A THEOREM ON FUNCTIONS HARMONIC IN A CIRCLE

C. Hewitt Dix

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This paper is to establish a theorem (1) relative to functions harmonic in the unit circle. The theorem has been proved in a similar manner for functions harmonic in the sphere by Bray and Evans(2).

Let $u(M)$ denote a function which is harmonic at every point M internal to the unit circle S with center O. Consider intervals on the circle and denote any one of them by s. s is then determined by two angles, $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$. We next set down the two definitions of $F(r,s)$ and $T(r,s)$,

$$F(r,s) \equiv F(r;\theta_1,\theta_2) \equiv \int_{\rho(r,s)} u(r,\theta)\,ds \qquad \text{(a)}$$

and

$$T(r,s) \equiv T(r;\theta_1,\theta_2) \equiv \int_{\rho(r,s)} \left| u(r,\theta) \right| ds \qquad \text{(b)}$$

where $\rho(r,s)$ denotes the projection from O of s on a circle of center O and radius r. In all that follows $F(r,s)$ is assumed to be of uniformly limited variation with respect to r, i.e., $T(r,s) < K$ for all $r < 1$. $F(r,s)$ and $T(r,s)$ are additive as functions of segments on S_r for all values of the parameter $r < 1$, and therefore for values of r given by the sequence $r_1 < r_2 < \cdots$ with $\lim_{i \to \infty} r_i = 1$.

(1) Theorem 7

(2) Bray and Evans, "A Class of Functions Harmonic within the Sphere," American Journal of Mathematics, Vol. XLIX (1927), pp. 153-180. See pp. 160-168. This paper is referred to hereafter as (2)

1. SOME THEOREMS ON CONVERGENCE IN THE MEAN

THEOREM 1. <u>If $\mathcal{J}(r_i, s) < K$ for all i, then the sequence $\{F(r_i; \theta_1, \theta_2)\}$ converges in the mean in a two dimensional region $\Theta(\theta_1, \theta_2)$</u>.

C Consider three concentric circles, S_{r_i}, S_{r_j}, and S of radii r_i, r_j, and 1 repectively, where $i < j$, i.e., $r_i < r_j < 1$. Then Poisson's integral can take the form

$$u(M_i) = \frac{1}{2\pi r_j} \int_{S_{r_j}} \frac{(r_j^2 - r_i^2)\, u(M_j)}{\overline{M_j M_i}^2}\, dM_j \tag{1}$$

where M_i and M_j are points on the circumferenees S_{r_i} and S_{r_j} respectively. Then (1) and (a) gives

$$F(r_i, s) = \int_{\rho(r_i, s)} \left\{ \frac{1}{2\pi r_j} \int_{S_{r_j}} \frac{r_j^2 - r_i^2}{\overline{M_j M_i}^2}\, u(M_j)\, dM_j \right\} dM_i$$

$$= \int_{S_{r_j}} p(M_j; r_j, r_i; s)\, u(M_j)\, dM_j$$

where

$$p(M_j; r_j, r_i; s) \equiv \frac{1}{2\pi r_j} \int_{\rho(r_i, s)} \frac{r_j^2 - r_i^2}{\overline{M_j M_i}^2}\, dM_i \tag{c}$$

by definition.

Now consider various positions of the point M_j with respect to $\rho(r_j, s)$.

If M_j is inside $\rho(r_j, s)$ and its smaller angular distance to the ends of $\rho(r_j, s)$ is β, then

$$p(M_j; r_j, r_i; s) > \frac{(r_j^2 - r_i^2)\, r_i}{2\pi r_j} \int_{-\beta}^{+\beta} \frac{d\alpha}{r_j^2 + r_i^2 - 2 r_i r_j \cos\alpha} \tag{i}$$

$$= \frac{2 r_i}{\pi r_j} \tan^{-1}\left(\frac{r_j + r_i}{r_j - r_i} \tan \tfrac{1}{2}\beta \right)$$

