POLYNOMIAL EXPANSIONS IN THE BOREL REGION

by

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This paper deals with the expansion of analytic functions into a series of polynomials. An explicit form for the polynomials is obtained in terms of the coefficients of a Taylor's expansion and of certain positive constants h_k. Conditions on the constants h_k are developed under which the series of polynomials will converge. In particular, conditions are imposed upon the h_k such that a region can be found within which the series will converge to the function. This region is in general larger than the circle of convergence of the Taylor's series.

Derivation. Let Γ be a regular curve enclosing the origin, within which $f(z)$ is analytic and on which $f(z)$ is continuous. For z within Γ we have from Cauchy's formula that

$$(1) \qquad f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)\,dt}{t-z}$$

Let $h_1, h_2, h_3, \ldots$ be a sequence of positive numbers. Writing $t-z = (t+h_1t) - (z+h_1t)$ gives

$$\frac{1}{t-z} = \frac{1}{t+h_1t} + \frac{z+h_1t}{t+h_1t}\,\frac{1}{t-z}$$

If in the right hand side we place $t-z = (t+h_2t) - (z+h_2t)$ it becomes

$$\frac{1}{t-z} = \frac{1}{t+h_1t} + \frac{z+h_1t}{(t+h_1t)(t+h_2t)} + \frac{(z+h_1t)(z+h_2t)}{(t+h_1t)(t+h_2t)}\,\frac{1}{t-z}$$

The repetition of this process m times, and the substitution in equation (1) gives

$$f(z) = P_0(z) + P_1(z) + \dots + P_{m-1}(z) + R_m(z)$$

where $$P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+h_1t)(z+h_2t)\dots(z+h_nt)}{(1+h_1)\dots(1+h_n)(1+h_{n+1})}\ \frac{f(t)\,dt}{t^{n+1}}$$

and

$$(2)\qquad R_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+h_1t)\dots(z+h_mt)}{(t+h_1t)\dots(t+h_mt)}\ \frac{f(t)\,dt}{t-z}$$

To find the polynomials $P_n(z)$ we apply again Cauchy's integral formula,

$$n!\,P_n(z) = \left[\prod_{k=1}^{k=n+1}\frac{1}{1+h_k}\right]\left[\frac{d^n}{dt^n}\left(f(t)\cdot\prod_{k=1}^{k=n}(z+h_kt)\right)\right]_{t=0}$$

In order to perform the indicated differentiation we make the following definitions,

$$(3)\qquad u(t) \equiv \prod_{k=1}^{k=n}(z+h_kt) \equiv z^n + b_{n,1}z^{n-1}t + \dots + b_{n,n}t^n$$

$$w(t) \equiv f(t)\,u(t)$$

The n-th derivative of $w(t)$ is

$$w^{(n)}(t) = u^{(n)}f + n\,u^{(n-1)}f' + \dots + \frac{n!\,u^{(n-k)}f^{(k)}}{(n-k)!\,k!} + \dots + u\,f^{(n)}$$

The k-th derivative of $u(t)$ is

$$u^{(k)}(t) = k!\,b_{n,k}\,z^{n-k} + (\text{terms in } t,\ t^2,\ \text{etc.})$$

or the (n-k)-th derivative which is needed in $w^{(n)}(t)$ is

$$u^{(n-k)}(t) = (n-k)!\,b_{n,n-k}\,z^k + (\text{terms in } t,\ t^2,\ \text{etc.})$$

Using the values for $t=0$ gives the $(k+1)$-th term in $w^{(n)}(0)$ as

$$\frac{n!}{k!} f^{(k)}(0)\; b_{n,n-k}\; z^k$$

Thus the polynomials $P_n(z)$ have the form

$$(4) \qquad P_n(z) = \left[\prod_{k=1}^{k=n+1} \frac{1}{1+h_k}\right] \cdot \sum_{k=0}^{k=n} \frac{f^{(k)}(0)}{k!}\, b_{n,n-k}\, z^k$$

