Systems of Equations in an Infinite Number of Unknowns, whose Solution involves an Arbitrary Parameter.

I. Linear Systems Reducible to Systems of Difference Equations.

Consider the infinite system of linear equations

$$\text{(I)} \qquad x_i + \lambda \sum_{j=i+1}^{\infty} x_j = c_i, \quad i = 1, 2, 3, \dots,$$

Where $|c_i| \leq MP^i, P < 1.$

Theorem 1: (I) has an infinity of solutions for each λ in $|\lambda - 1| > 1$, and a single solution for each λ in $|\lambda - 1| \leq 1$.

Proof: On subtracting successive equations we have

$$\text{(II)} \left\{ \begin{array}{l} x_1 + (\lambda - 1) x_2 = c_1 - c_2 \\ \dots\dots\dots\dots \\ x_n + (\lambda - 1) x_{n+1} = c_n - c_{n+1} \\ \dots\dots\dots\dots \end{array} \right.$$

Every solution of (I) satisfies (II). It remains to be seen when a solution of (II) is also a solution of (I).

Set $\lambda - 1 = \nu$, and suppose $\nu \neq 0$. Choose $x_1 = x_1^{(0)}$ arbitrarily. Then $x_2, x_3, \dots$ of (II) are uniquely determined. So there are infinitely many solutions of (II) when $\nu \neq 0$.

They are given by

$$\text{(1)} \qquad \left\{ \begin{array}{l} x_1 = x_1^{(0)} \\ x_2 = \dfrac{-x_1^{(0)}}{\nu} + \dfrac{c_1 - c_2}{\nu} \\ x_3 = \dfrac{x_1^{(0)}}{\nu^2} - \dfrac{c_1 - c_2}{\nu^2} + \dfrac{c_2 - c_3}{\nu} \\ \dots\dots\dots\dots \\ x_n = \dfrac{(-1)^{n-1} x_1^{(0)}}{\nu^{n-1}} + (-1)^{n-2} \dfrac{c_1 - c_2}{\nu^{n-1}} + (-1)^{n-3} \dfrac{c_2 - c_3}{\nu^{n-2}} + \dots + \dfrac{c_{n-1} - c_n}{\nu} \\ \dots\dots\dots\dots \end{array} \right.$$

$|c_i| \leq MP^i, P < 1.$

Therefore

$$|x_n| \leq \frac{|x_1^{(0)}|}{|\nu|^{n-1}} + \frac{MP(1+P)}{|\nu|^{n-1}} + \frac{MP^2(1+P)}{|\nu|^{n-2}} + \dots + \frac{MP^{n-1}(1+P)}{|\nu|}$$

$$= \frac{|x_1^{(0)}|}{|\nu|^{n-1}} + M(1+P)P^n \left[\frac{1}{|P\nu|^{n-1}} + \frac{1}{|P\nu|^{n-2}} + \dots + \frac{1}{|P\nu|} \right].$$

Assume temporarily that $|P\nu|>1$. (observe that then $|\nu|>1$.).

Then (2) $|x_n| \leq \frac{|x_1(0)|}{|\nu|^{n-1}} + \frac{M(1+P)}{|P\nu|-1} P^n \left[1-\frac{1}{|P\nu|^{n-1}}\right].$

$$\xi_i = x_i + \lambda \sum_{j=i+1}^{\infty} x_j \qquad \text{converges absolutely.}$$

For

$$|x_i| + |\lambda| \sum_{j=i+1}^{\infty} |x_j| \leq |x_i| + |\lambda| \left\{ \frac{|x_1(0)|}{|\nu|^i} \cdot \frac{1}{1-\frac{1}{|\nu|}} + \frac{M(1+P)}{P|\nu|-1} \cdot \frac{P^{i+1}}{1-P} - \frac{MP(1+P)}{(P|\nu|-1)|\nu|^i} \cdot \frac{1}{1-\frac{1}{|\nu|}} \right\}.$$

Let $\xi_{i,n} = x_i + \lambda(x_{i+1} + x_{i+2} + \cdots + x_n)$.

On adding the i^{th} to n^{th} equations of (II) we have

$$\xi_{i,n} + (\lambda-1)x_{n+1} = c_i - c_{n+1}.$$

Now by hypothesis, $\lim_{n\to\infty} c_n = 0$. And from (2),

$\lim_{n\to\infty} x_n = 0$. Hence $\xi_i = c_i$.

Therefore (x_i) is a solution of (I). We have thus shown that there are an infinity of solutions of (I) for each λ such that $P|\nu|>1$. This conclusion holds for every λ in $|\nu|>1$. For let ν_0 be any value of ν such that $|\nu_0|>1$. We can always choose P' satisfying $1>P'\geq P$ and such that $P'|\nu_0|>1$. And obviously $|c_i| \leq MP'^i$. We can therefore replace P by P', and so obtain an infinity of solutions for $\nu=\nu_0$. This proves the first part of the theorem.

Note: In the inequality for $|x_n|$, we must replace P by P' whenever we use P'.

Now assume a solution $x_i(\lambda)$ of (I) in the form of a power series:

$$(3) \qquad x_i = A_{0i} + A_{1i}(\lambda-1) + \cdots + A_{ni}(\lambda-1)^n + \cdots .$$

Substituting into (I) formally and equating coefficients, we obtain the following equations to determine A_{ij} :

$$(A) \qquad \begin{cases} \sum_{j=i}^{\infty} A_{0j} = c_i, \\ \sum_{j=i}^{\infty} A_{nj} = -\sum_{j=i+1}^{\infty} A_{n-1,j} \;, = c_i^{(n)}. \end{cases}$$

$A_{0i} = c_i - c_{i+1}$ is a solution of the first system, as is evident. Therefore

$$c_i^{(1)} = -\sum_{j=i+1}^{\infty} A_{0j} = -\sum_{j=i+1}^{\infty}(c_j - c_{j+1}) = -c_{i+1}.$$

Hence a solution of

$$\sum_{j=i}^{\infty} A_{1j} = -\sum_{j=i+1}^{\infty} A_{0j} \text{ is } A_{1i} = -(c_{i+1} - c_{i+2}).$$

