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TIME DOMAIN DESIGN OF RECURSIVE DIGITAL FILTERS WITH PRESPECIFIED POLES

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ABSTRACT

The problem of designing recursive digital filters for a particular case, that is, when some of the filter poles are prespecified, is considered. A modified version of the original Prony method is developed for the design of such filters. The procedure is very general and is applicable for any number of prespecified poles. An error anlysis for the proposed technique is made. Finally, an iterative procedure for least square error approximation is given. Also, some interesting features of Prony's method and their validity for the new scheme are discussed.

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TABLE OF CONTENTS

ABSTRACT

ACKNOWLEDGMENTS

INTRODUCTION 1

CHAPTER 1 - PRONY METHOD AND ITS VARIATION 3

1.1 Original Prony's Method 3

1.2 Padé Approximation 4

1.3 Tuttle's Method 5

1.4 Burrus's Method 7

CHAPTER 2 - ITERATIVE SCHEMES FOR BETTER APPROXIMATION 14

2.1 Steiglitz's Method (Mode 1) 14

2.2 Steiglitz's Iterative Scheme Applied to Burrus's Method 16

2.3 Another Iterative Scheme 20

2.4 Mode-2 Iteration of Steiglitz 23

CHAPTER 3 - SOME OBSERVATIONS 27

3.1 Computation of Autocorrelation Matrix 27

3.2 Error Interpretation in Terms of Correlation 28

3.3 Least Square Error Minimization 29

3.4 To Excite a Particular Mode of the Impulse Response 29

CHAPTER 4 - SOME MODIFICATIONS OF PRONY'S METHOD 30

4.1 To Make All Pole/Zeros Lie on a Circle 30

4.2 Fixing the Poles 32

CHAPTER 5 - PRONY'S METHOD FOR A FIXED NUMBER OF PRESPECIFIED POLES 35

5.1 General Procedure 35

5.2 Decoupling of Known and the Unknown Parts 37

5.3 Z-Domain Interpretation and Error Analysis 39
5.4 Error in Terms of Autocorrelation Function 41
5.5 Iterative Scheme for This Method 42
CONCLUSIONS 44
REFERENCES

INTRODUCTION

Time domain approximation in digital filter design is the problem of finding the filter parameters, so that the impulse response is an acceptable approximation of given desired function. Historically, Prony [1] was the first person to attempt such an approximation, but to a parameter identification problem in gas dynamics rather than filter design. He used a sum of N exponential terms to approximate measured data. Approximation with zero error was obtained when 2N equispaced samples of the actual function were known. Prony's approximation is of the form

$$\tilde{f}(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + \dots + a_n e^{s_n t}.$$

This could be considered as the response of a linear filter. The value of the residues a_i's and the poles s_i's have to be determined.

Later on more sophisticated procedures for the time domain approximation were developed. This includes works of Padé [3], Tuttle [4], Steiglitz [5], Burrus [6] and many more. However, all these techniques are in some way or the other variations of Prony's original method. This is shown in Chapters I and II of this thesis.

Moreover, there is one basic shortcoming of Prony's method, i.e., the inability to match the impulse responses which have some finite D. C. value when time t tends to infinity. This corresponds to having a pole at Z = 1.

Also in fitting the biological or chemical data, it is sometimes desirable to fix some of the pole positions of the approximating function, and find the remaining poles in the best possible manner.

This cannot be achieved by Prony's method.

A variation of Prony's method to allow fixing some of the pole positions is developed and discussed in this thesis. Also the error analysis for this scheme has been made. It is also shown that an iterative scheme similar to Steiglitz [5] can be applied to further reduce the data error. All this is explained in Chapter 5 of this thesis.

In Chapter 3, some interesting features of Prony's method are given. This includes efficient computation of the filter parameters, interpretation of error in terms of correlation, minimization of least square error with a particular input and excitation of a particular mode of the impulse response. Chapter 4 deals with modifications of Prony's method, namely placing of poles on a circle and fixing of the poles which is discussed in detail in Chapter 5.

CHAPTER 1

PRONY METHOD AND ITS VARIATIONS

1.1 Original Prony's Method

The basic problem in this case is to find

$$\{a_i, s_i\} \qquad i = 1,2,\ldots,N$$

in

$$\tilde{f}(t) = \sum_{i=1}^{N} a_i e^{s_i r} \tag{1}$$

such that $\tilde{f}(t)$ passes through 2N given points, which are the samples of the desired response f(t). This means

$$\tilde{f}(iT) = f(iT) = f(i), \; i = 0,1,2,\ldots,2N - 1 \tag{2}$$

where T is the sampling period. One of the methods to solve this problem was given by Prony [1]. The best interpretation of Prony is given by Gillmore [2] and will be discussed here. From (1) and (2) a set of 2N equations, with 2N unknowns can be found, which can be represented in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_n \\ z_1^2 & z_2^2 & z_n^2 \\ & & \\ z_1^{2N-1} & & z_N^{2N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \\ a_N \\ \\ \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ \\ f(2N-1) \end{bmatrix} \tag{3}$$

where $z_i = e^{s_i T}$.

These nonlinear equations can be solved easily if z_i is determined as the root of an algebraic equation,

$$\sigma_N z^N + \sigma_{N-1} z^{N-1} + \ldots + \sigma_1 z + \sigma_0 = 0 \qquad (4)$$

where $\sigma_N = 1$ with no loss of generality.

