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ON THE EXISTENCE OF KERNEL FUNCTIONS FOR THE HEAT EQUATION IN N DIMENSIONS

by

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ON THE EXISTENCE OF KERNEL FUNCTIONS

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Abstract:

Let Ω be a bounded open set in $R^n \times (t_0, t_1)$ such that each cross section $\Omega_t = \Omega \cap (R^n \times \{t\})$ is star-like. We define

the lateral boundary $\partial_L \Omega \equiv \bigcup_{t \in (t_0, t_1)} \partial \Omega_t$

and

the parabolic boundary $\partial_P \Omega \equiv \partial_L \Omega \cup \Omega_{t_0}$

where Ω_{t_0} denotes the base of Ω.

Theorem 1.1: Let Ω be as above, then there exists a function u such that u is continuous in $\Omega \cup \partial_L \Omega$, $u > 0$ in Ω, $u = 0$ on $\partial_L \Omega$, and u is caloric in Ω.

Theorem 122: Suppose the boundary of Ω extends continuously to a point (x', t_0) in the boundary of the base. Then there exists a kernel function in Ω at the point (x', t_0).

Theorem 1.3: There exists a kernel function at an interior point (x_0, t_0) of the base of Ω.

If we restrict our attention somewhat we obtain the following asymptotic relations. Suppose $\alpha \in C^2(0,1]$,

$\alpha\alpha'' \in L^1(0,1)$, $\alpha(t) \to 0$ as $t \to 0$, and $\alpha > 0$ on $(0,T)$. Then for the region $D = \{(x,t) \in R^n \times (0,T)\colon |x| < \alpha(t)\}$ there exists a kernel function u satisfying the asymptotic relation

$$u \sim \varphi\left(\frac{x}{\alpha(t)}\right) \alpha(t)^{-\frac{n}{2}} \exp\left(-\lambda \int_t \alpha^{-2}\right)$$

where λ, φ are the first eigenvalue and eigenfunction for the Laplacian in $B(0,1)$ in R^n and $\int_t \alpha^{-2}$ represents a function whose derivative with respect to t is $\alpha^{-2}(t)$.

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FOR ELLENE AND MIA

On the Existence of Kernel Functions for the Heat Equation in n dimensions

Introduction: In their paper [4], B. F. Jones and C. C. Tu have demonstrated the existence of kernel functions for the heat equation

$$Hu = u_{xx} - u_t = 0$$

for certain domains in the x-t plane. The type of domain they considered was of the form

$$D = \{(x,t): \eta_1(t) < x < \eta_2(t),\ T_0 < t < T_1\}$$

where η_1, η_2 are continuous functions on (T_0, T_1). They were specifically interested in including regions with a bottom cusp (that is, regions for which $\eta_1(T_0) = \eta_2(T_0)$) and in demonstrating the existence of kernel functions at such points. We seek to extend these results to n space variables.

For the general case, we have found it necessary to restrict our attention to regions of the following form: Let Ω be a bounded open set in $R^n \times \{t_0, t_1\}$ such that each cross section $\Omega_t = \Omega \cap (R^n \times \{t\})$ is star-like. The modulus of a boundary point in direction $\theta = (\theta_1, \ldots, \theta_{n-1})$ (with respect to some fixed coordinate system) at time t will be given by the function $\beta(\theta, t)$. We will denote the base of Ω by Ω_{t_0} and define it to be the set of points (r, θ, t_0)

such that $r \le \liminf_{t \to t_0} |\beta(\theta,t)|$.

The <u>lateral</u> boundary $\partial_L\Omega \equiv \bigcup_{t\in(t_0,t_1)} \partial\Omega_t$

and

the <u>parabolic</u> boundary $\partial_P\Omega \equiv \partial_L\Omega \cup \Omega_{t_0}$.

<u>Definition</u>: Let $P \in \partial_P\Omega$ and let u be a caloric function (that is $Hu = \Delta u - u_t = 0$) in Ω. Then u is a <u>kernel</u> <u>function</u> at P if $u \ge 0$ and $u \not\equiv 0$ in Ω, and for all $Q \in \partial_P\Omega$, $Q \ne P$,

$$\lim_{(x,t)\to Q} u(x,t) = 0.$$

To expedite the proofs of the existence theorems, Jones and Tu proved a special Harnack inequality. The classical Harnack inequality for the heat equation states that if $K \subset \Omega$ is compact, and (x',t') is a point in Ω with later time coordinate than any point in K, then there exists a constant C depending only upon Ω,K and (x',t') such that for any non-negative caloric function u in Ω

$$\sup_K u \le C\, u(x',t') .$$

The one directional aspect of the inequality is an obstacle to the flow of the existence proofs; the obstruction is removed by a hybrid Harnack inequality.

Our contribution has been chiefly the extension of this inequality to n space variables; the proofs of the existence theorems appear almost verbatim from the original with only minor modification for the change in dimension. Despite the lack of originality here, we of course include the proofs for the sake of completeness. Following the proof of the Harnack inequality we include two theorems which show that in certain restricted instances, there exist kernel functions which obey particular asymptotic laws.

