ABSTRACT

The Computation of Optimal Controls
in the Presence of
Equality Nondifferential Constraints

by

James R. Cloutier

An algorithm is developed to solve optimal control problems involving a functional $I$ subject to differential constraints, terminal constraints, and equality nondifferential constraints. The algorithm is composed of a sequence of cycles, each cycle consisting of two phases, a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the value of the functional while satisfying the constraints to first order. During this iteration, the gradient is projected onto the tangential hyperplane of the constraints, and a step is taken in the negative direction of the projection. The restoration phase involves one or more iterations and is designed to restore the constraints to a predetermined accuracy while minimizing the norm of the variations of the control and the parameter. To achieve this constraint satisfaction, quasilinearization (Newton's method) is employed.

The gradient stepsize $\alpha^g$ is chosen sufficiently small so that the restoration phase preserves the descent property of the gradient phase. This is possible due to the fact that the gradient corrections are of $O(\alpha^g)$ while the restoration corrections are of $O(\alpha^2)^g$. Thus, the value of the functional $\hat{I}$ at the end of any cycle is smaller than the value of the functional $I$ at the beginning of the cycle.
The restoration phase is terminated, or bypassed as the case may be, whenever the norm of the error in the constraints is less than its predetermined tolerance. Convergence is attained whenever both the above norm and the norm of the error in the first-order optimality conditions are less than their predetermined tolerances, respectively.

To facilitate numerical integrations, the interval of integration is normalized to unit length. Variable-time terminal conditions are transformed into fixed-time terminal conditions. The actual time $\tau$ at which the terminal boundary is reached then becomes a parameter to be optimized.

Two numerical examples substantiating the theory are presented.
ACKNOWLEDGEMENTS

The author is indebted to Dr. Angelo Miele for suggesting the topic and for stimulating discussions. He is also greatly indebted to the Naval Weapons Laboratory (NWL) for giving him financial support in his first year of graduate study and to Mr. Carlton Duke and Mr. Raymond Hughey of NWL for their efforts in regards to the obtainment of that support. In addition, he wishes to acknowledge the Office of Scientific Research, Office of Aerospace Research, United States Air Force for supplying support in the form of Grant No. AF-AFOSR-72-2185.

He would like to thank Miss Elaine Carter for typing the rough drafts and the master copy of the thesis. In addition he would like to thank Mr. Anil Aggarwal for proofreading the final copy and for useful suggestions.

Finally, and most importantly, special thanks go to his father, who always impressed upon him the importance of a good education, and to whom this thesis is dedicated.
# TABLE OF CONTENTS

1. **Introduction**
   Page 1

2. **Formulation of the Problem**
   Page 3

3. **Sequential Gradient-Restoration Algorithm**
   Page 7
   - 3.1. Gradient Phase
     Page 8
   - 3.2. Restoration Phase
     Page 10
   - 3.3. General Form of the Variations of a Cycle
     Page 12
   - 3.4. Linear, Two-Point Boundary-Value Problem
     Page 14
   - 3.5. Stepsize Determination
     Page 15
   - 3.6. Descent Property
     Page 18

4. **Numerical Computations**
   Page 19
   - Example 4.1
     Page 20
   - Example 4.2
     Page 25

5. **Summary**
   Page 30

References
Pages 33-35
1. **Introduction**

Many numerical techniques have been devised to solve the standard Bolza-type optimal control problem involving a functional $I$ subject to differential constraints and terminal constraints. Among those of the gradient type, the sequential gradient-restoration algorithm as defined by Miele et al. in Ref. 1 has proved to be very successful. This thesis extends the concepts developed in Ref. 1 to the case in which the state $x(\alpha)$, the control $u(\alpha)$, and the parameter $\pi$ must satisfy not only differential and terminal constraints, but also equality nondifferential constraints everywhere along the interval of integration.

The algorithm defined herein consists of a sequence of gradient-restoration cycles. The gradient portion of each cycle involves a single iteration and is designed to decrease the value of the functional while satisfying the constraints to first order. During this iteration, the gradient is projected onto the tangential hyperplane of the constraints, and a step is taken in the negative direction of the projection. The restoration portion of each cycle involves one or more iterations and is designed to restore the constraints to a predetermined accuracy while minimizing the norm of the variations of the control and the parameter. To achieve this constraint satisfaction, quasilinearization (Newton's method) is employed.

The algorithm is characterized by two main properties. First, at the end of each gradient-restoration cycle, the trajectories satisfy the constraints to a given accuracy. Thus, a sequence of feasible suboptimal solutions is produced. Second, due to the fact that the gradient corrections are of $O(\alpha_g)$ while the restoration corrections are of $O(\alpha_g^2)$, the gradient stepsize $\alpha_g$ can be chosen sufficiently small so that the restoration phase preserves the descent property...
of the gradient phase. Thus, the value of the functional $\hat{I}$ at the end of any cycle is smaller than the value of the functional $I$ at the beginning of the cycle.
2. **Formulation of the Problem**

The problem that we wish to solve numerically is the following. Minimize the functional

\[ I = \int_0^\tau f_*(x, u, \pi, \rho) \, d\rho + \left[ g_*(x, \pi, \rho) \right] \tau \]  

(1)

with respect to the state \( x(\rho) \), the control \( u(\rho) \), and the parameters \( \pi \) and \( \tau \), subject to the differential constraint

\[ \frac{dx}{d\rho} = \varphi_*(x, u, \pi, \rho), \quad 0 \leq \rho \leq \tau \]  

(2)

the equality nondifferential constraint

\[ E_*(x, u, \pi, \rho) = 0, \quad 0 \leq \rho \leq \tau \]  

(3)

and the boundary conditions

\[ (x)_{\rho} = \text{given}, \quad [\psi_*(x, \pi, \rho)]_\tau = 0 \]  

(4)

In the above equations, the functions \( f_* \) and \( g_* \) are scalar, the function \( \varphi_* \) is an \( n \)-vector, the function \( E_* \) is a \( k \)-vector, and the function \( \psi_* \) is a \( q \)-vector.

The independent variable is the actual time \( \rho \), while the dependent variables are the \( n \)-vector state \( x \), the \( m \)-vector control \( u \), the \( \bar{p} \)-vector parameter \( \pi \), and the scalar parameter \( \tau \). At the initial time \( \rho = 0 \), \( n \) scalar relations are specified. At the final time \( \rho = \tau \), \( q \) scalar relations are specified, where \( q \leq n + \bar{p} \) if \( \tau \) is fixed and \( q \leq n + \bar{p} + 1 \) if \( \tau \) is free.

