THE CAUCHY PROBLEM FOR LAPLACE'S EQUATION IN THREE DIMENSIONS.

A thesis
Presented to the Faculty of
the Rice Institute in
Partial Fulfillment of the
Requirements for the Degree of
Master of Arts.

Lee Harnie Johnson, Jr.
May
1931.

The Cauchy Problem for Laplaces Equation in Three Dimensions

1. Introduction

It is well known that the Cauchy data, if not given analytically, need not insure a solution of a partial differential equation; in fact if a harmonic function u(x,y,z) takes on zero values on a certain portion σ of the x,y plane, it may be extended harmonically across σ so as to be harmonic and analytic in a three dimensional region which includes in its interior part of σ. Hence $\frac{\partial u}{\partial z}$ will also be analytic on this part of σ and therefore cannot take on the values of an arbitrary continuous function. In this paper we seek conditions which we may impose on non-analytic data in this particular problem so as to insure a solution. We have the following theorem-

Theorem: If $f(x,y)$ and $f_1(x,y)$ are two functions continuous in a bounded region σ of the x,y plane and vanishing continuously on the boundary and outside of σ, and if the first and second partial derivatives of $f(x,y)$ are continuous, then

(1) there cannot be more than one function $u(x,y,z)$, harmonic for $z>0$ in the neighborhood[a] of σ for which

$$\lim u(x',y',z') = f(x,y)$$
$$\lim \frac{\partial u(x',y',z')}{\partial z'} = f_1(x,y) \qquad \ldots\ldots (1)$$

as (x',y',z') approaches $(x,y,0)$ in σ.

(2) and there will be one such function if and only if $f_1(x,y)$ has the value: $f_1(x,y) = F(x,y)$

$$-\frac{1}{4\pi}\int_\sigma \frac{\frac{\partial^2 f}{\partial x^2}(x-x_M)^2 + \frac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \frac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2+(y-y_M)^2} \cdot \log[(x-x_M)^2+(y-y_M)^2]\,dx\,dy$$

---

[a] see page 2

where $F(x,y)$ is an analytic function of x,y for x,y in σ.

2. Proof of Part (1).

It is almost immediately evident that there cannot be two different solutions which satisfy the given conditions. If there were two such solutions $u_1(x,y,z)$ and $u_2(x,y,z)$, their difference $U(x,y,z)$ would take on zero values in σ. But a harmonic function which takes on continuously zero values on a plane portion of the boundary of the region in which it is harmonic is in fact harmonic on this plane portion and can be extended across it uniquely as a harmonic function. Hence $U(x,y,z)$ is analytic at interior points of σ and vanishes there. Hence $\frac{\partial U}{\partial z}$ is also analytic at interior points of σ and vanishes there since $\frac{\partial U}{\partial z} = \frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z}$. But here we have the statement of the Cauchy problem with analytic data. There is one and only one $U(x,y,z)$ analytic in the neighborhood of σ such that

$$U(x,y,0) = 0 \quad , \quad \frac{\partial U(x,y,0)}{\partial z} = 0 \qquad \text{for } (x,y) \text{ in } \sigma.$$

and this function is evidently $U(x,y,z) \equiv 0$. Accordingly we cannot have $u_1(x,y,z)$ different from $u_2(x,y,z)$.

3. Determination of a Function $\bar{u}(x,y,z)$

In order to demonstrate the second part of our theorem, it is necessary to find a function related to $u(x,y,z)$ which can be extended across the plane $z=0$. Before we can do this though, we must find first a function $\bar{u}(x,y,z) = \bar{u}(M)$ which takes on continuously the values $\bar{g}(x,y) = \bar{g}(Q)$ as M approaches Q in the x,y plane, where $\bar{g}(x,y)$ is a function which is bounded and continuous over the x,y plane. For this purpose we employ the Green's function

---

[2] A neighborhood of σ is a region T such that if (x_0, y_0) is any point in the interior of σ, then a sphere can be drawn with center (x_0, y_0) and radius sufficiently small so that the whole sphere is contained in T.