To obtain the symmetric functions σ_j, each of the first (N + 1) equations in (3) is multiplied by the corresponding σ_i and then summed together to give

$$\sigma_0 f(0) + \sigma_1 f(1) + \ldots + \sigma_n f(n) = 0$$

The same procedure is applied to each consecutive group of (N + 1) equations in a similar manner. The result is a set of the following equations.

$$\begin{bmatrix} f(0) \cdots\cdots f(1) & f(N-1) \\ f(1) \cdots\cdots f(2) & f(N) \\ \vdots & \vdots \\ f(N-1) \cdots\cdots\cdots\cdots & f(2N-2) \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \\ \sigma_{N-1} \end{bmatrix} = \begin{bmatrix} -f(N) \\ -f(N+1) \\ \\ -f(2N-1) \end{bmatrix} \qquad (5)$$

Equations (5) give the solution of the set σ_i's which can be substituted in (4) to get z_i's. Once z_i's are found, (4) can be solved to get a_i's.

1.2 Padé Approximation [3]

In this case a rational function $\tilde{H}(s)$ is found in such a way that its Taylor series expansion about the origin has the same leading terms as those of H(s). H(s) is the function to be approximated.

Let $$H(s) = h_0 + h_1 s + h_2 s^2 + \ldots + h_k s^k + \ldots \tag{6}$$

Then Padé approximant is of the form

$$\tilde{H}(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}}{b_0 + b_1 s + \ldots + b_{n-1} s^{n-1}} \tag{7}$$

The coefficients of Padé approximant depend upon the degree of the numerator and denominator of the rational function.

The number of independent coefficients = (m + n - 1), i.e., there are (m + n - 1) degrees of freedom. The solution is such that $\frac{N(s)}{D(s)}$ has first (m + n - 1) terms identical to H(s). To get the solution, (6) and (7) are equated and (m + n - 1) leading terms are equated. This results in a set of equations similar to (5), which can be solved to find a_i's and b_i's. Weiss [7] has actually shown that Padé and Prony methods are the same. Also Padé approximation is the best least mean square approximation in frequency domain [3].

1.3 Tuttle's Method [4]

This method enables one to make an approximation when only one sample point is known, together with the first (2N - 1) derivatives of the function at that point. The procedure is closely related to that of Prony. Again, approximation is in the form

$$\tilde{f}(t) = \sum_{i=1}^{N} a_i e^{s_i t} \tag{8}$$

No loss of generality occurs if the known point with its first $(2N - 1)$ derivatives is taken at the origin $t = 0$.

Considering the system of equations by writing equation (8) and its $(2N - 1)$ derivatives at $t = 0$, a set of equations similar to (3) of Prony is obtained. This is given in matrix form as

$$\begin{bmatrix} 1 & 1 \cdots\cdots 1 \\ s_1 & s_2 \cdots\cdots s_n \\ \vdots & \\ s_1^{(2N-1)} & s_2^{(2N-1)} \cdots\cdots s_N^{(2N-1)} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{2N-1} \end{bmatrix} \tag{9}$$

where f_i is the i^{th} derivative of $f(t)$ at $t = 0$.

Again, s_i's are determined as the roots of an algebraic equation

$$\sum_{i=0}^{N} \sigma_i s^i = 0, \qquad \sigma_N = 1 \tag{10}$$

Tuttle's basic idea is that the approximation being a sum of N exponentials must be a solution of some linear constant coefficient N^{th} order differential equation, say

$$\sum_{i=0}^{N} \sigma_i f_i(t) = 0, \qquad \sigma_N = 1 \tag{11}$$

(10) is only a characteristic equation of (11), formed to solve for s_i's, i.e., roots of the system. Equations (9) and (10) show the similarity of this method to Prony's method. Once (10) is formed, it is clear that the rest of the procedure is the same as carried out in Prony's method.

1.4 Burrus's Method [6]

This method deals with the problem of the synthesis of recursive digital filter to give a desired impulse response over some finite interval. But this is essentially the same time domain approximation problem considered in previous sections. This can be verified by comparing Burrus's method to the Padé approximation [3].

This paper gives a clear insight into Prony's method, in a very systematic way. Also, the possibility of having a solution when more or less than 2N samples are given, is easily seen with the interpretation and procedure of this paper. Finally, the type of error minimized is discussed.

A brief discussion of Burrus's paper is given in this section. The procedure used is outlined and some of the important results, which will be constantly used in this thesis, are given.