To facilitate the implementation of the algorithm on a digital computer, we replace the actual time \( \rho \) with the normalized time \( t \). The latter is defined
in such a way that the interval of integration is of unit length. Thus, in normalized form, \( t = 0 \) denotes the time at which the initial boundary (4-1) is left and \( t = 1 \) denotes the time at which the terminal boundary (4-2) is reached. The following linear relation allows passage from the normalized time \( t \) to the actual time \( \tau \):

\[
\theta = \tau t, \quad 0 \leq t \leq 1
\] (5)

The fact that the normalized final time (\( t = 1 \)) is fixed does not cause any loss of generality in the problem. If the actual final time is free, \( \tau \) simply becomes a parameter to be optimized in the transformed problem. In view of this, we define the p-vector parameter

\[
\pi \in \pi \quad \text{or} \quad \pi = \begin{bmatrix} \pi \cr \tau \end{bmatrix}
\] (6)

where (6-1) holds if \( \tau \) is fixed and (6-2) holds if \( \tau \) is free.

In addition to the normalized time \( t \) and the parameter \( \pi \), we define the following functions:

\[
f(x, u, \pi, t) = \tau f_4(x, u, \pi, \theta), \quad 0 \leq t \leq 1
\] (7)

\[
E(x, u, \pi, t) = \tau E_4(x, u, \pi, \theta), \quad 0 \leq t \leq 1
\] (8)

\[
E(x, u, \pi, t) = \tau E_4(x, u, \pi, \theta), \quad 0 \leq t \leq 1
\] (9)

\[
g(x, \pi, t) = g_4(x, \pi, \theta)
\] (10)

\[
\psi(x, \pi, t) = \psi_4(x, \pi, \theta)
\] (11)

Under the above transformation and definitions, Problem (1)-(4) can be reformulated as follows: Minimize the functional
\[ J = \int_0^1 \left[ f(x, u, \pi, t) dt + [g(x, \pi, t)] \right] \]

with respect to the state \( x(t) \), the control \( u(t) \), and the parameter \( \pi \) subject to

the differential constraint

\[ \dot{x} = \varphi(x, u, \pi, t), \quad 0 \leq t \leq 1 \]

the equality nondifferential constraint

\[ E(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1 \]

and the boundary conditions

\[ (x)_0 = \text{given}, \quad [\psi(x, \pi, t)]_1 = 0 \]

The Lagrangian\(^1\) of problem (12)-(15) is given by

\[ J = \int_0^1 \left[ [f + \lambda^T (\dot{x} - \varphi) + \rho^T E] dt + (g + \mu^T \psi) \right] \]

where \( \lambda(t) \) is a variable Lagrange multiplier (an \( n \)-vector), \( \rho(t) \) is a variable Lagrange multiplier (a \( k \)-vector), and \( \mu \) is a constant Lagrange multiplier (a \( q \)-vector).

From Lagrangian theory, we know that problem (12)-(15) can be recast as that of minimizing \( J \) with respect to the state \( x(t) \), the control \( u(t) \), and the parameter \( \pi \) subject to (13)-(15). To free \( J \) of its explicit dependence upon \( \dot{x} \), we integrate Eq. (16) by parts. The Lagrangian (or augmented functional) then becomes

\[ J = \int_0^1 \left[ -\lambda^T x + f - \lambda^T \varphi + \rho^T E \right] dt - (\lambda^T x)_0 + (\lambda^T x + g + \mu^T \psi)_1 \]

\( ^1 \text{In Eq. (16), it is tacitly assumed that the initial condition (15-1) is satisfied.} \)
Taking the first variation of \( J \), we have

\[
\delta J = \int_0^1 (f_x - \varphi_x \lambda + E_x \rho - \lambda) T \Delta x dt + \int_0^1 (f_u - \varphi_u \lambda + E_u \rho) T \Delta u dt \tag{18}
\]

\[
+ \left[ \int_0^1 (f_{\pi} - \varphi_{\pi} \lambda + E_{\pi} \rho) dt + (g_{\pi} + \psi_{\pi} \mu) \right] T \Delta \pi + \left[ (\lambda + g_x + \psi_x \mu) T \Delta x \right]_1
\]

Since the variations \( \Delta x(t) \), \( \Delta u(t) \), and \( \Delta \pi \) are free, the following conditions must be satisfied for the first variation of \( J \) to vanish:

\[
\dot{\lambda} - f_x + \varphi_x \lambda - E_x \rho = 0, \quad 0 \leq t \leq 1 \tag{19}
\]

\[
f_u - \varphi_u \lambda + E_u \rho = 0, \quad 0 \leq t \leq 1 \tag{20}
\]

\[
\int_0^1 (f_{\pi} - \varphi_{\pi} \lambda + E_{\pi} \rho) dt + (g_{\pi} + \psi_{\pi} \mu) = 0 \tag{21}
\]

\[
(\lambda + g_x + \psi_x \mu) = 0 \tag{22}
\]

Thus, we seek functions \( x(t) \), \( u(t) \), \( \pi \) and multipliers \( \lambda(t) \), \( \rho(t) \), \( \mu \) which satisfy the constraints (13)-(15) and the optimality conditions (19)-(22).

**Approximate Methods**

In general, the differential system (13)-(15) and (19)-(22) is nonlinear. Therefore some type of iterative approximation must be employed in its solution.

For this purpose, let us define the scalar functionals \( P \) and \( Q \) which denote the error in the constraints and the error in the optimality conditions, respectively.

We have

\[
P = \int_0^1 N(\dot{x} - \varphi) dt + \int_0^1 N(E) dt + N(\psi)_1 \tag{23}
\]

\[\text{In Eq. (18), it is tacitly assumed that } (\Delta x)_0 = 0.\]
where $N(.)$ denotes the square of the vector norm, i.e.,

$$N(a) = a^T a$$

for given vector $a$.

For the optimal solution, $P = 0$ and $Q = 0$. For an approximation to the optimal solution,

$$P \leq \epsilon_1, \quad Q \leq \epsilon_2$$

where $\epsilon_1$ and $\epsilon_2$ are small, preselected numbers.

3. **Sequential Gradient-Restoration Algorithm**

The algorithm which is constructed to solve problem (12)-(15) is an extension of the sequential gradient-restoration algorithm of Ref. 1. The algorithm consists of a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase.

