ALEXANDER'S DUALITY THEOREM

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1. Introduction. Topology or Analysis Situs is usually defined as the study of properties of spaces or their configurations under continuous transformations. The invariants under these transformations play a leading role in the study of the subject, and it is with them that the duality theorems are concerned.

The first duality theorem was the theorem of Poincaré on the duality of the Betti numbers of orientable manifolds. This theorem is stated in terms of dual complexes and is quite different from the duality theorem of Alexander which concerns residual spaces. The latter theorem may also be stated in terms of Betti numbers, but it was proved for a different set of invariants, the connectivity numbers, by Alexander. The relation between the two kinds of invariants will be shown below.

The terminology and notation of Topology vary from one author to another. The definitions of such things as chains and circuits are not the same as given by different authorities. Wherever possible the notations and terminology of Veblen and Alexander have been followed.


Let \( \mathcal{N} \)-space be divided into two regions by \( \mathcal{N}+1 \) linearly independent \( (\mathcal{N}-1) \)-spaces. Then that region which does not contain the point at infinity is defined as an \( \mathcal{N} \)-dimensional simplex. The points of the boundary are
excluded. The interior of a tetrahedron, for example, constitutes a 3-dimensional simplex.

Any set of points in one-to-one correspondence with the points of an n-dimensional simplex constitutes an n-dimensional cell \( (n > 0) \). By this definition, which is a little more general than will be necessary in the latter part of this paper, the points of an arc of a curve joining two distinct points make up a one-cell.

The set of non-overlapping cells consisting of

\[
\begin{align*}
\alpha_0 & \quad \text{0-cells} \quad x_1^0 \quad x_2^0 \quad x_3^0 \quad \cdots \quad x_{\alpha_0}^0 \\
\alpha_1 & \quad \text{1-cells} \quad x_1^1 \quad x_2^1 \quad x_3^1 \quad \cdots \quad x_{\alpha_1}^1 \\
& \quad \vdots \\
\alpha_n & \quad \text{n-cells} \quad x_1^n \quad x_2^n \quad x_3^n \quad \cdots \quad x_{\alpha_n}^n
\end{align*}
\]

will constitute an n-dimensional complex \( C_n \) provided

1. The boundary of every \( i \)-cell \( (i > 0) \) is made up entirely of cells \( x_k^i \) of dimensionality less than \( i \).
2. Every \( i \)-cell \( (i < n) \) is on the boundary of some \((i+1)\)-cell \( x_k^{i+1} \).

The \((n-k)\)-cells on the boundary of an \( n \)-cell are said to be incident with it and it is said to be incident with them.

A complex \( C_n \) is called closed if every \((n-1)\)-cell is incident with an even number of \( n \)-cells. Otherwise it is open (or bounded) and the boundary consists of those \((n-1)\)-cells, together with their boundaries, upon which an odd number of \( n \)-cells abut.

If an \( n \)-dimensional complex \( C_n \) is such that

1. each \((n-1)\)-cell of \( C_n \) is incident with an even
number of n-cells, and

(2) no subset of the cells which constitute $C_n$ satisfies (1)
then the complex is said to define an n-circuit.

The set of all cells of a complex $C_n$ which are incident with an i-cell $a^i$ and of higher dimensionality than $a^i$ constitute, with $a^i$ itself, a star of cells. If the incidence relations between the $(i+j)$-cells $(j = 1, 2, \ldots, p-1)$ and the $(i+j+1)$-cells are the same as those between the $j$-cells and the $(j+1)$-cells of a p-dimensional sphere (see page 16), the star is said to be simply connected.

If $C_n$ is an n-circuit such that every star of its cells is simply connected, the set of points on $C_n$ is called an n-dimensional manifold.

3. Continuity and Homeomorphism.

A point $P$ is said to be a limit point of a set of points $X$ of a cell and its boundary if and only if the corresponding point $P'$ of the simplex by which the cell is defined is a limit point of the corresponding set $X'$ of the simplex and its boundary.

A transformation $T$ carrying a set of points $X$ into a set of points $X'$ is said to be continuous if it carries each limit point $P$ of $X$ into a limit point $P'$ of $X'$.

Two complexes are called homeomorphic if there exists a one to one continuous (both ways) transformation between them.


A set of cells of a complex $C_n$ will be called a chain
provided the set never contains a cell $E$ without containing all of the cells of its boundary. If a chain contains only $i$-cells and the cells of their boundaries, it is called an $i$-chain. Thus the points of a chain form a closed set.

The simplest $i$-chain is that which contains a single $i$-cell. It will be called a cellular $i$-chain.

The sum of two or more $i$-chains will be said to consist of those $i$-cells which belong to an odd number of the chains but no others. Hence the sum of two chains $K_1$ and $K_2$ would consist of the set of points given by $K_1 + K_2 - K_1K_2^t$.

For this reason any $i$-chain will be said to be the sum modulo 2 of the cellular $i$-chains determined by its individual cells. We write

\[(1) \quad K^i = K_1^i + K_2^i + K_3^i + \cdots + K_n^i \quad (\text{mod} \ 2)\]

In order to add two or more $i$-chains we express each in the form (1), add the corresponding components, and reduce coefficients modulo 2.

As in the case of a complex, an $i$-chain $K^i \ (i > 0)$ is called closed if each $(i-1)$-cell is on the boundary of an even number of $i$-cells. Otherwise it is open and the boundary $K^{i-1}$ will be the chain determined by the $(i-1)$-cells lying upon an odd number of $i$-cells. A 0-chain will be closed or open according as it consists of an even or odd number of points. In this paper a closed $i$-chain is the same as a set of $i$-circuits.

As an example of the addition of two $i$-chains we may consider the following simple diagram.
The boundary $K_i^i$ of an $i$-cell $E^i$ is a closed $(i-1)$-chain. For every $(i-2)$-cell of $K_i^{i-1}$ belongs to precisely two $(i-1)$-cells of $K_i^{i-1}$. The sum of two or more closed $i$-chains is a closed $i$-chain if it does not vanish.

5. **Boundaries of Chains. Congruences. Homologies**

The relation of the boundary $K_i^i$ of the chain $K^i$ to the chain $K_i^i$ may be expressed by the relation

\[(2) \quad K_i^i \equiv K_i^{i-1} \quad (\text{mod.} \ 2)\]

which is called a congruence and is read "$K_i^{i-1}$ bounds $K_i^i$". If $K_i^i$ is closed, $K_i^{i-1}$ does not exist and (2) becomes

\[K_i^i \equiv 0 \quad (\text{mod} \ 2)\]

It is clear that if two or more chains are added (mod 2) the boundary of the sum is the sum (mod 2) of the boundaries. Hence the relations

\[K_i^i \equiv K_i^{i-1} \quad (p = 1, 2, \ldots t) \quad (\text{mod} \ 2)\]

imply

\[\sum K_i^i \equiv \sum K_i^{i-1} \quad (\text{mod} \ 2)\]

An $i$-chain $K_i^i$ of a chain $C$ will be said to be homologous to zero, written

\[K_i^i \sim 0 \quad (\text{mod} \ 2)\]

provided $K_i^i$ is the boundary of some open $(i+1)$-chain $K_i^{i+1}$ of $C$. 
the i-chain is also said to bound on \( C \). The relations
\[ K^i_t \sim 0 \quad \text{and} \quad K^i_r \sim 0 \]
clearly imply \( K^i_t + K^i_r \sim 0 \).