If the Taylor's expansion of $f(z)$ at the origin is

$$f(z) = a_0 + a_1 z + a_2 z^2 + \quad \cdots$$

these polynomials take the form

$$(5) \qquad P_n(z) = \left[\prod_{k=1}^{k=n+1} \frac{1}{1+h_k}\right] \cdot \sum_{k=0}^{k=n} a_k\, b_{n,n-k}\, z^k$$

The coefficients $b_{n,p}$ are defined by the identity (3). They are the elementary symmetric functions of $h_1, h_2, \ldots, h_n$. That is: $b_{n,p}$ is the sum of the products taken p at a time without repetitions of the first n of the h_k. The sets of b's for the different polynomials are connected by a recurrence formula

$$(6) \qquad b_{n,p} = b_{n-1,p} + h_n\, b_{n-1,p-1}$$

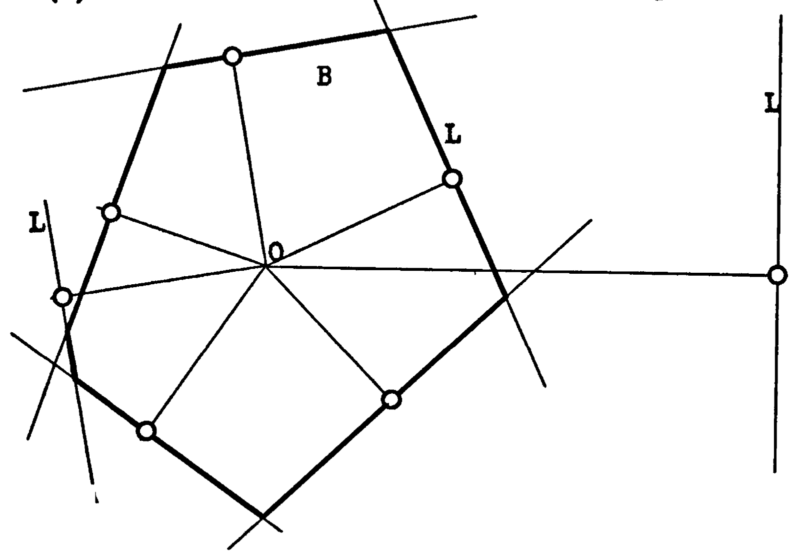
which holds in general if we set

$$b_{n,p} = 1 \quad \text{for } p=0$$
$$= 0 \quad \text{for } p<0, \text{ or } p>n.$$

<u>Definition.</u> The Borel region. Let a ray, that is a half line, issue from the origin; proceed along this line until a singularity of the function $f(z)$ is encountered, if any; thru this singular point draw a line L perpendicular to the ray. Do this for all rays.

The set of points not on any line L and which can be joined to the origin by line segments not meeting any of these lines L constitutes a region B enclosing the origin which will be called the Borel region of the function f(z). We shall refer to this as the region B.

Figure 1



If the function has only a finite number of singularities its Borel region is a convex polygon (Fig.1). The converse statement is not true.

We shall next consider the convergence of the infinite series of polynomials

(7) $$P(z) = P_0(z) + P_1(z) + P_2(z) + \cdots$$

This series will be called the series P(z).

<u>Theorem I.</u> <u>If</u> $\lim_{k\to\infty} h_k = \infty$ <u>and if</u> $\sum_{k=1}^{\infty} h_k^{-1}$ <u>diverges,</u> <u>then P(z) converges uniformly to f(z) in any region B' entirely within the Borel region of f(z).</u>

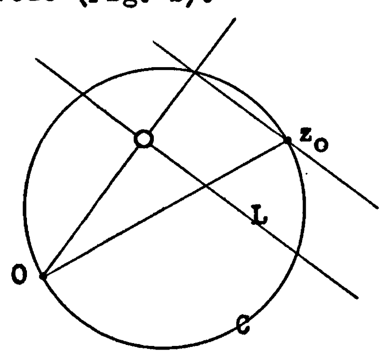
<u>Proof.</u> It is required to show that under the conditions of the theorem

$$\lim_{m\to\infty} R_m(z) = 0$$

uniformly in B'. The first step is to define the contour Γ.