For any iteration of the gradient phase or the restoration phase, let $x(t), u(t), \pi$ denote the nominal functions and let $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ denote the varied functions. Furthermore, let $\alpha$ denote the stepsize and let $A(t), B(t), C$ denote displacements per unit stepsize. Then,

$$\tilde{x}(t) = x(t) + \Delta x(t), \quad \tilde{u}(t) = u(t) + \Delta u(t), \quad \tilde{\pi} = \pi + \Delta \pi$$
\[ \Delta x(t) = \alpha A(t), \quad \Delta u(t) = \alpha B(t), \quad \Delta \pi = \alpha C \]  

which imply

\[ \tilde{x}(t) = x(t) + \alpha A(t), \quad \tilde{u}(t) = u(t) + \alpha B(t), \quad \tilde{\pi} = \pi + \alpha C \]

It will be seen that the variations \( \Delta x(t) \), \( \Delta u(t) \), \( \Delta \pi \) are selected so as to achieve a reduction in the functional \( I \) or the augmented functional \( J \) at the end of each cycle while satisfying the constraints to a predetermined accuracy.

**3.1. Gradient Phase.** The gradient phase is started only when Ineq. (26-1) is satisfied and involves a single iteration. In this iteration, the objective is to reduce the functional \( I \) or the augmented functional \( J \) while satisfying the constraints to first order. During this iteration, the gradient is projected onto the tangential hyperplane of the constraints, and a step is taken in the negative direction of the projection.

Suppose that nominal functions \( x(t) \), \( u(t) \), \( \pi \) satisfying (13)-(15) are available. Let \( \tilde{x}(t) \), \( \tilde{u}(t) \), \( \tilde{\pi} \) denote varied functions also satisfying (13)-(15). To first order, the perturbations \( \Delta x(t) \), \( \Delta u(t) \), \( \Delta \pi \) must satisfy the linearized constraint equations

\[ \dot{\Delta x} - \varphi_x^T \Delta x - \varphi_u^T \Delta u - \varphi_{\pi}^T \Delta \pi = 0, \quad 0 \leq t \leq 1 \]  

\[ E_x^T \Delta x + E_u^T \Delta u + E_{\pi}^T \Delta \pi = 0, \quad 0 \leq t \leq 1 \]

and the linearized boundary conditions.
In order to ensure a decrease in the augmented functional $J$, we must have $\Delta J < 0$, where $\Delta J$ denotes the total variation of $J$ caused by the perturbations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$. This inequality can be enforced at every iteration providing the stepsize $\alpha$ is sufficiently small and $\delta J < 0$, where $\delta J$ denotes the first variation of $J$. By inspection of (18), we see that $\delta J$ can be made negative if the variations $\Delta u(t)$, $\Delta \pi$ and the multipliers $\lambda(t)$, $\rho(t)$, $\mu$ satisfy the following equations:

\[
\Delta u = -\alpha(f_u - \phi_u \lambda + E_u \rho), \quad 0 \leq t \leq 1
\]

\[
\Delta \pi = -\alpha\left[\int_0^1 (f_\pi - \phi_\pi \lambda + E_\pi \rho)dt + (g_\pi + \psi_\pi \mu)\right] \quad (34)
\]

\[
\lambda - f_x + \phi_x \lambda - E_x \rho = 0, \quad 0 \leq t \leq 1 \quad (36)
\]

\[
(\lambda + g_x + \psi_x \mu) = 0
\]

Equations (31)-(33) imply that

\[
\delta I = \delta J
\]

while Eqs. (34)-(37) imply that

\[
\delta J = -\alpha Q
\]

where $Q$, the error in the optimality conditions, reduces to

\[
Q = \int_0^1 N(f_u - \phi_u \lambda + E_u \rho)dt + N\left[\int_0^1 (f_\pi - \phi_\pi \lambda + E_\pi \rho)dt + (g_\pi + \psi_\pi \mu)\right] > 0
\]

By noting that $Q > 0$, we can see from Eqs. (38)-(39) that a descent property on both the augmented functional $J$ and the functional $I$ is attained. Equations (31)-(33), in combination with Eqs. (34)-(37), uniquely define the variations of
the gradient phase.

At the end of the gradient phase, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are known and the varied constraint error $\tilde{P}$ can be computed with Eq. (23). If $\tilde{P}$ satisfies Ineq. (26-1), the gradient phase is repeated using the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ of the previous gradient phase for nominal functions. If Ineq. (26-1) is violated, a restoration phase must be employed prior to repeating the gradient phase.

3.2. Restoration Phase. As mentioned above, the restoration phase is started only when Ineq. (26-1) is violated. During this restoration phase, the objective is to reduce the constraint error $P$ to a level compatible with Ineq. (26-1) while preserving the descent property of the gradient phase. To achieve this constraint satisfaction, quasilinearization (Newton's method) is employed.

While the gradient phase involves a single iteration, the restoration phase may involve several iterations. This is due to the fact that the constraint equations (13)-(15) are considered only in linearized form during the restoration phase. In each restorative iteration, the objective is to reduce the functional $P$ while satisfying the constraints to first order and while minimizing the norm of the variations of the control and the parameter. The restoration phase is terminated whenever Ineq. (26-1) is satisfied.

For the first restorative iteration, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ of the previous gradient phase are used for nominal functions. For any subsequent restorative iteration, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ of the previous restorative iteration are used for nominal functions.
Let \( x(t), u(t), \pi \) denote nominal functions satisfying condition (15-1), and let \( \tilde{x}(t), \tilde{u}(t), \tilde{\pi} \) denote varied functions satisfying (13)-(15). To first order, the perturbations \( \Delta x(t), \Delta u(t), \Delta \pi \) must satisfy the linearized constraint equations

\[
\Delta \dot{x} - \dot{\varphi}_x \Delta x - \varphi_u \Delta u - \varphi_{\pi} \Delta \pi + \alpha (x - \varphi) = 0, \quad 0 \leq t \leq 1
\]

\[
E_T \Delta x + E_u \Delta u + E_{\pi} \Delta \pi + \alpha E = 0, \quad 0 \leq t \leq 1
\]

and the linearized boundary conditions

\[
(\Delta x)_0 = 0, \quad (\psi_T \Delta x + \psi_{\pi} \Delta \pi + \alpha \psi)_1 = 0
\]

where \( \alpha \) denotes a scaling factor (restoration stepsize) in the range \( 0 \leq \alpha \leq 1 \).