Preliminaries: We wish to utilize the Appell transform under which "caloricity" is preserved. If $E(x,t)$ is the fundamental solution for the heat equation, and u is caloric in a subset of $R^n \times (-\infty,0)$ then the Appell transform of u,

$$u^*(x,t) = E(x,t) \cdot u(\frac{x}{t}, -\frac{1}{t})$$

is caloric in the corresponding subset of $R^n \times (0,\infty)$.

Suppose u is defined on an open set $\Omega \subset (R^n \times (t_0,t_1))$ with continuous second partials in each space variable and continuous first derivative in t. Then u is supercaloric (subcaloric) if and only if $Hu \leq 0$ $(Hu \geq 0)$. (More precisely, we define $u \in \mathcal{D}'(\Omega)$ to be supercaloric (subcaloric) if $Hu \leq 0$ $(Hu \geq 0)$). If u_1, u_2 are subcaloric and supercaloric respectively, and if $u_1 \leq u_2$ then there exists a caloric function u, $u_1 \leq u \leq u_2$ which can be taken to be the greatest caloric minorant of u_2, or the least caloric majorant of u_1.

We shall assume the lateral boundary of Ω is smooth enough so that each boundary point is <u>regular</u> (see Petrovsky [5]). If $q \in \partial\Omega$ is regular, then there exists a <u>barrier</u> for q, that is, a function b which satisfies the following:

1) b is defined and supercaloric in $\Omega \cap B$, B some neighborhood of q
2) $b > 0$ in $\Omega \cap B$
3) $\lim_{p \to q} b(p) = 0$.

We shall require a second Harnack principle. If a sequence of caloric functions is uniformly bounded on each compact subset of Ω, then there exists a subsequence which converges uniformly on compact subsets of D to a caloric function.

1. <u>Harnack Inequality</u>: Let Ω be a bounded open set in $R^n \times (t_0, t_1)$ with smooth lateral boundary (C^2 is sufficient). We assume that each cross section Ω_t is star-like. Let $P_0 \in \Omega$ and $t_0 < \epsilon < t_1$. Then there exists a constant B, dependent only upon Ω, P_0, and ϵ, such that if u is continuous on $\Omega \cup \partial_p\Omega$, $u \geq 0$, $Hu = 0$ in Ω, and $u = 0$ on $\partial_L\Omega$, then

$$\sup_{\substack{(x,t)\in \Omega \\ \epsilon \leq t}} u(x,t) \leq Bu(P_0) \ .$$

Theorem 1.1: Let Ω be as above, then there exists a function u such that u is continuous in $\Omega \cup \partial_L \Omega$, $u > 0$ in Ω, $u = 0$ on $\partial_L \Omega$, and u is caloric in Ω.

Proof: Choose a point $P_0 \in \Omega$. For $s \in (t_0, t_1)$, we let u_s denote a solution of the first boundary value problem in $\Omega^s = \{(x,t) \in \Omega: t > s\}$ having zero boundary values on $\partial_L \Omega^s$, and non-negative initial values. We normalize each u_s so that $u_s(P_0) = 1$. Let $t_0 < r' < r < t_1$. The Harnack inequality implies that there exists a constant $B_{r,r'}$ dependent only upon Ω, P_0, r and r' such that

$$\sup_{\substack{(x,t) \in \Omega \\ r \leq t}} u_s(x,t) \leq B_{r,r'}$$

for all s such that $t_0 < s < r'$. The Harnack principle tells us that there exists a monotone sequence $\{s_n\}$ with $s_n \to t_0$ such that u_{s_n} converges locally uniformly on Ω to a caloric function u, with $u(P_0) = 1$ and $u \geq 0$ in Ω. If $Q \in \partial_L \Omega$ there exists a barrier b in $\Omega \cap G$, where G is some small neighborhood of Q. The barrier has a continuous extension to $\partial_P(\Omega \cap G)$ which is ≥ 0 except at Q and $b(Q) = 0$. Therefore b has a positive lower bound on $\partial_P(\Omega \cap G) - \partial_L \Omega$, and $u_s \leq B$ (B depends only on Q and Ω). Then by the maximum principle, there exists a constant B' independent of s such that

$$u_s \leq B' b \text{ in } \Omega \cap G$$

which implies

$$u \le B'b \text{ in } \Omega \cap G$$

and thus

$$\lim_{(x,t)\to Q} u(x,t) = 0 \ .$$

Since u extends continuously to $\Omega \cap \partial_L \Omega$ and $u = 0$ on $\partial_L \Omega$, the maximum principle implies $u > 0$ in Ω.

Theorem 1.2: In addition to the above hypotheses, we will suppose the boundary of Ω extends continuously to a point (x', t_0) in the boundary of the base. Then there exists a kernel function in Ω at the point (x', t_0).