Therefore

$$c_i^{(2)} = -\sum_{j=i+1}^{\infty} A_{1j} = c_{i+2} .$$

Therefore $A_{2i} = c_{i+2} - c_{i+3}$. And so on. In general, a solution of

$$\sum_{j=i}^{\infty} A_{nj} = c_i^{(n)}$$

is $A_{ni} = (-1)^n (c_{i+n} - c_{i+n+1})$.

$$\therefore \quad x_i = (c_i - c_{i+1}) - (\lambda-1)(c_{i+1} - c_{i+2}) + \cdots + (-1)^n (c_{i+n} - c_{i+n+1}) + \cdots ,$$

$$\text{i.e.,} \quad x_i = [c_i - c_{i+1}(\lambda-1) + \cdots + (-1)^n(\lambda-1)^n c_{i+n} + \cdots] - [c_{i+1} - c_{i+2}(\lambda-1) + \cdots + (-1)^n(\lambda-1)^n c_{i+n+1} + \cdots]$$

whenever the two series converge.

Since $|c_i| \leq MP^i$, $P < 1$, each series does converge for

$$|\lambda - 1| < \frac{1}{P} \quad ; \text{ and } \quad |x_i| \leq \frac{MP^i}{1-P|\nu|} + \frac{MP^{i+1}}{1-P|\nu|} = \frac{M(1+P)}{1-P|\nu|} P^i .$$

Therefore $|x_i| + |\lambda| \sum_{j=i+1}^{\infty} |x_j| \leq |x_i| + \frac{M(1+P)|\lambda|}{1-P|\nu|} \cdot \frac{P}{1-P} \cdot P^i$.

Therefore $x_i + \lambda \sum_{j=i+1}^{\infty} x_j$ converges uniformly in every circle of radius $< \frac{1}{P}$. It is therefore legitimate to sum by columns. Consequently

$$x_i + \lambda \sum_{j=i+1}^{\infty} x_j = c_i .$$

That is, the formal solution is a true solution. Denote it by $\bar{x}_i$.

Let $x_i = y_i + \bar{x}_i$. Then

$$(5) \qquad y_i + \lambda \sum_{j=i+1}^{\infty} y_j = 0.$$

If $\lambda = 1$ we see that the unique solution is $y_i = 0$. Suppose $\lambda \neq 1$.

Then (6) $\quad y_i + (\lambda-1) y_{i+1} = 0$, or $\quad y_i = \frac{(-1)^{i-1}}{(\lambda-1)^{i-1}} y_1$.

Add i^{th} to n^{th} equations in (6):

$$y_i + \lambda (y_{i+1} + \cdots + y_n) = -(\lambda-1) y_{n+1} .$$

Hence a solution $y_i \neq 0$ of (6) will be a solution of (5) if and only if $y_n \to 0$ as $n \to \infty$. i.e., if and only if $|\lambda - 1| > 1$. Hence for $|\lambda-1| \leq 1$ the only solution of (5) is $y_i = 0$, and therefore $\bar{x}_i$ is unique. This completes the proof. q.e.d.

<u>Corollary</u>: Every solution for $|\lambda-1|>1$ is obtained by giving all values to $x_1^{(0)}$.

Consider now the system

$$(\mathrm{I}) \qquad x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i , \quad i=1,2, \cdots .$$

On subtracting successive equations we obtain

$$(\mathrm{II}) \qquad x_{i-1} + (\lambda_i - 1) x_i = c_{i-1} - c_i .$$

Choose $x_1 = x_1^{(0)}$ arbitrarily. Then every solution of (II) is given by

$$(1) \quad \begin{cases} x_1 = x_1^{(0)} \\ \cdots\cdots \\ x_n = \dfrac{(-1)^{n-1} x_1^{(0)}}{\nu_2 \nu_3 \cdots \nu_n} + (-1)^{n-2} \dfrac{c_1 - c_2}{\nu_2 \nu_3 \cdots \nu_n} + \cdots + \dfrac{c_{n-1} - c_n}{\nu_n}, \end{cases}$$

where $\nu_i = \lambda_i - 1$.

Assume that

$$(2) \quad \begin{cases} |c_n| \leq M p^n, \quad p < 1; \\ |\nu_n| \geq \alpha > \frac{1}{p} \quad (\text{and therefore } \alpha > 1); \\ \sum_{j=1}^{\infty} |\lambda_j| p^j \qquad \text{converges.} \end{cases}$$

Then

$$|x_n| \leq \frac{|x_1^{(0)}|}{\alpha^{n-1}} + M(1+P)P^n\left[\frac{1-\left(\frac{1}{\alpha P}\right)^{n-1}}{\alpha P-1}\right]$$

$$\leq \frac{|x_1^{(0)}|}{\alpha^{n-1}} + \frac{M(1+P)}{\alpha P-1}\cdot P^n.$$

Therefore $\lim_{n\to\infty} x_n = 0.$

Also, $$|x_i| + \sum_{j=i+1}^{\infty}|\lambda_j x_j| \leq |x_i| + \sum_{j=i+1}^{\infty}|\lambda_j|\left\{\frac{|x_1^{(0)}|}{\alpha^{j-1}} + \frac{M(1+P)}{\alpha P-1}P^j\right\},$$

which converges, by (2).

Add i^{th} to n^{th} equations of (II):

$$x_i + \lambda_{i+1}x_{i+1} + \cdots + \lambda_n x_n = c_i - c_n + x_n.$$

Therefore

$$x_i + \sum_{j=i+1}^{\infty}\lambda_j x_j = c_i + \lim_{n\to\infty}(x_n - c_n) = c_i.$$

Therefore (x_i) is a solution of (I).

Since $x_1^{(0)}$ is arbitrary, there are an infinity of solutions. We have thus

<u>Theorem 2:</u> If in the system (I) $x_i + \sum_{j=i+1}^{\infty}\lambda_j x_j = c_i$,

$|c_n| \leq MP^n, P<1,$

then for each set (λ_i) satisfying

$$\begin{cases} |\lambda_i - 1| \geq \alpha > \frac{1}{P}, \\ \sum_{j=1}^{\infty}|\lambda_j|P^j \text{ converges,} \end{cases}$$

there are an infinity of solutions. (x_i).