$g(M,P)$ for the ~~upper half~~ domain $z>0$ plane and write

$$\bar{u}(M) = \frac{1}{4\pi}\int_W \frac{\partial g(M,P)}{\partial z_P}\,\bar{g}(P)\,d\sigma_P \qquad \ldots\ldots (3)$$

where W indicates integration over the whole x,y plane and P is the variable point in the x,y plane in terms of which the integration is effected. The integral is convergent since $\frac{\partial g}{\partial z_P}$ vanishes at infinity like $\frac{1}{MP^2}$. In fact we have

$$g(M,P) = \frac{1}{r} - \frac{1}{r'}$$

where $r = MP$, $r' = M'P$, and M' is the reflection of M in the plane $z=0$. To calculate $\frac{\partial g(M,P)}{\partial z}$ we write r and r' in terms of the coordinates of M, M', and P.

$$r = \sqrt{(x_P-x_M)^2+(y_P-y_M)^2+(z_P-z_M)^2}$$

$$r' = \sqrt{(x_P-x_{M'})^2+(y_P-y_{M'})^2+(z_P-z_{M'})^2}$$

$$\therefore \quad \frac{\partial g(M,P)}{\partial z_P} = -\frac{z_P-z_M}{r^3} + \frac{z_P-z_{M'}}{r'^3}$$

and since $z_{M'} = -z_M$,

$$\frac{\partial g(M,P)}{\partial z_P} = -\frac{z_P-z_M}{r^3} + \frac{z_P+z_M}{r'^3}$$

As z_P approaches zero, MP approaches $M'P$ and we have

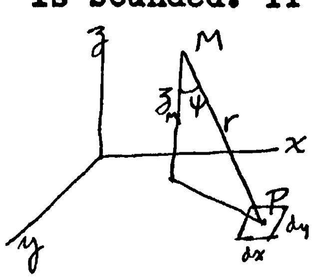
$$\lim \left.\frac{\partial g(M,P)}{\partial z_P}\right]_{z\to 0} = \frac{2z_M}{r^3} = \left.\frac{\partial g(M,P)}{\partial z_P}\right]_{z=0}$$

Therefore

$$\bar{u}(M) = \frac{1}{4\pi}\int_W \frac{2z_M}{r^3}\,\bar{g}(x,y)\,d\sigma_P \qquad \ldots\ldots (4)$$

We can now show that $\bar{u}(M)$ as given by the above integral is bounded. If ψ is the angle between MP and the dou vertical, we have

$$\cos\psi = \frac{z_M}{r}$$



Furthermore the projection of $d\sigma_P$ upon a plane perpendicular to MP is $d\sigma_P \cos\psi$. In fact if we consider the solid angle $d\omega$ subtended on a unit sphere at M by the element of area $d\sigma_P = dx\,dy$ at the point P, we have

$$d\omega = \frac{1}{r^2}\, d\sigma_P \cos\psi = \frac{\mathfrak{z}_M}{r^3}\, d\sigma_P$$

which upon substitution in (4) gives

$$\bar{u}(M) = \frac{1}{2\pi}\int_W \bar{g}(x,y)\, d\omega$$

By hypothesis $\bar{g}(x,y)$ is bounded over the x,y plane, which means that $|\bar{g}(x,y)| \leq K$, where K is an arbitrary constant. Therefore

$$|\bar{u}(M)| \leq \frac{1}{2\pi}\int_W |\bar{g}(x,y)|\, d\omega \leq \frac{K}{2\pi}\int_W d\omega \leq K$$

And further we can show that not only is $\bar{u}(M)$ bounded if $\bar{g}(x,y)$ is bounded, but also that $\bar{u}(M)$ takes on continuously the values $\bar{g}(Q)$ as point M approaches point Q in the x,y plane. If we let P be a variable point in the x,y plane and Q a fixed point, we write

$$\bar{g}(P) = \bar{g}(Q) + h(P)$$

where $\bar{g}(Q)$ is the value of the function $\bar{g}(x,y)$ at Q and where $|h(P)| \leq \epsilon$ if $\overline{QP} \leq \delta$, and is bounded $< H$ in W. Consider a circle with center Q and radius δ and call this region about Q the region σ_δ. Let $\sigma_{\delta'} = W - \sigma_\delta$. Keeping these notations in mind we write

$$\bar{u}(M) = \frac{1}{2\pi}\int_W \bar{g}(P)\, d\omega_P = \frac{1}{2\pi}\int_W \bar{g}(Q)\, d\omega_P + \frac{1}{2\pi}\int_W h(P)\, d\omega_P$$

$$\therefore\quad \bar{u}(M) - \frac{\bar{g}(Q)}{2\pi}\int_W d\omega_P = \frac{1}{2\pi}\int_{\sigma_\delta} h(P)\, d\omega_P + \frac{1}{2\pi}\int_{\sigma_{\delta'}} h(P)\, d\omega_P$$