In order to ensure a decrease in the functional \( P \), we must have \( \Delta P < 0 \). This inequality can be enforced at every iteration providing the stepsize \( \alpha \) is sufficiently small and \( \delta P < 0 \). The first variation of \( P \) is given by

\[
\delta P = 2 \int_0^1 (\dot{x} - \varphi)^T (\Delta \dot{x} - \dot{\varphi}_x \Delta x - \varphi_u \Delta u - \varphi_{\pi} \Delta \pi) dt
\]

\[
+ 2 \int_0^1 E_T (E_T \Delta x + E_u \Delta u + E_{\pi} \Delta \pi) dt + 2[\psi_T (\psi_T \Delta x + \psi_{\pi} \Delta \pi)]_1
\]

When the variations defined by (41)-(43) are employed, the first variation of the constraint error (23) becomes

\[
\delta P = -2\alpha P
\]

Since \( P > 0 \), Eq. (45) shows that \( \delta P < 0 \). Hence, for \( \alpha \) sufficiently small, a decrease in the constraint error \( P \) is attained.
In order to uniquely define the restorative iteration, an additional requirement must be introduced. We require that the restoration be accomplished while minimizing the norm of the variations of the control and the parameter.

Thus, we seek the minimum of the quadratic functional

\[ K = \int_0^1 \Delta u^T \Delta u dt + \Delta \pi^T \Delta \pi \]  

(46)

with respect to the perturbations \( \Delta x(t), \Delta u(t), \Delta \pi \) subject to the linearized constraints (41)-(42) and the linearized boundary conditions (43). From calculus of variations, we know that the solution minimizing (46) subject to (41)-(43) must satisfy the following optimality conditions:

\[ \Delta u = \alpha (\varphi_u \lambda - E_u \rho), \quad 0 \leq \tau \leq 1 \]  

(47)

\[ \Delta \pi = \left[ \int_0^1 (\varphi_\pi \lambda - E_\pi \rho) dt - (\psi_\pi \mu) \right] \]  

(48)

\[ \dot{\lambda} + \varphi_x \lambda - E_x \rho = 0, \quad 0 \leq t \leq 1 \]  

(49)

\[ (\lambda + \psi_x \mu)_1 = 0 \]  

(50)

Equations (41)-(43), in combination with Eqs. (47)-(50), uniquely define the variations of the restoration phase.

3.3. General Form of the Variations of a Cycle. Upon comparing the variations of the gradient phase [Eqs. (31)-(33) and (34)-(37)] with those of the restoration phase [Eqs. (41)-(43) and (47)-(50)] it can be seen that the two sets of variations are quite similar. The sequential gradient-restoration algorithm can be made computationally efficient by taking advantage of this similarity. These

\[ \text{Eqs. (41)-(43) admit an infinite number of solutions.} \]
two sets of variations can be embedded in a one-parameter family of variations
in which the parameter \( \beta \) has the following values:

\[
\begin{align*}
\beta &= 1 \quad \text{during the gradient phase} \\
\beta &= 0 \quad \text{during the restoration phase}
\end{align*}
\]

The variations of a cycle can then be represented in the following general form:

\[
\begin{align*}
\Delta x - \varphi_x T_{\Delta x} - \varphi_u T_{\Delta u} - \varphi_{\varpi} T_{\Delta \varpi} + \alpha(1 - \beta)(\dot{x} - \varphi) &= 0, & 0 \leq t \leq 1 \\
E_x T_{\Delta x} + E_u T_{\Delta u} + E_{\varpi} T_{\Delta \varpi} + \alpha(1 - \beta)E &= 0, & 0 \leq t \leq 1 \\
(\Delta x)_0 &= 0, & [\psi_x T_{\Delta x} + \psi_{\varpi} T_{\Delta \varpi} + \alpha(1 - \beta)\psi]_1 = 0 \\
\Delta u &= -\alpha(\beta f_x - \varphi_u \lambda + E_u \rho), & 0 \leq t \leq 1 \\
\Delta \varpi &= -\alpha \int_0^1 (\beta f_{\varpi} - \varphi_{\varpi} \lambda + E_{\varpi} \rho)dt + (\beta g_{\varpi} + \psi_{\varpi} u)_1, & 0 \leq t \leq 1 \\
\dot{\lambda} - \beta f_x + \varphi_x \lambda - E_x \rho &= 0, & 0 \leq t \leq 1 \\
(\lambda + \beta g_x + \psi_x u)_1 &= 0
\end{align*}
\]

Due to Eqs. (28), Eqs. (52)-(58) can be rewritten as

\[
\begin{align*}
\dot{A} - \varphi_x T_A - \varphi_u T_B - \varphi_{\varpi} T_C + (1 - \beta)(\dot{x} - \varphi) &= 0, & 0 \leq t \leq 1 \\
E_x T_A + E_u T_B + E_{\varpi} T_C + (1 - \beta)E &= 0, & 0 \leq t \leq 1 \\
(A)_0 &= 0, & [\psi_x T_A + \psi_{\varpi} T_C + (1 - \beta)\psi]_1 = 0 \\
B &= -(\beta f_u - \varphi_u \lambda + E_u \rho), & 0 \leq t \leq 1 \\
C &= [-\int_0^1 (\beta f_{\varpi} - \varphi_{\varpi} \lambda + E_{\varpi} \rho)dt + (\beta g_{\varpi} + \psi_{\varpi} u)_1] & 0 \leq t \leq 1 \\
\dot{\lambda} - \beta f_x + \varphi_x \lambda - E_x \rho &= 0, & 0 \leq t \leq 1 \\
(\lambda + \beta g_x + \psi_x u)_1 &= 0
\end{align*}
\]
\[(\lambda + \beta \, \varphi_x + \psi_x \mu)_1 = 0\]  

(65)

3.4. Linear, Two-Point Boundary-Value Problem. Equations (59)-(65) form a two-member family of linear, two-point boundary-value problems. The linear system obtained by setting \(\beta = 1\) defines the variations per unit stepsize of the gradient phase. The linear system obtained by setting \(\beta = 0\) defines the variations per unit stepsize of the restoration phase. These linear, two-point boundary-value problems can be solved in the same way.