Two chains \( K^i_t \) and \( K^i_r \) are said to be homologous, written
\[ K^i_t \sim K^i_r \], if \( K^i_t + K^i_r \sim 0 \). Because of the fact that a
linear combination of the homologies \( K^i_s \sim 0 \), \( i = 1, 2, \ldots, t \)
is an homology \( \sum K^i_s \sim 0 \), homologies can be handled like
linear equations modulo 2.

The notation "mod 2" will not be written hereafter
except where some ambiguity may arise.


The structure of any n-dimensional complex \( C_n \) may be
expressed by means of \( n \) matrices
\[ X_0, i, j, X_1, i, j, \ldots, X_{n-1}, n. \]
The matrix \( X_{i-1}, i \) is an array of \( a^i_\ell \) columns and \( a^i_i \) rows.
With each column is associated a distinct \( i \)-cell of \( C_n \)
and with each row a distinct \((i-1)\)-cell. In the \( j^{th} \) row
and \( k^{th} \) column there is a number \( v_{j, k} \) which is unity if the
cells which correspond to the \( j^{th} \) row and \( k^{th} \) column abut
and is zero otherwise. For example, the surface of a tetra-
hedron, a two-dimensional complex, may be described as
in the two matrices on the following page. The vertices,
faces, and edges of the figure may be numbered in any ar-
bitrary manner. These matrices are known as matrices of
incidence. Any two complexes \( C_n \) and \( \overline{C}_n \) which have the same
set of incidence matrices are in one-to-one correspondence
(continuous) with each other. For let the \( 0, 1, 2, 3, \ldots, n-1 \)
cells be given by the relations
\[ a^0_\ell, a^1_\ell, a^2_\ell, \ldots, a^n_\ell \]
and $l_i$, $l_j$, ..., $l^\pi$ respectively. Let the cells be numbered in such a way that whenever $a^i_j$ is incident with $a^i_1$ or $a^i_2$ or ... $a^i_\rho$ for any $i, j, k, ..., \rho$ then the $l^i_j$ for the same value of $j$ is incident with the $l^i_1$ or $l^i_2$ or ... or $l^i_\rho$ for the same values of $i, k, ..., \rho$. A one-to-one correspondence is then set up between $C_n$ and $\bar{C_n}$ by requiring

1. that $a^i$ correspond to $l^i$ for each $i$,
2. that $a^i_j$ and its ends correspond to $l^i_j$ and its ends for each $j$ in a one-to-one continuous correspondence such that the correspondence between the ends is that set up under (1).

\[ \vdots \]

(\(n+1\)) that $a^i_\rho$ and its boundary correspond to $l^i_\rho$ and its boundary in a one-to-one continuous fashion and such that the correspondence for the boundary shall be given
by (n).

The set of points on a complex whose matrices of incidence are all equal to
\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]
describes an n-sphere. We shall return to this later.

7. Sense of a Complex.

If a direction is assigned to every one-cell of a complex \( C_n \), the cell can be positively related to one end point and negatively related to the other. The sensed cell \( \chi'_i \) may be denoted by \( \pm a'_i \), where the + sign is to be taken with one notion of sense and the — sign with the other.

The method of choosing the sense may be related to the matrix \( \chi_{0,i} \). Leaving the zeros unchanged, we replace / by —/ wherever the relation of \{-cell to point is negative. Thus in the new matrix \( A_{0,i} \), the numbers / and —/ appear once in each column. Clearly the sign of one may be arbitrarily fixed, the sign of the other being then determined.

If two arcs (i.e., one-cells) abut on the same point \( P \), they are said to have the same sense if one is negatively related and the other positively related to \( P \). Otherwise they are said to be oppositely related or to have opposite sense.

By a sensed circuit is meant one in which each arc has the same sense as the two upon which it abuts. Thus two sensed one-circuits can be obtained from any one-
circuit by assigning either sense to any arc of the circuit. Hence the boundary of a 2-cell may have either of two senses.

A sense may also be assigned to the 2-cell itself. We say that it is positively related to the cells of the boundary taken in one sense and negatively related to the cells of the boundary taken in the other sense. Thus the sense of a 2-cell may be indicated by use of the matrix \( X_{i,2} \). The /'s are replaced by -/'s wherever it is desired to show that a 2-cell is negatively related to an adjacent one-cell, giving a matrix \( A_{i,2} \). If the sense of any one-cell is changed, then the signs of the \( \pm /'s \) in the corresponding columns must be altered. On the other hand, if the sense of a 2-cell is changed, the \( \pm /'s \) in the proper column must have their signs changed.

Two 2-cells which abut on a one-cell are said to have the same senses if they are oppositely related to the one-cell. The relation of a 2-cell to a one-cell is said to be positive if, when the one-cell is traversed in the positive direction, the 2-cell's boundary is traversed in the (assigned) positive direction. It is not true that, as in the linear case, we can always derive two sensed 2-circuits from an unsensed 2-circuit.

**Definition.** A two-circuit is said to be two-sided or orientable if two sensed 2-circuits can be derived from it. Otherwise it is called one-sided or non-orientable.

The projective plane is one-sided, whereas the surface of a tetrahedron is two-sided. This can be verified quite readily from the following two figures by assigning senses.
in order to the cells, determining each order from the preceding cell, and noting that we get a conflict in the first case but not in the second.

By proceeding in a similar fashion we may assign senses to the 3-cells, 4-cells, ... n-cells of the complex. Matrices $A_{i-1,i}$ can be set up which define the sense of each cell except the zero-cells, which have no sense. We thus get an n-dimensional oriented complex.

8. Betti Numbers.

Let us consider the equations

$$(A_{i-1,i}) \sum_{k=1}^{d_{i}} \epsilon_{j,k}^{i} X_{k}^{i} = 0 \quad (j=1,2,\ldots,d_{i-1})$$

which are related to the matrix $A_{i-1,i}$ in the following way. We consider the $X_{k}^{i}$ corresponding to the columns of $A_{i-1,i}$ as variables which can assume the values $0$, $1$, or $-1$. The numbers $\epsilon_{j,k}^{i}$ correspond to the elements of the matrix. We thus obtain an equation of the above form for each row—an equation related to each $(i-1)$-cell of $C_{n}$ and having as variables the numbers $X_{k}^{i}$ corres-
ponding to the $i$-cells which abut on the $(i-1)$-cell considered. The other $i$-cells have zero coefficients.

Every solution of the system $A_{i-1,i}$ marks every $i$-cell with $0$, $1$, or $-1$. If an $i$-cell appears twice in a circuit, let it be counted twice. Then the number of $1$'s and $-1$'s will always be equal for any solution, and the total number is either even or zero. It follows that a sensed circuit (or system of them) is defined since an even number of $i$-cells abuts on each $(i-1)$-cell. Conversely, of course, every sensed $i$-circuit is a solution of $A_{i-1,i}$. But the solutions of $A_{i-1,i}$ are each dependent upon a set of linearly independent solutions in the integers, so it may be said that the number of linearly independent sensed (and thus orientable) circuits is the same as the number of linearly independent solutions of $A_{i-1,i}$.

Because of the fact that the boundary of every $(i+1)$-cell taken in a definite sense is a sensed $i$-circuit, each column of the matrix $A_{i,i+1}$ must define a solution of the system $A_{i-1,i}$. Let the rank of $A_{i,i+1}$ be $\rho_{i,i+1}$. The number of such solutions which are linearly independent is then $\rho_{i,i+1}$. Combinations of these solutions give the sensed boundaries of open $(i+1)$-complexes. In general there are also non-bounding circuits which cannot be expressed linearly in terms of boundaries of cells.