Now let M approach Q and we see that

$$\left| \frac{1}{2\pi}\int_{\sigma_\delta} h(P)\, d\omega_P \right| \leq \frac{1}{2\pi}\int_{\sigma_\delta} |h(P)|\, d\omega_P \leq \frac{\epsilon}{2\pi}\cdot 2\pi \leq \epsilon$$

since $|h(P)| \leq \epsilon$ for $\overline{QP} \leq \delta$. Hence this integral can be made arbitrarily small, say $\leq \frac{\eta}{2}$. Also we have

$$\left| \frac{1}{2\pi} \int_{\sigma_{\delta'}} h(P)\, d\omega_P \right| \leq \frac{H}{2\pi} \int_{\sigma_{\delta'}} d\omega_P \leq \frac{H}{2\pi} \cdot \epsilon_1$$

where $|h(P)| < H$, and where ϵ_1 is the solid angle subtended by $\sigma_{\delta'}$ at M. But this becomes arbitrarily small as M approaches Q and therefore this integral also may be made $\leq \frac{\eta}{2}$. Thus we have

$$\left| \bar{u}(M) - \bar{g}(Q) \right| \leq \frac{\eta}{2} + \frac{\eta}{2} \leq \eta$$

$$\therefore \lim_{M \to Q} \bar{u}(M) = \bar{g}(Q)$$

since η is chosen arbitrarily small.

4. A Function which can be extended across the plane $z=0$

Now recalling the conditions of our theorem that

$$\lim u(x', y', z') = f(x,y)$$

$$\lim \frac{\partial u(x', y', z')}{\partial z'} = f_1(x,y)$$

as (x', y', z') approaches $(x, y, 0)$ in σ, where $f(x,y)$ and $f_1(x,y)$ are continuous in the bounded region σ of the x,y plane and vanish continuously outside of σ, we choose $\bar{g}(x,y) = f(x,y)$ in W so that $\bar{u}(M)$ approaches $f(Q)$ as point M approaches point Q in the x,y plane. We are now able to set up a function related to $u(x,y,z)$ which can be extended across the plane $z=0$.

We assume that we have a solution $u(x,y,z)$ and define

$$v(M) = u(M) - \bar{u}(M) \qquad \ldots\ldots (5)$$

and since $v(M)$ is the difference of functions harmonic in the upper half ~~plane~~ space, it is also harmonic in that region. Furthermore $v(x', y', z')$ approaches zero as (x', y', z') approaches $(x, y, 0)$, since

$$\lim \bar{u}(x', y', z') = f(x,y)$$

and by hypothesis, $\lim u(x', y', z') = f(x,y)$,

Also $v(M)$ is bounded in the neighborhood of σ for $z>0$ since it is continuous in a closed region about σ . Therefore by defining $v(x,y,-z) = -v(x,y,z)$, we can extend $v(x,y,z)$ harmonically across the plane $z=0$ in the neighborhood of σ. Moreover $v(x,y,z)$ is analytic on σ since it is harmonic in a region including σ. Hence $\frac{\partial v(x,y,z)}{\partial z}$ is analytic on σ and its values there determine $v(x,y,z)$ uniquely, for there is one and only one solution of Laplace's equation in the neighborhood of σ which with its normal derivative takes on given analytic values on σ. In this particular case the given value of $v(x,y,0)$ is identically zero. We may write

$$\left.\frac{\partial v(x,y,z)}{\partial z}\right]_{z=0} = F(x,y)$$

where $F(x,y)$ is a function analytic in σ .

5. <u>Necessary Relation between $f_1(x,y)$ and $f(x,y)$</u>.