Proof: Let u_s be as in the previous proof with the additional proviso that the initial values for u_s are zero for $(x,s) \in \Omega$ such that $|x'-x| > \varepsilon(s)$ where $\varepsilon(s) \to 0$ as $s \to t_0$. To demonstrate the existence of a kernel function, we must show that $u = \lim_{s\to t_0} u_s$ vanishes everywhere on the base excluding (x', t_0). If we extend the definition of u_s such that $u_s(x,t) = 0$ for $t < s$, it is a property of the heat equation that u_s is still a solution across that portion of the cross section where u_s vanishes.

Suppose $Q = (x, t_0)$ is an interior point of the base. Our Harnack inequality implies u_s is uniformly bounded near

Q, and an argument involving the barrier as above implies

$$\lim_{p \to Q} u(p) = 0 .$$

An auxiliary argument is necessary for points $P \neq (x', t_0)$ in the boundary of the base. Construct a smooth surface σ inside $\Omega \cap (R^n \times [t_0, t_0+\delta])$ such that σ intersected with the boundary of Ω encloses a region Ω' of $\Omega \cap (R^n \times [t_0, t_0+\delta])$ which excludes (x', t_0). The Harnack inequality can be used as previously to show that u_s is uniformly bounded by M on σ. The maximum principle then requires that $u_s \leq M$ on Ω'. Employment of the barrier once more shows that $u(P_0) = 0$ and thus u is in fact a kernel function at (x', t_0).

Theorem 1.3: There exists a kernel function at an interior point (x_0, t_0) of the base of Ω.

Proof: Without loss of generality we will assume $(x_0, t_0) = (0,0)$. By our hypothesis $\liminf_{t \to t_0} |\beta(\theta, t)| > 0$ for all θ, which implies there exist positive numbers ϵ, δ such that $|\beta(\theta, t)| \geq \epsilon$ for $0 < t \leq \delta$.

Let E be the fundamental solution

$$E(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} .$$

Then for $0 < t \leq \delta$

$$E(\beta(\theta,t),t) \leq \frac{1}{(4\pi t)^{n/2}} e^{\frac{-\epsilon^2}{4t}} \quad \text{for all } \theta.$$

Therefore there exists an increasing continuous function φ on $[0,t_1)$ such that $\varphi(0) = 0$ and $E(\beta(\theta,t),t) \leq \varphi(t)$ for all θ. For $s > 0$, let v_s be the solution of the first boundary value problem for the heat equation in Ω^s such that

$$v_s = \begin{cases} E & \text{on } \partial_L\Omega^s \\ 0 & \text{in } \Omega^s \text{ for } t = s \end{cases} .$$

As a consequence of the maximum principle, $v_s(x,t) \leq \varphi(t)$ for $s < t$, and if $0 < s' < s$ then $v_{s'} \geq v_s$ in Ω^s. Therefore the function

$$v = \lim_{s \to 0} v_s$$

exists in Ω, and $v(x,t) \leq \varphi(t)$. In addition, an argument using barriers implies $v = E$ on $\partial_L\Omega$, and by the maximum principle $v_s \leq E$ and thus $v \leq E$. Therefore the kernel function we are seeking is $u = E - v$.

<u>Proof of Harnack Inequality</u>: By assumption, the lateral boundary is smooth enough so that there exist positive numbers c, δ such that if $\Omega_0 = \Omega \cap (R^n \times \{t_0\})$

$$\Omega_1 = \{(x,t): (\tfrac{xc}{2c-t}, t_0) \in \Omega_0,\ t_0 < t < t_0+\delta\}$$