We integrate the differential system under consideration \(n + p + 1\) times using a forward integration scheme in combination with the method of particular solutions (Refs. 4-7). In each integration (subscript \(i\)), we assign a different set of values to the \(n\)-vector \(\lambda(0)\) and the \(p\)-vector \(C\), specifically,

\[
\lambda_i(0) = [\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}]^T, \quad C_i = [\delta_{i(n+1)} \delta_{i(n+2)} \ldots \delta_{i(n+p)}]^T
\]

(66)

where \(i = 1, 2, \ldots, n + p + 1\) and where the Kronecker delta \(\delta_{ij}\) is such that

\[
\delta_{ij} = 1 \quad \text{if} \quad i = j, \quad \delta_{ij} = 0 \quad \text{if} \quad i \neq j
\]

(67)

With the above vectors specified, the differential system is integrated forward employing (a) Eqs. (59), (60), (61-1), (62), (64) and bypassing (b) Eqs. (61-2), (63), (65). In this way, one obtains the particular solutions

\[
A_i(t), B_i(t), C_i, \quad i = 1, 2, \ldots, n + p + 1
\]

(68)

\[
\lambda_i(t), \rho_i(t), \quad i = 1, 2, \ldots, n + p + 1
\]

(69)

Now, consider the linear combinations
\[ A(t) = \sum_k A_k(t), \quad B(t) = \sum_k B_k(t), \quad C = \sum_k C_k \]

\[ \lambda(t) = \sum_k \lambda_k(t), \quad \rho(t) = \sum_k \rho_k(t) \]

(70) \hspace{1cm} (71)

where the summation is taken over the index \( i \). Here, the symbols \( k_i \) denote undetermined, scalar constants. These linear combinations satisfy conditions (a) providing

\[ \sum k_i = 1 \]  

(72)

and satisfy conditions (b) providing

\[
\sum k_i \left( \psi_x^T A_i + \psi_{\pi_i}^T C_i \right)_{1} + [(1 - \beta) \psi]_{1} = 0, \quad \sum k_i (\lambda_i)_{1} + (\beta g_{x_i})_{1} + (\psi_{x_i})_{1} \psi = 0 
\]

(73)

\[
\sum k_i \left[ \int_0^1 \left( \alpha_{\pi_i} \lambda_i - B_{\pi_i} \rho_i - \beta f \right) dt - C_i \right] - (\psi_{\pi_i})_{1} \psi - (\beta g_{\pi_i})_{1} = 0 
\]

(74)

The linear system (72)-(74) contains \( n + p + q + 1 \) equations in which the unknowns are the \( n + p + 1 \) constants \( k_i \) and the \( q \) components of the multiplier \( \mu \).

Upon determining the constants \( k_i \), two methods for constructing the solution of the linear, two-point boundary-value problem are possible. The first method requires the saving of the \( n + p + 1 \) particular solutions (68)-(69). In this case, the composite solution can be obtained directly via Eqs. (70)-(71).

The second method requires the saving of the initial conditions (66) employed to generate the particular solutions. In this case, the composite solution is obtained by first using the constants \( k_i \) to define \( \lambda(0) \) and \( C \) and then integrating the linear differential system forward once more. The latter technique requires less computer storage and is used in the examples of this thesis.

3.5. Step size Determination. From the solution of the linear, two-point
boundary-value problem, the functions \( A(t), B(t), C \) and the multipliers \( \lambda(t), \varphi(t), \mu \) are known. With these functions, one forms the one-parameter family of solutions (30) for which the augmented functional \( J \) and the constraint error \( P \) take the following form:

\[
J = f(\alpha), \quad P = \bar{P}(\alpha)
\]  

(75)

Then, a one-dimensional search scheme is employed. The type of search scheme used depends upon the phase.

In the gradient phase, \( \alpha \) must be selected in such a way that the inequality

\[
\bar{J}(\alpha) < \bar{J}(0)
\]  

(76)

is satisfied while keeping

\[
\bar{P}(\alpha) \leq \varepsilon_3 \quad \text{and} \quad \bar{\tau}(\alpha) \geq 0
\]  

(77)

Here, \( \varepsilon_3 \) is a small, preselected number. Satisfaction of (76) is guaranteed by the descent property of the gradient phase. Satisfaction of (77-1) is desirable in order to limit the constraint violation which is due to the use of the linearized constraint equations (31)-(33). Satisfaction of (77-2) is automatic in problems where the actual final time \( \tau \) is fixed and is required in problems where the actual final time is free.

Any violation of the above inequalities necessitates a reduction in the size of \( \alpha \). Such a reduction can be obtained by employing a bisection process, starting from a suitably chosen reference stepsize \( \alpha = \alpha_0 \).
The gradient phase reference stepsize is determined by finding the minimum of \( \tilde{J}(\alpha) \) with respect to \( \alpha \), under the assumption that \( \tilde{J}(\alpha) \) has a quadratic representation. Thus, let \( \tilde{J}(\alpha) \) be represented by

\[
\tilde{J}(\alpha) = K_0 + K_1 \alpha + K_2 \alpha^2 \tag{78}
\]

and let the coefficients of the quadratic be determined from the values of the ordinate and the slope at \( \alpha = 0 \) and the value of the ordinate at \( \alpha = 1 \). This yields the relations

\[
\tilde{J}(0) = K_0, \quad \tilde{J}(0) = K_1, \quad \tilde{J}(1) = K_0 + K_1 + K_2 \tag{79}
\]

Since \( \tilde{J}(0) = -Q \), we have

\[
K_0 = \tilde{J}(0), \quad K_1 = -Q, \quad K_2 = \tilde{J}(1) - \tilde{J}(0) + Q \tag{80}
\]

With the coefficients known, two possibilities exist; either \( K_2 > 0 \) or \( K_2 < 0 \).

If \( K_2 > 0 \), the quadratic function (78) has a minimum for the gradient stepsize value \( \alpha = -K_1 / 2K_2 \). If \( K_2 < 0 \), the quadratic function does not possess a minimum.

This suggests the use of the following reference stepsize in the gradient phase:

\[
\alpha_0 = -K_1 / 2K_2 \quad \text{if} \quad K_2 > 0
\]

\[
\alpha_0 = 1 \quad \text{if} \quad K_2 < 0 \tag{81}
\]

In the restoration phase, \( \alpha \) must be selected in such a way that the inequality

\[
\tilde{P}(\alpha) < \tilde{P}(0) \tag{82}
\]
Satisfaction of Ineqs. (82) and (83) is guaranteed for $\alpha$ sufficiently small. Any violation of the above inequalities necessitates a reduction in the size of $\alpha$. Again, such a reduction can be obtained by employing a bisection process, starting from a suitably chosen reference stepsize $\alpha = \alpha_0$.