If M_j is outside $\rho(r_j, s)$ and its smaller angular distance to the ends of $\rho(r_j, s)$ is β , then

$$0 < p(M_j ; r_j , r_i ; s) < \lim_{\epsilon \doteq 0} \left[\frac{r_i}{r_j \pi} \tan^{-1}\left(\frac{r_j + r_i}{r_j - r_i} \tan \tfrac{1}{2} \alpha \right) \left\{ \Big|_{\beta}^{\pi - \epsilon} + \Big|_{\pi + \epsilon}^{2\pi - \beta} \right\} \right]$$

$$= \frac{r_i}{\pi r_j} \left[\pi - 2 \tan^{-1} \left(\frac{r_j + r_i}{r_j - r_i} \tan \tfrac{1}{2} \beta \right) \right]$$

(ii)

For any position of M_j

$$0 < p(M_j ; r_j , r_i ; s) < \frac{r_i}{r_j} < 1$$

(iii)

Now let $\{ \delta_n \}$ be a non-increasing sequence associated with $\{ r_n \}$ such that $\lim_{n \doteq \infty} \delta_n = 0$ and

$$1 - r_i < \frac{\delta_n^2}{4} \quad \text{for} \quad i > n.$$

(iiii)

Also define a function $g(M_j, s)$ which is equal to 1 , $\frac{1}{2}$, 0 respectively, according as M_j is inside of, on an end of, outside of s . Furthermore define two intervals $\rho_{1n} = \rho_1 (r_i ; \theta_1 - \delta_n , \theta_1 + \delta_n)$ and $\rho_{2n} = \rho_2 (r_i ; \theta_2 - \delta_n , \theta_2 + \delta_n)$. Then it follows from inequalities (i), (ii), (iii), and (iiii) that if M_j is external to both ρ_{1n} and ρ_{2n}

$$\left| g(M_j , s) - p(M_j ; r_j , r_i ; s) \right| < \delta_n \quad \text{for} \quad j > i > n .$$

We see then that

$$\left| F(r_i;\theta_1,\theta_2) - F(r_i;\theta_1,\theta_2)\right|$$

$$= \left| \int_{S_{r_j}} \{ q(M_j,s) - p(M_j;r_i,r_i;s)\} u(M_j)\, dM_j \right|$$

$$\leq \int_{S_{r_j}} \left| q(M_j,s) - p(M_j;r_i,r_i;s)\right| \left| u(M_j)\right| dM_j$$

$$\leq \delta_n K + T(r_j;\theta_1-\delta_n,\theta_1+\delta_n) + T(r_j;\theta_2-\delta_n,\theta_2+\delta_n)$$

Hence

$$\int_{\Theta(\theta_1,\theta_2)} \left| F(r_j;\theta_1,\theta_2) - F(r_i;\theta_1,\theta_2)\right| d\Theta \leq 4\pi^2\delta_n K + 4\pi\int_0^{2\pi} T(r_j;\theta-\delta_n,\theta+\delta_n)\, d\theta$$

$$= 4\pi^2\delta_n K + 4\pi \int_0^{2\pi} \{ T(r_j;0,\theta+\delta_n) - T(r_j;0,\theta-\delta_n)\} d\theta$$

By changing the variable of integration the last integral can be written in the form

$$4\pi \left\{ \int_{\delta_n}^{2\pi+\delta_n} T(r_j;0,\theta')\, d\theta' - \int_{-\delta_n}^{2\pi-\delta_n} T(r_j;0,\theta')\, d\theta' \right\}$$

$$= 4\pi \left\{ \int_{2\pi-\delta_n}^{2\pi+\delta_n} T(r_j;0,\theta')\, d\theta' - \int_{-\delta_n}^{+\delta_n} T(r_j;0,\theta')\, d\theta' \right\}$$

$$\leq 4\pi \int_{2\pi-\delta_n}^{2\pi+\delta_n} T(r_j;0,\theta')\, d\theta' \leq 4\pi K 2\delta_n .$$