<u>The contour Γ.</u> Let $z_0 \neq 0$ be a point in the region B; then there are no singularities of f(z) within or on the circle C with diameter Oz_0. For suppose that there were a singular point in C, then as we proceed from the origin along some ray we would encounter this singularity before reaching a point on C. A perpendicular line thru this point on C will pass thru z_0 because we have a right angle inscribed in a semi-circle (Fig. 2).

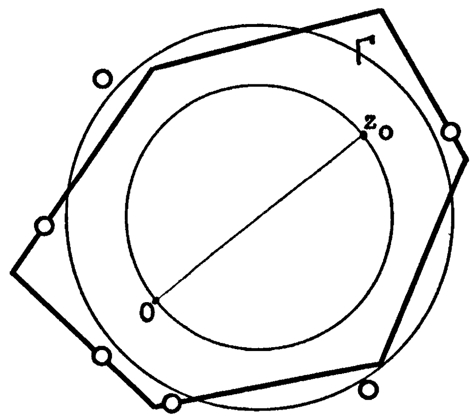
Figure 2



Therefore a perpendicular L thru the singular point will pass between z_0 and the origin O, and z_0 is not in the region B. Since this is contrary to hypothesis there can be no singularity within C. Likewise there can be no singularity of f(z) on C, for in this case the line L would pass thru z_0.

We may accordingly choose a circle, concentric with C and of slightly larger radius, within and on which f(z) is analytic. This circle, which need not lie wholly within B (Fig. 3), will be the contour Γ.

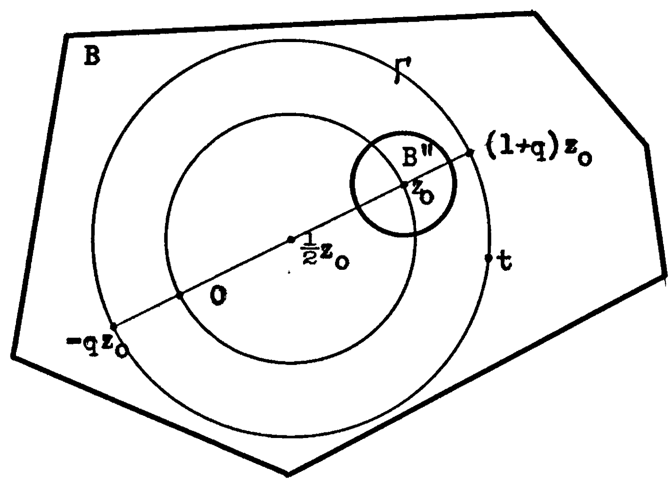
Figure 3



Had we chosen the point $z_0 = 0$ this argument must be modified to a slight extent. For the origin the contour Γ may be taken as a circle with center at the origin. By hypothesis $f(z)$ is analytic within and continuous on this circle for a sufficiently small radius. The remainder of the proof will assume that $z_0 \neq 0$; however, only slight and obvious changes need be made to treat the case $z_0 = 0$.

The center of Γ is the point $\frac{1}{2}z_0$ and we shall let the radius be $(\frac{1}{2}+q)|z_0|$, where $q > 0$ is sufficiently small (Fig. 4). In particular we shall require that $q < 1$. (If $z_0 = 0$ we can take the radius to be equal to q.)

Figure 4



Hence a point t on Γ satisfies the equation

$$(t - \tfrac{1}{2}z_0)(\bar{t} - \tfrac{1}{2}\bar{z}_0) = (\tfrac{1}{2}+q)^2 z_0\bar{z}_0$$

where as usual the bar denotes the conjugate imaginary quantity. This equation may be written as

$$(8) \qquad 2t\bar{t} - z_0\bar{t} - \bar{z}_0 t = 2q(1+q)\, z_0\bar{z}_0$$

The bounds on the variable t are very useful; and since t is on the circle Γ of radius $(\frac{1}{2}+q)|z_0|$, these bounds are obviously (Fig. 4)