<u>Corollary</u>: Every such solution of (I) is obtained by giving to $x_1^{(0)}$ all values.

Remarks: 1°: Observe that for any such solution (x_i) we have

$$|x_n| \leq \left[\frac{|x_1(0)|}{P} + \frac{M(1+P)}{P\alpha - 1} \right] P^n .$$

2°: $\sum_{j=1}^{\infty} |\lambda_j| P^j$ converges if $\lim_{j = \infty} \left| \frac{\lambda_{j+1}}{\lambda_j} \right| < \frac{1}{P}$,

and therefore λ_j need not be bounded as $j \to \infty$.

II. Method of Approximations Applied to a more General System.

We now consider the more general system

$$\text{(I)} \qquad x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i , \qquad i=1,2,\cdots .$$

Assume that
$$\begin{cases} |c_i| \leq M P^i, \ P<1, \\ |\lambda_i - 1| \geq \alpha > \frac{1}{P}, \ |b_{ij}| \leq N, \\ \sum_{j=1}^{\infty} |\lambda_j| P^j \quad \text{converges.} \end{cases}$$

Let $x_i^{(0,0)}$ be any solution of $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i$,

and set $x_i^{(1)} = x_i - x_i^{(0,0)}$. By Th. 2, a solution $x_i^{(0,0)}$ exists.

Then formally,

$$\begin{aligned} x_i^{(1)} &= \left\{ c_i - \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j \right\} - \left\{ c_i - \sum_{j=i+1}^{\infty} \lambda_j x_j^{(0,0)} \right\} \\ &= - \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij})(x_j - x_j^{(0,0)}) - \sum_{j=i+1}^{\infty} b_{ij} x_j^{(0,0)} . \end{aligned}$$

i.e.,

$$x_i^{(1)} + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j^{(1)} = c_i^{(1)},$$

where
$$c_i^{(1)} = - \sum_{j=i+1}^{\infty} b_{ij} x_j^{(0,0)} .$$

Let $x_i^{(0,1)}$ be a solution of

$$x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i^{(1)}.$$

It will be shown that $c_i^{(1)}$ satisfies the inequality required for Th.2, and consequently a solution exists. This remark applies also to the later $c_i^{(n)}$'s : $c_i^{(2)}, c_i^{(3)}, \cdots\cdots$.

Set $$x_i^{(2)} = x_i^{(1)} - x_i^{(0,1)} = - \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij})(x_j^{(1)} - x_j^{(0,1)}) - \sum_{j=i+1}^{\infty} b_{ij} x_j^{(0,1)} .$$

Then

$$x_i^{(2)} + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j^{(2)} = c_i^{(2)},$$

where

$$c_i^{(2)} = - \sum_{j=i+1}^{\infty} b_{ij} x_j^{(0,1)}.$$

And so on. In general,

let $x_i^{(0,n-1)}$ be a solution of $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i^{(n-1)}$,

and set $x_i^{(n)} = x_i^{(n-1)} - x_i^{(0,n-1)}$.

Then $$x_i^{(n)} + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j^{(n)} = c_i^{(n)},$$

where $$c_i^{(n)} = - \sum_{j=i+1}^{\infty} b_{ij} x_j^{(0,n-1)}.$$

We need to consider convergence. In solving the system $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i$, we obtained the inequality

$$|x_n| \leq \left[\frac{|x_1^{(0)}|}{P} + \frac{M(1+P)}{P\alpha - 1} \right] P^n.$$

In the present case this is

$$|x_n^{(0,0)}| \leq \left[\frac{|x_1^{(0,0)}|}{P} + \frac{M(1+P)}{P\alpha-1} \right] P^n.$$

Therefore

$$|c_i^{(1)}| \leq \sum_{j=i+1}^{\infty} |b_{ij}| |x_j^{(0,0)}| \leq N \left(\frac{|x_1^{(0,0)}|}{P} + \frac{M(1+P)}{P\alpha-1} \right) \sum_{j=i+1}^{\infty} P^j$$
$$= N \left(\frac{|x_1^{(0,0)}|}{P} + \frac{M(1+P)}{P\alpha-1} \right) \frac{P}{1-P} \cdot P^i.$$

Let
$$\begin{cases} Q = \dfrac{N}{1-P}, \quad R = \dfrac{PN(1+P)}{(1-P)(P\alpha-1)}, \\ M^{(1)} = Q|x_1^{(0,0)}| + RM. \end{cases}$$

Then

$$|c_i^{(1)}| \leq M^{(1)} P^i.$$

<u>Assume</u> α chosen so that $R<1$. This is equivalent to assuming

that $$\alpha > \frac{NP^2+(N-1)P+1}{P(1-P)}.$$

(Observe that this inequality implies $P\alpha>1$.).

In the solution $x_i^{(0,1)}$ of $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i^{(1)}$, $x_1^{(0,1)}$

can be taken arbitrarily. (See Th. 2).

Therefore $$|x_n^{(0,1)}| \leqq \left[\frac{|x_1^{(0,1)}|}{P} + \frac{M^{(1)}(1+P)}{P\alpha-1} \right] P^n.$$

Therefore

$$|c_i^{(2)}| \leqq \sum_{j=i+1}^{\infty} |b_{ij}||x_j^{(0,1)}| \leqq N\left(\frac{|x_1^{(0,1)}|}{P} + \frac{M^{(1)}(1+P)}{P\alpha-1} \right)\frac{P}{1-P} P^i$$

$$= [Q|x_1^{(0,1)}| + RM^{(1)}]P^i$$

$$= M^{(2)}P^i \quad \text{, say.}$$

And so on. It is clear that in general

$$|x_n^{(0,\lambda)}| \leqq \left[\frac{|x_1^{(0,\lambda)}|}{P} + \frac{M^{(\lambda)}(1+P)}{P\alpha-1} \right] P^n,$$

and $$|c_i^{(\lambda+1)}| \leqq \sum_{j=i+1}^{\infty} |b_{ij}||x_j^{(0,\lambda)}|$$

$$\leqq [Q|x_1^{(0,\lambda)}| + RM^{(\lambda)}]P^i = M^{(\lambda+1)}P^i,$$

where $x_1^{(0,\lambda)}$ is chosen arbitrarily, and $M^{(\lambda)}$ is defined by the

relation $$\begin{cases} M^{(\lambda+1)} = Q|x_1^{(0,\lambda)}| + RM^{(\lambda)}, \\ M^{(0)} = M. \end{cases}$$