The problem is to find the parameters of a recursive filter,

$$G(z) = \frac{a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-N+1}}{1 + b_1 z^{-1} + \dots + b_{M-1} z^{-M+1}}$$

having a unit impulse response

$$g(k) = g_0, g_1, g_2, \dots g_{k-1} \dots$$

which matches the desired impulse response h(k) over the range of $k = 0,1,\dots k-1$. The solution is obtained by setting the inverse transform of $G(z) = h(k)$, i.e.,

$$\frac{A(a)}{B(z)} = \frac{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{-(N-1)}}{1 + b_1 z^{-1} + \ldots + b_{M-1} z^{-(M-1)}}$$

$$= h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots \tag{12}$$

In terms of convolution (12) becomes

$$\sum_{n-0}^{M-1} b_n h_{i-n} = \begin{cases} a_i & i = 0,1,\ldots,N-1 \\ 0 & i \geq N \end{cases} \tag{13}$$

or in matrix form

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & \\ \vdots & & & \\ h_{N-1} & \cdots & h_0 & \\ \hline h_{N-2} & & & \\ \vdots & & & \\ h_{K-1} & \cdots & \cdots & b_{M-1} \end{bmatrix} \begin{bmatrix} 1 \\ b_2 \\ \vdots \\ b_{M-1} \end{bmatrix} \tag{14}$$

The partitioning of (14) is done, such that

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 \\ \vdots & \ddots & \\ h_{N-2} & & \\ \hline h_N & & \\ \vdots & & \\ h_{K-1} & \cdots & h_{K-M} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{M-1} \end{bmatrix} \tag{15}$$

or

$$\begin{bmatrix} \underline{a} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} H_1 \\ \hline H_2 \end{bmatrix} [\underline{b}] \tag{16}$$

where

$$[\underline{a}] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{M-1} \end{bmatrix}$$

$$H_1 = \begin{bmatrix} h_0 & 0 & 0 \\ \vdots & \ddots & \\ h_{N-1} & \cdots\; h_0 & 0 \end{bmatrix} \qquad H_2 = \begin{bmatrix} h_N \cdots & \\ \vdots & \\ h_{K-1} & h_{K-M} \end{bmatrix}$$

(16) can further be written as

$$\begin{bmatrix} \underline{a} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} H_1 \\ \hline h^1 \,\vdots\, H_3 \end{bmatrix} \begin{bmatrix} 1 \\ b^* \end{bmatrix} \tag{17}$$

where

$$h^1 = \begin{bmatrix} h_N \\ \vdots \\ h_{K-1} \end{bmatrix}, \quad H_3 = \begin{bmatrix} h_{N-1} & \\ \vdots & \\ h_{K-2} & h_{K-M} \end{bmatrix}$$

and

$$b^* = \begin{bmatrix} b_1 \\ b_2 \\ b_{M-1} \end{bmatrix}$$

The lower part of (17) gives the solution of $\underline{b}^*$ as

$$b^* = -H_3^{-1} h^1 \quad \text{and} \quad b_0 = 1 \tag{18}$$

This is trivial solution if $K = M + N - 1$, and when H_3 is non-singular. Other cases, when H_3 is singular, are discussed in the paper. Once the denominator coefficients are determined, $\underline{a}$ is obtained by

$$\underline{a} = [H_1]\underline{b} \tag{19}$$

The above procedure shows that realization of the filter, i.e., determination of the coefficients, is exact as long as $K = M + N - 1$, because no error is involved anywhere in the procedure. K, M and N can have any value. Normally, $N = M - 1$ is taken. Usually, the given value of $K > M + N - 1$. In this case, an approximate procedure as shown next is used.

For approximate techniques, the exact equation (14) is modified to define the error terms:

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \\ \hline \varepsilon_N \\ \vdots \\ \varepsilon_{K-1} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_1 & h_0 & & \\ \vdots & & & \\ h_{N-1} & & & \\ \hline h_N & & & \\ \vdots & & & \\ h_{K-1} & \cdots & \cdots & h_{K-M} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ \vdots \\ b_{M-1} \end{bmatrix} \tag{20}$$

or as before

$$\begin{matrix} \underline{a} \\ \underline{\varepsilon} \end{matrix} = \begin{matrix} H_1 & \\ \hline h^1 & H_3 \end{matrix} \quad \begin{matrix} 1 \\ b^* \end{matrix} \tag{21}$$

Also

$$\varepsilon = H_3 b^* + h^1 \tag{22}$$

The solution to (22) is given by

$$b^* = -[H_3^T H_3]^{-1} H_3^T h^1$$

Again $$\underline{a} = [H_1][b] \tag{23}$$

This solution of b^* by taking the pseudo-inverse minimizes $\varepsilon^T\varepsilon$, i.e., $||\varepsilon||$. In this case, any N values of the impulse response can be matched exactly. Equation (22) will match only first N values exactly. Also, another modified procedure considers the error $\underline{\varepsilon}$ distributed uniformly. Then

$$\begin{matrix} \underline{a} \\ 0 \end{matrix} + [\underline{\varepsilon}] = H\underline{b} \tag{24}$$

where $\underline{\varepsilon}$ is now a (K x 1) vector. The above equation (24) is now written as

$$\begin{bmatrix} \underline{a} \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon \end{bmatrix} = \begin{bmatrix} b_0 & 0 & & 0 \\ b_1 & b_0 & & 0 \\ \vdots & & \ddots & \\ b_{M-1} & & & \\ 0 & \ddots & & \ddots \\ 0 & & b_{M-1} & b_0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \\ h_{K-1} \end{bmatrix} = Bh \tag{25}$$

where B is a [K x K] triangular matrix, and is nonsingular. Therefore, $[\underline{a}]$ can be obtained by using equation (25) instead of (23).