Suppose that $\rho^{i}_{i-1}$ is the number of sensed $i$-circuits which must be added to the bounding $i$-circuits in order to obtain a complete set of linearly independent solutions.
Definition. \( \rho_l \) is the \( l \)-th Betti number of the complex \( C_n \).

9. Dual Complexes.

If two complexes \( C_n \) and \( C'_n \) are such that the points, arcs, \ldots, \( k \)-cells of one are in one-to-one correspondence with the \( n \)-cells, \((n-1)\)-cells, \ldots, \((n-k)\)-cells of the other in such a way that two cells of \( C_n \) abut on a common cell only if the two corresponding cells of \( C'_n \) do, and conversely, the two complexes are said to be dual to each other.

Let \( C_n \) define a manifold \( M_n \). Then there always exists a dual complex which defines the same manifold. This may be shown as follows. In the interior of each cell \( a^k_l \), \( k \geq 1 \), choose a point \( \rho(a^k_l) \). Join by an arc each \( \rho(a^2_l) \) to the vertices \( a^o_l \) and to the points \( \rho(a^1_l) \). We thus have the 2-cell divided into triangular regions or two-dimensional pyramids. We then join the points \( \rho(a^2_l) \) to the vertices of the triangle just formed, giving a tetrahedral region. The process is continued in this fashion, and we join \( \rho(a^n_l) \) to the vertices of the \((n-1)\)-dimensional pyramid formed in the induction. The result is a complex \( C'_n \) with every 2-cell bounded by three arcs, every 3-cell by four triangles, and every \( n \)-cell by \( n+1 \) \((n-1)\)-cells.

Since \( C_n \) defines a manifold, the \( n \)-cells of \( C'_n \) which abut on an \( a^l \) together with the \((n-1)\)-cells which separate them from each other, constitute an \( n \)-cell \( E_n \). Let us call it \( \mathcal{E}_n \). Upon the boundaries of the \( n \)-cells
$E_n$ are the points $a_i^1$. The $(n-1)$-cells of $C_n$ which abut on $a_i^1$, together with the $(n-2)$-cells which separate them constitute an $(n-1)$-cell $E_n-1$ which we call $l_i^{n-1}$. In general, then, the $(n-k)$-cells of $C_n$ which abut on an $a_i^k$, together with their boundaries, constitute an $(n-k)$-cell $E_n-k$. Clearly the $(n-k)$-cells of $C_n'$ are then in one-to-one correspondence with the $k$ cells of $C_n$, and if two cells $a_{i}^{j}$ and $a_{i}^{j+1}$ have a cell $a_{i}^{j+1}$ in common, then the two cells $l_{i}^{n-j}$ and $l_{i}^{n-j-1}$ have a cell $l_{i}^{n-j-1}$ in common. Because of the one-to-one correspondence and the nature of the relations between the cells of $C_n$ and $C_n'$, it follows that the matrix $X_{i,j+1}$ in the representation of $C_n$ is the same as the matrix $X'_{n-i,j-1}$; i.e. as the transposed of the matrix $X'_{n-i,j-1}$.

Thus the matrices $X_{i,j+1}$ and $X'_{n-i,j-1}$ have the same rank.

Because of the fact that many complexes may give rise to the same manifold, the problem arose to determine the invariants, if any, which characterize the matrices of all complexes associated with the same manifold. Poincaré found that the Betti numbers are such invariants and proved that if the manifold is two-sided, then the Betti numbers satisfy the duality relation

$$P^{i} = P^{n-i}$$

10. Connectivity Numbers.

In § 8 we considered a set of equations associated with the matrices $A_{i,j}$. A similar set may be associated with the unsensed matrices $X_{i,j}$. 
\[ \left( X_{i-1,i} \right) \sum_{k=1}^{\alpha_i} \eta_{j,k}^i X_k^i = 0 \quad (j = 1, 2, \ldots, \alpha_i) \]

The variables \( X_k^i \) can take on only the values 0 and 1 in this case and are combined by reducing modulo 2. The numbers \( \eta_{j,k}^i \) correspond to the elements of the matrix \( X_{i-1,i} \) in the same way that the \( \xi_{j,k}^i \) of \( \S \) 8 did to the elements of \( A_{i-1,i} \). A solution of the above system marks each i-cell of \( C_\eta \) with 0 or 1, and no equation of the set is satisfied unless an even number (or none) of the variables take on the value 1. Thus, as before, we see that each solution defines an i-circuit (not sensed now, however) and that every column of \( X_{i-1,i} \) defines a solution of the system \( (X_{i-1,i}) \). Combinations of these solutions give the boundaries of open \((i+1)\)-complexes. The number of such linearly independent solutions is \( v_{i,i+1} \), the rank of \( X_{i-1,i} \).

*Definition.* If \( R_i - 1 \) is the number of i-circuits which must be added to the bounding i-circuits before we can obtain a complete set of linearly independent solutions, \( R_i \) is known as the \( i \)th connectivity number of the complex.

The connectivity numbers are related to the Betti numbers through the coefficients of torsion. It is well known, from the elementary theory of matrices, that a matrix of integers \( \| a_{ij} \| \) can be reduced by means of elementary transformations (which can be combined into a linear substitution on the columns or rows, the substitution having integer coefficients and a determinant equal to one) to the canonical form. In this form all of the
elements not on the main diagonal are zero, whereas the elements of the main diagonal are the invariant factors of $\|a_{ij}\|$. If we thus reduce the matrix $A_{i,j+1}$, which describes a manifold, those invariant factors of the reduced matrix $\overline{A}_{i,j+1}$ which are not zero or unity have absolute values which are called the coefficients of torsion of the manifold. The rank of $\overline{A}_{i,j+1}$ is the number of non-zero elements in $\overline{A}_{i,j+1}$.

Since the elements of the matrices $X_{i,j+1}$ and $A_{i,j+1}$ are all either zeros or ones and differ from the corresponding ones at most in sign, then if we reduce them mod 2 the ranks of the reduced matrices will be equal. But reducing the matrix $A_{i,j+1}$ mod 2 is the same as reducing the elements of the diagonal of $\overline{A}_{i,j+1}$ mod 2. The reduction mod 2 does not affect $X_{i,j+1}$. Therefore the rank of $X_{i,j+1}$ is equal to the rank of a canonical matrix which has only zeros or ones in the principal diagonal. We thus have

(2) \[ v_{i,j+1} - \rho_{i,j+1} = \lambda_{i,j+1} \]

where $t_{i,j+1}$ is the number of even coefficients of torsion, this being the number of new zeros introduced by the reduction mod 2 in the matrix $\overline{A}_{i,j+1}$.

Let the number of linearly independent solutions of the system $(A_{i-1,i})$ be $\mu_{i-1,i}$. Then

\[ \mu_{i-1,i} = \alpha_i - v_{i-1,i} \]

From the definition of the Betti number we have

\[ \rho_{i-1} = \mu_{i-1,i} - v_{i,i+1} \]
and, combining the two results, we get

\[(3) \quad \alpha^i - \nu^i - 1 = \nu^i, i + 1 + P^i - 1.\]

Also, if \(\sigma^i, i \) denotes the maximum number of linearly independent solutions of \((\chi^i, i)\), we get a similar relation

\[(4) \quad \alpha^i - \rho^i - 1, i = \rho^i, i + 1 + R^i - 1.\]

From the relations (3) and (4) and (2) it follows that

\[R^i - P^i = (\nu^i - 1, - \rho^i - 1, i) + (\nu^i, i + 1 - \rho^i, i + 1)\]

and

\[R^i = P^i + \tau^i, i + 1 + \tau^i, i + 1.\]

This is the relation which shows the connection between the Betti numbers and the connectivity numbers. The connectivity numbers satisfy a relation analogous to that previously stated for the duality of the Betti numbers. It is more general, however, in that it is valid for both one-sided and two-sided manifolds.