We therefore have

$$\int_{\Theta} | F(r_j;\theta_1,\theta_2) - F(r_i;\theta_1,\theta_2) | \, d\Theta \leq \delta_n K 4\pi(\pi+2).$$

Now choose n so large that $\delta_n K 4\pi[\pi+2] < \epsilon$. Since $\int_{\Theta} |F(r_j;\theta_1,\theta_2) - F(r_i;\theta_1,\theta_2)| \, d\Theta < \epsilon$ implies convergence in the mean of $\{F(r_i;\theta_1,\theta_2)\}$ on a region Θ, our theorem is proved.

THEOREM 2. -- There exists a subsequence $\{\bar{r}_i\}$ of $\{r_i\}$ and a function $\Phi(\theta_1,\theta_2)$, defined nearly everywhere in Θ, such that

$$\lim_{i \doteq \infty} F(\bar{r}_i;\theta_1,\theta_2) = \Phi(\theta_1,\theta_2)$$

except at a set of Θ of superficial measure zero.

Theorem 1. and a theorem of Hobson (3) allow us to infer Theorem 2.

Theorem 3. -- There exists a subsequence $\{r'_i\}$ of $\{\bar{r}_i\}$ and a function $\Phi(0,\theta)$, defined at nearly every point of the interval $0 \leq \theta \leq 2\pi$, such that

$$\lim_{i \doteq \infty} F(r'_i;0,\theta) = \Phi(0,\theta)$$

nearly everywhere in a set of points $E(\theta)$ of linear measure 2π.

The proof is omitted since it is similar to that of Theorem 1 with Theorem 2.

(3) Hobson, "Theory of Functions of a Real Variable," 2nd ed. Vol. II, # 168.

THEOREM 4. -- There exists a set $E'(\theta)$ dense in the interval $0, 2\pi$ such that if θ_1, θ_2 belong to $E'(\theta)$ then

$$\lim_{i \doteq \infty} F(r_i'; \theta_1, \theta_2) = \Phi(\theta_1, \theta_2)$$

exists.

Since $F(r_i'; \theta_1, \theta_2)$ is additive we have

$$F(r_i'; \theta_1, \theta_2) = F(r_i'; 0, \theta_2) - F(r_i'; 0, \theta_1)$$

Theorem 3 allows us to infer that both terms on the right can only diverge, as $i \doteq \infty$, on a set of zero measure. Hence $F(r_i'; \theta_1, \theta_2)$ can only diverge, as $i \doteq \infty$, for θ_1, θ_2's in a subset $E''(\theta)$ of $E(\theta)$ of zero linear measure. By setting $E'(\theta) = E(\theta) - E''(\theta)$ we obtain the desired result.

2. AN ALTERNATIVE PROOF OF THE RESULTS OF THEOREMS 2, 4 COMBINED

THEOREM 5. $F(r_i, \Delta)$ is of uniformly limited variation with respect to r_i as a function of intervals, Δ, implies the existence of a sequence of r_i's $\{r_i^*\}$ with $\lim r_i^* = 1$, such that $\lim F(r_i^*, \Delta)$ exists and is additive on a dense set $E(\theta)$ of θ_1, θ_2's which determine the Δ's.

The hypothesis of the theorem is equivalent to: the upperbound, for all k modes of subdivision of (θ_1, θ_2), of

$$\sum_{j=1}^{k} \left| F(r_i; \theta_j^{(k)}, \theta_{j+1}^{(k)}) \right|$$

is less than B for every r_i in $(0,1)$. Hence $|F(r_i, \Delta)| < B$.