$$q^2 z_0 \bar{z}_0 \leqslant t\bar{t} \leqslant (1+q)^2 z_0 \bar{z}_0 \tag{9}$$

The next step is to prove the uniform convergence of P(z) in a sufficiently small region B" about the point z_0. For convenience B" will be taken as a circle with center at z_0 and radius $\frac{1}{2}q^2|z_0|$. It is clear that this circle lies entirely within Γ. Now for z in the region B" we may place

$$z = z_0 + \eta \qquad |\eta| < \tfrac{1}{2} q^2 |z_0| \tag{10}$$

To prove the uniform convergence of P(z) in the region B" we must show that : Given an arbitrary $\epsilon > 0$, an M can be found such that for all $m > M$, and for all z in B", $|R_m(z)| < \epsilon$. In order to demonstrate this we investigate an upper bound to the absolute value of the product

$$\prod_{k=1}^{k=m} \left| \frac{z+h_k t}{t+h_k t} \right|$$

which appears in the expression (2) for the remainder $R_m(z)$.

To this end we consider the above product taken to infinity. This infinite product converges or diverges with

$$\prod_{k=1}^{\infty} \left| \frac{z+h_k t}{t+h_k t} \right|^2$$

which is easier to study. We write this last product as

$$\prod_{k=1}^{\infty} \left| \frac{z+h_k t}{t+h_k t} \right|^2 = \prod_{k=1}^{\infty} \left\{ 1 - \left(1 - \left| \frac{z+h_k t}{t+h_k t} \right|^2 \right) \right\} \tag{11}$$

In this form the infinite product converges or diverges with the summation

$$(12) \qquad \sum_{k=1}^{\infty}\left(1 - \left|\frac{z+h_k t}{t+h_k t}\right|^2\right)$$

provided the terms of this sum have ultimately the same sign. In particular, if $\left|\frac{z+h_k t}{t+h_k t}\right|^2 < 1$ for all $k > N$, and if (12) diverges, then the infinite product (11) diverges to the value zero. We shall prove this to be the case for all z in B'' and all t on Γ.

The terms of the summation (12) can be written as

$$1 - \left|\frac{z+h_k t}{t+h_k t}\right|^2 = \frac{t\bar{t} - z\bar{z} + h_k(2t\bar{t} - z\bar{t} - \bar{z}t)}{t\bar{t}\ (1+h_k)(1+h_k)}$$

$$= \frac{h_k^{-1}(t\bar{t} - z\bar{z}) + 2t\bar{t} - z\bar{t} - \bar{z}t}{t\bar{t}\ (1+h_k)(1+h_k^{-1})}$$

If we are to have $\left|\frac{z+h_k t}{t+h_k t}\right|^2 < 1$ this expression must be positive for large enough values of k. Now

$$2t\bar{t} - z\bar{t} - \bar{z}t = 2t\bar{t} - z_o\bar{t} - \bar{z}_o t - (\eta\bar{t}+\bar{\eta}t)$$

$$= 2q(1+q)\ z_o\bar{z}_o \quad - (\eta\bar{t}+\bar{\eta}t)$$

by relations (10) and (8). Since

$$|\eta\bar{t}+\bar{\eta}t| \leqslant 2|\eta t|$$

which, by (9) and (10), is

$$\leqslant q^2(1+q)\ z_o\bar{z}_o$$

we find that

$$q(1+q)(2-q)\ z_o\bar{z}_o \leqslant 2t\bar{t} - z\bar{t} - \bar{z}t \leqslant q(1+q)(2+q)\ z_o\bar{z}_o$$

These bounds are definitely positive because $q < 1$. Similarly we can show that

$$-z_0\bar{z}_0(1+\tfrac{1}{4}q^4) \leq t\bar{t} - z\bar{z} \leq q(2+2q-\tfrac{1}{4}q^3)\, z_0\bar{z}_0$$

or better that $$|t\bar{t} - z\bar{z}| < 4\, z_0\bar{z}_0$$

Upon the substitution of these results and the use of equation (9), the terms of the summation (12) are found to be bounded by

$$\frac{-4\,h_k^{-1} + q(1+q)(2-q)}{(1+q)^2(1+h_k)(1+h_k^{-1})} < 1 - \left|\frac{z+h_k t}{t+h_k t}\right|^2 < \frac{4\,h_k^{-1} + q(1+q)(2+q)}{q^2(1+h_k)(1+h_k^{-1})}$$