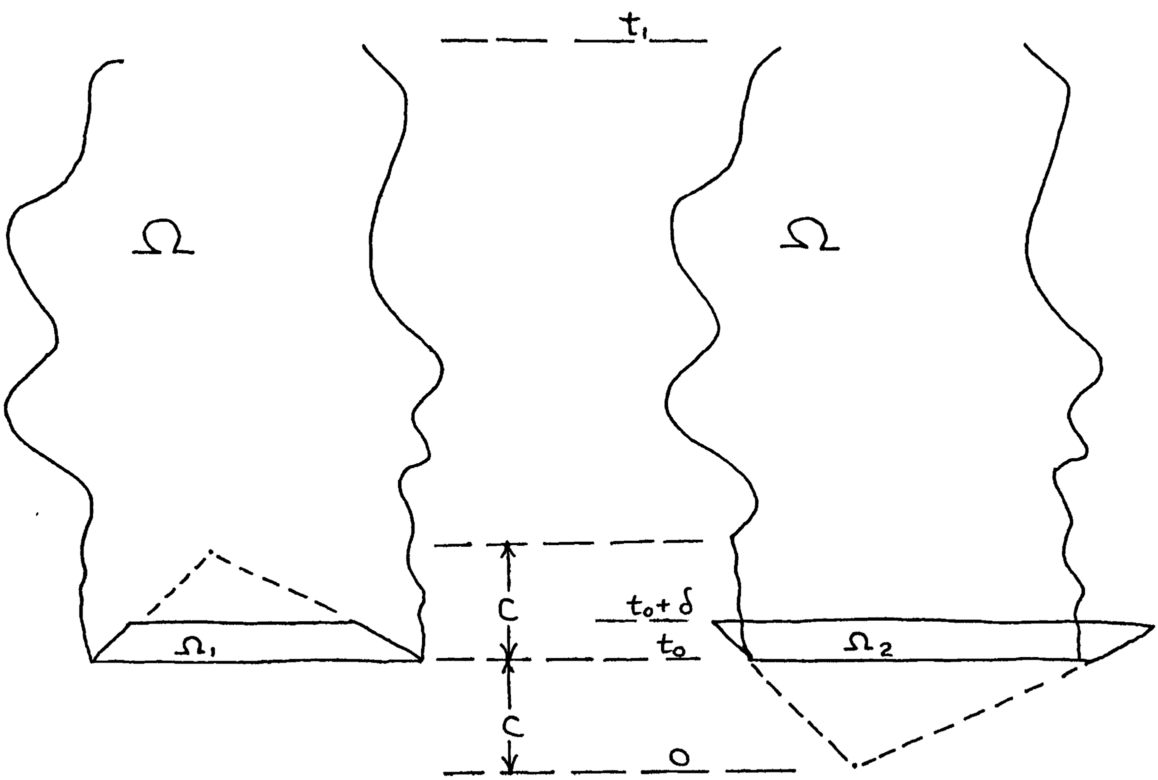
and

$$\Omega_2 = \{(x,t): (\tfrac{xc}{t}, t_0) \in \Omega_0,\ t_0 < t < t_0+\delta\}$$

then

$$\Omega_1 \subset \Omega \cap (R^n \times (t_0, t_0+\delta)) \subset \Omega_2 \ .$$

(Here we are assuming that $0 < t_0$ and that $c = t_0$.)



We will let u_i denote the solution of the first boundary value problem for the heat equation in Ω_i which vanishes on the lateral boundary and satisfies $u_i(x,0) = u(x,0)$ for $x \in \Omega_0$. The maximum principle implies $u_1 \leq u$ in Ω_1 and $u \leq u_2$ in $\Omega \cap (R^n \times (t_0, t_0+\delta))$.

Let P_1 be any point in the interior of the upper base of Ω_1 (i.e., $\Omega_1 \cap \{t_0 + \delta\}$) and let $(x, t_0+\delta)$ be any point in the upper base of Ω_2. We will eventually prove the inequality.

$$u_2(x, t_0+\delta) \leq B_1 u_1(P_1) \tag{1}$$

where B_1 is a constant depending only upon c, δ, P_1 and Ω_0. Now (1) implies

$$u(x, t_0+\delta) \leq B_1 u(P_1) \tag{2}$$

for all $(x, t_0+\delta) \in \Omega$.

If we choose δ small enough so that P_1 lies below P_0, then the classical Harnack inequality implies

$$u(P_1) \leq B_2 u(P_0)$$

where B_2 depends only upon Ω, P_0 and P_1. The inequality (2) thus implies

$$u(x, t_0+\delta) \leq B_1 B_2 u(P_0) ,$$

which by the maximum principle implies $u(x,t) \leq B_1 B_2 u(P_0)$ for all $(x,t) \in \Omega$ such that $\delta \leq t$. If δ is chosen smaller than the ϵ in our hypothesis, the inequality is proven modulo the proof of (1).

Derivation of (1):

Suppose $U(x,t)$ is a non-negative solution of the heat equation in a cylinder set $D \times (0,T)$, where $D \subset R^n$ such that

U is continuous on $\bar{D}\times[0,T]$, and $U=0$ on the lateral boundary of $D\times(0,T)$. Then U has the following representation:

$$U(x,t) = \int_{D\times\{0\}\equiv D_0} K(x,t,y)U(y,0)dy$$

where

$$K(x,t,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\ \varphi_n(y)\ , \tag{3}$$

where the λ_n's are the eigenvalues and the φ_n's the orthonormal eigenfunctions associated with

$$\begin{cases} \Delta\varphi_n = -\lambda_n\varphi_n & \text{on } D_0 \\ \varphi_n = 0 & \text{on } \partial D_0\ . \end{cases}$$

It is easily seen that $HK = 0$ for each fixed y, and $K > 0$ for $x,y \in D\times(0,T)$ by the maximum principle.

We would now like to estimate K in terms of the first eigenfunction which we assume to be positive since the first eigenfunction does not change signs; (see [1], p. 143-144) therefore we need an estimate of the form

$$|\varphi_k| \le C_k\varphi_1\ .$$

(In order to make the estimate useful, we will need some control over C_k; we in fact will have $C_k = 0(\lambda_k^m)$.)
We have

$$\varphi_k(x) = \lambda_k \int_{D_0} G(x,y)\varphi_k(y)dy \tag{4}$$

where $G(x,y)$ is the Green's function for D_0. If $D \subset R^n$,

$n \le 3$, then by the Schwarz inequality

$$\varphi_k(x) \le \lambda_k [\int_{D_0} (G(x,y))^2 \, dy]^{\frac{1}{2}} [\int_{D_0} (\varphi_k(y))^2 \, dy]^{\frac{1}{2}} .$$

For $n \le 3$, $G(x,y)$ is square integrable and $[\int_{D_0} (\varphi_k(y))^2 \, dy] = 1$.