If it is desired to find $[\underline{a}]$, given $\underline{b}$ so that $||\varepsilon||$ is minimized, the solution is

$$a = [\beta_1^T \beta_1]^{-1} \beta_1^T h. \tag{26}$$

Error analysis:

Actual error is given by the difference between the actual and desired output, or

$$e(k) = g(k) - h(k)$$

or in vector form

$$\underline{e} = \underline{g} - \underline{h} \tag{27}$$

where $\underline{e}$, $\underline{g}$ and $\underline{h}$ are [K x 1] vectors. The filter equation in terms of actual output g is

$$\begin{bmatrix} a \\ \hdashline 0 \end{bmatrix} = B\underline{g} \tag{28}$$

and in terms of desired output h and the two measures of error, i.e., ε and e, is

$$\begin{bmatrix} a \\ \hdashline 0 \end{bmatrix} + \varepsilon = B\underline{h} \quad \text{and} \quad \begin{bmatrix} a \\ \hdashline 0 \end{bmatrix} = B[\underline{h} + e] \tag{29}$$

From (29), it is clear that

$$\varepsilon = B\underline{e} \tag{30}$$

Since B is non-singular, an inverse exists. Let $B^{-1} = \beta$. Then,

$$e = \beta\varepsilon \tag{31}$$

Also, according to a theorem proved in [6], the i^{th} column of H_2 can be written as

$$h^i = [c_i(\lambda_i)^{M-i}]\underline{\Lambda i} \tag{32}$$

where

$$\underline{i} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \\ \lambda_i^{K-N-1} \end{bmatrix}$$

and λ_i's are distinct eigen values of the system.

CHAPTER 2

ITERATIVE METHODS FOR BETTER APPROXIMATION

2.1 Steiglitz's Method (Mode 1)

Steiglitz has proposed an iterative scheme for identification of filter parameters when the input to the filter and the desired output is given. Thus, it is slightly different from other methods discussed so far. Let X(z) and H(z) be the z-transforms of the given input and the output. Then

$$G(z) = \frac{A(z)}{B(z)} = \frac{a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)}}{1 + b_1 z^{-1} + \dots + b_{M-1} z^{-(M-1)}} = \frac{H(z)}{X(z)} \tag{33}$$

has to be found. A(z) and B(z) as obtained by Steiglitz are using Kalman's [5] estimate. This solution minimizes

$$\bar{\varepsilon}(z) = X(z)A(z) - H(z)B(z) \tag{34}$$

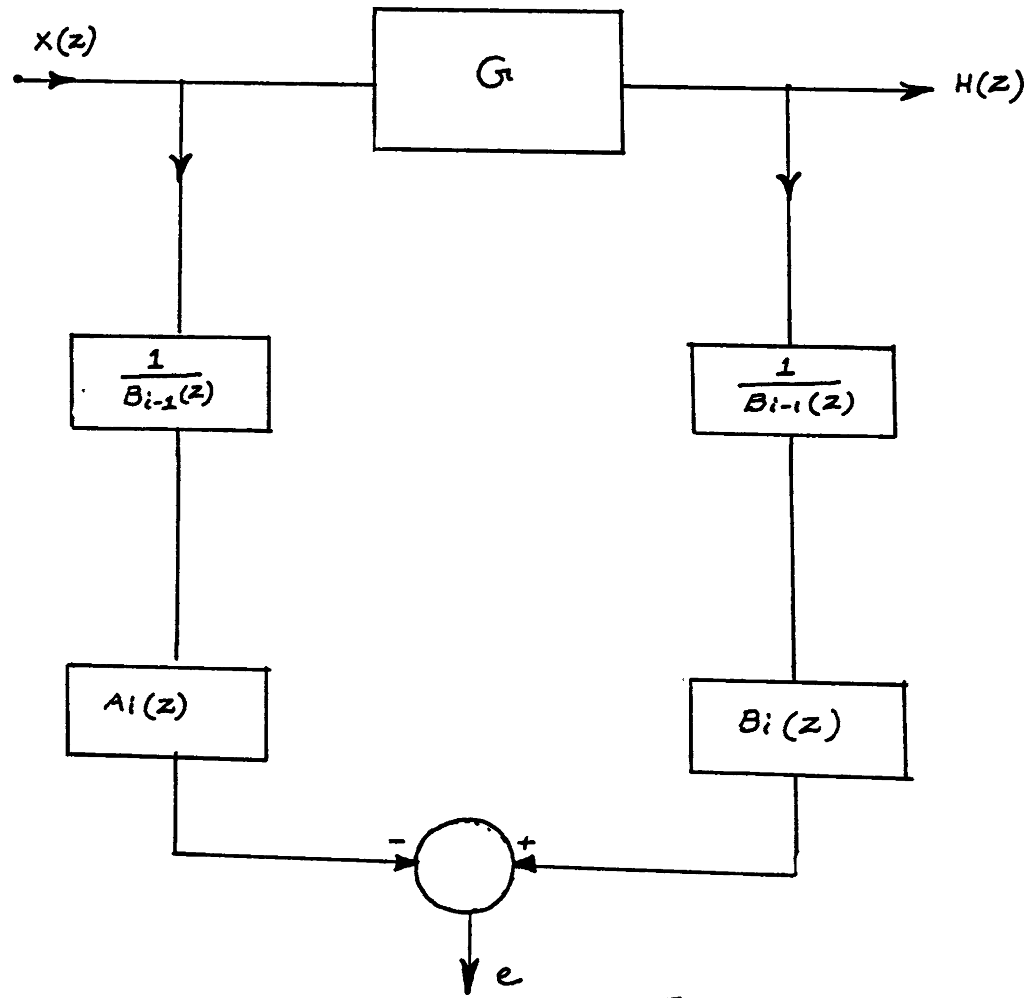
But actual error is

$$\bar{e}(z) = X(z) \cdot \frac{A(z)}{B(z)} - H(z) \tag{35}$$

Comparing (34) and (35)

$$\bar{e}(z) = \frac{\bar{\varepsilon}(z)}{B(z)} = \beta(z)\bar{\varepsilon}(z) \quad \text{where} \quad \beta(z) = B^{-1}(z) \tag{36}$$

Thus, (35) is not minimized if the procedure is carried out as given by Kalman. To minimize $\bar{e}(z)$, Steiglitz's iterative scheme is used.