These bounds are independent of z and t. They are definitely positive for h_k sufficiently large; so if $\lim_{k\to\infty} h_k = \infty$, then $\left|\frac{z+h_k t}{t+h_k t}\right|^2 < 1$ for all $k > N$. It is also clear that we can choose c_1 and c_2, say $c_1 = q/8$ and $c_2 = 6/q$, such that

$$\frac{c_1}{h_k} < 1 - \left|\frac{z+h_k t}{t+h_k t}\right|^2 < \frac{c_2}{h_k}\text{, for all } k > N.$$

Therefore the summation (12), and consequently the infinite product (11), converges or diverges with $\sum_{k=1}^{\infty} h_k^{-1}$. If this last sum diverges, then (11) diverges to the value zero.

This result means that under the hypotheses of the theorem the product in the remainder $R_m(z)$ is bounded; and, given an $\epsilon > 0$ an M exists such that for all $m > M$, all z in B'', and all t on Γ

$$\prod_{k=1}^{k=m}\left|\frac{z+h_k t}{t+h_k t}\right| < \epsilon' < \frac{\epsilon}{(\frac{1}{2}+q)\,|z_0|\;K}$$

where K is an upper bound of the function $f(t)\cdot(t-z)^{-1}$ on Γ,

$\left|f(t)\cdot(t-z)^{-1}\right| < K$. The length of the contour is $2\pi(\frac{1}{2}+q)|z_0|$. Therefore for all $m > M$

$$\left|R_m(z)\right| = \frac{1}{2\pi}\left|\int_{\Gamma}\prod_{k=1}^{k=m}\frac{z+h_k t}{t+h_k t}\cdot\frac{f(t)\ dt}{t-z}\right| < \frac{1}{2\pi}\cdot\epsilon'\cdot K\cdot 2\pi(\tfrac{1}{2}+q)|z_0| < \epsilon$$

Thus the series P(z) converges uniformly to f(z) in the small region B" about the point z_0, which was any point in the Borel region of the function f(z).

Such a region B" exists about every point of any region B' which lies entirely within B. By the Heine-Borel theorem B' can be covered by a finite number of such regions, and hence the series P(z) converges uniformly in B' to the function f(z).

In theorem I we have given the sufficient conditions that P(z) represent the function f(z) in the Borel region. We shall now prove that the condition that $\sum_{k=1}^{\infty} h_k^{-1}$ should diverge is also a necessary condition. In order to do this we shall assume that this sum converges, and show that the series P(z) does not converge to the function.

Theorem II. If $\sum_{k=1}^{\infty} h_k^{-1}$ converges, then P(z) converges absolutely and uniformly in any finite region to an analytic function, but does not converge to f(z), provided that f(z) is not identically zero, except at most on an isolated set of points.

<u>Proof.</u> 1). The convergence of P(z). We shall show that the series P(z) converges absolutely and uniformly in a circle $|z| \leq M$ arbitrarily large.

The coefficients a_k in the Taylor's expansion of f(z) at the origin are bounded

$$|a_k| \leq \frac{A}{\rho^k}$$

where ρ is less than the radius of the circle of convergence of the Taylor's series; and A is the maximum value of $|f(z)|$ on a circle with center at the origin and radius ρ. Then, with the use of the identity (3), we can write

$$\left|P_n(z)\right| \leq \left[\prod_{k=1}^{k=n+1} \frac{1}{1+h_k}\right] \cdot \sum_{k=0}^{k=n} A\, b_{n,n-k} \left|\frac{z}{\rho}\right|^k = \frac{A}{1+h_{n+1}} \prod_{k=1}^{k=n} \frac{\left|\frac{z}{\rho}\right|+h_k}{1+h_k}$$

$$\leq \frac{A}{1+h_{n+1}} \prod_{k=1}^{k=n} \frac{1+M(\rho h_k)^{-1}}{1+h_k^{-1}}$$

for all z in the circle $|z| \leq M$. Since M is arbitrary we may choose $M > \rho$.

