A RAY MODEL
FOR HEAD WAVES IN A
FLUID-FILLED BOREHOLE

by

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Abstract

A model, suggested by the ray expansion of Roever et al., is constructed to rapidly generate the compressional and shear refracted arrivals, known as head waves, received from a point source on the axis of an ideal fluid-filled borehole. An impulse response is derived, and its frequency characteristics are investigated.

The waves are compared to those obtained by the real axis integration method of Tsang and Rader, which results in a complete waveform, including the modes as well as the refracted arrivals. The ray model gives accurate results for the compressional head wave. The shear region of the complete waveform contains strong modal interference, making it difficult to evaluate the quality of the ray model shear wave.

A useful filter results from insight provided by the basic structure of the model. This filter can be used to estimate the borehole diameter or formation compressional velocity. It can also
remove the second and later compressional arrivals, thus providing an accurate estimate of the source pulse and a relatively uncorrupted view of the initial arrival in the shear region.
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This study is an attempt to better understand the waveforms obtained from sonic borehole logging. Sonic logging is a technique whereby a transmitter in the borehole fluid emits an acoustic pulse which interacts with the fluid and rock formation before being received at another point within the fluid. The purpose of these logs is to obtain, by analysis of the measured acoustic waveforms, properties of the geological formation surrounding the borehole. This "inverse problem" consists of finding certain parameters of the system, given a response from the system. The corresponding input may also be known. The solutions of such inverse problems are the ultimate goal of most geophysical signal processing.

In order to obtain the acoustic parameters the output must be related back to the system. The inputs and outputs considered are stresses(forces) or strains(displacements), which are small enough for a linear model to be accurate[1]. This allows the system to be characterized by its impulse response \( h(\Theta, t, z) \), where \( \Theta \) is the set of parameters whose values are to be estimated and \( z \) is the receiver position along the borehole axis. The output \( y(\Theta, t, z) \) is then related to the input \( x(t) \) through temporal convolution with \( h \):

\[
y(\Theta, t, z) = h(\Theta, t, z) * x(t).
\]

Note that the source \( x(t) \) is assumed to be at the origin \( z = 0 \). In the second chapter an exact integral
expression for \( y(t,z) \) is derived, following closely the development by Tsang and Rader[2] and Schoenberg et al.[3], using 2-D transform techniques. The response is found to be a complex nonlinear function of the parameters \( \Theta \).

This process of obtaining the output as a function of \( \Theta, t \) and \( z \) is a "forward problem". Its solution is useful in making progress with the inverse problem. Knowing \( h(\Theta, t, z) \) can provide a starting point for obtaining \( \Theta \). However, the complexity of \( h \) precludes direct inversion, so algorithms must be developed to estimate the parameters. Using the solution to the forward problem, waveforms from formations with specified properties can be generated to test these algorithms.

A simple model of the sonic logging technique is a point source and point receiver, both on the axis of a fluid-filled cylinder in an otherwise homogeneous, isotropic solid. Tsang and Rader[2] have developed a program to numerically obtain \( y(t, z) \) via inversion of the 2-D transform \( Y(\omega, k_z) \), where \( \omega \) and \( k_z \) are the temporal and spatial frequencies respectively. The transform \( Y(\omega, k_z) \) is found analytically. The complete response is computed for each of an array of points distributed along the borehole axis. For realistic geometries this program takes many hours on a VAX computer to produce the waveforms. Kurkjian[4] has developed a table lookup technique and used an FFT to reduce this time; however, it is still on the order of hours. The program also restricts the shape of the source pulse.
The work described in Chapter 3 is a response to this problem. There a simple ray model is developed for the compressional and shear refraction components of the waveforms. The model provides a 2-D impulse response (over time and source-receiver distance) as a function of the formation parameters. From this model the impulse response as a function of time at a fixed source-receiver distance can be quickly calculated.

The frequency characteristics are then examined. An important property of the frequency response is the interleaving of the peaks from the shear and compressional components. The spectrum of the input pulse windows the frequency response to determine the relative frequencies of the shear and compressional arrivals.

In Section 2 waves obtained from the ray model are compared to those from the "real axis integration" program of Tsang and Rader. Excellent agreement is found for the compressional component. This correspondence breaks down in the region of the shear arrivals. A slowness vs. time plot is presented to show that, as predicted by Tsang and Rader[2], a strong modal arrival occurs along with the shear wave. This could account for an apparent failure to model the shear refractions.

Finally, Section 3 presents some methods of filtering and parameter estimation based on the structure of the model.
CHAPTER 2

A FORMAL SOLUTION

In this chapter the 2-dimensional (space-time) Fourier transform for the fluid displacement potential along the borehole axis will be derived, following the technique of Tsang and Rader [2] and Schoenberg et al. [3]. The inverse transform of this function is a formal solution to the idealized system. The solution is investigated to show where various properties of the received waveforms arise in this frequency-wavenumber representation. Tsang and Rader have developed a program to numerically compute the inverse transform, and waveforms from this program will be used to verify those generated by the model developed in the next chapter.

When an elastic body is subjected to a force, particles within the body are displaced. This displacement $u = u(x,y,z,t)$ is a continuous vector field and so it can be represented as

$$u = \nabla \phi + \nabla \times \xi , \quad (2.1)$$

where $\phi$ is a scalar potential and $\xi$ is a vector potential [5].

The system to be considered here is axisymmetric, as shown in Figure 1. A cylindrical coordinate system $(r, \theta, z)$ is used, with the source and receiver on the borehole axis, which is coincident with the $z$ axis. Since the pulse transmitted at the origin is spherically
FIGURE 1 Axisymmetric Coordinate System
symmetric, the displacement field will be axially symmetric. In terms of \( u \) this means that \( u_\theta = 0 \) and neither \( u_r \) nor \( u_z \) depends on \( \Theta \).

This leaves

\[
  u_r = \frac{\partial \phi}{\partial r} - \frac{\partial \xi_\theta}{\partial z}
\]

and

\[
  u_z = \frac{\partial \phi}{\partial z} + \frac{\xi_\theta}{r} + \frac{\partial \xi_\phi}{\partial r}
\]

A volume element in the elastic body can undergo two types of displacement: A compression, which causes a change in volume; and a twisting, or rotation. The change in volume of an infinitesimal element is given by \( \nabla \cdot u \), the divergence of the displacement field. Since \( \nabla \cdot (\nabla \xi) = 0 \), the vector potential does not contribute to this form of displacement. The scalar potential \( \phi \) satisfies the scalar wave equation

\[
  \alpha^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2}, \tag{2.4}
\]

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \) and \( \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \). Here \( \lambda \), \( \mu \) and \( \rho \) are parameters of the material: \( \lambda \) is Lamé's constant, \( \mu \) is the modulus of rigidity, and \( \rho \) is the density. The units for \( \alpha \) are those of speed, and \( \alpha \) is the rate at which the compressional waves travel through the material. Because the direction of particle motion is the same as the direction of propagation for these waves, they are also called longitudinal waves.
The rotational form of displacement is proportional to $\nabla x u$, the curl of the vector displacement\[5\]. And since $\nabla (\nabla \phi) = 0$, the scalar potential makes no contribution to this motion. Define $\psi$ such that $\xi_\theta = -\frac{\partial \psi}{\partial r}$ \[6\]. Then Equations (2.2) and (2.3) become

$$u_r = \frac{\partial \phi}{\partial x} + \frac{\partial^2 \psi}{\partial r \partial z} \quad (2.2a)$$

and

$$u_z = \frac{\partial \phi}{\partial z} - \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2} \quad (2.3a)$$

and $\psi$ also satisfies a scalar wave equation:

$$\beta^2 \nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2} \quad (2.5)$$

where $\beta = \frac{c}{\sqrt{\mu}}$ is a speed. Particle motion for these waves is perpendicular to the direction of wave propagation, so they are often called transverse waves. Because they involve a shearing motion in the material they are also known as shear waves. Note that in a rectangular coordinate system there are two components perpendicular to the direction of propagation, so shear waves can be broken down into shear-horizontal (S-H) and shear-vertical (S-V) components.