Graphical Representation of the Iterative Scheme.

Figure 1.

A graphical representation of the iterative scheme is shown in Figure 1. First of all, a pre-estimate of $A_0(z)$ and $B_0(z)$ is made using the following formulae, i.e.,

$$\delta = Q^{-1}c \tag{37}$$

where
$$\delta' = [a_0,\ldots a_{N-1},-b_1,\ldots -b_N] \tag{38}$$

assuming $N = M - 1$

$$Q = \Sigma q_j q_j', \quad c = \Sigma h_j q_j \tag{39}$$

where
$$q_j' = [x_j,\ldots,x_{j-N+1},h_{j-1},\ldots h_{j-n}] \tag{40}$$

Also, $h_k = [H(z)]^{-1}$. As said before, (37) minimizes

$$\varepsilon(z) = X(z) \cdot A_0(z) - H(z)B_0(z) \tag{41}$$

To start the iteration, further both sides of (41) can be multiplied by $\frac{1}{B(z)}$, or in other words, input X(z) and output H(z) can be prefiltered with 1/B(z). Then the above procedure is repeated so as to find next the set of $A_1(z)$ and $B_1(z)$, such that

$$\varepsilon_1(z) = \frac{X(z)}{B_0(z)} \cdot A_1(z) - \frac{H(z)}{B_0(z)} \cdot B_1(z)$$

Again the solution is given by

$$\delta_1 = \hat{Q}^{-1}\hat{c} \tag{41}$$

where $\hat{Q}$ and $\hat{c}$ are as before, but with filtered elements. The same process is repeated at the next iteration to get $A_2(z)$ and $B_2(z)$.

Thus, at i^{th} iteration, $A_i(z)$ and $B_i(z)$ are found, such that

$$\varepsilon_i(z) = \frac{X(z)}{B_{i-1}(z)} A_i(z) - \frac{H(z)}{B_{i-1}(z)} B_i(z) \tag{42}$$

is minimized. This process is repeated till the convergence is obtained, i.e., when

$$B_{i-1}(z) = B_i(z) \tag{43}$$

Then, as seen from equations (42) and (43),

$$\varepsilon_i(z) = X(z) \frac{A_i(z)}{B_i(z)} - H(z) \tag{44}$$

is minimized, which is also that actual error e(z) as defined in equation (35).

Thus, using this iteration, actual error can be minimized. This sort of iteration is called Mode 1 by Steiglitz.

2.2 Steiglitz's Iterative Scheme Applied to Burrus's Method

According to equation (31), actual error

$$e = \beta\varepsilon \quad \text{or} \quad e(z) = \beta(z)\varepsilon(z) \tag{45}$$

where ε is the error minimized by Burrus's method. This is similar to equation (36) in Steiglitz's method. But in that case, the input and the output are given while in this case, the impulse response is given.

Now Burrus's method can be generalized to the case when the input and the output are given. Furthermore, it is seen that the expression for the first estimate in this case comes out to be the same as used by Steiglitz. Also, the iterated version turns out to be the same as Mode 1 of Steiglitz. This is shown next.

As before, it is assumed that first K terms of input $X = x_0, x_1, \ldots$ and the output $H = h_0, h_1, h_2, \ldots$ are known. The problem is again to find the best A(z)/B(z), so that

$$\varepsilon = H(z)B(z) - X(z) \cdot A(z) \tag{46}$$

is minimized. Equation (46) can also be written as

$$h * b \approx x * a \tag{47}$$

writing equation (47) in matrix form for the first K terms

$$\begin{bmatrix} x_0 & 0 & & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & x_1 & x_0 & \\ & & & x_0 \\ x_{K-1} & x_{K-2} & & x_{K-N} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \\ a_{N-1} \end{bmatrix} \approx \begin{bmatrix} h_0 & 0 & & 0 \\ h_1 & h_0 & 0 & 0 \\ h_2 & h_1 & h_0 & \\ & & & h_0 \\ h_{K-1} & h_{K-2} & & \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \\ b_N \end{bmatrix} \tag{48}$$

or
$$\underline{Xa} \approx \underline{h} + \underline{Hb}^*$$

or
$$\underline{Xa} = \underline{h} + \underline{Hb}^* - \varepsilon \tag{49}$$

where
$$H = \begin{bmatrix} 0 & & \\ h_0 & & \\ h_1 & h_0 & \\ & & h_0 \\ h_{K-2} & & h_{K-N-1} \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \\ \varepsilon_{K-1} \end{bmatrix}$$