Therefore $\varphi_k \le c$. For $n \ge 4$ we define

$$\begin{aligned} G^1(x,y) &= G(x,y) \\ &\vdots \\ G^{k+1}(x,y) &= \int_{D_0} G(x,y) G^k(x,y) \, dy . \end{aligned}$$

$G^m \le A$ for large enough m since

$$G^2(x,y) \le \int_{D_0} \frac{1}{|x-y|^{n-2}} \frac{1}{|y-z|^{n-2}} \, dy \le \frac{A}{|x-z|^{n-4}} , \text{ etc.}$$

Thus we have

$$\begin{aligned} \varphi_k &= \lambda_k G \circ \varphi_k \\ G \circ \varphi_k &= G \circ (\lambda_k G \circ \varphi_k) \\ \varphi_k = \lambda_k G \circ \varphi_k &= \lambda_k G \circ (\lambda_k G \circ \varphi_k) = \lambda_k^2 G^2 \varphi_k \\ &\vdots \\ \varphi_k &= \lambda_k^m G^m \varphi_k \le C \lambda_k^m . \end{aligned}$$

Employing this estimate in (4) implies

$$\varphi_k \le \lambda_k \int_{D_0} G(x,y) \, C \, \lambda_k^m \, dy \le C_k \int_{D_0} G(x,y) \, dy .$$

We now claim that

$$g(x) \equiv \int_{D_0} G(x,y)dy \le C'\varphi_1(x).$$

We know that $G(x,y)$, $\varphi_1(x)$ are equal to zero $(x \neq y)$ on the boundary of D_0 and > 0 in D_0. Let

$$D_{0,\eta} = \{x \in D_0 : d(x,\partial D_0) > \eta\} \qquad (\eta \ll \text{diam } D_0) .$$

There exist positive constants μ, ν such that $\varphi_1 \ge \mu$ on $D_{0,\eta}$ and $G(x,y) \le \nu$ on $D_0 \sim B(x,\eta)$. For $x \in D_{0,\eta}$

$$g(x) = \int_{D_0} G(x,y)dy = \int_{B(x,\eta)} G(x,y)dy + \int_{D_0 \sim B(x,\eta)} G(x,y)dy .$$

The first integral on the right is less than

$$c \int_{B(x,\eta)} \frac{dy}{|x-y|^{n-2}} \qquad \text{for } n \ge 3$$

$$c \int_{B(x,\eta)} \log \frac{1}{|x-y|} dy \qquad \text{for } n = 2 .$$

The second integral is bounded by ν times the measure of $D_0 \sim B(x,\eta)$. Therefore we have

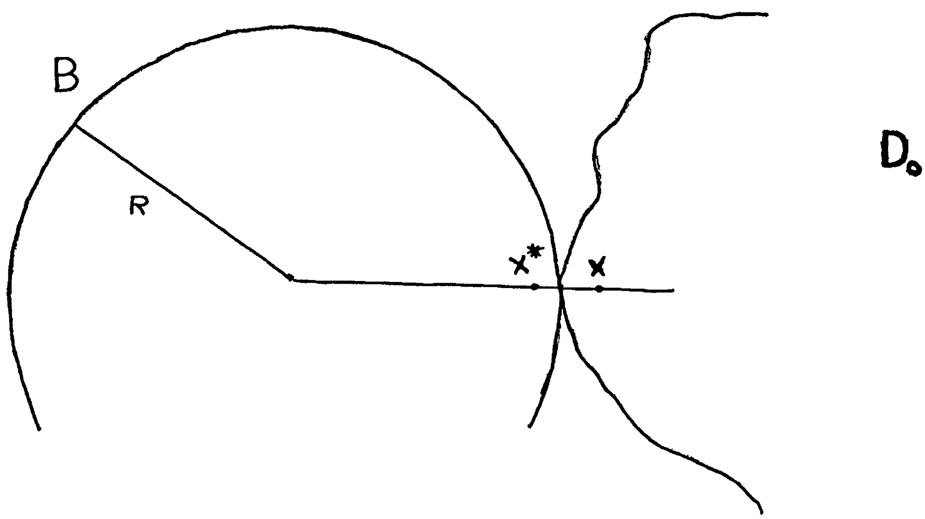
$$g(x) \le \text{constant} \le C\mu \le C\varphi_1(x) \quad \text{for some constant } C.$$

If $x \in D_0 \sim D_{0,\eta}$ we claim

$$g(x) \le C'd(x,\partial D_0) \le C''\varphi_1(x) .$$

The second inequality is clear because the inward normal derivative of φ_1 at the boundary of D_0 is positive [see

Protter and Weinberger [6], p. 67). To prove the first inequality, we draw the shortest normal to the boundary from x, and then construct the largest possible exterior ball B, tangent to D_0 at the intersected boundary point.



We know that G(x,y) is less than or equal to $G_{\sim B}(x,y)$ the Green's function for the complement of the ball.