In sonic well logging, waveforms are measured at an array of points along the borehole axis. The result is a two-dimensional array distributed over space ($z$) and time. Following the success of one-dimensional frequency analysis, investigation of the two-
dimensional (frequency and wavenumber) spectral characteristics is also expected to lend insight into properties of the signals. Schoenberg et al.[3] have used this method to study the modal characteristics. The transform technique can also be used to generate the waveforms from knowledge of the formation and fluid parameters, as Tsang and Rader have done[2].

The space-time frequency response was briefly derived in both of these papers. A more complete derivation will be presented here in order to fill in the details, and because the waveforms are used as a test standard for a model to be described in the next chapter. The development begins by considering a point source in an infinite fluid medium. The effect of the formation is added, and the boundary conditions at the interface then allow the resulting 2-D spectrum to be calculated.

First consider an infinite fluid with wave speed $\alpha_f$. Since the ideal fluid has no rigidity (no viscosity), $\mu_f = 0$ and $\beta = \sqrt{\frac{\mu_f}{\rho_f}} = 0$; shear waves cannot be propagated, and the compressional velocity $\alpha_f = \sqrt{\frac{\lambda_f}{\rho_f}}$.

It is useful to define the spherically symmetric point source in terms of pressure, since this quantity can be related to the displacement field $\phi_{tr}$ through the relation
The propagating pressure wave is described by\[^2\]

\[ p(t) = \frac{p(t-R)}{R} = \frac{p(t-R)}{R} \]  

(2.7)

where \( P_0 \) is the magnitude at radius \( R_0 \), and \( R = (r^2+z^2)^{1/2} \). Taking the Fourier transform of \( p(t) \):

\[ P(\omega) = \frac{P_0}{R} F(\omega) e^{\frac{-j\omega R}{\alpha_f}} \]  

(2.8)

where \( F(\omega) \) is the transform of \( f(t) \). Transforming (2.6) gives

\[ P(\omega) = -p_f (j\omega)^2 f_{tr}(\omega) = p_f \omega^2 f_{tr}(\omega) \]  

(2.9)

Combining (2.8) and (2.9) gives

\[ f_{tr}(\omega) = \frac{P_0}{p_f} \frac{-j\omega R}{\alpha_f} \]  

(2.10)

Next the "spherical wave" factor can be expanded as follows\[^7\]:

\[ e^{\frac{-j\omega R}{\alpha_f}} = i \int_0^\infty H_0^1(k_x r) e^{jk_x z} dk_x, \quad r > 0 \]  

(2.11)

where \( k_x = \left[ \frac{\omega^2}{\alpha_f^2} - k_z^2 \right]^{1/2} \). This allows the transform of the potential to be written as
\[ \varphi_{tr}(\omega, r, z) = \frac{jP}{2\rho_f} \left( \omega - \omega_0 \right) \int \frac{F(\omega)}{\omega^2} H_0^{(1)}(kr) e^{jkz} \, dk \, d\omega, \quad r > 0. \quad (2.12) \]

Taking the inverse transform gives the displacement potential for the source in an infinite fluid:

\[ \varphi_{tr}(t, r, z) = \frac{jP}{2\rho_f} \left( \omega - \omega_0 \right) \int \frac{F(\omega)}{\omega^2} e^{j\omega t} \int H_0^{(1)}(kr) e^{-jkz} \, dk \, d\omega, \quad (2.13) \]

for \( r > 0 \). The positive parameter \( \Delta \) is chosen so that the integration path passes below the singularities of the integrand, to satisfy causality requirements[8].

Next a term must be included to reflect the fact that the interaction of the pulse with the formation creates an additional field in the fluid. Any axisymmetric solution to the wave equation for the fluid, \( \frac{\partial^2 \varphi_f}{\partial t^2} - \nabla^2 \varphi_f = 0 \), can be expressed as a linear combination of elementary wave functions of the form[9]

\[ j(\omega t - k z) \]

\[ J_0(kr)e^{-jkz}. \quad (2.14) \]

Note that this expansion applies in a region including the axis \( r=0 \). For a radius extending outward from the fluid boundary the compressional and shear potentials are a superposition of the wave functions

\[ H_0^{(1)}(k_r(c)r)e^{j(\omega t - k_r(z)z)} \quad \text{and} \quad H_0^{(1)}(k_r(s)r)e^{j(\omega t - k_r(z)z)}. \quad (2.15) \]

Here \( k_r(c) = \left[ \frac{\omega^2}{\alpha^2} - k_z^2 \right]^{1/2} \) and \( k_r(s) = \left[ \frac{\omega^2}{\beta^2} - k_z^2 \right]^{1/2} \) are the same notation as used in [2] and [3].
In order to more easily apply the boundary conditions, the potentials in the fluid and solid are expressed in the form given for $\varphi_{tr}$ in Eq. (2.13). Using (2.13) and (2.14) the complete potential in the fluid becomes

$$
\varphi_f(t, r, z) = \frac{jP_o R_o}{4\pi \rho_f} \int_{-\infty}^{\infty} \frac{1}{\omega^2} \left[ H_0^1(k r) + A(\lambda, k_z) J_0(k r) \right] e^{-jk_z z} \, dk_z \, d\omega,
$$

for $0 < r < r_b$, which incorporates $\varphi_{tr}$. Similarly, the compressional and shear potentials for the solid are written as

$$
\varphi_s(t, r, z) = \frac{jP_o R_o}{4\pi \rho_f} \int_{-\infty}^{\infty} \frac{1}{\omega^2} \left[ C(\lambda, k_z) H_0^1(k r) \right] e^{-jk_z z} \, dk_z \, d\omega
$$

for $r > r_b$, and

$$
\varphi_s(t, r, z) = \frac{jP_o R_o}{4\pi \rho_f} \int_{-\infty}^{\infty} \frac{1}{\omega^2} \left[ D(\lambda, k_z) H_0^1(k r) \right] e^{-jk_z z} \, dk_z \, d\omega
$$

for $r > r_b$. It has been assumed the point source at the origin is the only energy source in the system.