Take a particular Δ, say Δ_1. Then we have the result that the sequence, $\{F(r_i, \Delta)\}$ has an upper and a lower limit which we may denote by $\overline{F}(\Delta_1)$ and $\underline{F}(\Delta_1)$ respectivel . Now out of the $\{r_i\}$ choose a subsequence $\{r_{i,1}\}$, where $r_{1,1} = r_1$, on which the sequence $\{F(r_{i,1}, \Delta_1)\}$ has a unique limit, say $\bar{F}(\Delta_1)$. Next choose another $\Delta = \Delta_2$. Then $\{F(r_{i,1}, \Delta_2)\}$ will have an upper and a lower limit which we may denote by $\bar{F}(\Delta_2)$ and $\underline{F}(\Delta_2)$ respectively. As before select from $\{r_{i,1}\}$ a subsequence $\{r_{i,2}\}$ such $r_{1,2} = r_{1,1}$, $r_{2,2} = r_{2,1}$, and such that the sequence $\{F(r_{i,2}, \Delta_2)\}$ has a unique limit, say $\bar{F}(\Delta_2)$. Continue in this manner. Select Δ_n from $0, 2\pi$, and choose a subsequence $\{r_{i,n}\}$ from $\{r_{i,n-1}\}$ such that $r_{1,n} = r_{1,1}$, $r_{2,n} = r_{2,2}$, - - - - - - , $r_{n-1,n} = r_{n-1,n-1}$, $r_{n,n} = r_{n,n-1}$, $r_{n+1,n} = r_{n+1,n}$---. Now the sequence $\{F(r_{i,n-1}, \Delta_n)\}$ will have an upper and a lower limit, and $\{F(r_{i,n}, \Delta_n)\}$ will have a unique limit, say $\bar{F}(\Delta_n)$, since I can choose out of $\{r_{i,n-1}\}$ that portion (for late terms of the sequence) that will give $\bar{F}(\Delta_n)$. Continue this process without end.

Now I have a sequence $\{\Delta_j\}$ and corresponding sequences $\{r_{i,j}\}$ on which the members of the sequence of sequences $\{\{F(r_{i,j}, \Delta_j)\}\}$ have unique limits $\bar{F}(\Delta_j)$ as $r_{i,j} \doteq 1$. Write $\lim\limits_{j=\infty}\{r_{i,j}\} = \{r_i^*\}$. Then ${}^i\{\{F(r_i^*, \Delta_j)\}\}$ all have unique limits, since $\{r_i^*\}$ is contained in all the $\{r_{i,j}\}$'s. Furthermore the $\theta_1\theta_2$'s which determine the Δ's may be selected from a denumerable dense set $E(\theta)$ of θ's in the interval $(0, 2\pi)$.

$\bar{F}(\Delta)$ is additive, for $F(r_i^*, \Delta)$ is additive and bounded and hence $F(r_i^*, \Delta_1+\Delta_2) = F(r_i^*, \Delta_1) + F(r_i^*, \Delta_2)$. If $\bar{F}(\Delta)$ were not additive, $\lim F(r_i^*; \Delta_1+\Delta_2) \neq \lim F(r_i^*, \Delta_1) + \lim F(r_i^*, \Delta_2)$ in which case we would have to have $F(r_i^*, \Delta_1+\Delta_2) \neq F(r_i^*, \Delta_1) + F(r_i^*, \Delta_2)$ for i's greater than some finite integer, I, which is impossible. Hence $\bar{F}(\Delta)$ is additive.

3. APPLICATION OF THE FOREGOING THEOREMS TO TWO THEOREMS ON HARMONIC FUNCTIONS

THEOREM 6. -- If $F(r_i, s)$ is of uniformly limited variation with respect to i as a function of intervals where

$$F(r,s) = \int_{\rho(r,s)} u(M)\,dM$$

and $\rho(r,s)$ denotes that portion of S_r which is bounded by the projection from O of s on S, then, there exists a sequence of values of r, $r_1 < r_2 < r_3 \cdots$ with $\lim r_i = 1$ such that for any interval, s, $\lim F(r_i, s)$ exists.