From this recurrent relation for $M^{(\lambda)}$ we have:

$$M^{(\lambda)} = Q|x_1^{(0,\lambda-1)}| + QR|x_1^{(0,\lambda-2)}| + QR^2|x_1^{(0,\lambda-3)}| + \cdots\cdots + QR^{\lambda-1}|x_1^{(0,0)}| + R^{\lambda}M.$$

The choice of $x_1^{(0,0)}, x_1^{(0,1)}, \ldots, x_1^{(0,n)}, \ldots$ is arbitrary. Now assume them so chosen that

$$|x_1^{(0,n)}| \leq \beta T^n, \qquad R < T < 1.$$

Then

$$M^{(n)} \leq \beta Q T^{n-1}\left[1 + \frac{R}{T} + \frac{R^2}{T^2} + \cdots + \frac{R^{n-1}}{T^{n-1}}\right] + MR^n$$

$$= \beta Q T^{n-1}\left[\frac{1-(\frac{R}{T})^n}{1-\frac{R}{T}}\right] + MR^n$$

$$\leq \left[\frac{\beta Q}{1-\frac{R}{T}} + MR\right]T^{n-1} = ET^{n-1}, \text{ say.}$$

Therefore $\quad |c_i^{(n+1)}| \leq ET^n P^i,$

and

$$|x_n^{(0,n)}| \leq \left[\frac{\beta T^n}{P} + \frac{(1+P)}{P\alpha - 1}ET^{n-1}\right]P^n$$

$$= \left[\beta\frac{T}{P} + \frac{(1+P)E}{P\alpha-1}\right]T^{n-1}P^n = HT^{n-1}P^n, \text{ say.}$$

Observe that

$$\lim_{n\to\infty} c_i^{(n)} = 0.$$

Let $\quad X_i = \sum_{n=0}^{\infty} x_i^{(0,n)}.$

The series converges absolutely, since

$$\sum_{n=0}^{\infty}|x_i^{(0,n)}| \leq HP^i\sum_{n=0}^{\infty}T^{n-1}, \qquad T<1.$$

Define $Y_i^{(n)}$ by

$$X_i = x_i^{(0,0)} + x_i^{(0,1)} + \cdots + x_i^{(0,n-1)} + Y_i^{(n)}.$$

$$|Y_i^{(n)}| \leq \sum_{n=n}^{\infty}|x_i^{(0,n)}| \leq HP^i\sum_{n=n}^{\infty}T^{n-1}.$$

Therefore $\quad \lim_{n\to\infty} Y_i^{(n)} = 0.$

$$\sum_{j=i+1}^{\infty}|\lambda_j + b_{ij}||x_j^{(0,n)}| \leq NHT^{n-1}\sum_{j=i+1}^{\infty}P^j + HT^{n-1}\sum_{j=i+1}^{\infty}|\lambda_j|P^j.$$

Both series on the right converge, and

Therefore $\sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})x_j^{(0,r)}$ converges absolutely for every r.

$$\sum_{j=i+1}^{\infty}|\lambda_j+b_{ij}||Y_j^{(n)}| \leq NH\left(\sum_{r=n}^{\infty}T^{r-1}\right)\sum_{j=i+1}^{\infty}P^j + H\left(\sum_{r=n}^{\infty}T^{r-1}\right)\sum_{j=i+1}^{\infty}|\lambda_j|P^j ,$$

and both series on the right converge.

Therefore $\sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})\,Y_j^{(n)}$ converges absolutely .

Moreover,

$$\lim_{n\to\infty}\sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})Y_j^{(n)} = 0.$$

Therefore

$$X_i + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})X_j$$

$$\begin{aligned}
&= x_i^{(0,0)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})\,x_j^{(0,0)} \\
&\quad + x_i^{(0,1)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})\,x_j^{(0,1)} \\
&\quad + \cdots \\
&\quad \cdots + x_i^{(0,n-1)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})x_j^{(0,n-1)} \\
&\quad + Y_i^{(n)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})Y_j^{(n)}
\end{aligned}$$

$$\begin{aligned}
&= (c_i-c_i^{(1)}) + (c_i^{(1)}-c_i^{(2)}) + \cdots + (c_i^{(n-1)}-c_i^{(n)}) \\
&\quad + Y_i^{(n)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})\,Y_j^{(n)}
\end{aligned}$$

$$= c_i + \left[-c_i^{(n)} + Y_i^{(n)} + \sum_{j=i+1}^{\infty}(\lambda_j+b_{ij})Y_j^{(n)}\right].$$

As $n \to \infty$, the bracket approaches 0.

Now the left hand member is independent of n.

Therefore $$X_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) X_j = c_i .$$

i.e., (X_i) is a solution of (I).

Except for the condition $|x_1^{(0,\lambda)}| \leqq \rho T^{\lambda}$, $R < T < 1$,

the $x_1^{(0,\lambda)}$'s are arbitrary.

Therefore X_r is arbitrary. Consequently there are an infinity of solutions. We thus have:

<u>Theorem 3:</u> If in the system

$$\text{(I)} \qquad x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i ,$$

the conditions $$\begin{cases} |c_n| \leqq MP^i, \; P<1, \\ \qquad |b_{ij}| \leqq N, \end{cases}$$ hold,

then for every set (λ_i) for which

$$\begin{cases} |\lambda_i - 1| \geqq \alpha > \dfrac{NP^2 + (N-1)P + 1}{P(1-P)}, \\ \qquad \displaystyle\sum_{j=1}^{\infty} |\lambda_j| P^j \end{cases}$$ converges, there exist an infinity of solutions of (I).

<u>Corollary:</u> From the inequality

$$|X_i| \leqq HP^i \sum_{\lambda=0}^{\infty} T^{\lambda-1}$$

we have the result:

An infinity of the solutions (for a given set λ_i).

satisfy the inequality $|x_i| \leq A p^i$,

where **A** is a constant (depending on α).