Consider the product in the right hand member. It may be written in the form

$$\frac{\prod_{k=1}^{k=n} (1+M(\rho h_k)^{-1})}{\prod_{k=1}^{k=n} (1+h_k^{-1})}$$

Each factor in numerator and denominator is positive and greater than unity, so if $\sum_{k=1}^{\infty} h_k^{-1}$ converges, these products taken to infinity converge to values greater than 1. Evidently the numerator is larger than the denominator. Thus

$$\lim_{n\to\infty} \prod_{k=1}^{k=n} \frac{1+M(\rho h_k)^{-1}}{1+h_k^{-1}} = H > 1$$

the approach to H being from below. We have accordingly that the terms of the series P(z) are less numerically than the corresponding terms of a convergent series of positive terms,

$$|P_n(z)| < A\,H\,(1+h_{n+1})^{-1} < A\,H\,h_{n+1}^{-1}$$

Therefore the series P(z) converges absolutely and uniformly in the circle $|z| \leq M$, and to an analytic function.

2). To show that $P(z) \not\equiv f(z)$. If we assume that $P(z) = f(z)$ on any infinite set of points having a finite cluster point in a region D within which f(z) is analytic, then $P(z) \equiv f(z)$ in D. We shall prove that this cannot happen in any such region D. Since f(z), by hypothesis, is analytic at the origin, and any D may be enclosed in a larger region which includes the origin and within which f(z) is analytic, it is sufficient to prove that $P(z) \neq f(z)$ in some small region about the origin. And as $\frac{P(z)}{f(z)}$ is an analytic function for $f(z) \neq 0$, we shall prove this by showing that $\lim_{z\to 0} \frac{P(z)}{f(z)} \neq 1$.

We shall first assume that $f(0) \neq 0$, whence it follows that $f(z) \neq 0$ for sufficiently small values of z. At the origin

$$P_n(0) = \left[\prod_{k=1}^{k=n+1}(1+h_k)^{-1}\right] b_{n,n}\, f(0) = f(0)\,(1+h_{n+1})^{-1}\prod_{k=1}^{k=n}\frac{h_k}{1+h_k}$$

where the definition of $b_{n,n}$ has been used to obtain the final expression. The sum of the first two of these polynomials is

$$P_o(0) + P_1(0) = f(0)\left[\frac{1}{1+h_1} + \frac{1}{1+h_2}\,\frac{h_1}{1+h_1}\right]$$

$$= f(0)\left[1 - \frac{h_1h_2}{(1+h_1)(1+h_2)}\right]$$

We have here a suggestion as to the form for the sum of the first n polynomials. We shall prove by induction that this form is the correct one. Assuming that

$$\sum_{k=o}^{k=n-1} P_k(0) = f(0)\left[1 - \prod_{k=1}^{k=n}\frac{h_k}{1+h_k}\right]$$

we see immediately that

$$\sum_{k=o}^{k=n} P_k(0) = f(0)\left[1 - \prod_{k=1}^{k=n}\frac{h_k}{1+h_k} + \frac{1}{1+h_{n+1}}\prod_{k=1}^{k=n}\frac{h_k}{1+h_k}\right]$$

$$= f(0)\left[1 - \prod_{k=1}^{k=n+1}\frac{h_k}{1+h_k}\right]$$

We conclude that $P(0)$, the sum to infinity, is

$$P(0) = f(0)\left[1 - \prod_{k=1}^{\infty}(1+h_k^{-1})^{-1}\right]$$

As we have assumed that $\sum_{k=1}^{\infty} h_k^{-1}$ converges, the infinite product will converge to a value $h > 0$ and

$$P(0) = f(0)\ (1 - h)$$

which is to say that

$$\lim_{z \to 0} \frac{P(z)}{f(z)} = \frac{P(0)}{f(0)} = 1 - h \neq 1$$

For this case then the theorem is proved.