The restoration phase reference stepsize is obtained by setting $\alpha_0 = 1$.

3.6. Descent Property. After completing a gradient-restoration cycle, we must check to see if the restoration phase has preserved the descent property of the gradient phase. Let $I$ denote the value of the functional (12) at the beginning of the gradient phase and let $\hat{I}$ denote the value of (12) at the end of the restoration phase. We would like to have

$$\hat{I} < I$$  \hfill (84)

If Ineq. (84) is satisfied at the end of the restoration phase, the next gradient phase is started. If Ineq. (84) is violated, one returns to the previous gradient phase and reduces the gradient stepsize $\alpha_g$ until, after restoration, Ineq. (84) is satisfied.

That the above procedure leads to satisfaction of Ineq. (84) is guaranteed by the fact that the gradient corrections are of $O(\alpha_g)$ while the restoration corrections are $O(\alpha_g^2)$. Hence, if the gradient stepsize $\alpha_g$ is sufficiently small, the restoration phase preserves the descent property of the gradient phase.

---

*For example, one can use a bisection process.*
4. **Numerical Computations**

In order to substantiate the theory, two numerical examples were considered. The algorithm was programmed in FORTRAN IV. Double-precision arithmetic was used.

The problems were solved on the Rice University IBM 370/155 digital computer. In each case, the interval of integration was divided into 50 steps. The differential equations were integrated by Hamming's modified predictor-corrector method using a special Runge-Kutta starting procedure. The definite integrals $I, J, P, Q$ were computed by a modified Simpson's rule.

The determination of the gradient stepsize or the restoration stepsize was performed in accordance with Section 3.5. For the gradient phase, the stepsize was subject to the inequalities:

$$\tilde{f}(\alpha) < \tilde{J}(0), \quad \tilde{F}(\alpha) \leq 1, \quad \tilde{\tau}(\alpha) \geq 0$$  \hspace{1cm} (85)

For the restoration phase, the stepsize was subject to the inequalities

$$\tilde{F}(\alpha) < \tilde{F}(0), \quad \tilde{\tau}(\alpha) \geq 0$$  \hspace{1cm} (86)

The restoration phase was terminated whenever

$$\tilde{F}(\alpha) \leq 10^{-8}$$  \hspace{1cm} (87)

Convergence of the algorithm was defined by

$$P \leq 10^{-8}, \quad Q \leq 10^{-4}$$  \hspace{1cm} (88)
Nonconvergence of the algorithm was defined by

\[(a) \quad N > 50 \quad (89-1)\]
\[(b) \quad N_S > 10 \quad (89-2)\]

where \(N\) is the number of iterations and \(N_S\) is the number of bisections of the stepsize \(\alpha\) required to satisfy Ineqs. (85) or (86).

**Example 4.1.** Consider the problem of minimizing the functional\(^5\)

\[I = \tau \quad (90)\]

with respect to the state variables \(x(\theta), y(\theta), z(\theta)\), the control variables \(u(\theta), v(\theta)\), and the parameter \(\tau\) which satisfy the differential constraints

\[
\frac{dx}{d\theta} = zu, \quad \frac{dy}{d\theta} = zv, \quad \frac{dz}{d\theta} = v \quad (91)
\]

the equality nondifferential constraint

\[u^2 + v^2 - 1 = 0 \quad (92)\]

and the boundary conditions

\[
x(0) = 0, \quad y(0) = 0, \quad z(0) = 0 \quad (93)
\]
\[
x(\tau) = 1 \quad (94)
\]

Introduce the normalized time \(t = \theta/\tau\). Then, problem (90)-(94) becomes that of minimizing the functional

\[I = \tau \quad (95)\]

\(^5\)This is a reformulation of the classical brachistochronic problem.
with respect to the state variables \( x(t) \), \( y(t) \), \( z(t) \), the control variables \( u(t) \), \( v(t) \), and the parameter \( \tau \) subject to

\[
\begin{align*}
\dot{x} &= \tau z u, \\
\dot{y} &= \tau z v, \\
\dot{z} &= \tau v \\
u^2 + v^2 - 1 &= 0 \\
x(0) &= 0, \\
y(0) &= 0, \\
z(0) &= 0 \\
x(1) &= 1 \\
\end{align*}
\]

In this problem,

\[
\begin{align*}
n &= 3, & m &= 2, & p &= 1 \\
k &= 1, & q &= 1 \\
\end{align*}
\]

Since \( n + p + 1 = 5 \), five particular solutions are needed for each gradient or restorative iteration.

This problem is characterized by the following analytical solutions:

\[
\begin{align*}
x &= t - (t/\pi) \sin (\pi t), & y &= (2/\pi) \sin^2 (\pi t/2), & z &= (2/\sqrt{\pi}) \sin (\pi t/2) \\
u &= \sin(\pi t/2), & v &= \cos(\pi t/2) \\
\tau &= \sqrt{\pi} \\
\end{align*}
\]

and the minimum value of the functional is

\[
I = \sqrt{\pi} = 1.7724
\]

Assume the nominal functions
which are consistent with the boundary conditions (98)-(99) but violate the constraints (96)-(97). Since these nominal functions do not constitute a feasible solution, the algorithm starts with a restoration phase. After a total of $N = 10$ iterations the algorithm has converged to the required accuracy. Tables 1-3 show the convergence history, the optimal control variables, and the optimal state variables, respectively.
Table 1. Convergence history for Example 4.1.