Consider a linear net H of intervals s' determined by radii θ = const. in $E'(\theta)$. Then

$$F(r_i, s) = \lim_{j \to \infty} \int_{S_{r_j}} p(M_j; r_j, r_i; s)\,dF(r_j, s'_{M_j}) = \int_S p(M; 1, r_i; s)\,d\Phi(s'_M)$$

since (1) the inegrand is bounded; (2) the integration is with respect to a function of bounded variation and hence we have a Daniell S - integral (4) which gives the above results. Now $\Phi(s')$ is a bounded additive function of intervals on the net H.

Therefore

$$\lim_{i \to \infty} F(r_i, s) = \lim_{i \to \infty} \int_S p(M; 1, r_i'; s)\,d\Phi(s'_M)$$
$$= \int_S q(M, s)\,d\Phi(s'_M) = F_1(s)$$

since we have again a Daniell S - integral. The theorem of paper (2) page 157 allows us to infer the result.

(4) P. J. Daniell, "A General Form of Integral, " Annals of Mathematics, Vol. 19 (1918) pp. 279-294. See section 7.7 and 8.

THEOREM 7. *If u is a function, harmonic at every interior point of the unit circle S, then, (a) there exists on the circumference of S a bounded additive function of intervals, $F(s)$ such that*

$$u(M) = \frac{1}{2\pi} \int_S \frac{1-r^2}{\overline{MP}^2}\, dF(s_P)$$

if and only if, (b) $F(r_i, s)$ is of uniformly limited variation as a function of intervals for all i, where, by definition

$$F(r,s) = \int_{\rho(r,s)} u(M)\, dM$$

and $\rho(r,s)$ denotes the part of S_r which is bounded by the projection from O of the interval s on S.

(a) implies (b), can be established directly. From the definition of the total variation function $T(s)$ of $F(s)$ and of the Stieltjes integral,

$$\sum_j \left| F(r_i, s_j) \right| \leq \frac{1}{2\pi} \int_{\rho(r_i, \Sigma s_j)} dM \int_S \frac{1-r_i^2}{\overline{MP}^2}\, dT(s_P) \leq T(S) < B$$

where s_j 's are non-overlapping intervals on S_{r_i}.

(b) implies (a), may also be easily obtained. Let M_0 be a fixed point, internal to a circle $S^*_{r_i^*}$ of radius r_i^*. Then since u is harmonic within S, $u(M_0)$ may be expressed by means of a Poisson integral over $S_{r_i^*}$. Thus

$$u(M_0) = \frac{1}{2\pi r_i^*} \int_{S_{r_i^*}} \frac{(r_i^{*2} - r_0^2)\, u(M)\, dM}{\overline{M_0 M}^2} = \frac{1}{2\pi r_i^*} \int_{S_{r_i^*}} \frac{(r_i^{*2} - r_0^2)}{\overline{M_0 M}^2}\, dF(r_i^*, s_M)$$

$$= \lim \frac{1}{2\pi r_i^*} \int_{S_{r_i^*}} \frac{r_i^{*2} - r_0^2}{\overline{M_0 M}^2}\, dF(r_i^*, s_M) = \frac{1}{2\pi} \int_S \frac{1 - r_0^2}{\overline{M_0 P}^2}\, d\overline{F}(s_P).$$

by Theorem 6.

Furthermore

$$F(r_i, s) = \int_S p(P; r_i, 1; s)\, d\bar{F}(s'_P)$$

and

$$\lim_{r_i \doteq 1} F(r_i, s) = \int_S q(P; s)\, d\bar{F}(s'_P) = F_1(s)$$

$q(P, s)$, being the limit of a sequence of continuous **functions** is of class 1 of Baire and is therefore measurable Borel. Apply a theorem (5) of (1). We have as a result

$$u(M_0) = \frac{1}{2\pi} \int_S \frac{1 - r_0^2}{\overline{M_0 P}^2}\, d F_1(s_P) \quad ,$$

which completes the proof of the theorem.

It is very important to notice that as a consequence of Theorem 5 the last integral is defined, not merely for a set of s_P's corresponding to a dense set of θ's, but also for every s_P on the circle.

(5) (1) See p. 157