The preceding work is a special case of the more general proof which will now be given. This special case, $f(0) \neq 0$, was presented because of its simplicity and to illustrate the method. The methods used were exactly those which are needed for the general case.

At the origin we have used the Taylor's expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \quad \cdots$$

In this series let a_s be the first coefficient that is not zero. Then

$$f(z) = z^s\ (a_s + a_{s+1}\ z\ + \cdots\ ) \qquad a_s \neq 0$$

and the polynomials of $P(z)$ become

$$P_n(z) = z^s \left[\prod_{k=1}^{k=n+1} (1 + h_k)^{-1} \right] \cdot \sum_{k=s}^{k=n} a_k\, b_{n,n-k}\, z^{k-s}$$

where the first polynomial which is not identically zero is $P_s(z)$. That is,

$$P(z) = P_s(z) + P_{s+1}(z) + \quad \cdots\cdot$$

This result follows naturally from the definition of the $P_n(z)$.

For $z = 0$ we have that

$$\left[z^{-s} f(z)\right]_0 = a_s$$

$$\left[z^{-s} P_n(z)\right]_0 = a_s\, b_{n,n-s} \prod_{k=1}^{k=n+1} (1+h_k)^{-1}$$

The sum of the first three of these terms is

$$\sum_{n=s}^{n=s+2} \left[z^{-s} P_n(z)\right]_0 = a_s \left[\prod_{k=1}^{k=s+1} (1+h_k)^{-1} + b_{s+1,1} \prod_{k=1}^{k=s+2} (1+h_k)^{-1} + b_{s+2,2} \prod_{k=1}^{k=s+3} (1+h_k)^{-1} \right]$$

$$= a_s \prod_{k=1}^{k=s+3} (1+h_k)^{-1} \left[1 + h_{s+3} + h_{s+2} + b_{s+1,1} + h_{s+3}(h_{s+2} + b_{s+1,1}) + b_{s+2,2} \right]$$

From the definition of $b_{s+1,1}$ we see that

$$h_{s+3} + h_{s+2} + b_{s+1,1} = b_{s+3,1}$$

$$h_{s+2} + b_{s+1,1} = b_{s+2,1}$$

With these expressions and the recurrence formula (6), the sum becomes

$$\sum_{n=s}^{n=s+2} \left[z^{-s} P_n(z)\right]_0 = a_s \prod_{k=1}^{k=s+3} (1+h_k)^{-1} \left[1 + b_{s+3,1} + b_{s+3,2} \right]$$

We are now in a position to establish by induction the form for the sum of m terms. Assume that

$$\sum_{n=s}^{n=s+m-2} \left[z^{-s} P_n(z)\right]_0 = a_s \prod_{k=1}^{k=s+m-1} (1+h_k)^{-1} \sum_{k=0}^{k=m-2} b_{s+m-1,k}$$

then

$$\sum_{n=s}^{n=s+m-1}\left[z^{-s}P_n(z)\right]_o = a_s \prod_{k=1}^{k=s+m}(1+h_k)^{-1}\left[(1+h_{s+m})\cdot\sum_{k=0}^{k=m-2} b_{s+m-1,k} + b_{s+m-1,m-1}\right]$$

$$= a_s \prod_{k=1}^{k=s+m}(1+h_k)^{-1}\sum_{k=0}^{k=m-1}(b_{s+m-1,k} + h_{s+m}\, b_{s+m-1,k-1})$$

$$= a_s \prod_{k=1}^{k=s+m}(1+h_k)^{-1}\sum_{k=0}^{k=m-1} b_{s+m,k}$$

In the last two steps we have used the fact that $b_{s+m-1,-1} \equiv 0$, and also the recurrence formula (6).