From equation (5) for a point M we have

$$\left.\frac{\partial v(M)}{\partial z_M}\right]_{z_M>0} = \left.\frac{\partial u(M)}{\partial z_M}\right]_{z_M>0} - \left.\frac{\partial \bar{u}(M)}{\partial z_M}\right]_{z_M>0}$$

$$= \left.\frac{\partial u(M)}{\partial z_M}\right]_{z_M>0} - \frac{1}{2\pi}\int_W \frac{\partial}{\partial z_M}\frac{z_M}{r^3} f(x,y)\, d\sigma_P \qquad \ldots\ldots (6)$$

Differentiating under the last integral sign and taking the limit as z_M approaches zero, equation (6) becomes

$$F(x,y) = f_1(x,y) - \lim_{z_M\to 0}\frac{1}{2\pi}\int_W \frac{(x_P-x_M)^2+(y_P-y_M)^2-2z_M^2}{r^5} f(x,y)\, d\sigma_P \qquad \ldots\ldots (7)$$

When $z_M=0$ the integral is improper since the denominator approaches zero to a higher order than the numerator, ~~and~~ the fraction becomes infinite like r^3. ~~It~~ The integral is moreover not convergent.

Before letting z_M approach zero, we make the transformation

$$x = x_M + r_1 \cos\theta$$
$$y = y_M + r_1 \sin\theta$$
$$(x_P - x_M)^2 + (y_P - y_M)^2 = r_1^2$$

where r_1 and θ are the polar coordinates in the plane referred to (x_M, y_M). Substituting in the integral, we have

$$I = \int_W \frac{(x_P - x_M)^2 + (y_P - y_M)^2 - 2z_M^2}{[(x_P - x_M)^2 + (y_P - y_M)^2 + (z_M)^2]^{5/2}} f(x,y)\, d\sigma_P$$

$$= \int_0^{2\pi}\int_0^{\infty} f(x_M + r_1\cos\theta, y_M + r_1 \sin\theta) \frac{r_1^2 - 2z_M^2}{(r_1^2 + z_M^2)^{5/2}}\, r_1\, d\theta\, dr_1$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} f(x_M + r_1\cos\theta, y_M + r_1\sin\theta) \left[\frac{r_1(r_1^2 + z_M^2)}{(r_1^2 + z_M^2)^{5/2}} - \frac{3r_1 z_M^2}{(r_1^2 + z_M^2)^{5/2}}\right] dr_1 \quad \ldots\ldots(8)$$

Furthermore we have

$$\int_0^{r_1} \left[\frac{r_1}{(r_1^2 + z^2)^{3/2}} - 3z^2 \frac{r_1}{(r_1^2 + z^2)^{5/2}}\right] dr_1 = -\frac{r_1^2}{(r_1^2 + z^2)^{3/2}},$$

since

$$\frac{\partial}{\partial r_1}\left(-\frac{r_1^2}{(r_1^2 + z^2)^{3/2}}\right) = \frac{r_1^3 - 2r_1 z^2}{(r_1^2 + z^2)^{5/2}}.$$

~~and~~ Therefore if we integrate by parts, we obtain

$$I = \int_0^{2\pi} d\theta \int_0^{\infty} (f_{10}\cos\theta + f_{01}\sin\theta) \frac{r_1^2}{(r_1^2 + z^2)^{3/2}}\, dr_1 \quad ;$$

in fact, ~~for~~ the part outside the integral vanishes, since $f(M, r_1, \theta) = 0$ for $r_1 = \infty$ and $\frac{r_1^2}{(r_1^2 + z^2)^{3/2}} = 0$ when $r_1 = 0$. The f_{10} and f_{01} denote respectively $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$. But the denominator still approaches zero to too high an order.

We integrate again by parts, first writing

$$\int_0^{r_1} \frac{r_1^2}{(r_1^2+z^2)^{3/2}}\,dr_1 = \int_0^{r_1}\left(\frac{1}{(r_1^2+z^2)^{1/2}} - \frac{z^2}{(r_1^2+z^2)^{3/2}}\right)dr_1$$

$$= \log\left(r_1+\sqrt{r_1^2+z^2}\right) + C - \frac{r_1}{(r_1^2+z^2)^{1/2}} .$$