11. Regular Subdivisions. n-Spheres.

The notion of a regular subdivision is fundamental in what is to follow. It is useful in arguments where it is desirable to subdivide a given complex into "arbitrarily small" cells. Instead of using the usual Euclidean space, however, the domain of definition will be the n-sphere of spherical geometry

\[
\mathbb{H}^n: \quad \chi_0^2 + \chi_1^2 + \chi_2^2 + \cdots + \chi_n^2 = 1
\]

set in a space of \(n+1\) dimensions.
An n-plane \( a_0 x_0 + a_1 x_1 + \cdots + a_n x_n = 0 \) through the origin will divide the n-sphere into two n-regions separated by an \((n-1)\)-sphere. This in turn may be separated into two \((n-1)\)-regions separated by an \((n-2)\)-sphere, and so on until we have two one-regions (i.e. semi-circles) separated by a 0-sphere or pair of points. Such a subdivision is known as an elementary subdivision. It is seen that each i-cell of this subdivision is the boundary of two \((i+1)\)-cells, and that therefore the incidence matrices would be as we have previously mentioned.

It is evident that any \( k \)-region of an elementary subdivision can be re-subdivided into arbitrarily small regions of dimensionality zero to \( k \), so that the n-sphere may be so divided. One of the arbitrarily small \( p \)-dimensional regions will be called a \( p \)-cell. A repartitioning which is done in such fashion that only complete cells are left upon the boundaries of higher dimensional cells is called a regular subdivision or merely a subdivision. If a subdivision \( S \) can be obtained from a subdivision \( S' \) by repartitioning only, it is said to be derived from \( S' \). Obviously any subdivision is derived from an elementary subdivision.

A chain \( C' \) will be said to be derived from a chain \( C \) if it is one of a sequence of chains beginning with \( C \) and such that each member of the sequence is obtainable from the preceding by dividing a single \( k \)-cell \( E^k \) into a pair of \( k \)-cells \( E'^k \) and \( E^2_k \) separated by a \((k-1)\)-cell \( E^{k-1} \).
An analytical expression for an elementary subdivision of a two-sphere is given by the following equalities and inequalities.

The 2-sphere \( X_0^2 + X_1^2 + X_2^2 = 1 \) is divided into two 2-regions

\[
\begin{align*}
\mathcal{A}_1^2 & \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
-1 \leq X_0 \leq 0
\end{array} \right. \\
\mathcal{A}_2^2 & \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
0 < X_0 \leq 1
\end{array} \right.
\end{align*}
\]

by the plane \( X_0 = 0 \). The regions \( \mathcal{A}_1^2 \) and \( \mathcal{A}_2^2 \) are separated by the one-sphere

\[
\mathcal{A}_1 \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
X_0 = 0
\end{array} \right.
\]

The one-sphere \( \mathcal{A}_1 \) may be separated into two one-regions

\[
\begin{align*}
\mathcal{A}_1' & \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
X_0 = 0 \\
-1 \leq X_1 \leq 0
\end{array} \right. \\
\mathcal{A}_2' & \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
X_0 = 0 \\
0 < X_1 \leq 1
\end{array} \right.
\end{align*}
\]

by the plane \( X_1 = 0 \) which separates \( \mathcal{A}_1' \) and \( \mathcal{A}_2' \) by the zero-sphere or pair of points

\[
\mathcal{A}_0 \left\{ \begin{array}{l}
X_0^2 + X_1^2 + X_2^2 = 1 \\
X_0 = 0
\end{array} \right. \\
X_1 = 0
\]

A region will be called convex if any two points in it can be connected by precisely one geodesic arc made up of points of the region. From the form of the relations that determine a cell, which would be similar to those
given above, it is evident that a cell is convex. Consequently two \( i \)-cells are homeomorphic. And since an \( (i-1) \)-sphere bounds each of two \( i \)-cells on an \( i \)-sphere, it follows that the boundary of any \( i \)-cell is homeomorphic with an \( (i-1) \)-sphere.

The connectivity numbers of a chain may be defined in a manner different from that of \( \S \) 10 by using homologies. A number of \( i \)-chains \( K_s^i \) are said to be independent with respect to homologies if there exists no relation of the form

\[
\sum S \epsilon_S K_s^i = 0
\]

unless all of the \( \epsilon_S^i \) are zero.

Let \( K^i \) denote an \( i \)-chain of a chain \( C \). Then if \( R_i^i \) denotes the maximum number of closed non-bounding \( i \)-chains \( K_1^i, K_2^i, \ldots \) of \( C \) which are independent with respect to homologies, \( R^i \) is called the \( i \)th connectivity number of \( C \).

This definition is also expressible in terms of incidence matrices if we associate variables with the \( i \)-cells, \((i-1)\)-cells, and \((i+1)\)-cells as we have done before. The method is suggested in the special case below and has already been carried out in \( \S \) 8.

Let \( i \) be the highest dimensionality of cells of a chain \( C \). Suppose the number of \( i \)-cells to be \( \alpha \) and that associated with each \( i \)-cell \( \Lambda^i_p \) \( (p=1,2,\ldots \alpha) \) is a symbol \( X^i_p \) which takes on the values zero or unity. Then to every choice of a set of values \( X^i_p \) with at least one \( X^i_p \) not zero can be associated an \( i \)-chain of \( C \), and
conversely. Then corresponding to every \((i-1)\)-cell \((i>0)\) \(A_{\bar{q}}^{i-1}\) of \(C\) there is an equation of the form

\[
(5) \quad \sum_{\rho=1}^{\alpha} \varepsilon_{g\rho} X_{\rho}^i = 0 \quad \text{(mod 2)} \quad (g = 1, 2, \ldots, \alpha')
\]

where \(\varepsilon_{g\rho}\) is zero if \(A_{\bar{q}}^{i-1}\) is not on the boundary of \(A_{\rho}^i\) and is unity if it is. Hence to a solution of the system (5) will be associated a closed \(i\)-chain, this being a set of modular equations. Consequently if \(\rho\) represents the number of linearly independent equations in (5), the maximum number of independent solutions is

\[
(6) \quad R^i - 1 = \alpha - \rho
\]

since the \(i\)-chains are non-bounding. If \(i=0\), (6) reduces to the form \(R^0 - 1 = \alpha - 1\) and \(R^0 = \alpha\).

The number \(R^0\) denotes the number of separate connected parts of the chain \(C\). This may be seen by an inductive method. If \(R^0 = 1\), \(R^{i-1} = 0\) and the maximum number of independent closed 0-chains is zero. Hence the chain \(C\) cannot have more than one connected part. For if there were two parts \(C_1\) and \(C_2\), a closed 0-chain \(q_1^0, q_2^0\) belonging to both \(C_1\) and \(C_2\) would fail to bound since \(q_1^0\) and \(q_2^0\) could not be joined by a chain of arcs of \(C\). Conversely, if \(C\) is simply connected, a closed zero-chain bounds because the cellular zero-chains may be joined two by two.

If \(R^0 = 2\), the number of connected parts must be at least two. It cannot be three since then we can find a pair of independent non-bounding closed zero-chains. In fact, if \(q_1, q_2, q_3, q_4\) are in \(C_1\); \(q_3, q_4\) in \(C_2\); and
The connectivity numbers of any subdivision
\( S^n \) of an \( n \)-sphere are all unity except the \( n^{th} \), which
is two.