<u>Corollary:</u> If $\lambda_j \equiv \lambda$, then we have:

The system $x_i + \sum_{j=i+1}^{\infty} (\lambda + b_{ij}) x_j = c_i$ has an infinity

of solutions for each λ in $|\lambda - 1| > \dfrac{Np^2 + (N-1)p + 1}{p(1-p)}$,

if $\begin{cases} |c_i| \leq M p^i, \ p<1, \\ |b_{ij}| \leq N, \end{cases}$

and an infinity of these solutions

satisfy $|x_i| \leq A p^i$, A depending on λ .

<u>Corollary:</u> Consider the system

$$y_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) y_j = c_i ,$$

where $\begin{cases} |b_{ij}| \leq N ; \ p \text{ exists}, \ 0 < p < 1 \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| p^j \text{ converges;} \\ |\lambda_i - 1| \geq \alpha > \dfrac{Np^2 + (N-1)p + 1}{p(1-p)} . \end{cases}$

If this system has one solution, then it has an infinite number.

For: Let $\bar{y}_i$ be a solution, and set $y_i = x_i + \bar{y}_i$.

Then $x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = 0$, which is a system satisfying the

conditions of Theorem 3.

Therefore an infinity of solutions x_i , and consequently y_i , exist.

III. Solutions Satisfying the Inequality $|x_i| \leq AP^i$:

In obtaining Theorem 3, we assumed that

$$|x_1^{(0,\lambda)}| \leq \beta T^\lambda , \quad R < T < 1.$$

Let us now make a more particular choice: take

$$x_1^{(0,\lambda)} = 0 , \quad \lambda = 0, 1, 2, \cdots\cdots .$$

Then

$$\begin{cases} M^{(\lambda)} = MR^\lambda , \\ |c_i^{(\lambda)}| \leq MR^\lambda P^i , \\ |x_m^{(0,\lambda)}| \leq \dfrac{M(1+P)}{P\alpha - 1} R^\lambda P^m . \end{cases}$$

Therefore

$$|X_i| = \left|\sum_{\lambda=0}^{\infty} x_i^{(0,\lambda)}\right| \leq \frac{M(1+P)}{P\alpha - 1} \left(\sum_{\lambda=0}^{\infty} R^\lambda\right) P^i = BMP^i ,$$

where $\quad B = \dfrac{1+P}{(P\alpha-1)(1-R)}$.

We shall use these inequalities.

Consider again System (I) of Theorem 3. We found that for a given permissible set (λ_i) an infinity of solutions exist satisfying

$$|X_i| \leq AP^i .$$

We shall now show that <u>every</u> solution satisfying

$$|X_i| \leq AP^i ,$$

A a constant, is obtained by the method given in Theorem 3. We shall however require a modification in the inequality for α .

Let $X_i^{(0,0)}$ be a solution for which $|X_i^{(0,0)}| \leqq A p^i$.

$$X_i^{(0,0)} + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) X_j^{(0,0)} = c_i .$$

Consider the system $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i$.

Define $x_i^{(1)} = x_i - X_i^{(0,0)}$.

Then $$x_i^{(1)} + \sum_{j=i+1}^{\infty} \lambda_j x_j^{(1)} = c_i^{(1)} ,$$

where $$c_i^{(1)} = \sum_{j=i+1}^{\infty} b_{ij} X_j^{(0,0)} .$$

The series for $c_i^{(1)}$ obviously converges absolutely, and

$$|c_i^{(1)}| \leqq A \frac{Np}{1-p} p^i \quad , \quad = M^{(1)} p^i \qquad , \text{ say.}$$

Now let $X_i^{(0,1)}$ be a solution of

$$x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i^{(1)}$$

corresponding to the choice $x_1^{(0,n)} = 0$, $n = 0, 1, 2, \cdots\cdots$ in

Theorem 3.

Then $$| X_i^{(0,1)} | \leqq B M^{(1)} p^i .$$

Define $x_i^{(2)} = x_i^{(1)} - X_i^{(0,1)}$.

Then $$x_i^{(2)} + \sum_{j=i+1}^{\infty} \lambda_j x_j^{(2)} = c_i^{(2)},$$

where $$c_i^{(2)} = \sum_{j=i+1}^{\infty} b_{ij} X_j^{(0,1)} .$$

$$|c_i^{(2)}| \leqq \frac{N B M^{(1)} p}{1-p} p^i = B A \left(\frac{Np}{1-p}\right)^2 p^i = M^{(2)} p^i .$$

In general,

$$x_i^{(n)} + \sum_{j=i+1}^{\infty} \lambda_j x_j^{(n)} = c_i^{(n)} ,$$

Where

$$c_i^{(n)} = \sum_{j=i+1}^{\infty} b_{ij} X_j^{(0,n-1)} , \qquad x_i^{(n)} = x_i^{(n-1)} - X_i^{(0,n-1)},$$

and $X_i^{(0,n-1)}$ is a solution of

$$x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij})\, x_j = c_i^{(n-1)}$$

corresponding to the choice $x_1^{(0,n)} = 0, \; n = 0,1,2,\cdots\cdots$.

$$|c_i^{(n)}| \leqq M^{(n)} p^i ,$$

where

$$\begin{cases} M^{(n)} = \dfrac{NBP}{1-P} M^{(n-1)}, \\ M^{(1)} = \dfrac{NAP}{1-P} . \end{cases}$$

Hence $M^{(n)} = \dfrac{A}{B} \left(\dfrac{BNP}{1-P}\right)^n = \dfrac{A}{B} W^n ,$

where $W = \dfrac{BNP}{1-P}$.

Also, $|X_i^{(0,n)}| \leqq B M^{(n)} p^i = A W^n p^i$.

Assume $W < 1$. This is equivalent to assuming that $R < \frac{1}{2}$;

i.e., to $\alpha > \dfrac{2NP^2 + (2N-1)P + 1}{P(1-P)}$.

(Observe that this inequality implies the previous one:

$$\alpha > \frac{NP^2 + (N-1)P + 1}{P(1-P)} . \quad)$$

Since $W < 1$, $\lim_{n \to \infty} c_i^{(n)} = 0$.