Case 1. n = 2. We assume for notational convenience that B is a ball of radius R with center at the origin. The inverse of x with respect to B is denoted as x^*.

$$G_{\sim B}(x,y) = \log \frac{|x|\ |z-x^*|}{R|z-x|}$$

We have $d = d(x, \partial D_0) \sim (|x| - |x^*|)$. Let Q be a compact set containing D_0. Then

$$g(x) \le \int_{|z-x|<100d} \log \frac{|x|\ |z-x^*|}{R|z-x|}\, dz$$

$$+ \int_{z\in Q\sim(B\cup\{|z-x|<100d\})} \{\log \frac{|x|}{|z-x|} - \log \frac{|x^*|}{|z-x^*|}\}\, dz$$

since $\log \frac{|x^*|}{|z-x^*|} \le \log \frac{R}{|z-x^*|}$. The first integral is less than or equal to

$$\int_{|z-x|<100d} \log \frac{2R(|z-x|+|x-x^*|)}{R\ |z-x|}\, dz \le \int_{|z-x|<100d} \log\ (2+\frac{cd}{|z-x|})dz$$

(let $z-x = dw$)

$$= d \int_{|w|<100} \log(2+\frac{c}{|w|})dw = \text{const}\cdot d \ .$$

Let $f(y) = \log \frac{|y|}{|z-y|}$. Then by the mean value theorem

$$\log \frac{|x|}{|z-x|} - \log \frac{|x^*|}{|z-x^*|} \le |x-x^*|\ |\nabla f(y)|$$

$$= |x-x^*|\ |\frac{y}{|y|^2} + \frac{(z-y)}{|z-y|^2}|$$

$$\le |x-x^*|\ (\frac{1}{|y|} + \frac{1}{|z-y|})$$

where y is between x and x^*. Therefore the second integral is less than

$$|x-x^*| \int_{z\in Q\sim(B\cup\{|z-x|<100d\})} (\frac{1}{|y|} + \frac{1}{|z-y|})\, dz$$

$$\le |x-x^*| \int_{z\in Q\sim(B\cup\{|z-x|<100d\})} (\frac{1}{|x^*|} + \frac{1}{50d})\, dz$$

$$= C\ |x-x^*|$$

and thus

$$g(x) \le c'd(x, \partial D_0) \ .$$

Case 2: $n \ge 3$.

$$G_{\sim B}(x,y) = \frac{1}{|z-x|^{n-2}} - \frac{R^{n-2}}{(|x|\ |z-x^*|)^{n-2}} \ .$$

As above

$$g(x) \le \int_{z\in Q\sim B} \{\frac{1}{|z-x|^{n-2}} - \frac{R^{n-2}}{|x|^{n-2}|z-x^*|^{n-2}}\}\, dz$$

$$\le \frac{1}{|x|^{n-2}} \int_{z\in Q\sim B} \{(\frac{|x|}{|z-x|})^{n-2} - (\frac{|x^*|}{|z-x^*|})^{n-2}\}\, dz$$

$$\le \int_{|z-x|<100d} \frac{dz}{|z-x|^{n-2}}$$

$$+ \frac{1}{|x|^{n-2}} \int_{z\in Q\sim(B\cup\{|z-x|<100d\})} \{(\frac{|x|}{|z-x|})^{n-2} - (\frac{|x^*|}{|z-x^*|})^{n-2}\}dz \ .$$

The first integral is less than cd (let $z-x = dw$ as above).

Now let $f(y) = \frac{|y|^{n-2}}{|z-y|^{n-2}}$, then as before

$$(\frac{|x|}{|z-x|})^{n-2} - (\frac{|x^*|}{|z-x^*|})^{n-2} \le |x-x^*|\ |\nabla f(y)|$$

$$= |x-x^*|\ \frac{n-2}{|z-y|^n}\ |y|y|^{n-4}|z|^2 + (z-y)|y|^{n-2}|$$

$$\le |x-x^*|\ \frac{n-2}{|z-y|^n}\ (|x|^{n-3}\ |z|^2 + |x|^{n-2}(z-y)) \ .$$

The second integral is less than

$$\frac{(n-2)|x-x^*|}{|x|^{n-2}} \int_{z\in Q\sim(B\cup\{|z-x|<100d\})} \frac{|x|^{n-3}|z|^2 + |x|^{n-2}(z-y)}{|z-y|^n}\, dz$$

$$\le C\ |x-x^*|$$

and again $g(x) \le c'd(x,\partial D_0)$ which implies $g(x) \le c''\varphi_1(x)$. By (3)

$$K(x,t,y) \le \sum_{k=1}^{\infty} c_k^2 e^{-\lambda_k t} \varphi_1(x)\, \varphi_1(y) = M(t)\varphi_1(y)\,.$$

(We note that by our estimates $c_k = 0(\lambda_k^m)$ and thus $M(t) < \infty$.)