<table>
<thead>
<tr>
<th>N</th>
<th>Phase</th>
<th>P</th>
<th>Q</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Rest</td>
<td>0.3000 E+01</td>
<td></td>
<td>1.0000</td>
</tr>
<tr>
<td>1</td>
<td>Rest</td>
<td>0.1139 E+01</td>
<td></td>
<td>1.7500</td>
</tr>
<tr>
<td>2</td>
<td>Rest</td>
<td>0.7271 E-02</td>
<td></td>
<td>1.8499</td>
</tr>
<tr>
<td>3</td>
<td>Rest</td>
<td>0.2194 E-04</td>
<td></td>
<td>1.8898</td>
</tr>
<tr>
<td>4</td>
<td>Grad</td>
<td>0.3076 E-09</td>
<td>0.4403 E+00</td>
<td>1.8930</td>
</tr>
<tr>
<td>5</td>
<td>Rest</td>
<td>0.3489 E+00</td>
<td></td>
<td>1.5137</td>
</tr>
<tr>
<td>6</td>
<td>Rest</td>
<td>0.1078 E-02</td>
<td></td>
<td>1.7493</td>
</tr>
<tr>
<td>7</td>
<td>Rest</td>
<td>0.5449 E-07</td>
<td></td>
<td>1.7737</td>
</tr>
<tr>
<td>8</td>
<td>Grad</td>
<td>0.1405 E-15</td>
<td>0.6377 E-02</td>
<td>1.7739</td>
</tr>
<tr>
<td>9</td>
<td>Rest</td>
<td>0.4785 E-05</td>
<td></td>
<td>1.7709</td>
</tr>
<tr>
<td>10</td>
<td>Grad</td>
<td>0.2511 E-11</td>
<td>0.7819 E-04</td>
<td>1.7724</td>
</tr>
</tbody>
</table>
Table 2. Optimal control variables for Example 4.1.

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1546</td>
<td>0.9879</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3058</td>
<td>0.9520</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4500</td>
<td>0.8930</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5837</td>
<td>0.8119</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7038</td>
<td>0.7103</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8070</td>
<td>0.5904</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8905</td>
<td>0.4548</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9517</td>
<td>0.3067</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9887</td>
<td>0.1497</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9999</td>
<td>-0.0118</td>
</tr>
</tbody>
</table>

\( \tau = 1.7724 \)

Table 3. Optimal state variables for Example 4.1.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0016</td>
<td>0.0155</td>
<td>0.1765</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0127</td>
<td>0.0608</td>
<td>0.3488</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0420</td>
<td>0.1314</td>
<td>0.5126</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0965</td>
<td>0.2205</td>
<td>0.6640</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1805</td>
<td>0.3194</td>
<td>0.7992</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2959</td>
<td>0.4184</td>
<td>0.9148</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4413</td>
<td>0.5076</td>
<td>1.0076</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6121</td>
<td>0.5781</td>
<td>1.0753</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8013</td>
<td>0.6225</td>
<td>1.1158</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0000</td>
<td>0.6363</td>
<td>1.1281</td>
</tr>
</tbody>
</table>
Example 4.2. Consider the problem of minimizing the functional \(^6\)

\[
I = -\int_{0}^{1} (u+v) dt
\]

(109)

with respect to the state variables \(x(t), z(t)\), and the control variables \(u(t), v(t)\) which satisfy the differential constraints

\[
\dot{x} = -x^2, \quad \dot{z} = -5z^2
\]

(110)

the nondifferential constraint

\[
(u + v)/2 - 3xu/(u + 3x) - 6zv/(v + 6z) = 0
\]

(111)

and the boundary conditions

\[
x(0) = 1, \quad z(0) = x(1)
\]

(112)

Introduce the auxiliary parameter \(\pi = z(0)\) and the auxiliary state variable \(y = z/\pi\). Then, problem (109)-(112) becomes that of minimizing the functional

\[
I = -\int_{0}^{1} (u + v) dt
\]

(113)

with respect to the state variables \(x(t), y(t)\), the control variables \(u(t), v(t)\), and the parameter \(\pi\) subject to

\[
\dot{x} = -x^2, \quad \dot{y} = -5\pi y^2
\]

(114)

\[
(u + v)/2 - 3xu/(u + 3x) - 6\pi yv/(v + 6\pi y) = 0
\]

(115)

\[
x(0) = 1, \quad y(0) = 1
\]

(116)

\[
x(1) = \pi
\]

(117)

\(^6\)This problem occurs in the study of some chemical engineering processes (Ref. 8).
In this problem,

\[ n = 2, \quad m = 2, \quad p = 1 \]  \hspace{1cm} (118)

\[ k = 1, \quad q = 1 \]  \hspace{1cm} (119)

Since \( n + p + 1 = 4 \), four particular solutions are needed for each gradient or restorative iteration.

This problem is characterized by the following analytical solutions:

\[ x = 1/(1 + t), \quad y = 2/(2 + 5t) \]  \hspace{1cm} (120)

\[ u = 3/(1 + t), \quad v = 6/(2 + 5t) \]  \hspace{1cm} (121)

\[ \tau = 1/2 \]  \hspace{1cm} (122)

and the minimum value of the functional is

\[ I = -(6/5) \log 7 - (9/5) \log 2 = -3.5827 \]  \hspace{1cm} (123)

Assume the nominal functions

\[ x = 1/(1 + t), \quad y = 2/(2 + 5t) \]  \hspace{1cm} (124)

\[ u = 10, \quad v = 10 \]  \hspace{1cm} (125)

\[ \tau = 1/2 \]  \hspace{1cm} (126)

which are consistent with the differential equations (114) and the boundary conditions (116)-(117), but violate the nondifferential constraint (115). Since these nominal functions do not constitute a feasible solution, the algorithm starts with a restoration phase. After a total of \( N = 8 \) iterations the algorithm has converged to the required accuracy. Tables 4-6 show the convergence history, the
optimal control variables, and the optimal state variables, respectively.
Table 4. Convergence history for Example 4.2.

<table>
<thead>
<tr>
<th>N</th>
<th>Phase</th>
<th>P</th>
<th>Q</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Rest</td>
<td>0.4949 E+02</td>
<td>-20.0000</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Rest</td>
<td>0.2815 E+00</td>
<td>-5.3148</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Rest</td>
<td>0.2714 E-02</td>
<td>-3.7703</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Rest</td>
<td>0.1816 E-05</td>
<td>-3.5762</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Grad</td>
<td>0.1601 E-11</td>
<td>0.1548 E-01</td>
<td>-3.5710</td>
</tr>
<tr>
<td>5</td>
<td>Rest</td>
<td>0.1384 E-04</td>
<td>-3.5942</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Grad</td>
<td>0.3010 E-09</td>
<td>0.5002 E-03</td>
<td>-3.5824</td>
</tr>
<tr>
<td>7</td>
<td>Rest</td>
<td>0.2704 E-07</td>
<td>-3.5831</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Grad</td>
<td>0.4544 E-14</td>
<td>0.3640 E-04</td>
<td>-3.5827</td>
</tr>
</tbody>
</table>
Table 5. Optimal control variables for Example 4.2.