In this equation, if we let r_1 approach zero, holding z fixed, ~~and as M approaches P so that r_1 approaches zero~~, all terms vanish except $\log\sqrt{z^2} + C$, ~~therefore~~ so that $C = -\log z$ and we have

$$\int_0^{r_1} \frac{r_1^2}{(r_1^2+z^2)^{3/2}}\,dr_1 = \log\left(\frac{r_1+\sqrt{r_1^2+z^2}}{z}\right) - \frac{r_1}{(r_1^2+z^2)^{1/2}} .$$

Also

$$\frac{d}{dr_1}\left(f_{10}\cos\theta + f_{01}\sin\theta\right) = f_{20}\cos^2\theta + 2f_{11}\sin\theta\cos\theta + f_{02}\sin^2\theta$$

where

$$f_{20} = \frac{\partial^2 f}{\partial x^2}\ ,\quad f_{11} = \frac{\partial^2 f}{\partial x\,\partial y}\ ,\quad f_{02} = \frac{\partial^2 f}{\partial y^2} .$$

As before in the integration by parts the part outside of the integral vanishes, on account of the continuity of f_{10}, f_{01} and their vanishing outside σ, and there is left

$$I = -\int_0^{2\pi} d\theta \int_0^{\infty} \left\{f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta\right\}$$

$$\left\{\log\left[r_1 + (r_1^2+z^2)^{1/2}\right] - \log z - \frac{r_1}{(r_1^2+z^2)^{1/2}}\right\} dr_1$$

The integral involving $-\log z$ vanishes, for we have

$$\log z \int_0^{2\pi} d\theta \int_0^{\infty} \left\{f_{20}\cos^2\theta + 2f_{11}\sin\theta\cos\theta + f_{02}\sin^2\theta\right\} dr_1$$

$$= \log z \int_0^{2\pi} d\theta \left[f_{10}\cos\theta + f_{01}\sin\theta\right]_0^{\infty} = 0 ,$$

since

$$\int_0^{2\pi} \cos\theta\, d\theta = 0\ ,\quad \int_0^{2\pi} \sin\theta\, d\theta = 0,\quad f_{10}\Big]_{r_1=\infty} = f_{01}\Big]_{r_1=\infty} = 0 .$$

Thus we have

$$I = \int_0^{2\pi} d\theta \int_0^{\infty} \left\{f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta\right\}$$

$$\left\{\frac{r_1}{(r_1^2+z^2)^{1/2}} - \log\left(r_1 + (r_1^2+z^2)^{1/2}\right)\right\} dr_1 \qquad \ldots\ldots(9)$$

The integral is now convergent and we may write(see Appendix 3)

$$\lim_{z=0} I = \int_0^{2\pi} d\theta \int_0^{\infty} \left\{ f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta \right\} \left\{ 1 - \log 2r_1 \right\} dr_1 \qquad \cdots\cdots (10)$$

which reduces in the manner which we have seen to

$$\lim_{z=0} I = \int_0^{2\pi} d\theta \int_0^{\infty} \left\{ f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta \right\} \left\{ -\log r_1 \right\} dr_1 \, .$$

~~and now~~ Substituting rectangular coordinates, we have

$$\lim_{z=0} I = -\iint_W \left\{ \frac{\frac{\partial^2 f}{\partial x^2}(x-x_M)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \frac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2+(y-y_M)^2} \right\} \left\{ \frac{1}{2}\log\left[(x-x_M)^2+(y-y_M)^2\right] \right\} dx\,dy \qquad \cdots\cdots (11)$$

Therefore we may write equation (7) in the form

$$f_1(x,y) = F(x,y) - \frac{1}{4\pi}\iint_\sigma \left\{ \frac{\frac{\partial^2 f}{\partial x^2}(x-x_M)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \frac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2+(y-y_M)^2} \right\} \left\{ \quad \log\left[(x-x_M)^2+(y-y_M)^2\right] \right\} dx\,dy \qquad \cdots\cdots (12)$$

since by hypothesis $f(x,y) \equiv 0$ outside of the region σ and therefore $f_{ij}(x,y)$ vanishes outside of σ and the region of integration becomes the region σ.

The condition on $f(x,y)$ and $f_1(x,y)$ stated in the theorem is therefore necessary.

6. Sufficiency of the Condition on f(x,y) and $f_1(x,y)$.

Suppose that $F(x,y)$ is an arbitrary analytic function and that the condition (2) holds. We show that there is a function $u(x,y,z)$ harmonic in the neighborhood of σ which satisfies the Cauchy conditions (1).