$\underline{a}' = [a_0, a\ , \ldots, a_{N-1}]$, $\underline{b}^{*\prime} = [b_1, \ldots b_N]$, and $\underline{h}' = [h_0, h_1, \ldots h_{K-1}]$.
Equation (49) can be written as

$$X\underline{a} - H\underline{b}^* = \underline{h} - \underline{\varepsilon} \tag{50}$$

or

$$[X \mid H] \begin{bmatrix} \underline{a} \\ -b^* \end{bmatrix} = \underline{h} - \underline{\varepsilon} \tag{51}$$

or

$$Z\underline{\delta} = \underline{h} - \underline{\varepsilon} \tag{52}$$

where

$$\underline{\delta} = \begin{bmatrix} a_0 \\ a_1 \\ \\ a_{N-1} \\ -b_1 \\ -b_2 \\ \\ -b_N \end{bmatrix} = \begin{vmatrix} \underline{a} \\ -\underline{b}^* \end{vmatrix} \quad \text{and} \quad Z = [X \mid H]$$

Also x_j and h_j are zero for negative j. Normally $K > 2N$. Therefore, matrix [Z] will be a rectangular matrix which may or may not be of full rank. In any case of pseudoinverse can also be found. In that case, the best solution of $\underline{\delta}$, which will minimize $\underline{\varepsilon}$ in equation (52), is given by

$$\delta = [Z^T Z]^{-1} Z^T h \tag{53}$$

Now the columns of matrix Z^T are nothing else but q_j's as given in equation (40). Therefore,

$$[Z^TZ] = [q_0 \quad q_1 \quad \ldots q_{k-1}] \begin{bmatrix} q_0' \\ q_1' \\ \\ q_{k-1}' \end{bmatrix}$$

or

$$Z^TZ = \sum_{j=0}^{K-1} q_j q_j' = Q \tag{54}$$

as in equation (39). Also,

$$Z^T\underline{h} = \sum_{j=0}^{K-1} h_j q_j = \underline{c}$$

in equation (39). Therefore, equation (53) can be written as $\delta = Q^{-1}\underline{c}$, which is the same as equation (37). This proves the equivalence of Steiglitz and Prony.

Next the iterative scheme applied to this case is considered. The procedure is as before. First, an estimate of $\delta_0 = [\underline{a}_0/\underline{b}_0{}^*]$ is obtained and the vector $\underline{b}_0$ is used to find β_0. Then $\underline{\delta}_2$ is obtained which minimizes the norm of $\beta_0\varepsilon_1$ and so on. At the i^{th} state β_i is used to obtain δ_{i+1} which minimizes $||\beta_i\varepsilon_{i+1}||$.

Now according to equation (52), $Z\underline{\delta} = \underline{h} - \underline{c}$. Both sides of the above equation are multiplied by β. Then

$$\beta Z\underline{\delta} = \beta h - \beta\underline{\varepsilon} \tag{55}$$

The solution to equation (55), so that the norm of $\beta\underline{\varepsilon}$ is minimized, is given by

$$\underline{\delta} = [(\beta Z)^T\beta Z](\beta Z)^T\beta\underline{b} = [Z^T\beta^T\beta Z]Z^T\beta^T\beta\underline{h} \tag{56}$$

or in particular

$$\delta_{i+1} = [Z^T\beta_i{}^T\beta_i Z]Z^T\beta_i{}^T\beta_i\underline{h} \tag{57}$$

Also, according to Steiglitz, $\hat{X}(z) = X(z)/B(z) = X(z)\beta(z)$ and $\hat{H}(z) = H(z)/B(z) = H(z)\beta(z)$

or

$$\hat{x} = x * \beta$$
$$\hat{h} = h * \beta \tag{58}$$

which gives prefiltered input and output. Therefore, equation (57) can be written as

$$\delta = [\hat{Z}^T\hat{Z}]^{-1}\hat{Z}^T\underline{\hat{h}} \tag{59}$$

Since $\hat{Z}$ and $\hat{h}$ are as before, but with filtered elements x and h, equation (59) can be written in Steiglitz's notations as $\delta = [\hat{Q}]^{-1}c$, which proves the equivalence of this iterative scheme with Mode 1 of Steiglitz, where for subsequent iteration, $\hat{x}$ and $\hat{h}$ are used in place of x and h for the computation of matrix [Q] and the vector $\underline{c}$.

2.3 Another Iterative Scheme

This iterative scheme can be used on the original Burrus's method, i.e., when the impulse response is given. The calculation of the a's and b's is done in two different steps. First a pre-estimate is obtained as given by equation (23) in Section 1.4 and then the iteration is carried out by finding the optimum $\underline{b}^*$ first and then an optimum $\underline{a}$ to minimize $||e||$. Thus, calculation of a's at every step is avoided. Given $\underline{a}$, to find $\underline{b}$ so as to minimize $e = -\beta\varepsilon$, now

$$\begin{bmatrix} \underline{a} \\ 0 \end{bmatrix} \approx \begin{bmatrix} H\underline{b} \end{bmatrix} \tag{60}$$