The boundary conditions occur where fluid meets formation and there are three (subscripts f and s refer to displacements in the fluid and solid respectively) [11]:

i) Continuity of normal displacement:

$$
u_{fr} \bigg|_{r=r_b} = u_{sr} \bigg|_{r=r_b}
$$

ii) Continuity of normal stress:
\[
\lambda_f \left[ \frac{\partial u_r}{\partial r} + u_r \frac{\partial}{\partial z} + \frac{\partial u_z}{\partial z} \right] \bigg|_{r=r_b} = 2\mu_s \frac{\partial u_s}{\partial r} \bigg|_{r=r_b} + \lambda_s \left[ \frac{\partial u_r}{\partial r} + u_r \frac{\partial}{\partial z} + \frac{\partial u_z}{\partial z} \right] \bigg|_{r=r_b}
\] (2.20)

iii) Vanishing tangential stress:

\[
\left[ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \bigg|_{r=r_b} = 0
\] (2.21)

To apply the boundary conditions, only the parts of the potentials which depend on \( r \) and \( z \) need to be used. Define

\[
\hat{\phi}_f(r,z) = \left[ H^{(1)}_0(k_r) + A(\omega, k_z) J_0(k_r) \right] e^{jk_z z}
\] (2.22)

\[
\hat{\phi}_s(r,z) = C(\omega, k_z) H^{(1)}_{0}(k_c) e^{jk_z z}
\] (2.23)

\[
\hat{\phi}_s(r,z) = D(\omega, k_z) H^{(1)}_{0}(k_s) e^{jk_z z}
\] (2.24)

Using Equations (2.2a) and (2.3a) and noting that \( \Psi = 0 \) within the fluid, the displacements can be calculated. Inside the borehole:

\[
\hat{\mathbf{u}}_r = \frac{\partial \hat{\phi}_f}{\partial r} = -\left[ k_r H^{(1)}_{1}(k_r) + A k J_1(k_r) \right] e^{-jk_z z}
\] (2.25)

and

\[
\hat{\mathbf{u}}_z = \frac{\partial \hat{\phi}_f}{\partial z} = -j k_z \left[ H^{(1)}_{0}(k_r) + A J_0(k_r) \right] e^{-jk_z z}
\] (2.26)

In the formation:
\begin{align}
\hat{u}_{sr} &= \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r \partial z} \\
&= -\left[k_{r}^{(c)} B_1(k_{r}^{(c)} r) - jk_{z} k_{r}^{(s)} C_1(k_{r}^{(s)} r)\right] e^{-jk_{r} z}
\end{align}

and

\begin{align}
\hat{u}_{sz} &= \frac{\partial \Phi}{\partial z} - \frac{1}{r} \frac{\partial \Phi}{\partial r} - \frac{\partial^2 \Phi}{\partial r^2} \\
&= -\left[jk_{z} B_1^{(1)}(k_{r}^{(c)} r) - (k_{r}^{(s)})^2 C_1(k_{r}^{(s)} r)\right] e^{-jk_{r} z}
\end{align}

Applying the three boundary conditions gives

From condition i)

\begin{align}
k_{r} B_1^{(1)}(k_{r} r_b) + A k_{r} J_1(k_{r} r_b) = k_{r}^{(c)} B_1^{(1)}(k_{r}^{(c)} r_b) - jk_{z} k_{r}^{(s)} C_1(k_{r}^{(s)} r_b)
\end{align}

From condition ii)

\begin{align}
\lambda_{r} \left[-k_{r}^{(c)} B_1(k_{r} r_b) - k_{r}^{(c)} A J_1(k_{r} r_b) - k_{z}^{(c)} B_1(k_{z} r_b) - k_{z}^{(c)} A J_1(k_{z} r_b)\right] \\
= 2\mu_{s} \left[-(k_{r}^{(c)})^2 B_1(k_{r}^{(c)} r_b) + \frac{k_{r}^{(c)}}{r_b} B_1(k_{r}^{(c)} r_b)\right] \\
+ jk_{z} (k_{r}^{(s)})^2 C_1(k_{r}^{(s)} r_b) - jk_{z} k_{r}^{(s)} C_1(k_{r}^{(s)} r_b) \\
+ \lambda_{s} \left[ -(k_{r}^{(c)})^2 B_1(k_{r}^{(c)} r_b) + \frac{k_{r}^{(c)}}{r_b} B_1(k_{r}^{(c)} r_b) - \frac{k_{z}^{(c)}}{r_b} B_1(k_{z}^{(c)} r_b)\right]
\end{align}

From condition iii)

\begin{align}
jk_{z} k_{r}^{(c)} B_1(k_{r}^{(c)} r_b) + k_{z}^{(c)} k_{r}^{(s)} C_1(k_{r}^{(s)} r_b) \\
+ jk_{z} k_{r}^{(c)} B_1(k_{r}^{(c)} r_b) - (k_{r}^{(s)})^2 C_1(k_{r}^{(s)} r_b) = 0
\end{align}
If these three equations are written as

\[ X_1A + X_2B + X_3C = K_1 \]
\[ Y_1A + Y_2B + Y_3C = K_2 \]
\[ Z_2B + Z_3C = 0 \]

then

\[ A = \frac{(Z_3Y_2 - Z_2Y_3)K_1 + (X_3Z_2 - X_2Z_3)K_2}{(Z_3Y_2 - Z_2Y_3)X_1 + (X_3Z_2 - X_2Z_3)Y_1} \] \hspace{1cm} (2.32)

where

\[ X_1 = k_x J_1(k_x r_b) \]
\[ X_2 = -k_x^{(c)} H_1^{(1)}(k_x^{(c)} r_b) \]
\[ X_3 = jk_z k_x^{(s)} H_1^{(1)}(k_x^{(s)} r_b) \]
\[ Y_1 = -\rho_x^2 J_0(k_x r_b) \]
\[ Y_2 = -\mu_s \left[ k_x^2 - (k_x^{(s)})^2 \right] H_0^{(1)}(k_x^{(c)} r_b) - 2\mu_s k_x^{(c)} H_1^{(1)}(k_x^{(c)} r_b) \]
\[ Y_3 = -2j\mu_s k_x^{(s)} k_x^{(s)} H_1^{(1)}(k_x^{(s)} r_b) + 2j\mu_s k_x^{(s)} H_1^{(1)}(k_x^{(s)} r_b) \]
\[ Z_2 = 2jk_x k_x^{(c)} H_1^{(1)}(k_x^{(s)} r_b) \]
\[ Z_3 = \left[ k_x^2 - (k_x^{(s)})^2 \right] k_x^{(s)} H_1^{(1)}(k_x^{(s)} r_b) \]
\[ K_1 = -k_r H_1^{(1)}(k_r r_b) \]
\[ K_2 = \rho_f \omega^2 H_0^{(1)}(k_r r_b) . \]

So
\[ X_3 Z_2 - X_2 Z_3 = -k_s^2 k_r^2 H_1^{(1)}(k_r r_b) H_1^{(1)}(k_r r_b) , \quad (2.33) \]

and
\[ Z_3 Y_2 - Z_2 Y_3 = -\mu_s \left[ k_z^2 - (k_r^2)^2 \right] H_1^{(1)}(k_r r_b) H_1^{(1)}(k_r r_b) \\
- 4\mu_s k_z^2 k_r^2 (k_r^2)^2 H_1^{(1)}(k_r r_b) H_0^{(1)}(k_r r_b) \\
+ 2\mu_s k_z^2 k_r^2 (k_r^2) H_1^{(1)}(k_r r_b) H_1^{(1)}(k_r r_b) , \quad (2.34) \]

where \( k_s = \frac{\omega}{\beta} \) and \( k_c = \frac{\omega}{\alpha} \).