Since φ_1 and k are both > 0 in D_0 and both have positive inward normal derivatives at each boundary point of D_0 (see [6], p. 170), and since neither can have a positive minimum in D_0 (by the maximum principle), there exist positive constants $m(y,t)$, $m(x,t)$ such that

$$K(x,t,y) \ge m(y,t)\varphi_1(x) \qquad K(x,t,y) \ge m(x,t)\varphi_1(y)$$

These inequalities yield

$$(*) \qquad m(x,t) \int_{D_0} U(y,0)\varphi_1(y)\, dy \le U(x,t)$$
$$\le M \int_{D_0} U(y,0)\varphi_1(y)\, dy.$$

We now return to Ω_1 and Ω_2. If we make the change of variable $t' = 2c - t$, then

$$\Omega_1 = \{(x,t'): (\tfrac{xc}{t'},t_0) \in \Omega_0\ ,\ \ t_0 - \delta < t' < t_0\}\ .$$

If u_1 is a solution to the heat equation in Ω_1 then u_1^*, the Appell transform of u_1 is a solution in

$$\Omega_1^* = \{(z,s'): (zc,t_0) \in \Omega_0\ ,\ \ \frac{1}{\delta - t_0} < s' < \frac{1}{t_0}\ \}\ .$$

Let (z_1, s_1') in Ω_1^* correspond to the point P_1 in Ω_1. By (*)

$$m(z_1, s_1') \int_{\{z:\ (zc, t_0) \in \Omega_0\}} u_1^*(z_1 - \frac{1}{t_0}) \varphi_1(cz) dz \le u_1^*(z_1, s_1').$$

By definition of the Appell transform, this implies

$$m(z_1, s_1') \int_{\{y \in \Omega_0\}} E(\frac{y}{c}, -\frac{1}{t_0}) u(y, t_0) \varphi_1(y) dy \le E(z_1, s_1') u_1(P_1)$$

and therefore

$$m' \int_{\{y \in \Omega_0\}} u(y, t_0) \varphi_1(y) dy \le u_1(P_1) \ ,$$

where m' is a constant dependent only on Ω_0, δ, c and P_1.

If u_2 is a solution in Ω_2, then u_2^* satisfies the heat equation in

$$\Omega_2^* = \{(z,s):\ (zc, t_0) \in \Omega_0 \ , \ -\frac{1}{t_0} < s < -\frac{1}{t_0+\delta}\} \ .$$

By (*)

$$u_2^*(z, -\frac{1}{t_0+\delta}) \le M \int_{\{\bar{z}:\ (\bar{z}c, t_0) \in \Omega_0\}} u_2^*(\bar{z}, -\frac{1}{t_0}) \varphi_1(c\bar{z}) d\bar{z}$$

or

$$E(z, -\frac{1}{t_0+\delta})\ u_2(x, t_0+\delta) \le M \int_{\{y \in \Omega_0\}} E(\frac{y}{c}, -\frac{1}{t_0})\ u\ (y, t_0) \varphi_1(y) dy \ ,$$

which implies

$$u_2(x, t_0+\delta) \le M' \int_{\{y \in \Omega_0\}} u_2(y, t_0) \varphi_1(y) dy$$

where M' depends only upon Ω_0, c, δ. Thus $u_2(x, t_0+\delta) \le \frac{M'}{m'} u_1(P_1)$ proving (1).

Theorem 2.1: Let $\alpha \in C^2(0,1]$ and let $\alpha\alpha'' \in L^1(0,1)$. Assume $\alpha(t) \to 0$ as $t \to 0$ and that $\alpha > 0$ on $(0,T)$. Let $D = \{(x,t) \in R^n \times (0,T): |x| < \alpha(t)\}$. Then there exists a kernel function u satisfying the asymptotic relation

$$u \sim \varphi(\frac{x}{\alpha(t)})\alpha(t)^{-\frac{n}{2}}\exp(-\lambda \int_t \alpha^{-2}) \qquad (2.1)$$

where λ, φ are the first eigenvalue and eigenfunction for the Laplacian in $B(0,1)$ in R^n, and $\int_t \alpha^{-2}$ represents a function whose derivative with respect to t is $\alpha(t)^{-2}$.

Note: By lemma 4.2 of [4] we have $\lim_{t\to 0} \alpha\alpha' = 0$.

Proof: Mimicking the proof of theorem 4.1 of [4], we will construct a supercaloric function u_1 and a subcaloric function u_2 with $u_1 \geq u_2$ such that both functions obey the asymptotic relation. This will of course imply the existence of a caloric function u with $u_1 \geq u \geq u_2$. Let

$$w = \varphi(\frac{x}{\alpha(t)}) \exp(-\lambda g(t) - \frac{h(t)|x|^2}{4})$$

where g(t), h(t) will be determined below. Applying the heat operator, we have

$$\frac{Hw}{w} = \frac{\varphi'}{\varphi}\left[-\frac{hx}{\alpha} + \frac{\alpha' x}{\alpha^2}\right] + \frac{|x|^2}{4}[h^2 + h'] - \frac{\lambda}{\alpha^2} + \lambda g' - \frac{nh}{2} \qquad (2.2)$$

where φ' denotes $\frac{d\varphi}{dy}$ with $y = \frac{x}{\alpha(t)}$. Choose $h = \frac{\alpha'}{\alpha}$. Then by (2.2) we have

$$\frac{Hw}{w} = \frac{1}{4}|x|^2(\frac{\alpha''}{\alpha}) - \frac{\lambda}{\alpha^2} - \frac{n\alpha'}{2\alpha} + \lambda g' .$$