<table>
<thead>
<tr>
<th>t</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3.0000</td>
<td>3.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>2.7126</td>
<td>2.4145</td>
</tr>
<tr>
<td>0.2</td>
<td>2.4852</td>
<td>2.0146</td>
</tr>
<tr>
<td>0.3</td>
<td>2.2971</td>
<td>1.7248</td>
</tr>
<tr>
<td>0.4</td>
<td>2.1369</td>
<td>1.5059</td>
</tr>
<tr>
<td>0.5</td>
<td>1.9976</td>
<td>1.3356</td>
</tr>
<tr>
<td>0.6</td>
<td>1.8747</td>
<td>1.2002</td>
</tr>
<tr>
<td>0.7</td>
<td>1.7646</td>
<td>1.0909</td>
</tr>
<tr>
<td>0.8</td>
<td>1.6648</td>
<td>1.0018</td>
</tr>
<tr>
<td>0.9</td>
<td>1.5734</td>
<td>0.9285</td>
</tr>
<tr>
<td>1.0</td>
<td>1.4889</td>
<td>0.8681</td>
</tr>
</tbody>
</table>

\[ \pi = 0.5000 \]

Table 6. Optimal state variables for Example 4.2.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9090</td>
<td>0.8000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8333</td>
<td>0.6666</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7692</td>
<td>0.5714</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7142</td>
<td>0.5000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6666</td>
<td>0.4444</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6250</td>
<td>0.4000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5882</td>
<td>0.3636</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5555</td>
<td>0.3333</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5263</td>
<td>0.3076</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5000</td>
<td>0.2857</td>
</tr>
</tbody>
</table>
5. **Summary**

An algorithm has been developed to solve optimal control problems in the presence of equality nondifferential constraints. In order to facilitate numerical integrations, the algorithm has been constructed based on an interval of integration of unit length. This has been done without loss of generality.

The algorithm is composed of a sequence of cycles, each cycle consisting of two phases, a gradient phase and a restoration phase. The objective of each cycle is to decrease the functional $I$ while satisfying the constraints to the predetermined accuracy of (26-1).

The gradient phase involves a single iteration and is designed to decrease the augmented functional $J$ while satisfying the constraints to first order. During this iteration, the gradient is projected onto the tangential hyperplane of the constraints, and a step is taken in the negative direction of the projection. The gradient phase is started only when the nominal functions $x(t), u(t), \pi$ satisfy the constraints (13)-(15) within the preselected accuracy of (26-1). When this occurs, the nominal functions are used to compute the vectors $f_x, f_u, f_\pi$ and the matrices $c_x, c_u, c_\pi$ and $S_x, S_u, S_\pi$ along the interval of integration and to evaluate the vectors $g_x, g_\pi$ and the matrices $\phi_x, \phi_\pi$ at the final time $t = 1$. The linear, two-point boundary-value problem (59)-(65) is then solved by the method of particular solutions with $\beta = 1$. In this way, the functions $A(t), B(t), C$ are obtained. Using these functions, the gradient stepsize is computed in accordance with Section 3.5. The perturbations $\Delta x(t), \Delta u(t), \Delta \pi$ are then determined from Eq. (28). Finally, the varied functions are calculated from Eq. (27).
At the end of the gradient phase, the constraint error (23) is computed. If Ineq. (26-1) is satisfied, the restoration phase is bypassed and the next cycle of the algorithm is started, using the varied functions of the previous gradient iteration for nominal functions. Otherwise, the restoration phase is begun.

The restoration phase involves one or more iterations and is designed to restore the constraints to a level compatible with (26-1). Each iteration is designed to decrease the constraint error $P$ while minimizing the norm of the variations of the control and the parameter. In achieving constraint satisfaction, quasilinearization (Newton's method) is employed.

For the first restorative iteration the nominal functions are identical with the varied functions of the previous gradient iteration. For any subsequent restorative iteration the nominal functions are identical with the varied functions of the previous restorative iteration. In either case, the nominal functions satisfy condition (15-1) but violate conditions (13)-(14), (15-2).

These nominal functions are used to compute the vector $\dot{x} - \varphi$, the vector $S$, and the matrices $\omega_x, \omega_u, \omega_{\pi}$ and $S_x, S_u, S_{\pi}$ along the interval of integration and to evaluate the vector $\psi$ and the matrices $\psi_x, \psi_{\pi}$ at the final time $t = 1$. The linear, two-point boundary-value problem (59)-(65) is then solved by the method of particular solutions with $\beta = 0$. In this way, the functions $A(t), B(t), C$ are obtained. Using these functions, the restoration stepsize is computed in accordance with Section 3.5. The perturbations $\Delta x(t), \Delta u(t), \Delta \pi$ are then determined from Eq. (28). Finally, the varied functions are calculated from Eq. (27).

At the end of each restorative iteration, the constraint error (23) is computed. If Ineq. (26-1) is still violated, one or more restorative iterations
are performed until Ineq. (26-1) is satisfied. Once this occurs, the functional (12) is evaluated. If Ineq. (84) is satisfied, the next cycle of the algorithm is started. If not, the previous gradient phase is returned to and the previous gradient stepsize $\alpha_g$ is bisected until, after restoration, Ineq. (84) is satisfied.

The above bisection procedure is guaranteed to lead to satisfaction of Ineq. (84). This is due to the fact that the gradient corrections are of $O(\alpha_g)$ while the restoration corrections are of $O(\alpha_g^2)$.

At the end of each cycle, the error in the optimality conditions (24) is computed. The algorithm is terminated whenever Ineqs. (26-1) and (26-2) are satisfied simultaneously.

It should be remarked that the algorithm can be started with nominal functions satisfying condition (15-1) but violating conditions (13),(14),(15-2). If Ineq. (26-1) is satisfied, the algorithm starts with a gradient phase. If Ineq. (26-1) is violated, the algorithm starts with a restoration phase.

A final remark, concerning the solution of the linear, two-point boundary-value problem should be made. For given values $A, C, \lambda$, relations (60) and (62) constitute a system of $m+k$ equations in the $m+k$ components of the vectors $B$ and $p$. The system admits a unique solution providing

$$\det \begin{bmatrix} I & B u \\ E u & 0 \end{bmatrix} = (-1)^k \det [E_u^T E_u] \neq 0$$

where $I$ denotes the $m \times n$ identity and $O$ denotes the $k \times k$ null matrix.

Two numerical examples were solved. In each example, the results agreed very closely with the analytical solution.
References


Additional Bibliography