Let

$$\begin{bmatrix} a \\ 0 \end{bmatrix} = \underline{A}, \quad b^* = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{M-1} \end{bmatrix}$$

$$H = \begin{bmatrix} h_0 & & \\ h_1 & \ddots & h_0 \\ \vdots & & \\ h_{K-1} & \cdots & h_{K-M} \end{bmatrix}, \quad \text{also} \quad H = [h \vdots H_4], \quad h = \begin{bmatrix} h_0 \\ \vdots \\ h_{K-1} \end{bmatrix} \tag{61}$$

Also

$$\beta A = [\beta_1 \vdots \beta_2] \begin{bmatrix} \beta \underline{a} \\ 0 \end{bmatrix} = \beta_1 \underline{a} \tag{62}$$

Then equation (60) implies that

$$\underline{\varepsilon} + \underline{A} = Hb$$

or

$$\varepsilon = -\underline{A} + Hb$$

or

$$e = -\beta\varepsilon = \beta A - \beta Hb = \beta_1 a - \beta Hb \tag{63}$$

Now let

$$\bar{\bar{H}} = \beta H = \begin{bmatrix} \beta_0 & & \\ \beta_1 & \ddots & \\ \beta_{K-1} & & \beta_0 \end{bmatrix} [H] \tag{64}$$

This is equivalent to prefiltering of H(z) by $1/\beta(z)$. Therefore, let the prefiltered data be $\bar{\bar{H}}$. Then

$$\bar{\bar{H}} = \beta H = \begin{bmatrix} \bar{\bar{h}}_0 & & \\ \bar{\bar{h}}_1 & \bar{\bar{h}}_0 & \\ \vdots & & \bar{\bar{h}}_0 \\ \vdots & & \\ \bar{\bar{h}}_{K-1} & \cdots\cdots & \bar{\bar{h}}_{K-N} \end{bmatrix}$$

Then equation (63) becomes

$$e = \beta_1 \underline{a} - \bar{\bar{H}}b = \beta_1 \underline{a} - \bar{\bar{h}} - \bar{\bar{H}}b^* \tag{65}$$

where $[\bar{\bar{h}}'] = [\bar{\bar{h}}_0, \bar{\bar{h}}_1, \ldots \bar{\bar{h}}_{k-1}]$. The solution to (65), to minimize $||e||$, is given by

$$\underline{b}^* = [\bar{\bar{H}}^T\bar{\bar{H}}]^{-1}\, \bar{\bar{H}}(\beta_1 a - \bar{\bar{h}}) \tag{66}$$

This process of finding $\underline{b}^*$ is continued till the convergence is obtained, i.e., when $b_{i-1}^* = b_i^*$, within some limits.

Next given $\underline{b}^*$ optimum, $\underline{a}$ is obtained to minimize $||e||$. For this equation (60) is considered, i.e.,

$$\underline{h} = \beta \begin{bmatrix} a \\ 0 \end{bmatrix} = [\beta_1 \quad \beta_2] \begin{bmatrix} a \\ 0 \end{bmatrix} = \beta_1 \underline{a}$$

or

$$\underline{a} = (\beta_1^T \beta_1)^{-1}\, \beta_1^T h \tag{67}$$

Thus, $\underline{a}$ can be found. Z-domain interpretation of this scheme is given next.

Now $e(z) = \varepsilon(z)/B_i(z)$ is to be minimized for given $A(z)$, i.e., $\beta_{k+1}(z)$ is found so that $e(z)$ is minimized. Since actual $\beta(z)$ is not known, the last approximation is used. Now $\beta_{i+1}(z)$ is found so that

$$\frac{H(z)}{B_i(z)} B_{i+1}(z) = \frac{A_i(z)}{B_i(z)} + \frac{\varepsilon(z)}{B_i(z)}$$

or
$$\frac{H(z)}{B_i(z)} B_{i+1}(z) = \frac{A_i(z)}{B_i(z)} + e(z)$$

or
$$\bar{\bar{H}}(z)B_{i+1}(z) = A_i(z)\beta_i(z) + e(z) \tag{68}$$

The procedure is again as before repeated till the convergence is obtained. Similarly, for given $\beta_{i+1}(z)$, $A_{i+1}(z)$ can be found to minimize $e(z) = e(z)/B_{i+1}(z)$. In this case,

$$H(z) = \frac{A_{i+1}(z) + \varepsilon(z)}{B_{i+1}(z)}$$

or
$$H(z) = \beta_{i+1}(z)A_{i+1}(z) + \frac{\varepsilon(z)}{B_{i+1}(z)} \tag{69}$$

or
$$\underline{h} = \beta_{i+1} * a_{i+1} + \underline{e}$$

A pseudoinverse will find a_{i+1}, to minimize $||e||$.

2.4 Mode-2 Iteration of Steiglitz

At convergence in the Mode-1 iteration, although actual error $e = \varepsilon$, where $e = \Sigma e_j^2$, $\varepsilon = \Sigma \varepsilon_j^2$. But [gradient e] $\neq$ [gradient ε]. This is because the partial derivatives of $\underline{e}$ and $\underline{\varepsilon}$ are not equal. Now

$$\frac{\partial \varepsilon}{\partial a_i} = \frac{\partial}{\partial a_i}[\hat{H}(z)b(z) - A(z)\hat{X}(z)]$$
$$\frac{\partial \varepsilon}{\partial b_i} = \frac{\partial}{\partial b_i}[\hat{H}(z)B(z) - A(z)\hat{X}(z)] \tag{70}$$

$$\frac{\partial e}{\partial a_i} = \frac{\partial}{\partial a_i}[\hat{H}(z) - \frac{A(z)}{B(z)} \hat{X}(z)]$$

$$\frac{\partial e}{\partial b_i} = \frac{\partial}{\partial b_i}[\hat{H}(z) - \frac{A(z)}{B(z)} \hat{X}(z)] \qquad (71)$$