Zeros of the denominator of (2.32),
\[ (Z_3 Y_2 - Z_2 Y_3) X_1 + (X_3 Z_2 - X_2 Z_3) Y_1 = 0 , \quad (2.35) \]
cause singularities in the integrand of the equation for the fluid potential (2.16), giving rise to the "normal" or guided modes for the borehole[3,10,12,13]. There are two basic types, both dispersive. The first is called the Stoneley mode, and it has a phase speed less than the fluid speed. The Stoneley wave is a surface wave, since its spatial dependence is in the form of an exponential decay away from the interface (both inside and outside the fluid).

The second type of guided wave has a phase speed greater than the fluid speed \( \alpha_f \), but less than the formation shear speed \( \beta \). These modes are known variously as pseudo-Rayleigh waves[2], reflected
conical waves[10], and trapped fluid modes[3]. They decay exponen-
tially, radially outward from the interface, but have the form of an
exponentially decaying sinusoid inward from the interface in the
fluid. These modes have a low frequency cutoff. At higher frequen-
cies their number increases, unlike the Stoneley wave for which a
single mode exists.

If the numerator and denominator of (2.32) are divided by

\[(X_3 Z_2 - X_2 Z_3) \rho_f \omega^2,\]

then

\[A(\omega, k_z) = - \frac{F H_1^{(1)}(k r_b) + H_0^{(1)}(k r_b)}{F J_1(k r_b) + J_0(k r_b)}, \tag{2.36}\]

where

\[F = \frac{\mu_s}{\rho_f \omega^2} \left[ \frac{4k_s^2 k_r(s)}{k_s^2 H_1^{(1)}(k r_b)} - \frac{H_1^{(1)}(k r_b)}{H_0^{(1)}(k r_b)} \right. \]

\[\left. + \frac{[k_z^2 - (k(s))^2]^2}{k_s^2 H_0^{(1)}(k r_b)} - \frac{H_0^{(1)}(k r_b)}{H_1^{(1)}(k r_b)} \right], \tag{2.37}\]

The expression for \(A(\omega, k_z)\) can be written as[12]

\[A(\omega, k_z) = 2 \sum_{m=1}^{\infty} g^m, \tag{2.38}\]

where

\[g = - \frac{F H_1^{(1)}(k r_b) + H_0^{(1)}(k r_b)}{F H_1^{(2)}(k r_b) + H_0^{(2)}(k r_b)}. \tag{2.39}\]

To see this write
\[ \sum_{m=1}^{\infty} g^m = \frac{1}{1-g} - 1 = \frac{\alpha}{1-g} = - \frac{\frac{FH_1(1) + H_0(1)}{FH_1(2) + H_0(2)}}{1 + \frac{FH_1(1) + H_0(1)}{FH_1(2) + H_0(2)}} \]

\[ = \frac{FH_1(1) + H_0(1)}{FH_1(2) + H_0(2)} \frac{\left[ \frac{H_1(1) + H_0(2)}{H_0(1) + H_0(2)} \right]}{\left[ J_1^2 + J_0^2 \right]} = \frac{1}{2} \frac{FH_1(1) + H_0(1)}{J_1 + J_0} \]  

(2.40)

where the identity \( H_n^{(1)}(kr_b) + H_n^{(2)}(kr_b) = 2J_n(\lambda kr_b) \) was used in the final step.

The expansion in (2.38) has been termed a "ray expansion". To see this, divide through (2.39) by \( kr_b \):

\[ g = - \frac{\frac{FH_1(1)}{kr_b} + \frac{1}{kr_b} H_0(1)}{\frac{FH_1(2)}{kr_b} + \frac{1}{kr_b} H_0(2)} \]  

(2.41)

The quantity \( kr_b \) will be large for either high frequencies or a large borehole radius. For \( kr_b \to \infty \), \( \frac{F}{kr_b} \to 0 \), and so

\[ g \to - \frac{H_1^{(1)}(kr_b)}{H_0^{(2)}(kr_b)} = \frac{\left[ \frac{2}{\pi kr_b} \right]^{1/2} e^{-j(kr_b \cdot \frac{\pi}{4})}}{\left[ \frac{2}{\pi kr_b} \right]^{1/2} e^{-j(kr_b \cdot \frac{\pi}{4})}} = j \frac{\pi}{2} e^{j2kr_b} \]  

(2.42)

This corresponds to a phase shift of \( m \frac{\pi}{2} \) and a delay of \( \frac{2kkr_b}{\omega} \) m for the \( m^{th} \) "ray". The delay \( \frac{2kkr_b}{\omega} \) is the time required for a wave to
cross the borehole at a given angle. Note that $k_r = \omega p$, where $p$ is the radial slowness, which depends on the angle the ray is traveling across the borehole.

Rays for which this delay and shift would occur are shown in Figure 2. The ray paths for $m = 0$ and $m = 1$ are shown. While the path shown is the only one possible for $m = 0$, only one of an infinite number of possible paths is shown for $m = 1$. The phase shifts arise as the cone of rays passes through the axis to form a caustic. Since the wave can travel as either a P wave (compressional) or an S wave (shear) after being refracted into the formation, the shear wave will arrive later, reflecting the lower shear speed. These rays will be discussed in greater detail in the next chapter.

The full received waveform is mainly made up of these four basic components: The compressional and shear refracted arrivals, the Stoneley surface wave, and the waveguide modes. In addition there is a direct arrival through the fluid, although this is usually not as strong as the modal arrivals[2]. Figure 3 is an example of a complete waveform at an 8 ft. source-receiver spacing. These were generated using the real axis integration program of Tsang and Rader[2], as modified by Kurkjian[4].

Because (2.16) is not valid for receivers on the borehole axis, the received pressure is calculated by performing the integration

\[ p(t,0,z) = \frac{jP R_o}{4\pi} \int_{-\infty}^{\infty} \int_{-\Delta}^{\Delta} \int_{\infty}^{\infty} F(\omega) e^{j\omega t} A(\omega, k_z) e^{-jk_z z} dk_z d\omega , \]  \hspace{1cm} (2.43)
FIGURE 2 Two General Ray Paths
FIGURE 3
Real Axis Integration Waveform
and then adding the direct fluid arrival, given by Eq. (2.7). Using the Laplace contour for the frequency integration moves the poles of the integrand of the wavenumber integral off of the path of integration (the real \( k_z \) axis). Tsang and Rader then use Simpson's method to evaluate the double integral. Kurkjian achieves the same effect by adding small complex perturbations to \( \omega \) and \( k_z \) in the integrand. He then uses an FFT to perform the integrations along the real axis of both frequency and wavenumber.