For the subdivision \( S^n \) can be obtained from an elementary subdivision and will thus have the same connectivity numbers. But in an elementary subdivision every closed chain bounds a cell except the \( n \)-chain determined by the two \( n \)-cells.

**Corollary 1.** Any closed \((n-1)\)-chain \( K^{n-1} \) of \( S^n \) bounds exactly two open \( n \)-chains, and these chains have only the points of \( K^{n-1} \) in common.

For, since \( R^{n-1} = 1 \), there is an open \( n \)-chain \( K^n \) such that
\[
K^n \equiv K^{n-1}
\]
Then \( K^{n-1} \) must also bound a second open chain \( K^n + S^n \) composed of the \( n \)-cells of \( S^n \) which do not lie in \( K^n \) and such that \( K^n + S^n \) meets \( K^n \) only in the points of \( K^{n-1} \). If there were a third chain \( L^n \) bounded by \( K^{n-1} \), we should have \( L^n \equiv K^{n-1} \). Thus there would be two independent closed chains
\[
L^n + K^n \equiv 0, \quad L^n + K^n + S^n \equiv 0
\]
so that \( R^{n-1} = 3 \) at least. This is a contradiction with the hypothesis.

**Definition.** If \( H^m \) and \( H^n \) are \( m \) and \( n \)-spheres, respectively, and if \( \bar{C} \) is a chain of a subdivision of \( \bar{H}^m \) then a set of points \( C \) of \( H^n \) which is homeomorphic with \( \bar{C} \) is called a chain immersed in \( H^n \).

A simple closed curve of \( H^n \) is homeomorphic with the boundary of a 2-cell and is thus immersed in \( H^n \).
The chains \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) will have the same cellular structure and thus the same connectivity numbers. By the uniform continuity of the correspondence of the closed sets it follows that there exists a derived chain of \( \mathcal{C} \) made up of arbitrarily small cells.

**Definition.** That part of \( H^n \) not filled by \( \mathcal{C} \) is denoted by \( H^n - \mathcal{C} \) and is called the domain residual to \( \mathcal{C} \). If a chain of \( H^n \) lies in \( H^n - \mathcal{C} \), it is called a chain of \( H^n - \mathcal{C} \).

We set up the following homologies on \( H^n - \mathcal{C} \):

1. Each closed \( i \)-chain will be homologous to the derived chains.

2. Each closed \( i \)-chain which bounds an open \((i+1)\)-chain of \( H^n - \mathcal{C} \) will be called homologous to zero.

**Definition.** If \( R^n_i \) denotes the maximum number of linearly independent (with respect to homologies) closed \( n \)-chains of \( H^n - \mathcal{C} \), then \( R^n_i \) is the \( n \)th connectivity number of \( H^n - \mathcal{C} \).

We shall consider only chains related by homologies and therefore no distinction will be made between equivalent chains—i.e. chains with a common derived chain. A closed \( i \)-chain which bounds in any of its derived forms will be said to bound.

**Theorem 2.** If \( \mathcal{C}^i \) is a cellular \( i \)-chain immersed in \( H^n \), the connectivity numbers of \( H^n - \mathcal{C}^i \) are all unity. That is, every closed chain \( L^k \) of \( H^n - \mathcal{C}^i \) bounds.

The proof will be by induction and will depend upon a lemma which involves the induction. We suppose the theorem...
to be valid for the index $i-1$.

If $i=0$, $C^i$ is a point $C^o$. There is only one closed n-chain of $H^n$ and consequently none in $H^n C^o$. By the corollary to Theorem 1 each closed $(n-1)$-chain of $H^n C^o$ bounds twice in $H^n$. Hence it must bound at least once in $H^n C^o$ since at least one of the open n-chains it bounds is in $H^n C^o$. Every chain of dimension $n-i$, ($i=2,3,\ldots n$) bounds as often as desired since there are as many open chains of dimensions $n-2, n-3,\ldots 0$ as we please.

We now prove the lemma mentioned above.

**Lemma 1.** Let $C^i$ be divided into two cellular 1-chains $A$ and $B$ meeting in a cellular $(i-1)$-chain $C^{i-1}$. Then every k-chain $L^k$ of $H^n C^i$ which bounds in $H^n A$ and in $H^n B$ also bounds in $H^n C^i$.

The proof is considered in two parts.

1. $k=n-1$. Then, from Corollary 1, $L^k$ bounds precisely two open n-chains of $H^n$ meeting in $L^k$. The connected set $C^i = A+B$ must thus lie completely inside one of them because it does not meet $L^k$. Hence the other chain must lie in $H^n C^i$, and $L^k$ bounds there.

2. $k<n-1$. Then there exist chains of dimensionality as great as $k+2$. By hypothesis there exist two open chains $L_{A}^{k+1}$ and $L_{B}^{k+1}$ such that

$$L_{A}^{k+1} = L^k \quad (H^n A)$$
$$L_{B}^{k+1} = L^k \quad (H^n B)$$

and thus

$$L_{A}^{k+1} + L_{B}^{k+1} \equiv O \quad (H^n C^{i-1})$$
where the parentheses at the right indicate the region.

The adjacent parallel lines represent superimposed lines.

By Theorem 2 for the case \( i = 1 \) there is an open \((k+2)\)-chain \( M^{k+2} \) such that
\[
M^{k+2} = L^{k+1}_A + L^{k+1}_B \quad (H_1 - C_{i-1})
\]

Since this cuts \( A \) and \( B \) in mutually exclusive closed sets, it may be broken into small cellular \((k+2)\)-chains with none meeting both \( A \) and \( B \). Let \( M^{k+2} \) be the sum of the cellular \((k+2)\)-chains that meet \( A \) (and hence not \( B \)), and let \( L^{k+1} \) be its boundary. Then
\[
M^{k+2} + M^{k+2} = (L^{k+1}_B + L^{k+1}) + L^{k+1}_A = 0 \quad (H_1 - C_{i-1})
\]
i.e., the chain \( M^{k+2} + M^{k+2} \) is bounded by \( L^{k+1}_A \) and that part of \( L^{k+1}_B \) which does not coincide with \( L^{k+1} \) (shown dotted).

Since \( L^{k+1}_B + L^{k+1} \) meets neither \( A \) nor \( B \) it lies in \( H^n - C^c \). Then, from (7) and (9),
\[
L^{k+1}_B + L^{k+1} = L^{k+1} \quad (H^n - C^c)
\]
which proves the lemma.

Theorem 2 can now be proved. We suppose the theorem to be false for the chain \( C^i \); that is, that some closed \( L^{k_i} \) of \( H^\eta C^i \) does not bound. By the lemma \( L^{k_i} \) must fail to bound in either \( H^\eta A \) or \( H^\eta B \), say \( H^\eta A \). We repeat the argument, dividing \( A \) in a similar fashion, and note that \( L^{k_i} \) must fail to bound in \( H^\eta C^i \) where \( C^i_\alpha \) \( (\alpha = 1, 2, \ldots) \) is a sequence of cellular chains shrinking to a point \( C^0 \). But by Theorem 2 for \( i=0 \) every closed \( k \)-chain \( L^{k_i} \) of \( H^\eta C^0 \) bounds a chain \( L^{k_i+1} \) which does not meet \( C^0 \). Thus it bounds a chain which fails to meet some of the \( C^i_\alpha \), and it must therefore bound in some \( H^\eta - C^i_\alpha \). This is a contradiction, and the theorem is proved.

Corollary 2. A cellular \( i \)-chain immersed in \( H^\eta \) cannot fill \( H^\eta \).