We let $\bar{u}(x',y',z')$ be a function harmonic in the upper half plane which takes on continuously the values $f(x,y)$ in W as (x',y',z') approaches $(x,y,0)$ where $f(x,y)$ is bounded and continuous in W. The function $\bar{u}(x',y',z')$ given by (3), with $\bar{g}(P)=f(x,y)$, in fact satisfies these conditions. Moreover $\dfrac{\partial \bar{u}(x',y',z')}{\partial z'}$ takes on continuously the values $J/2\pi$ where J is given by the second member of (11). And we let $v(x',y',z')$ be the harmonic function which satisfies the analytic Cauchy data

$$v(x,y,0) = 0$$

$$\frac{\partial v(x,y,0)}{\partial z} = F(x,y)$$

We define

$$u(x',y',z') = \bar{u}(x',y',z') + v(x',y',z')$$

Letting (x',y',z') approach $(x,y,0)$, we have

$$\lim u(x',y',z') = \lim \bar{u}(x',y',z') + \lim v(x',y',z')$$

$$= f(x,y) + 0 \qquad = f(x,y)$$

Furthermore differentiating $u(x',y',z')$ and letting (x',y',z') approach $(x,y,0)$ we have

$$\lim \frac{\partial u(x',y',z')}{\partial z'} = \lim \frac{\partial \bar{u}(x',y',z')}{\partial z'} + \lim \frac{\partial v(x',y',z')}{\partial z'}$$

$$= J + F(x,y)$$

$$= f_1(x,y)$$

Thus the condition on $f(x,y)$ and $f_1(x,y)$ in the theorem is sufficient.

Appendix 1. Proof of Extension Theorem

If $U(M)$ is harmonic for $z > 0$ and takes on continuously the value zero on the bounded region σ of the x,y plane, then $U(M)$ can be extended across the plane $z = 0$ so as to be harmonic in the neighborhood of σ, and there is only one such extension.

Take a sphere S with center O in σ and radius less than the distance to any boundary point of σ. We write, for M within S,

$$V(M) = \frac{1}{4\pi a}\int_S \frac{a^2-\rho^2}{r^3}\bar{U}(P)\,dS,$$

where P is a point on the surface of the sphere, $\rho = OM$, ~~and~~ S denotes integration over the whole surface of the sphere, ~~provided that point M is in the hemisphere above the plane $z=0$~~ and ~~that~~

$$\bar{U}(P) = U(P), \quad z_P > 0$$

and

$$\bar{U}(P') = -U(P), \quad z_{P'} < 0$$

~~where~~ P' being ~~is~~ the image of P in the x,y plane. In the following, we let M' similarly be the image of M.

From the definition of the integral we may write

$$V(M) = \lim_{\Delta S\to 0}\sum_{\text{upper half}}\frac{a^2-\rho^2}{r_i^3}\bar{U}(P_i)\Delta S_i + \lim_{\Delta S\to 0}\sum_{\text{lower half}}\frac{a^2-\rho^2}{r_i^3}\bar{U}(P_i')\Delta S_i$$

and for each element in the upper half sphere, there corresponds an equal element ΔS_i in the lower half ~~plane~~ sphere. Therefore, letting S_1 denote the surface of the sphere above the x,y plane and S_2 the other half of the sphere, we have for $z_M = 0$

$$V(M) = \frac{1}{4\pi a}\int_{S_1}\frac{a^2-\rho^2}{r^3}\bar{U}(P)\,dS + \frac{1}{4\pi a}\int_{S_2}\frac{a^2-\rho^2}{r^3}\bar{U}(P')\,dS$$

$$= \frac{1}{4\pi a}\int_{S_1}\frac{a^2-\rho^2}{r^3}U(P)\,dS - \frac{1}{4\pi a}\int_{S_1}\frac{a^2-\rho^2}{r^3}U(P)\,dS$$

$$= 0$$

Moreover for $z_M \neq 0$ we have $V(M') = -V(M)$ for

$$V(M) = \frac{1}{4\pi a}\int_S \frac{a^2-\rho^2}{r^3}\,\bar{U}(P)\,dS_P$$

$$V(M') = \frac{1}{4\pi a}\int_S \frac{a^2-\rho^2}{r'^3}\,\bar{U}(P)\,dS_P$$

$$= -\frac{1}{4\pi a}\int_S \frac{a^2-\rho^2}{r'^3}\,\bar{U}(P')\,dS_{P'} = -V(M)$$