Define $\bar{x}_i^{(0,0)}$ by the series

$$\bar{x}_i^{(0,0)} = \sum_{n=0}^{\infty} X_i^{(0,n)} .$$

The series converges absolutely, and

$$|\bar{x}_i^{(0,0)}| \leqq A p^i \sum_{r=0}^{\infty} W^r = \frac{A}{1-W} p^i .$$

Let

$$y_i^{(n)} = X_i^{(0,n+1)} + X_i^{(0,n+2)} + \cdots\cdots .$$

$$|y_i^{(n)}| \leqq A p^i \sum_{r=n+1}^{\infty} W^r . \qquad \therefore \lim_{n\to\infty} y_i^{(n)} = 0 .$$

$$\sum_{j=i+1}^{\infty} |\lambda_j||X_j^{(0,r)}| \leqq A W^r \sum_{j=i+1}^{\infty} |\lambda_j| p^j ,$$

which converges.

$$\sum_{j=i+1}^{\infty} |\lambda_j||y_j^{(n)}| \leqq A\Big(\sum_{r=n+1}^{\infty} W^r\Big)\sum_{j=i+1}^{\infty} |\lambda_j| p^j .$$

Therefore $\lim_{n\to\infty} \sum_{j=i+1}^{\infty} \lambda_j y_j^{(n)} = 0.$

$$\begin{aligned}
\bar{x}_i^{(0,0)} &+ \sum_{j=i+1}^{\infty} \lambda_j \bar{x}_j^{(0,0)} \\
&= X_i^{(0,0)} + \sum_{j=i+1}^{\infty} \lambda_j X_j^{(0,0)} \\
&\quad + \cdots\cdots + X_i^{(0,n)} + \sum_{j=i+1}^{\infty} \lambda_j X_j^{(0,n)} \\
&\quad + y_i^{(n)} + \sum_{j=i+1}^{\infty} \lambda_j y_j^{(n)} \\
&= (c_i - c_i^{(1)}) + (c_i^{(1)} - c_i^{(2)}) + \cdots + (c_i^{(n)} - c_i^{(n+1)}) \\
&\quad + y_i^{(n)} + \sum_{j=i+1}^{\infty} \lambda_j y_j^{(n)} \\
&= c_i + \Big[- c_i^{(n+1)} + y_i^{(n)} + \sum_{j=i+1}^{\infty} \lambda_j y_j^{(n)}\Big] .
\end{aligned}$$

The bracket $\to 0$ as $n \to \infty$.

Therefore $\quad \bar{x}_i^{(0,0)} + \sum_{j=i+1}^{\infty} \lambda_j \bar{x}_j^{(0,0)} = c_i$.

i.e. , $\quad (\bar{x}_i^{(0,0)})$

is a solution of

$$x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = c_i \quad .$$

Now we shall start with the solution $\bar{x}_i^{(0,0)}$ and show that one of the solutions X_i of

$$x_i + \sum_{j=i+1}^{\infty} (\lambda_j + \ell_{ij}) x_j = c_i$$

resulting from the method of Theorem 3 is precisely $X_i^{(0,0)}$.

Define $\bar{x}_i^{(0,1)}, \bar{x}_i^{(0,2)}, \ldots$ as solutions of the systems

$$\bar{x}_i^{(0,1)} + \sum_{j=i+1}^{\infty} \lambda_j \bar{x}_j^{(0,1)} = -\sum_{j=i+1}^{\infty} \ell_{ij} \bar{x}_j^{(0,0)} = e_i^{(1)},$$
$$\bar{x}_i^{(0,2)} + \sum_{j=i+1}^{\infty} \lambda_j \bar{x}_j^{(0,2)} = -\sum_{j=i+1}^{\infty} \ell_{ij} \bar{x}_j^{(0,1)} = e_i^{(2)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

respectively.

There are of course an infinite number of solutions of each of these systems, by Theorem 2. The precise solutions which we choose will be made evident later.

$X_i^{(0,0)}$ is a solution of $\quad x_i + \sum_{j=i+1}^{\infty} (\lambda_j + \ell_{ij}) x_j = c_i$,

and

$$|X_i^{(0,0)}| \leqq A\tau^i.$$

Let

$$X_i^{(1,0)} = -\sum_{n=1}^{\infty} X_i^{(0,n)} . \qquad |X_i^{(1,0)}| \leq A\frac{W}{1-W}P^i .$$

Then $X_i^{(1,0)}$ satisfies $x_i + \sum_{j=i+1}^{\infty}(\lambda_j + b_{ij})x_j = e_i^{(1)}$.

For, $X_i^{(1,0)} = X_i^{(0,0)} - \bar{x}_i^{(0,0)}$,

and

$$\begin{cases} X_i^{(0,0)} + \sum_{j=i+1}^{\infty}(\lambda_j + b_{ij})X_j^{(0,0)} = c_i , \\ \bar{x}_i^{(0,0)} + \sum_{j=i+1}^{\infty}\lambda_j \bar{x}_j^{(0,0)} = c_i . \end{cases}$$

$$|e_i^{(1)}| \leq \frac{NAP}{(1-W)(1-P)}P^i .$$

Now define $X_i^{(1,1)}, X_i^{(1,2)}, \dots, X_i^{(1,n)}, \dots$ with respect to $X_i^{(1,0)}$ in the same way that $X_i^{(0,1)}, \dots, X_i^{(0,n)}, \dots$ were defined with respect to $X_i^{(0,0)}$.

$$|X_i^{(1,n)}| \leq A\left(\frac{W}{1-W}\right)W^n P^i .$$

Choose $\bar{x}_i^{(0,1)}$ as that solution of

$$x_i + \sum_{j=i+1}^{\infty}\lambda_j x_j = e_i^{(1)}$$

corresponding to the set $X_i^{(1,n)}$;

$$\bar{x}_i^{(0,1)} = \sum_{n=0}^{\infty} X_i^{(1,n)} .$$

Now define $X_i^{(2,0)} = -\sum_{n=1}^{\infty} X_i^{(1,n)}$, $= X_i^{(1,0)} - \bar{x}_i^{(0,1)}$.