For let \( C^i \) be separated into \( A \) and \( B \) as in Lemma 1, with \( P_A \) and \( P_B \) points of \( A \) and \( B \), respectively, but not of \( C^{i-1} \). By Theorem 2 for \( i=1 \) the \( 0 \)-chain \( P_A + P_B \) bounds a one-chain of \( H^\eta C^{i-1} \) which contains a broken line of geodesics joining \( P_A \) and \( P_B \) and meeting \( A \) and \( B \) in mutually exclusive point sets. Hence it contains points not in \( A \) or \( B \). Therefore there are points of \( H^\eta \) not in \( C^i \).

Corollary 3. Let \( C \) be the sum and \( C^{i-1} \) the intersection of two closed point sets \( A \) and \( B \). Then every closed \( k \)-chain \( L^{k_i} \) \( (k<n-1) \) of \( H^\eta C \) which bounds a chain \( L^{k_i+1} \) of \( H^\eta - A \) and a chain \( L^{k_i+1} \) of \( H^\eta B \) bounds in \( H^\eta - C \)
if we can choose \( L_A \) and \( L_B \) such that \( L_A + L_B \) bounds in \( H^{n} - C^{i-1} \). If \( C^{i-1} \) is not null, this is valid for \( k=n-1 \).

Clearly, if \( k=n-1 \) the proof holds as in Lemma 1 since the set \( C=A+B \) is connected if \( C^{i-1} \) is not null.

We recall that a chain \( C \) immersed in \( H^{n} \) is a closed set, and we note that the set \( C^{i-1} \) need not be of lower dimensionality than \( C \). If, then, the \( i \)-chain \( C^{i} \) of Theorem 2 and Lemma 1 is replaced by a closed point set \( C = \overline{C^{i}} \) where \( \overline{\cdot} \) denotes dimensionality, the proof goes as before.

13. The Duality Theorem.

The duality theorem of Alexander will be proved first for a special case which we use in the more general theorem.

Theorem 3. If \( C^{i} \) is an \( i \)-sphere immersed in an \( n \)-sphere \( H^{n} \), the connectivity numbers \( R^{s} \) of \( C^{i} \) are related to those \( \overline{R}^{s} \) of \( H^{n} C^{i} \) by

\[
R^{i} = \overline{R}^{n-i-1} = 2 \\
R^{s} = \overline{R}^{n-s-1} = 1 \quad (i \neq s)
\]

Paraphrase. There exists but one independent closed non-bounding chain in \( H^{n} C^{i} \). It has dimensionality \( n-i-1 \).

Definition. The above-mentioned chain is said to link the \( i \)-sphere \( C^{i} \).

The theorem will be proved by induction. It is clear at once if \( i=0 \). We suppose that it is valid for the index \( i-1 \); that is, that \( R^{i-1} = \overline{R}^{n-i}=2 \), and divide \( C^{i} \) as in Lemma 1. Then by Theorem 2 each closed chain \( L^{k} \) of
\( H^n-C^i \) bounds two open chains \( L_A^{k+1} \) (of \( H^n-A \)) and \( L_B^{k+1} \) (of \( H^n-B \)). Thus \( L^k \) bounds in \( H^n-C^i \) unless \( L_A^{k+1} + L_B^{k+1} \) does not bound in \( H^n-C^i \) (by Corollary 3.)

By the case for \( i-1 \) this is possible only if \( L_A^{k+1} + L_B^{k+1} = L^n-i \) is the \((n-i)\)-chain linking \( C^{i-1} \). Hence \( L^k \) fails to bound only if \( k+1 = n-i \) and thus if \( k = n-i-1 \). Therefore it follows that

\[
\overline{R} n-s-1 = 1 \quad (s \neq i')
\]

and since \( R^s = 1 \) if \( s \neq i \) by Theorem 1, we have

\[
R^s = \overline{R} n-s-1 = 1 \quad (s \neq i')
\]

For the second part, we note that the chain \( L^{n-i} \) meets \( A \) and \( B \) in mutually exclusive sets of points. For if it did not meet \( A \), say, it would bound in \( H^n-A \) (Theorem 2) and thus in \( H^n-C^{i-1} \), contrary to hypothesis. Thus \( L^{n-i} \) may be written as

\[
(10) \quad L^{n-i} = L_A^{n-i} + L_B^{n-i}
\]

where \( L_A^{n-i} \) and \( L_B^{n-i} \) are open chains lying in \( H^n-A \) and \( H^n-B \) respectively, and having a common boundary \( L^{n-i} \).

This is the reverse of the method of Lemma 1. The chain \( L^{n-i-1} \) links \( C^i \). For if not there is an open chain \( L^{n-i} \) with

\[
\overline{L}^{n-i} \equiv L^{n-i-1} \quad (H^n-C^i)
\]

and thus one of the closed chains \( \overline{L}^{n-i} + L_A^{n-i} \), \( \overline{L}^{n-i} + L_B^{n-i} \) must link \( C^{i-1} \) since their sum \( L^{n-i} \) would. This is impossible, for if, say, the first linked \( C^{i-1} \) it would have to meet \( A \) and it cannot because neither of its components
does.

There is not another independent chain $\mathcal{M}^{n-i-1}$ linking $C^i$. For then a closed chain $\mathcal{M}^n_{A} + \mathcal{M}^n_{B}$ analogous to (10) links $C^{i-1}$. Then associated with $L^{n-i-1} + M^{n-i-1}$ is a chain

$$(L^n_{A} + M^n_{A}) + (L^n_{B} + M^n_{B})$$

which cannot link $C^{i-1}$ for the same reason as given above. Thus, by Corollary 3, $L^{n-i-1} + M^{n-i-1}$ bounds in $H^n - C^i$ so that

$$L^n_{A} + M^n_{A} \sim 0 \quad (H^n_{-} C^i)$$

and $M^{n-i-1}$ depends upon $L^{n-i-1}$. It follows that

$$R^n_{n-i-1} = 2$$

and, since $R^i = 2$ from Theorem 1, we have

$$R^i = R^n_{n-i-1} = 2.$$ 

This completes the proof.

The chain of $H^n - C^i$ which links $C^i$ may be chosen as irreducible because if it consists of several irreducible parts one would have to link

Theorem 4. (Alexander's Duality Theorem)

Let $C$ be any chain immersed in an $n$-sphere $H^n$. Then between the connectivity numbers $R^n_i$ of $C$ and the connectivity numbers $R^n_i$ of $H^n C$ there exists the duality relation

$$R^n_i = R^n_{n-i-1} \quad (0 \leq i \leq n-1)$$

The theorem will be proved by induction. We distinguish two cases.
1. \( C \) consists of 0-cells only. We have noted before that a closed chain of \( H^0 C \) of dimensionality less than \( n-1 \) bounds as often as we please. Thus

\[
R^{n-i-1} = R^i = 1 \quad (i > 0)
\]

In order to find the connectivity numbers of the residual space, let \( H^n \) be subdivided in such a way that each point \( A^0 \) of \( C \) appears in the interior of a cellular \( n \)-chain \( M^n_S \) of the subdivision and such that no two \( A^0 \) lie in the same cell of a chain. Let \( L^{n-i}_S \) be the closed \((n-1)\)-chain which constitutes the boundary of \( M^n_S \) so that

\[
(11) \quad M^n_S \equiv L^{n-i}_S \quad (H^n)
\]