Accordingly the function $V(M)$ is harmonic in the upper hemisphere, is bounded, and takes on continuously on the upper hemisphere and on the portion of σ bounding this hemisphere the same values as $U(M)$. Hence $V(M) \equiv U(M)$, for otherwise the difference $V(M) - U(M)$ would have a positive maximum or a negative minimum inside the hemisphere.

Also $V(M)$ is analytic and harmonic on $z = 0$, because $V(M)$ is analytic and harmonic inside S. Hence $V(M)$ provides an analytic and harmonic extension for $U(M)$, namely the extension $U(M') = -U(M)$

The extension id moreover unique. If there were two functions harmonic in the neighborhood of σ and identical for $z \geq 0$, they would have to be identical also for $z < 0$, since they would both be analytic functions of x, y, z and their difference would therefore be an analytic function of x, y, z, identically zero for $z \geq 0$.

Appendix 2. Determination of g(M,P).

The Green's function $g(M,P)$ for the upper half plane must be chosen so that as a function of P, it is a solution of

Laplace's equation when $z_M > 0$, $z_P > 0$ and so that

(a) $g(M,P) = 0$ for $z_M > 0$, $z_P = 0$

(b) $g(M,P) - \frac{1}{MP}$ remains bounded as P approaches point M for $z_M > 0$

(c) $[g(M,P)] \cdot MP$, $\frac{\partial g(M,P)}{\partial x_P} \cdot \overline{MP}^2$, $\frac{\partial g(M,P)}{\partial y_P} \cdot \overline{MP}^2$, $\frac{\partial g(M,P)}{\partial z_P} \cdot \overline{MP}^2$ all remain bounded as P approaches infinity.

If r is the distance between any two points, we know that $1/r$ is harmonic throughout a three-dimensional space. In order to have $g(M,P) = 0$ for $z_M > 0$, $z_P = 0$, we may write $g(M,P)$ as the difference of two harmonic functions $\frac{1}{MP}$ and $\frac{1}{M'P}$ where MP is the distance between M and P ,and $M'P$ is the distance between P and M' ,the reflection of M in the x,y plane. We see that for $z_P = 0$, $MP = M'P$ and that

$$g(MP) = \frac{1}{MP} - \frac{1}{M'P} = 0$$

Furthermore as P approaches M , $z_M > 0$,we have

$$g(M,P) - \frac{1}{MP} = \frac{1}{MP} - \frac{1}{M'P} - \frac{1}{MP} = -\frac{1}{M'P}$$

which remains bounded as P approaches M for $z_M > 0$.

Finally we see that as P approaches infinity

$$g(M,P) \cdot MP = \left[\frac{1}{MP} - \frac{1}{M'P}\right] MP = 1 - \frac{MP}{M'P}$$

and $\frac{MP}{M'P}$ remains bounded sinde MP approaches $M'P$ as P goes to infinity. And since

$$\frac{\partial g(M,P)}{\partial x_P} = \frac{\partial}{\partial x_P} \frac{1}{[(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2]^{1/2}} - \frac{\partial}{\partial x_P} \frac{1}{[(x_P - x_{M'})^2 + (y_P - y_{M'})^2 + (z_P - z_{M'})^2]^{1/2}}$$

$$= -\frac{x_P - x_M}{\overline{MP}^3} + \frac{x_P - x_{M'}}{\overline{M'P}^3}$$

we have

$$\frac{\partial g(MP)}{\partial x_P} \cdot \overline{MP}^2 = -\frac{x_P - x_M}{\overline{MP}} + \frac{x_P - x_{M'}}{\overline{M'P}}$$

and this also remains bounded since $|x_P - x_M| < MP$ and $|x_P - x_{M'}| < M'P$.

Therefore our Green's function chosen as

$$g(M,P) = \frac{1}{MP} - \frac{1}{M'P}$$

satisfies all conditions.

Appendix 3. Proof of Existence and of Limit of Integral (9).