Then $X_i^{(2,0)}$ is a solution of

$$x_i + \sum_{j=i+1}^{\infty}(\lambda_j + b_{ij})x_j = e_i^{(2)} .$$

$$|e_i^{(2)}| \leq \frac{NAP}{(1-W)(1-P)}\left(\frac{W}{1-W}\right)P^i ; \qquad |X_i^{(2,0)}| \leq A\left(\frac{W}{1-W}\right)^2 P^i .$$

Define the corresponding set $X_i^{(2,1)}, X_i^{(2,2)}, \dots$.

$$|X_i^{(2,n)}| \leq A\left(\frac{W}{1-W}\right)^2 W^n P^i .$$

Then choose $\bar{x}_i^{(0,2)}$ as that solution of

$$x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = e_i^{(2)}$$

which is determined by $X_i^{(2,n)}$:

$$\bar{x}_i^{(0,2)} = \sum_{n=0}^{\infty} X_i^{(2,n)} .$$

And so on. In general, we define

$$X_i^{(k,0)} = -\sum_{n=1}^{\infty} X_i^{(k-1,n)} \quad , = \quad X_i^{(k-1,0)} - \bar{x}_i^{(0,k-1)} .$$

Then $X_i^{(k,0)}$ satisfies $x_i + \sum_{j=i+1}^{\infty} (\lambda_j + l_{ij}) x_j = e_i^{(k)}$,

where $e_i^{(k)} = -\sum_{j=i+1}^{\infty} l_{ij} \bar{x}_j^{(0,k-1)}$.

By induction we have $|X_i^{(k,0)}| \leq A\left(\frac{W}{1-W}\right)^k P^i$.

We then define $X_i^{(k,1)}, X_i^{(k,2)}, \ldots\ldots,$

and obtain $|X_i^{(k,n)}| \leq A\left(\frac{W}{1-W}\right)^k W^n P^i$.

Also, $|e_i^{(k)}| \leq \frac{NAP}{(1-P)(1-W)}\left(\frac{W}{1-W}\right)^{k-1} P^i$.

Choose $\bar{x}_i^{(0,k)}$ as that solution of $x_i + \sum_{j=i+1}^{\infty} \lambda_j x_j = e_i^{(k)}$ which

is given by $\bar{x}_i^{(0,k)} = \sum_{n=0}^{\infty} X_i^{(k,n)}$.

Observe that

$$|\bar{x}_1^{(0,k)}| \leq \sum_{n=0}^{\infty} A\left(\frac{W}{1-W}\right)^k W^n P = \frac{AP}{1-W}\left(\frac{W}{1-W}\right)^k .$$

This is in the form $|\bar{x}_1^{(0,k)}| \leq \beta T'^k$, $T' < 1$,

provided $\frac{W}{1-W} < 1$; i.e., $W < \frac{1}{2}$, or $R < \frac{1}{3}$, which is

equivalent to the inequality $\alpha > \frac{3NP^2 + (3N-1)P + 1}{P(1-P)}$.

We <u>assume</u> this condition on α. (It implies the previous condition $R < \frac{1}{2}$.)

Consequently, by the method of Theorem 3,

$$X_i = \sum_{n=0}^{\infty} \bar{x}_i^{(0,n)}$$

is a solution of $\quad x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i$.

It remains to identify X_i with $X_i^{(0,0)}$.

$$\bar{x}_i^{(0,0)} + \bar{x}_i^{(0,1)} + \cdots + \bar{x}_i^{(0,n)}$$

$$\begin{aligned} = \; & X_i^{(0,0)} + \{ X_i^{(0,1)} + X_i^{(0,2)} + \cdots \} \\ & + X_i^{(1,0)} + \{ X_i^{(1,1)} + X_i^{(1,2)} + \cdots \} \\ & + \cdots \\ & \cdots + X_i^{(n,0)} + \{ X_i^{(n,1)} + \cdots \}, \end{aligned}$$

which from the definition of $X_i^{(k,0)}$, $k=1,2,\cdots,n+1$, reduces to

$$X_i^{(0,0)} - X_i^{(n+1,0)}.$$

Now $\quad \lim_{n\to\infty} X_i^{(n+1,0)} = 0.$

Therefore $\quad \lim_{n\to\infty} [\bar{x}_i^{(0,0)} + \cdots + \bar{x}_i^{(0,n)}] = X_i^{(0,0)}.$

i.e., $\quad X_i = X_i^{(0,0)}.$

We have thus proved

<u>Theorem 4:</u> If in the system

$$\text{(I)} \qquad x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i \quad ,$$

$$\begin{cases} |c_i| \leq M p^i, \; p<1; \\ |b_{ij}| \leq N; \\ |\lambda_i - 1| \geq \alpha > \dfrac{3Np^2 + (3N-1)p + 1}{p(1-p)}; \\ \sum_{j=i+1}^{\infty} |\lambda_j| p^j \text{ converges;} \end{cases}$$

then <u>every</u> solution x_i for which $|x_i| \leq A\rho^i$,

(of which there are an infinite number), is contained in the set obtained by the method of Theorem 3, assigning to $x_1^{(0,0)}, x_1^{(0,1)}, \ldots\ldots$ suitable values in the range $|x_1^{(0,n)}| \leq \beta\tau^n,\ \tau < 1$.

IV. The Power Series Method:

In Theorem 1. we made use of a Taylor series in λ, and found a solution by substituting the assumed series for x_i into the system and equating coefficients. This method canbe applied to other systems.

For example, consider the system

$$\text{(I)} \qquad x_i + \lambda \sum_{j=i+1}^{\infty} a_{ij} x_j = c_i .$$

Assume a solution $\quad x_i = A_{0i} + \cdots + A_{ni}\lambda^n + \cdots$.

We find for the A_{ij}'s the relations

$$\begin{cases} A_{0i} = c_i, \\ A_{ni} = -\sum_{j=i+1}^{\infty} a_{ij} A_{n-1,j}, \quad n > 0. \end{cases}$$

It is readily shown that

if $\begin{cases} |c_i| \leqq C, \\ \sum_{j=i+1}^{\infty} |a_{ij}| \leqq P, \quad i \geqq i_0, \end{cases}$ then (I) has a bounded solution (x_i) for every λ in $|\lambda| < \frac{1}{P}$.