Let \( L^{n-1} \) be a closed \((n-1)\)-chain of \( H^n - C \). Then \( L^{n-1} \) is homologous to some combination of the chains \( L^{n-i}_S \). For

\[
(12) \quad M^n \equiv L^{n-1} \quad (H^n)
\]

since \( L^{n-1} \) bounds in \( H^n \). Then if \( M^n \) contains any points of \( C \) we add the relations of (11) which involve these points and get an open chain not containing any points of \( C \) which is bounded by \( L^{n-1} \) and a linear combination of the \( L^{n-i}_S \). There can be but one homology between the \( L^{n-i}_S \) since, as we have seen, any linear combination of the \( L^{n-i}_S \) bounds precisely two \( n \)-chains of \( H^n \). One of these must be free of points of \( C \) if the combination is to bound in \( H^n - C \). If this is so, all of the chains \( L^{n-i}_S \) must appear in the linear combination. Let the number of points \( A^0_S \) be
denoted by $\alpha$. Hence the maximum number of independent non-bounding $(n-1)$-chains of $H^n C$ must be $\alpha - 1$. Therefore

$$\overline{R}^{n-1} - 1 = \alpha - 1$$

From the relation (6) for $i=0$ we get

$$R^0 - 1 = \alpha - 1$$

and thus

$$R^0 = \overline{R}^{n-1}$$

2. $C$ contains at least one cell of dimensionality greater than zero. We let $A$ be the cellular $i$-chain found by removing an $i$-cell of maximum dimension from $C$ and denote the residual chain $C-A$ by $B$. The theorem will be supposed valid for $B$, and it will be shown that any change of connectivity caused by replacing $A$ is balanced by a dual change in the residual space so that the theorem still holds for $C$.

Let the boundary of $A$ be $C^{i-1}$. Then any change in the connectivity of $B$ caused by the addition of $A$ will be either from the appearance of new independent $i$-chains containing $A$ or the disappearance of old independent $(i-1)$-chains. We consider two cases.

I. The chain $C^{i-1}$ does not bound on $B$.

II. $C^{i-1}$ bounds an open $i$-chain $A^i$ of $B$.

In Case I no new closed $i$-chains are created since they must have the form $A+ A^i$. Then
\[ A + A^i = 0 \quad (B) \]

so that
\[ A^i \equiv C^{i-1} \quad (B) \]

which means that \( C^{i-1} \) bounds on \( B \), a contradiction.

However, an independent non-bounding \((i-1)\)-chain is lost because we have
\[ A \equiv C^{i-1} \quad (C) \]

This is the only \((i-1)\)-chain lost. For if \( D^i \) is a second
open \( i \)-chain of \( B \), the relation
\[ A + D^i \equiv D^{i-1} \quad (C) \]

implies that

\[ (13) \quad D^i \equiv C^{i-1} + D^{i-1} \quad (B) \]

as is seen by adding the above relations and reducing mod 2.

The congruence (13) shows that \( D^{i-1} \) depends upon \( C^{i-1} \).
A schematic diagram to illustrate this is the following.

\[ \require{cancel} \]
\[ \text{In Case II a closed } i \text{-chain } A + A^i \text{ is gained. For} \]

since

\[ (14) \quad A \equiv C^{i-1} \quad (C), \quad A^i \equiv C^{i-1} \quad (B) \]
we have

\[ A + A^i \equiv O \quad (C) \]

If \( C^{i-1} \) bounds more than one open i-chain of \( B \), the closed chains formed from these are not independent. If \( D^i \) is a chain of \( B \) not bounded by \( C^{i-1} \), then \( D^i \) cannot form a closed i-chain with \( A \). For if

\[ A + D^i \equiv O \quad (C) \]

then

\[ D^i \equiv C^{i-1} \quad (B) \]

which contradicts the hypothesis. On the other hand, no independent \( (i-1) \)-chain of \( B \) is lost. For suppose \( D^{i-1} \) to be an independent non-bounding chain of \( B \) such that

\[ A + D^i \equiv D^{i-1} \quad (C) \]

From this and the relations (14) it follows that

\[ A^i + D^i \equiv D^{i-1} \quad (B) \]

which contradicts the hypothesis that \( D^{i-1} \) is non-bounding.

We now seek the compensating changes in the space residual to \( C \).

Let an irreducible \( (n-i) \)-chain \( L^{n-i} \) of \( H^\eta C^{i-1} \) be chosen linking the boundary of \( A \). This is possible since the boundary of \( A \) is homeomorphic with an \( (n-1) \)-sphere. \( L^{n-i} \) will meet \( A \) in a closed set of points and may or may not contain a point not of \( A \). If possible we let it do so. If \( L^{n-i} \) meets \( A \), we separate it, as before, into two open chains bounded by a closed chain \( L^{n-i-1} \) of \( H^\eta C^{i-1} \).
and such that one of the open chains (call it $M^{n-i}$) contains all of the points of intersection with $A$ but no point of $B$:

$$M^{n-i} = L^{n-i-1} \quad (H^n \setminus B)$$

If $L^{n-i}$ contains no point of $H^n \setminus A$, then $M^{n-i} = L^{n-i}$ and the boundary of $M^{n-i}$ is a null chain.

**Definition.** The chain $L^{n-i}$ is the chain dual to $A$.

Since it is found precisely by the method of Theorem 3, it links any $i$-sphere of $C$ which contains $A$.

If $\varepsilon$ is any positive constant, the chain $M^{n-i}$ may be so chosen that each point of it is within a distance $\varepsilon$ of some point of intersection of $M^{n-i}$ and $A$. The other open $(\gamma-i)$-chain is then what is left in $L^{n-i}$ — that is, $M^{n-i} + L^{n-i}$.

**Lemma 2.** If a closed chain $L^{k+1}$ of $H^n \setminus B$ does not link $C^{i-1}$, the boundary of $A$, there is always some chain in $H^n \setminus C$ which is homologous to $L^{k+1}$ in $H^n \setminus B$. If $L^{k+1}$ does link $C^{i-1}$, there is no chain of $H^n \setminus C$ homologous to $L^{k+1}$.

If $L^{k+1}$ links $C^{i-1}$, it must cut the $i$-chain $A$.

For if not it lies in $H^n \setminus A$ and must bound there (Theorem 2), then a fortiori it must bound in $H^n \setminus C^{i-1}$. This is a contradiction with the hypothesis that $L^{k+1}$ links $C^{i-1}$. Therefore $L^{k+1}$ does not lie in $H^n \setminus C$ and consequently no chain homologous to it will.

If $L^{k+1}$ does not link $C^{i-1}$, an open chain $M^{k+2}$ exists such that
\[ M^{k+2} \equiv L^{k+1} \quad (H^n C^{c-1}) \]

The following schematic diagram may be used.

It is conceivable that \( M^{k+2} \) may meet \( B \), though \( L^{k+1} \) may not.

Since the intersections of \( M^{k+2} \) with \( A \) and \( B \), if any, must consist of mutually exclusive point sets, the method of Lemma 1 is applicable and we can find a chain \( L^{k+1} + \overline{L}^{k+1} \) not meeting \( A \) or \( B \) and hence in \( H^n C \).

Thus, using the notation of Lemma 1, we have

\[ \overline{M}^{k+2} \equiv \overline{L}^{k+1} \quad (H^n - B) \]

or

\[ \overline{L}^{k+1} \sim 0 \]

Then, since \( L^{k+1} \cup L^{k+1} \), we have

\[ L^{k+1} + \overline{L}^{k+1} \sim L^{k+1} \]
which proves the lemma.