To transform this result we use the identity (3). This gives

$$\prod_{k=1}^{k=n}(1+h_k) \equiv \sum_{k=0}^{k=n} b_{n,k}$$

and enables us to write

$$\sum_{k=0}^{k=m-1} b_{s+m,k} = \prod_{k=1}^{k=s+m}(1+h_k) - \sum_{k=m}^{k=s+m} b_{s+m,k}$$

We have accordingly that

$$\sum_{n=s}^{n=s+m-1}\left[z^{-s}P_n(z)\right]_o = a_s\left[1 - \prod_{k=1}^{k=s+m}(1+h_k)^{-1}\sum_{k=m}^{k=s+m} b_{s+m,k}\right]$$

By definition this summation carried to infinity is $\left[z^{-s}P(z)\right]_o$. Now $\left[z^{-s}f(z)\right]_o = a_s$, and since

$$\frac{z^{-s}P(z)}{z^{-s}f(z)} = \frac{P(z)}{f(z)}$$

so that

$$\lim_{z\to o}\frac{P(z)}{f(z)} = \frac{\left[z^{-s}P(z)\right]_o}{\left[z^{-s}f(z)\right]_o}$$

it is required to show that $\left[z^{-s}P(z)\right]_o \neq a_s$; that is, that

$$\lim_{n\to\infty} \prod_{k=1}^{k=n} (1+h_k^{-1}) \sum_{k=n-s}^{k=n} b_{n,k} \neq 0 .$$

in order to prove that $\lim_{z\to 0} \frac{P(z)}{f(z)} \neq 1.$

For this purpose consider the coefficients $b_{n,k}$. They have been defined as the elementary symmetric functions in $h_1, h_2, \ldots, h_n$. Let us define, in an exactly similar fashion, quantities $d_{n,k}$ as the elementary symmetric functions in $h_1^{-1}, h_2^{-1}, \ldots, h_n^{-1}$. That is :

$$d_{n,1} \equiv \sum_{k=1}^{k=n} h_k^{-1}$$

$$d_{n,2} \equiv \sum_{i,k=1}^{n} h_k^{-1} h_i^{-1} \qquad i \neq k$$

etc.

We notice immediately a useful inequality

$$d_{n,k} \leqslant d_{n,1}^{k}$$

which holds for all finite n and k. From this definition it is clear that the relation between these new quantities and the $b_{n,k}$ is of the form

$$b_{n,n}\, d_{n,n-k} \equiv b_{n,k}$$

This is the relation which could have been obtained directly from the identity (3) by division thru-out by the factor $h_1 h_2 \ldots h_n = b_{n,n}$.

In terms of these new symmetric functions we can write

$$\prod_{k=1}^{k=n} (1+h_k)^{-1} \sum_{k=n-s}^{k=n} b_{n,k} \equiv \prod_{k=1}^{k=n} (1+h_k^{-1})^{-1} \sum_{k=n-s}^{k=n} d_{n,n-k}$$

$$\equiv \prod_{k=1}^{k=n} (1+h_k^{-1})^{-1} \sum_{k=0}^{k=s} d_{n,k}$$

We have assumed that the summation

$$\sum_{k=1}^{\infty} h_k^{-1} = \lim_{n\to\infty} \sum_{k=1}^{k=n} h_k^{-1} = \lim_{n\to\infty} d_{n,1} = d_1 \neq 0$$

converges; therefore the following infinite product converges

$$\prod_{k=1}^{\infty} (1+h_k^{-1})^{-1} = h \neq 0$$

Also $$\lim_{n\to\infty} d_{n,k} = d_k \neq 0$$

This limit exists because $d_{n,k} \leq d_{n,1}^{k}$; and is different from zero because $d_{n,k}$ has for all values of n a term $h_1^{-1} h_2^{-1} \cdot\cdot\, h_k^{-1}$.

With these results we find that

$$\lim_{n\to\infty} \left[\prod_{k=1}^{k=n} (1+h_k)^{-1} \right] \cdot \sum_{k=n-s}^{k=n} b_{n,k} = h \sum_{k=0}^{k=s} d_k \neq 0$$

As this was the relation needed to prove that

$$\lim_{z\to 0} \frac{P(z)}{f(z)} \neq 1$$

our theorem is proved.