The principle characteristics of sonic well logging waveforms can be observed in this example. The different components of the waveform can be separated by time of arrival, frequency characteristics, magnitude, and velocity. For example, the refracted compressional arrival is the first component to arrive, appears to have the least magnitude, and has the highest velocity present in the waveform.
CHAPTER 3

A MODEL FOR HEAD WAVES

The real axis integration technique, used to generate the waveform shown in Figure 3, has two disadvantages. First, the program will typically take hours to create the output for a system with borehole size and source-receiver distance typical of sonic logging. Second, the program severely restricts the input pulse that may be used. Specifically, it will reject many of the pulses measured as outputs from actual logging transmitters.

These difficulties motivate a search for more simple ways to construct the waveforms, or equivalently, a more simple physical model for the wave propagation. Because the waves are a summation of components, each of which arises from a different mode of propagation along the borehole, a single model could not be expected to represent the full complement of wave types. Here attention is focused on the compressional and shear refraction arrivals, for three reasons: They are the first components to arrive at the receiver, and so should initially be observed with less interference than the later arrivals. Second, the axial propagation takes place almost entirely within the surrounding solid, so that the formation properties strongly influence the wave characteristics. And finally, this is the portion of the waveform predominantly used in present velocity logging measurements.
The model to be used is one suggested by the ray expansion of the previous chapter. The various ray paths for the compressional and shear refracted waves are developed. This first step can best be viewed in terms of a 2-D "delay response" over time and receiver position along the axis. Phase shifts and magnitudes, which are a function of ray path, the number of borehole crossings, and wave type are then included to give a complete impulse response.
Consider the rays (normals to the wavefront) existing in a plane passing through the axis of the borehole. A view from perpendicular to this plane is shown in Figure 4. Since the source and system are axisymmetric, this view is the same for any other plane through the axis. Let the angle of incidence be the angle $\theta$ between the ray and a normal to the cylinder wall.

These rays will be reflected and refracted at the boundary, with the angles of incidence and transmission related to the velocity in each medium by Snell’s Law. The rays which are refracted parallel to the boundary are incident with the critical angle

$$\theta_c = \sin^{-1}\left[\frac{c_f}{v_s}\right],$$

(3.1.1)

where $v_s$ is the speed in the solid. As they travel along the boundary in the solid the waves will displace the fluid, and this disturbance will propagate back into the fluid at the same angle $\theta_c$. These waves are also called "head waves".

The waves can travel at two different velocities in the solid, $\alpha$ or $\beta$, depending on whether they are of the P or S type. This gives two separate critical angles according to equation (3.1.1). Since $\alpha > \beta$, the angle at which the refracted rays enter and leave the wall will be less for P wave than for S wave propagation. The rock speeds are generally greater than the fluid speed, and the source-receiver...
FIGURE 4  A General m=1 Ray in a Plane Through the Borehole Axis
distance much greater than the borehole radius. Note that sometimes \( a_f \beta \), and these are termed "slow formations"[3]. However, in this case there will be no shear refraction along the borehole wall. Here it will be assumed that \( \beta > a_f \), as is normally the case.

As a first step in creating an impulse response for these head waves the phase shifts and magnitudes will not be considered. The source will be an impulse, and will arrive at the receiver as impulses delayed by the various travel paths. These impulses will essentially be pointers for arrival times. The delays can be found by first noting that all ray paths that cross the borehole the same number of times travel the same distances both in the fluid and in the formation (considering P and S waves separately). So the delay is not a function of where the crossings occur, only of their number.

These delays can be calculated, with reference to the quantities labeled in Figure 4:

- \( z \) = Source-to-receiver distance.
- \( \Theta_c \) = Critical angle.
- \( d = 2r_b \) = Borehole diameter.
- \( l_s \) = The axial distance the ray travels in crossing the borehole at angle \( \Theta_c \).
- \( l_b \) = The total distance traveled in crossing the borehole at angle \( \Theta_c \).
- \( t_f \) = The time required to travel across the borehole in the fluid at angle \( \Theta_c \).
- \( m \) = The number of times the ray crosses completely across the borehole.

All of these except \( z \), \( d \) and \( m \) depend on whether the wave travels as a P or S wave in the solid.
The difference in travel time for an additional crossing is the time it takes to cross minus the time it takes to travel the equivalent axial distance in the formation. For any ray that crosses the borehole \( m \) times, the arrival time is

\[
t_a = (m+1) l_b \frac{1}{v_f} + \left[ z - (m+1) l_s \right] \frac{1}{v_s}.
\]  

(3.1.2)

There is a minimum source-receiver distance \((m+1) l_s\) below which no refracted arrivals which have crossed \( m \) or more times can be received. For a fixed number of crossings, the change in travel time is linearly proportional to changes in \( z \).

These observations can be combined into a 2-D "delay response" \( f(z,t) \), as a function of time and source-receiver distance. The function can be written as

\[
f(z,t) = \sum_{m=0}^{\infty} u[z-(m+1) l_s] u[t-(m+1) t_s] \delta[z-v_s t+(m+1) l_d].
\]  

(3.1.3)

Here

- \( u = \) The unit step function.
- \( \delta = \) The unit impulse function.
- \( l_d = \) The distance the wave will travel in the rock in the time it takes to make one crossing of the borehole at angle \( \theta_c \).

Figure 5 is a graph of \( f(z,t) \), where

- \( t_d = \) The difference in the time it takes to travel across the borehole at angle \( \theta_c \) and the time it takes to travel the same axial distance along the borehole in the solid.

The quantities \( \theta_c, l_b, l_s, l_d, t_s \) and \( t_d \) can all be determined from knowledge of \( v_f, d \) and \( v_s \) (either \( \alpha \) or \( \beta \)).
FIGURE 5  A 2-D Delay Response

\[ \text{Slope} = v_s \]
The delay function \( f(z,t) \) obtained for the P arrivals will be different from the one given by the S arrivals, assuming that \( \alpha \neq \beta \). The complete delay response from a single source pulse would be given by a superposition of the P and S responses. Most importantly, the slope of the lines would be different for the two propagation modes. The arrival times at an array of receivers are the intersections of the \( f(z,t) \) lines with horizontal lines at heights \( z_0, z_1, \ldots, z_n \) given by the receiver positions. The different slopes (velocities) for the compressional and shear lines would give a different "moveout" for the arrivals along the receiver array.

The next step in the development of the impulse response is to include magnitude factors. That is, factors which will vary the magnitude of the impulses in the delay response. Two such parameters will be included in this model. The first, \( \gamma \), is used to indicate a loss of energy associated with an additional crossing of the borehole. Thus the factor \( \gamma^m, 0 < \gamma < 1 \), is included in the
expression for the magnitude of an arrival after \( m \) crossings. This loss can be accounted for in at least two ways. Only a portion of the energy in the wave which has crossed \( m \) times will be radiated out to cross \((m+1)\) times. Also, losses due to other factors such as imperfect coupling of wave types at the borehole wall should be expected.