**Lemma 3.** If \( L^k \) is a closed chain of \( H^n - C \) which bounds in \( H^n B \):
\[
L_{B}^{k+1} = L^k \quad (H^n - B)
\]
and thus in \( H^n A \):
\[(16) \quad L_{A}^{k+1} = L^k \quad (H^n - A) \]
then \( L^k \) fails to bound in \( H^n C \) if and only if the closed chain
\[
(17) \quad L_{A}^{k+1} + L_{B}^{k+1} = 0 \quad (H^n - C_{i-1})
\]
links \( C_{i-1} \) for every possible choice of \( L_{B}^{k+1} \).

For the chain \( L_{A}^{k+1} + L_{B}^{k+1} = L_{A}^{k+1} \) must cut \( A \) if it links \( C_{i-1} \). Then \( L_{A}^{k+1} \) must cut \( A \), for \( L_{A}^{k+1} \) cannot.
(See figure on page 35). But if this occurs for every \( L_{B}^{k+1} \), \( L^k \) cannot bound in \( H^n - C \).

On the other hand, if \( L_{B}^{k+1} \) is chosen so that \( L_{A}^{k+1} + L_{B}^{k+1} \) does not link \( C_{i-1} \), \( L^k \) bounds in \( H^n - C \) by Corollary 3. The lemma is proved.

Let us now consider the changes in the residual space.
We distinguish two cases analogous to those on page 31.

I'. \( H^n - B \) contains an \((n-1)\)-chain \( L \) linking \( C_{i-1} \).

II'. \( H^n - B \) contains no \((n-1)\)-chain linking \( C_{i-1} \).

In Case I' no new independent \((n-i-1)\)-chain can be
created in \( H^{n-C} \). For, since \( L = 0 \), we have

\[
L + L_{A}^{k+1} = L_{A}^{k} \quad (H^{n-A})
\]

where \( k+1 = n-\ell \), and thus in place of (17)

\[
(18) \quad L + L_{A} + L_{B}^{k+1} = O \quad (H^{n-C}^{i-1})
\]

But (17) and (18) cannot both link \( C^{i-1} \). For then their sum \( L \) would not. (Clearly, if two chains each link \( C^{i-1} \) their sum cannot because it will not meet \( A \) if each chain does.) Hence \( L_{A}^{k} = L_{A}^{n-C} \) must bound in \( H^{n-C} \) by Corollary 3, and we see that by adding \( A \) to \( B \) we reduce by one the number of independent chains of dimensionality \( k+1 = n-\ell \). That is, \( \overline{R}^{n-\ell} \) is reduced by one. The other connectivity numbers of \( H^{n-C} \) are left invariant.

In case II' none of the numbers \( \overline{R}^{s} \) are diminished. This comes from Lemma 2, which shows that in \( H^{n-C} \) there is always a member of the family of chains homologous to \( L_{A}^{k+1} \) in \( H^{n-B} \), and from Theorem 3 for index \( \ell-1 \), which tells us that \( R^{i-1} = \overline{R}^{n-C} = 2 \) and \( \overline{R}^{n-s-1} = 1 \) if \( s \neq \ell-1 \). Thus a chain of \( H^{n-C}^{i-1} \) which links \( C^{i-1} \) must meet \( A \) and \( B \). We can now use a literal transcription of the proof of Theorem 3 and we find that a single new independent \((n-C-1)\)-chain is created. This, by the definition given previously, is the dual of \( A \). Hence \( \overline{R}^{n-\ell-1} \) is increased by unity.

It also follows that every independent non-bounding
\((n-i-1)\)-chain of \(H^n C\) is homologous to some linear combination of the duals of the \(i\)-cells of \(C\).

We return now to the connectivity changes of the immersed figure. The changes in the residual space are those which compensate for the changes in the immersed figure given by I and II on page 31. In other words, when \(A\) is added to \(B\) the differences \(R^i - R^{i-1}\) and \(\overline{R}^{n-i-1} \overline{R}^{n-i}\) both increase by one. They thus remain equal since they were so assumed for \(B\).

In order to complete the proof it is necessary to show that \(R^i = \overline{R}^{n-i-1}\). Let \(C'\) be the chain left if all of the cells of highest dimensionality in \(C\) are removed. Then, as was previously pointed out, corresponding to each cellular \((i-1)\)-chain \(A'\) of \(C'\) may be found a dual \((n-i)\)-chain \(L^{n-i}\) of \(H^n C'\) so close to \(A'\) that it meets only those \(i\)-cells that have \(A'\) on their boundaries. These it must meet since it links their boundaries. Then \(L^{n-i}\) may be broken into a set of open \((n-i)\)-chains each of which meets but one of the \(i\)-cells of \(C\) and each of which is bounded by the dual of the chain determined by that \(i\)-cell. Let the dual of \(A^\prime_p\) be \(\overline{L}^{n-i}\). Then we can set up the set of homologies

\[
(19) \quad \sum_{\rho=-i}^{\infty} c_{\rho} L^{n-i} \sim 0
\]

such that the incidence relations are the same as those in (5). That is, \(c_{\rho}\) is equal to unity or zero according as \(A^\prime_p\) is or is not on the boundary of \(A^\prime_p\). Any other
homology among the $L_p^{n-i-1}$ can be expressed in terms of the set (19). For let one of them be

$$M_n^{n-i} \equiv \sum \alpha_n L_n^{n-i-1} = 0$$

Then if $M_P^{n-i}$ is bounded by $L_P^{n-i-1}$ (see relation 15) we have

$$\sum P \alpha_P M_P^{n-i} = \sum P \alpha_P L_P^{n-i-1},$$

and combining with (20) we get

$$M_n^{n-i} + \sum \alpha_n M_n^{n-i} = 0$$

Since this closed chain is expressible in terms of the duals $L_n^{n-i}$ of the cellular $(i+1)$-chains of $C'$, the $M_n^{n-i}$ being parts of $L_n^{n-i}$, the homology (20) is expressible in terms of (19). Then every independent non-bounding $(n-i-1)$-chain of $H^{n-1}C$ can be expressed in terms of (19). Thus, as in (6),

$$\overline{R}^{n-i-1} = \alpha_P$$

and $R_i = \overline{R}^{n-i-1}$. This completes the proof.

This theorem establishes the topological character of $R_i$ and $\overline{R}^{n-i-1}$ (which we may now call invariants) since it is seen that they depend not upon the cellular structure of $C$ but upon the set of points determined by $C$.

14. **The Jordan Curve Theorem**.

Perhaps the most important application of Theorem 4 is in the proof of the Jordan-Brouwer separation theorem. This theorem is a generalization to $n$ dimensions of the Jordan Curve Theorem. It may be stated as follows. The $(n-1)$-dimensional manifold $M^{n-1}$ immersed in a Euclidean $n$-
space $E^n$ separates $E^n$ into just two domains of which it is the common boundary. Brouwer also showed that the points of the manifold are accessible from each of the complementary domains. Alexander's work is an extension of this in that he not only showed that the residual set consists of just two connected domains, but also showed the duality between the manifold and the residual domain.

The proof of the Jordan-Brouwer theorem follows at once from Theorem 4. It is clear, of course, that the number $\overline{R}^o$ represents the number of connected parts of $H^n - C$ just as $R^o$ represents the number of connected parts of $C$. Then if $H^n$ is the n-space and $M^{n-1}$ is the manifold, we know that $\overline{R}^o = R^{n-1}$. But $M^{n-1}$ is a manifold, so that $R^{n-1} = 2$ and hence $\overline{R}^o = 2$. Thus the theorem is proved. The Jordan Curve Theorem is given when $n$ is equal to 2. The surface of a sphere is homeomorphic with the plane.
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