The only question is in regard to the terms corresponding to $f(r,\theta,z) = \frac{1}{r}\log\left(r+\sqrt{r^2+z^2}\right)\left\{f_{20}\cos^2\theta + \cdots + f_{02}\sin^2\theta\right\}$

since $\frac{r}{(r^2+z^2)^{1/2}}$ is bounded for all z. Take a circle of radius δ in the x,y plane and center point M when $z_M = 0$. We call this region σ_δ and the rest of the region of integration $\sigma - \sigma_\delta$. We have from (9)

$$I_z = \int_{\sigma-\sigma_\delta} f(r,\theta,z)\,d\sigma + \int_{\sigma_\delta} f(r,\theta,z)\,d\sigma \,.$$

We write

$$I_0 = \lim_{\delta\to 0}\int_{\sigma-\sigma_\delta} f(r,\theta,0)\,d\sigma$$

and wish to show that:

1) the improper integral I_0 exists

2) $\lim_{z=0} I_z = I_0$

In order to prove 1) we take a circle concentric with δ and with radius $\delta' < \delta$ and show that given ϵ the integral

$$\int_{\delta'}^{\delta} |f(r,\theta,0|\,d\sigma$$

can be made $<\epsilon$ if $\delta < \Delta$. In fact we have

$$\int_{\delta'}^{\delta} |f(r,\theta,0)|\,d\sigma \leq \int_0^{2\pi} d\theta \int_{\delta'}^{\delta} \left|\log\left(r+\sqrt{r^2+z^2}\right)\right| \cdot M_1\, dr$$

since $|f_{20}\cos^2\theta + \cdots + f_{02}\sin^2\theta| < M_1$, and further we can write

$$\int_{\delta'}^{\delta} |\xi(r,\theta,z)|\,d\sigma \le M_1 \cdot 2\pi \int_{\delta'}^{\delta} |\log 2r|\,dr = M_1 \cdot 2\pi(\delta-\delta')\log 2 + M_1 \cdot 2\pi \left|\left[r\log r - r\right]_{\delta'}^{\delta}\right|$$

$$\le 2\pi M_1 \{\delta \log 2 + \delta \log \delta + \delta\}$$

$$\le \epsilon$$

irrespective of z and δ' if δ is small enough. For $z = 0$ this result yields the convergence of I_0. Hence I_0 exists and we have

$$I_0 = \int_{\sigma-\sigma_\delta} \xi(r,\theta,0)\,d\sigma + \int_{\sigma_\delta} \xi(r,\theta,0)\,d\sigma$$

for the integral is said to exist if it is convergent.

In proving that $\lim_{z=0} I_z = I_0$ we write

$$|I_z - I_0| = \left| \int_{\sigma-\sigma_\delta} [\xi(r,\theta,z) - \xi(r,\theta,0)]\,d\sigma + \int_{\sigma_\delta} \xi(r,\theta,z)\,d\sigma - \int_{\sigma_\delta} \xi(r,\theta,0)\,d\sigma \right|$$

$$\le \int_{\sigma-\sigma_\delta} |\xi(r,\theta,z) - \xi(r,\theta,0)|\,d\sigma + \int_{\sigma_\delta} |\xi(r,\theta,z)|\,d\sigma + \int_{\sigma_\delta} |\xi(r,\theta,0)|\,d\sigma$$

By taking δ small enough,

$$|I_z - I_0| \le \int_{\sigma-\sigma_\delta} |\xi(r,\theta,z) - \xi(r,\theta,0)|\,d\sigma + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

With δ fixed, take M near enough to Q in σ_δ so that

$$|\xi(r,\theta,z) - \xi(r,\theta,0)| < \frac{\epsilon}{3(\sigma_\delta - \sigma_\delta)}$$

for (r,θ) in $\sigma-\sigma_\delta$. This can be done since r remains greater than some positive number for (r,θ) in $\sigma-\sigma_\delta$ as z approaches zero.

Hence

$$\left| I_{\zeta} - I_0 \right| \leq \epsilon$$

That is, given we can take near enough to in

so that $\left| I_{\zeta} - I_0 \right| < \epsilon$

or so that

$$\lim_{M = Q} I_{\zeta} = I_0$$

Therefore the integral given by (9) exists and its limit as $\zeta = 0$ is the integral (11).