The second parameter used to describe the energy distribution in the head waves is \( e \), in the form \( 1 \frac{e}{r} \). Here \( e > 0 \) and \( 1 \frac{e}{r} \) is the distance the ray has traveled in the formation. \( 1 \frac{e}{r} \) is a function of the source-receiver distance \( z \) and the number of crossings \( m \). The \( -e \frac{1}{r} \) part of this factor is to account for the fact that as the wave travels along the wall it is constantly radiating energy (in the form of head waves) into the fluid.

The inclusion of the multiplier \( 1 \frac{e}{r} \) is explained with reference to the \( m=1 \) rays in Figure 6(a)-(c). Suppose the source and receiver are just \( 21_s \) apart. Then there is only a single thin ray path that crosses the borehole once. As \( z \) increases, the region available for rays to cross once begins to increase, and so more energy would be expected. However, this increase cannot continue forever, and the \( -e \frac{1}{r} \) will take over to give the general form of Figure 6(d). The same arguments also hold for \( m > 1 \) crossings, with \( 1 \frac{e}{r} = z-(m+1)1_s \).

It is apparent from the discussion of the mechanisms that give rise to the parameters \( \gamma \) and \( \varepsilon \) that they will take on one set of
FIGURE 6 Explaining the $l_r e^{-\epsilon l_r}$ Magnitude Factor
values for the rays representing P waves, and another set for those representing S waves. Estimated values for the magnitude decay (as a function of z) of the first arrivals also indicate that they should be different for the two cases [12].

The final characteristics needed to complete the impulse response for the head waves are the phase shifts associated with the ray paths. One type is created as the cone of rays passes through the axis of the borehole. This phenomena of a surface of rays converging at a point (called a "caustic") occurs in other geophysical problems [14,15], and the result is a phase shift of \( \frac{\pi}{2} \text{sgn}(\omega) \). This gives the negative Hilbert transform (or "allied function") of the original wave shape. It can be obtained by convolving the pulse with the time function \(-\frac{1}{\pi t}\). The phase change occurs at each crossing, and has the property that the negative of the original pulse results upon application a second time. The phase shift applied to the ray is \( \text{e}^{\frac{j\pi}{2} m \text{sgn}(\omega)} \) for a ray that has made \( m \) borehole crossings.

The other phase shift was suggested by White [16], and will apply to rays representing shear wave propagation only. Consider a positive pressure pulse impinging on the borehole, creating an outward bulge in the wall. The shear refracted component travels along the wall as particle displacement radially outward from the wall. As this propagates, the widening of the hole creates a decrease in the fluid pressure. So the positive pressure ray entering the wall and travelling as a shear wave emerges as a negative pressure. This
corresponds to a $\pi$ phase shift.

The compressional wave has axial particle displacement, along the borehole wall. This ring of pressure causes the wall to bulge inward, compressing the fluid. So a positive pressure entering the wall and refracted as a compressional component radiates positive pressure waves back into the fluid.

The inclusion of the magnitude and phase factors along with the delay function $f(z, t)$ results in a 2-D impulse response for the head wave portion of the waveforms. Letting $H$ denote the Hilbert transform operator and $H^n$ the $n^{th}$ application of the operator, the 2-D impulse response for the compressional arrivals is

$$
a(z, t) = \sum_{m=0}^{\infty} \left[ \gamma^m \left( z-(m+1)l_s \right) \right] e^{-\alpha[z-(m+1)l_s]} (-H)^m \left[ u[z-(m+1)l_s] u[t-(m+1)t_s] \delta[z-v_s t+(m+1)d] \right].
$$

The shear response has the same form, with the inclusion of the factor $(-1)^{m+1}$ within the summation. The parameters $\gamma$, $s$, $l_s$, $t_s$, $v_s$ and $d$ will generally be different for the P and S waves.

It is instructive to look at the response as a function of time at a fixed source-receiver distance $z$. Figure 7 shows such a 1-D impulse response for typical parameters: $a = 16,667$ ft/s, $\beta = 9615$ ft/s, $a_f = 5263$ ft/s, $d = 9 \text{ in.}$, $z = 10$ ft.; $\gamma = 0.9$ and $\varepsilon = 0.25$ for the compressional wave; $\gamma = 0.9$ and $\varepsilon = 0.1$ for the shear wave.
FIGURE 7  1-D Impulse Response
Figure 7(a) is the magnitude response, without any of the phase shifts. The larger decaying series represents the S arrivals, and the smaller series the P arrivals. An important characteristic to note is the smaller delay $t_d$ between the shear impulses. If the expression for $t_d$ in Eq. (4) is written in terms of inverse speed, or slowness, $S_s = \frac{1}{v_s}$ and $S_f = \frac{1}{a_f}$,

$$t_d = d \sqrt{\frac{S_f^2 - S_s^2}{S_s^2}} .$$

(3.1.6)

This shows clearly that a greater formation velocity (smaller slowness $S_s$) gives a greater interarrival time $t_d$. The complete impulse response for these head waves at $z = 10$ ft. is shown in 7 (b).

The larger magnitude for the shear arrivals is typical of hard formations. Here the critical angle $\theta_c$ is $18^\circ$ for the P waves and $33^\circ$ for the S waves. The steep angle at which the fluid wave impinges on the wall would tend to favor a more direct coupling of energy to the radial shear motion than to the axial compressional wave motion.

The impulse response can be observed separately for the P and S arrivals, before being combined as in Figure 7(b). Figure 8(a) and (b) are the magnitude of the frequency response for these individual arrivals.

The placement of the frequency peaks in the general case can be predicted by observing the spectral magnitudes of the basic sequences of length $4t_d$ (Figures 9(a) and (c)) which make up the impulse
FIGURE 8  Frequency Response (Magnitude)
response for the compressional and shear waves. Figures 9(b) and (d) are the resulting spectra when the attenuation factors are equal to 1. The spectra are periodic with period $f = \frac{1}{t_d}$, and so only one period is shown. These show that the spectral peaks for the P wave should occur at $\frac{1}{4t_d} + \frac{n}{t_d}$ and those for the S wave at $\frac{3}{4t_d} + \frac{n}{t_d}$, $n = 0, 1, \ldots$. In addition, the repetition of the length $4t_d$ sequences will cause a peak at $f = \frac{1}{4t_d}$.

The spacing between the spectral peaks is $\frac{1}{t_d}$, the inverse of the interarrival time for the impulses. The spacing is 7.4 KHz for the P and 8.4 KHz for the S spectrum in this example. Figure 8 shows the peak positions to be as predicated.

The magnitude spectrum of the received refracted waves will be the product of the frequency response and the spectrum of the input pulse. Because of the interleaving of the compressional and shear spectral peaks, the position and shape of the source spectrum are critical factors in determining the relative frequencies of the P and S waves. Information concerning these frequencies is important in some algorithms for velocity estimation.

