-19</td>
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<td>5</td>
<td>0.42E-08</td>
<td>0.18E-06</td>
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<td>4</td>
<td>0.63E-16</td>
<td>0.92E-29</td>
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References