Choose

$$g_1(t) = \int_t \alpha^{-2} + \frac{\frac{n}{2}\log\alpha}{\lambda} - \frac{1}{4\lambda}\int_0^t \alpha|\alpha''|$$

$$g_2(t) = \int_t \alpha^{-2} + \frac{\frac{n}{2}\log\alpha}{\lambda} + \frac{1}{4\lambda}\int_0^t \alpha|\alpha''| \quad .$$

Let u_i be the function w with $g = g_i$. Then

$$\frac{Hu_1}{u_1} = \frac{1}{4}|x|^2(\frac{\alpha''}{\alpha}) - \frac{\lambda}{\alpha^2} - \frac{n}{2}\frac{\alpha'}{\alpha} + \lambda[\frac{1}{\alpha^2} + \frac{n\alpha'}{2\lambda\alpha} - \frac{1}{4\lambda}\alpha|\alpha''|]$$

$$\frac{Hu_2}{u_2} = \frac{1}{4}|x|^2(\frac{\alpha''}{\alpha}) - \frac{\lambda}{\alpha^2} - \frac{n}{2}\frac{\alpha'}{\alpha} + \lambda[\frac{1}{\alpha^2} + \frac{n\alpha'}{2\lambda\alpha} + \frac{1}{4\lambda}\alpha|\alpha''|] \quad .$$

Since $|\frac{x^2\alpha''}{4\alpha}| \leq \frac{1}{4}\alpha|\alpha''|$, it follows that

$$Hu_1 \leq 0 \quad \text{and} \quad Hu_2 \geq 0 \; .$$

Now $\int_0^t \alpha|\alpha''| \to 0$ as $t \to 0$, and the right side of

$$|\frac{h(t)x^2}{4}| \leq \frac{1}{4}\alpha(t)\;|\alpha'(t)|$$

tends to zero by our note. Thus both u_1 and u_2 satisfy the asymptotic relation (2.1), and because $u_1 \geq u_2$, the theorem is proved.

<u>Theorem 2.2</u>: Let $D = \{(x,t) \in R^n \times (-\infty,T): |x| < \alpha(t)\}$ where $\alpha \in C^2$, α is positive on $(-\infty,T)$ and

$$\lim_{t\to-\infty}\frac{\alpha(t)}{t} = 0 \quad , \quad \alpha\alpha'' \in L^1(-\infty,T) \; . \tag{2.3}$$

Then there exists a kernel function at P_∞ (an ideal point used when the region has no base) satisfying the asymptotic relation

$$u \sim \varphi\left(\frac{x}{\alpha(t)}\right)\alpha(t)^{-\frac{n}{2}} \exp\left(-\lambda \int_t \alpha^{-2}\right) \quad .$$

<u>Proof</u>: Let u^* be the Appell transform. Then $u^*(z,s)$ is defined in the domain

$$D^* = \{(z,s) : |z| < s\alpha(-\tfrac{1}{s})\} \; .$$

By hypothesis

$$\lim_{s\to 0} \alpha^*(s) \equiv \lim_{\substack{s\to 0 \\ s\to 0}} s\alpha(-\tfrac{1}{s}) = \lim_{t\to\infty} -\frac{\alpha(t)}{t} = 0 \; .$$

It can be shown that

$$\alpha^*(s)\alpha^*(s)'' \, ds = \alpha(-\tfrac{1}{s})\alpha''(-\tfrac{1}{s}) \; d(-\tfrac{1}{s})$$

so that (2.3) implies that $\alpha^*(s)$ satisfies the hypothesis of theorem 2.1. Therefore, there exists a kernel function u^* for D^* at (0,0) which satisfies

$$u^*(z,s) \sim \varphi\left(\frac{z}{\alpha^*(s)}\right) \alpha^*(s)^{-\frac{n}{2}} \exp\left(-\lambda \int_s \alpha^*(s)^{-2}\right) \; .$$

By definition of the Appell transform

$$u(x,t) \sim \varphi\left(\frac{x}{\alpha(t)}\right)(-t)^{-\frac{n}{2}} \bar{\alpha}(t)^{-\frac{n}{2}} E\left(-\frac{x}{t}, -\frac{1}{t}\right)^{-1} \exp\left(-\lambda \int_t \alpha^{-2}\right).$$

Now

$$(-\ t)^{-\frac{n}{2}}\ E(-\ \frac{x}{t}\ ,\ -\ \frac{1}{t})^{-1} = \exp\ (\frac{|x|^2}{4t})$$

and for some constant c

$$|\frac{x^2}{4t}| \le \frac{c\alpha(t)^2}{|t|} = \frac{c\alpha^*(s)^2}{s} \le c \int_0^s [(\alpha^*(s))']^2$$

by remark 4.3 of [4]. Therefore the term $\exp\ (\frac{|x|^2}{4t})$ does not affect the asymptotic relation as $t \to -\infty$.

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