Figure 10 shows a typical source and its magnitude spectrum. The pulse in 10(a) was convolved with the compressional and shear impulse responses, and the resulting spectra of these two waveforms are shown in Figure 11(a) and (b). These last two figures could also
FIGURE 9  Spectral Energy for Basic P and S Wave Sequences
FIGURE 10  A Transmitter Pulse and its Magnitude Spectrum
FIGURE 11  Complete Magnitude Response
be obtained by multiplying 8(a) and (b) by 10(b). The complete time
response is shown in Figure 12.

In this case, the source spectrum picks out two of the spectral
peaks for both the P and S waves, although one of the shear peaks is
down about 6 dB from the other. Each of the shear peaks is lower in
frequency than the corresponding nearby compressional peak. Note
also that the higher frequency shear peak occurs at the peak of the
source spectrum, whereas the compressional peaks straddle this max-
imum. The source spectrum drops off quite rapidly above 15 Khz. Had
the pass band been slightly narrower, or shifted slightly lower, the
high frequency compressional peak would have been greatly attenuated,
and the P wave would have been a "lower frequency" wave than the S
wave. The example in Figure 11 shows that the interleaving of the
spectral lines can cause complications when frequency-based filtering
is used to separate the P and S components for use in velocity esti-
mation algorithms. Even if properly positioned, a narrowband filter
attempting to reject the S arrival would capture only about half of
the energy in the P component.
FIGURE 12 Ray Model Head Waves

5 μs/sample

512

n
Section 3.2: Comparison With Real Axis Integration Waveforms

Now that the model has been derived and some of its properties investigated, the waveforms it generates will be compared with a standard to test their accuracy. The acoustic parameters are unknown in actual well logging waveforms, so these will not be used. Also, the environment is not ideal as is assumed in the model, and these irregularities would make for an unreasonable comparison.

Waveforms obtained by the real axis integration method discussed in Chapter 2 fulfill the requirements of known parameters in an ideal model. Figure 13(a) is a detail of approximately the first millisecond of the waveform in Figure 3. This includes the compressional and shear regions. A vertical line has been drawn to indicate the start of the secondary (non-compressional) arrivals. The marked time was found by comparing this waveform, with a source-receiver distance of 8 ft., to the arrivals at 10 ft. Since the moveout of the secondary wave is greater than for the P wave, the point where these two waves begin to differ is the time of first arrival for the secondary wave in the waveform taken at 8 ft. (Note that this technique will work only for synthetic waves, where the compressional arrivals are almost identical, except for moveout, for closely spaced receivers).

Figure 13(b) is a waveform obtained from the ray model using the same borehole parameters, with $\gamma = 1.0$, $s = 0.25$ for the P wave, and $\gamma = 1.0$, $s = 0.1$ for the S wave. As can be seen by comparing 13(a)
FIGURE 13 Comparison of Ray Model and Real Axis Waves
and (b), the compressional arrivals have been quite accurately modeled. The greatest difference is in the magnitude of the first P arrival with respect to the later P arrivals. In the real axis wave, this first arrival has only about two-thirds the magnitude of the later ones. In the ray model they are approximately of equal magnitude. Recall that when the various ray paths were discussed it was noted that the first arrival was unique in that it had only one generalized ray path, whereas the later arrivals were composed of an infinite number of generalized paths. Figure 14 shows the spectral characteristics of the two compressional waves, and again these are indicative of the quality of the approximation.

The secondary(shear) wave arrival time is seen to be accurate. However, the ray model does not seem to be capable of reproducing the wave in this region. The frequency characteristics in Figure 15 show that the waves lack even a common major spectral peak.

Although this seems to indicate a failure to properly model the shear waves, there is more in this region than shear waves. Figure 16 is a slowness vs. time contour plot for the real axis wave, produced using waveforms from an array of receiver locations. This plot shows that at the beginning of the shear region a wave of comparatively low magnitude travels at the shear wave slowness of 105 $\text{us/ft}$ but that this wave is soon overshadowed by a much larger, slower wave. This is the beginning of the modal arrivals. Knowing this, the accuracy of the model in this region is difficult to assess.
FIGURE 14 Comparison of Magnitude Spectra (Compressional)
FIGURE 15  Comparison of Magnitude Spectra (Shear)
FIGURE 16 Slowness Vs. Time Plot for Real Axis Wave
As a final comment on the composition of these waveforms, the form of the initial compressional arrival will be considered. The theory of head waves in plane geometry predicts that the first arrival will be in the form of the integral of the transmitted pulse[1]. Roever et al. predict this same behavior for the fluid-filled borehole[12].

Figure 17(b) is the time integral of the source pulse (a) used in generating the real axis waves. Figure 18 is the same as 13 except that the ray model for 18(b) uses the integrated pulse. Inspection shows 18(b) to be an inferior model for the real axis wave. In particular, the first two large positive peaks should be close to equal in magnitude, whereas the integrated pulse gives them significantly different values.

Also in support of using the non-integrated pulse, Figure 19 is presented. This the beginning of a real axis wave using a larger borehole diameter and a higher frequency input pulse, still in the shape of Figure 17(b). The effect is to create an interarrival time such that the initial compressional arrivals are separate and distinct. The first arrival is clearly represented more accurately by the symmetric input pulse than by the integrated one.

Further evidence for this result will be obtained in Section 3 where, in the case of overlapping arrivals, the second and later ones are removed.
FIGURE 17 Real Axis Transmitter Pulse and its Time Integral
FIGURE 18  Wave Comparison Using Integrated Pulse
FIGURE 19 Real Axis Integration Waveform

Large Borehole / High Frequency
Section 3.3: Applications

In this section the structure developed earlier to describe the refracted waves will be used to create techniques to investigate the properties of the waveforms. Two applications will be described, both relying on the fact that the model is a linear system. That is, that the waveform is the convolution of the source pulse with the system impulse response. Thus the degree of success of these techniques provides a measure of the efficacy of the model.

The first technique is an attempt to time-concentrate the compressional wave by finding a filter that will leave the first arrival and remove the later arrivals. One of the problems which arises when processing sonic logging waveforms is that earlier, high velocity wave types interfere with the slower, later ones. This makes velocity estimation of the later waves more difficult. Removal of the compressional interference would help to establish when the next component arrives.

This problem will be derived in terms of estimating an unknown source pulse, $P(f)$, given a compressional wave: Let $S(f)$ be the transform of the compressional wave, and find $\hat{P}(f)$ to make the compressional wave the model predicts equal to $S(f)$. This formulation can be written as
\[
\hat{P}(f) + \hat{P}(f) e^{j2\pi ft_d} + \hat{P}(f) e^{j4\pi ft_d} + \cdots = S(f),
\]

where the amplitude factor involving \( \gamma \) has been neglected, and the wave is assumed to continue indefinitely. Recall that \( t_d \) is the interarrival distance in the delay response.