But this method does not appear to yield uniqueness properties.

This method also gives the Theorem:

If $\begin{cases} |c_i| \leqq \alpha; \\ \sum_{j=i+1}^{\infty} |a_{ij}| \leqq \beta, \quad i=1,2,\cdots; \\ \varphi_{ij}(x) \text{ is analytic}, \ |x| \leqq M, \ \left(\begin{smallmatrix} i=1,2,\cdots \\ j=i+1,i+2,\cdots \end{smallmatrix}\right); \quad M > \alpha \\ |\varphi_{ij}(x)| \leqq N, \ |x| \leqq M; \end{cases}$

then the system

$$x_i + \lambda \sum_{j=i+1}^{\infty} \alpha_{ij} \varphi_{ij}(x_j) = c_i$$

has a bounded solution for every λ in $|\lambda| \leq \frac{M-\alpha}{5NS}$.

It is an interesting fact that if we assume a power series solution in $\frac{1}{\lambda}$, we obtain in many cases an infinite number of solutions. Let us consider again the system of Theorem 3, where however we take $\lambda_i \equiv \lambda$:

$$x_i + \sum_{j=i+1}^{\infty} (\lambda + b_{ij}) x_j = c_i \quad , \quad \begin{cases} |c_i| \leq M p^i, \ p<1, \\ |b_{ij}| \leq N. \end{cases}$$

Assume a solution $x_i = A_{0i} + \cdots + \frac{A_{ni}}{(\lambda-1)^n} + \cdots$.

On substituting in and equating coefficients we get

$$\begin{cases} \sum_{j=i+1}^{\infty} A_{0j} = 0, \quad A_{0i} + \sum_{j=i+1}^{\infty} (b_{ij}+1) A_{0j} + \sum_{j=i+1}^{\infty} A_{1j} = c_i, \\ A_{ni} + \sum_{j=i+1}^{\infty} (b_{ij}+1) A_{nj} + \sum_{j=i+1}^{\infty} A_{n+1,j} = 0, \ n > 0. \end{cases}$$

A solution of the first system is $A_{0i} = 0, \ i > 1$.

Observe that A_{01} does not enter, and therefore can be taken arbitrarily. And in the general system, A_{n1} does not enter and is therefore arbitrary. This makes x_1 arbitrary, and we therefore obtain an infinity of solutions. Let us determine the A_{ij}'s:

$$\sum_{j=i+1}^{\infty} A_{1j} = c_i - A_{0i} = c_i^{(1)}. \text{ Therefore a solution is } A_{1,i+1} = c_i^{(1)} - c_{i+1}^{(1)}.$$

$$\sum_{j=i+1}^{\infty} A_{2j} = -A_{1i} - \sum_{j=i+1}^{\infty} (b_{ij}+1) A_{1j} = c_i^{(2)},$$

and a solution is

$$A_{2,i+1} = c_i^{(2)} - c_{i+1}^{(2)}.$$

And generally, $A_{n,i+1} = c_i^{(n)} - c_{i+1}^{(n)}$,

where $c_i^{(n)} = -A_{n-1,i} - \sum_{j=i+1}^{\infty} (\ell_{ij}+1) A_{n-1,j}$.

$c_i^{(1)} = c_i - A_{0i}$.

Choose A_{01} so that $|c_1 - A_{01}| \leqq MP$;

otherwise arbitrary.

Then $|c_i^{(1)}| \leqq MP^i = M^{(1)} P^i$.

Therefore $|A_{1,i+1}| \leqq M^{(1)}(1+P)P^i$.

Choose A_{11} arbitrary except for the condition $|A_{11}| \leqq M(1+P)$.

Then $|A_{1i}| \leqq M(1+P)P^{i-1} = \frac{M(1+P)}{P} P^i$.

Therefore $|c_i^{(2)}| \leqq \frac{M^{(1)}(1+P)}{P}\left[1 + \frac{(N+1)P}{1-P}\right]P^i = M^{(2)}P^i$.

Therefore $|A_{2,i+1}| \leqq M^{(2)}(1+P)P^i$.

Choose $|A_{2,1}| \leqq M^{(2)}(1+P)$;

then $|A_{2i}| \leqq \frac{M^{(2)}(1+P)}{P} P^i$.

And so on. We obtain finally: $|c_i^{(n)}| \leqq M^{(n)}P^i$,

where $\begin{cases} M^{(n)} = RM^{(n-1)} = \cdots = MR^{n-1}, \\ R = \left(1 + \frac{P(N+1)}{1-P}\right)\left(\frac{1+P}{P}\right). \end{cases}$

Therefore $\begin{cases} |c_i^{(n)}| \leqq MR^{n-1}P^i; \\ |A_{ni}| \leqq \frac{M(1+P)}{P} R^{n-1} P^i, \end{cases}$

on choosing

$$|A_{n1}| \leqq MR^{n-1}(1+P).$$

Therefore $|x_i| \leq p^i \frac{M(1+p)}{Rp} \sum_{n=0}^{\infty} \frac{R^n}{|\lambda-1|^n} = T_i$,

which converges if $|\lambda-1| > R$;

i.e., if $|\lambda-1| > \frac{(1+p)(Np+1)}{p(1-p)}$.

Also, $T_i + \sum_{j=i+1}^{\infty} |\lambda+b_{ij}| T_j$ converges. Hence we can sum by columns. We thus have the theorem:

If
$$\begin{cases} |c_i| \leq Mp^i, \ p<1, \\ |b_{ij}| \leq N, \\ |\lambda-1| > \frac{(1+p)(Np+1)}{p(1-p)}, \end{cases}$$

then the system $x_i + \sum_{j=i+1}^{\infty} (\lambda+b_{ij}) x_j = c_i$

has an infinity of solutions.

So we obtain some of the results of Theorem 3. But the power series method does not extend to the case where the λ_j's are not all equal; and even in the case just treated, the λ- region for which the proof is valid is not as extended as the region found by the method of Theorem 3.

FINIS