Equation (3.3.1) can be written as

\[
\hat{P}(f) \left[ \sum_{n=0}^{\infty} (K(t_d, f))^n \right] = S(f),
\]

where \( K(t_d, f) = \gamma e^{j2\pi ft_d} \). Since \( |K| = |\gamma| < 1 \), Equation (3.3.2) can be written as

\[
P(f) \left[ \frac{1}{1-K} \right] = S(f).
\]

This gives the estimate

\[
\hat{P}(f) = S(f)[1-k] = S(f) \left[ 1 - \gamma e^{j2\pi ft_d} \right].
\]

In the time domain this corresponds to convolving the received waveform \( s(t) \) with an impulse and a delayed Hilbert transform sequence, the latter being delayed by \( t_d \) and attenuated by \( \gamma \).

Figure 20 shows the result of applying the technique to the real axis integration waveform discussed in the previous section. A 1.25 \( \mu s \) sample rate was used for these signals. Plot (a) is 512 samples of the original signal and includes a portion of the secondary
1.25 μs/sample

FIGURE 20 Filtering a Real Axis Wave
arrival. Plot (b) is the unit sample and delayed Hilbert transform operator to be used in the convolution, shown with $\gamma = 1.0$ and $t_d = 84$ samples. Figure 21(a)-(g) are the results of convolving 20(a) with 20(b) for $t_d = 80, 82, 83, 84, 85, 86$ and 88 samples.

Based on the velocities and borehole diameter used to generate the signals, the ray model would have $t_d = 84.5$ samples. The fact that the least error in removing the later compressional arrivals occurs at $t_d = 84$ samples lends credence to the basic structure of the model.

This series of waveforms show that the method can also be used to estimate the P wave velocity given the borehole diameter $d$, or vice versa, using Equation (3.1.6) (The fluid velocity $\alpha_f$ is assumed known). A higher sample rate increases the accuracy of this estimate: Here $t_d$ was found to within $\approx 1$ $\mu$s.

In observing plot 20(a) it is not obvious where the secondary arrival first occurs. By comparison to a wave received at a longer source-receiver distance, it was shown (in the previous section) to arrive at the point marked with the vertical line in 20(a). The plot in 21(d), for $t_d = 84$ samples, also clearly indicates this as the correct arrival time.

These results also show that the simple magnitude parameter $\gamma^m$ in the ray model does not completely describe the actual response in this case. This was discussed in the previous section, where it was briefly mentioned that the first P arrival seemed to have a magnitude
FIGURE 21  Filtered Waves with $t_d$ Varying ($\gamma=1$)
FIGURE 21 (Cont.)

\( t_d = 85 \)

\( t_d = 86 \)

\( t_d = 88 \)
of approximately two-thirds that of the second and later arrivals. Although $\gamma > 1$ invalidates the use of the closed form (3.3.3) to represent the sum in (3.3.2), in reality the sum has a finite number (say N) of terms, and so a closed form may still be found. The result is that the filter is the same as that given here, only it is repeated every $Nt_d$ samples. Figure 22 shows the results of using $\gamma = 1.0$, 1.1, 1.25 and 1.5 when performing the filtering. As $\gamma$ increases, the second P arrival becomes less evident. However, for $\gamma > 1.1$, the later compressional arrivals are not so well removed. This again indicates that the underlying structure is correct, but that the magnitude of the first arrival is smaller than the later ones, which seem to be slightly increasing in magnitude over the time range visible. Using a magnitude factor of the form $(1 + \eta m)\gamma^m$, with $\eta > 0$ would allow more latitude in modelling the "envelope" of the arrivals, and would correspond to a more complex view of the energy distribution among the ray paths than the simple $\gamma^m$ factor. Finally, the plot for $\gamma = 1.5$ shows the first arrival to be in the shape of the source pulse, not its time integral, confirming the results in Section 2 on this question.

The second application uses an actual sonic logging waveform to show that the relative frequencies of the shear and compressional regions can change, under the linear model, depending on the source pulse used. A logging tool using the transmitter pulse shown in Figure 23(a) was used in a well and the first portion of the received
FIGURE 22  Filtered Wave with $\gamma$ Varying ($t_d=84$)
waveform is shown in Figure 24(a). The frequency characteristics of the compressional (low magnitude) and shear region (large magnitude) are shown in Figures 25(a) and (b), respectively. Note that the compressional wave has its energy concentrated at higher frequencies.

Another logging tool operating in the same location and using the transmitter pulse shown in Figure 23(d) obtained waveforms for which this relationship was reversed. That is, the compressional waves were of lower frequency than the waves in the shear region. One explanation is that the different acoustical characteristics of the two logging tools could affect this reversal. Another possibility is that the spectral characteristics of the inputs are causing the difference.

To test this last theory, let the transform of the first waveform be \( B(f) = V(f) \cdot H(f) \), where \( V(f) \) is the transform of the input (Fig. 23(a)), and \( H(f) \) is the system transfer function. Let \( S(f) \) be the transform of the other input (Fig. 23(d)). Then if the first received waveform is filtered with \( \frac{S(f)}{V(f)} \), the result is

\[
C(f) = \frac{S(f)}{V(f)} \cdot B(f) = S(f)H(f),
\]

which gives the output with the second pulse applied to the same system.

The length 512 time domain version of the filter \( \frac{S(f)}{V(f)} \) is shown in Figure 23(b). Figure 23(c) was obtained by convolving \( v(t) \) by the filter (b). The quality of the filter may be assessed by comparing 23(c) and (d). Figure 24(b) is the result of applying this filter to \( b(t) \). Frequency plots of the compressional and shear regions of this
FIGURE 23 Filtering Between Transmitter Pulses
FIGURE 24  Head Waves with Filtered Source Pulse
FIGURE 25 Magnitude Spectra - Before Filtering
filtered wave are given in Figure 26(a) and (b). As these plots show, the P wave energy is at a predominantly lower frequency than the wave in the shear region. This result does not, of course, rule out other reasons for the frequency shift in the actual waveforms.
FIGURE 26  Magnitude Spectra - After Filtering
CHAPTER 4

CONCLUSIONS

The ray model developed here has been shown to produce waveforms which are a good approximation to the compressional head waves propagating in an ideal fluid-filled borehole. The refracted shear waves suffer from large amplitude modal interference, and so evidence of the ability to model shear waves was inconclusive. One way to overcome this would be to remove the modal contribution from the complete waveform, thereby exposing the shear arrivals. This procedure would require great accuracy since the modes can be of much larger magnitude.

The success of the model in the case of the compressional refractions allowed effective filtering techniques to be developed which take advantage of the specific structural characteristics of the wave. These methods can be used to estimate the borehole diameter or formation compressional velocity, the shape of the source pulse, and the time of arrival of the secondary wave.

Extending this model to the case of horizontally or cylindrically layered formations or varying borehole diameter should be considered. The work of Schoenberg et al. [3] would allow a comparison to an accurate result for the case of cylindrical layers, but it is questionable if the results can yet be verified in the other
instances. It is clear that a more detailed analysis of the distribution of energy among the ray paths will be a critical part of any such study.

The wide variation in waveforms from adjacent receivers in actual borehole logs causes even greater difficulty. Even if the basic structure of the model continues to hold with the logging sonde present, it remains to be seen whether the filtering techniques described here are robust enough to overcome less than ideal conditions such as off-axis tools and irregular boreholes.
REFERENCES