Comparing (70) and (71), it is easy to see that [gradient e] $\neq$ [gradient ε]. So at convergence, the gradient of error $\underline{e}$ is not equal to zero. Thus, the actual error $\underline{e}$ minimized by Mode-1 iteration is not a true minima. Therefore, Mode 2 iteration is used where at every step of iteration $\delta = \begin{bmatrix} a \\ b^* \end{bmatrix}$ is found in such a way, so that the gradient of $\underline{e}$ is equal to zero. This is shown next. Now

$$e(z) = X(z)\frac{A(z)}{B(z)} - H(z)$$

and

$$\frac{\partial e(z)}{\partial a_i} = \frac{X(z)}{B(z)} z^{-i} = \hat{X}(z) z^{-i}$$

$$\frac{\partial e(z)}{\partial b_i} = \frac{-X(z)}{B^2(z)} N(z) z^{-i} = -\hat{H}(z) z^{-i} \qquad (72)$$

where X(z) and H(z) represent prefiltered input and output. From equations (72),

$$\frac{\partial e_j}{\partial a_i} = \hat{x}_{j-i} \quad \text{and} \quad \frac{\partial e_j}{\partial b_i} = -\hat{h}_{j-i} \qquad (73)$$

Also, error $\underline{e} = \sum_{j}^{K-1} ej^2$, where K is assumed = M + N - 1 and $M = N^{-1}$. The above implies

$$\frac{\partial e}{\partial a_i} = \sum_j 2e_j \frac{\partial e_j}{\partial a_i} = \sum_j 2e_j \hat{X}_{j-1}$$

$$\frac{\partial e}{\partial b_i} = \sum_j 2e_j \frac{\partial e_j}{\partial b_i} = -\sum_j 2e_j \hat{h}_{j-1} \tag{74}$$

Therefore,

$$\text{gradient } \underline{e} \triangleq \begin{bmatrix} \frac{\partial e}{\partial a_i} \\ \\ \frac{\partial e}{\partial a_{n-1}} \\ \frac{\partial e}{\partial b_1} \\ \\ \frac{\partial e}{\partial b_N} \end{bmatrix} = \sum 2e_j \begin{bmatrix} x_j \\ x_{j-1} \\ \\ x_{j-n+1} \\ -h_{j-1} \\ \\ -h_{j-n} \end{bmatrix} \triangleq \sum 2e_j b_j \tag{75}$$

Now the actual e_j in (75) are nonlinear in terms of a's and b's. Therefore, e_j is again approximated by ε_j as in Mode-1 of Steiglitz where

$$\varepsilon_j = \hat{q}_j'\delta - \hat{h}_j \tag{76}$$

where $q_j' = [\hat{x}_j,\ldots,\hat{x}_{j-n+1},\hat{h}_{j-1},\ldots\hat{h}_{j-n}]$. Now the gradient of $\underline{e}$ is to be made zero. From equations (75) and (76), it can be written that

$$\text{gradient } (\underline{c}) = 2\Sigma\, p_j e_j = 2\Sigma(p_j \hat{q}_j \delta - \hat{h}_j p_j) = 0 \tag{77}$$

From equation (77)

$$\delta = (\Sigma p_j \hat{q}_j')^{-1}(\Sigma p_j \hat{h}_j)$$

or

$$\delta = Q^{-1}\hat{\underline{c}} \qquad (78)$$

where $Q = \Sigma p_j q_j'$ and $\hat{\underline{c}} = \Sigma p_j \hat{h}_j$. Now equation (78) is similar to equation (37) in Mode-1 of Steiglitz, but $\underline{Q}$ and $\underline{c}$ are different. The iteration process is the same as in Mode 1; the only difference is that equation (78) is used to calculate $\underline{\delta}$ at every iteration, which will make the gradient of the actual error equal to zero at the convergence.

Since the calculation of $\underline{Q}$ and $\underline{c}$ is more tedious in Mode-2, sometimes the first convergence is obtained using Mode-1 and then switching over to Mode-2 is done to obtain actual minima of the error.

CHAPTER 3

SOME OBSERVATIONS

In this section some interesting results which give more insight into Prony's method are given.

3.1 Computation of the Autocorrelation Matrix

As given in equation (23), b^* is computed as

$$b^* = -[H_3^T H_3]^{-1} H_3^T h^1$$

Now computation of $[H_3^T H_3]$ and $H_3^T h^1$ can be simplified, if an autocorrelation function is defined as

$$R_{hn}(i,j) = \sum_{\ell=0}^{K-M} h_{i+\ell} h_{i+j+\ell} \triangleq R(i,j) \qquad (79)$$

With this notation $H_3^T H_3$ can be written as

$$H_3^T H_3 = Q = \begin{bmatrix} R(M-2,0) & R(M-3,1) & R(0,M-2) \\ R(M-3,1) & R(M-3,0) & R(0,M-3) \\ & & \\ R(0,M-2) & & R(0,0) \end{bmatrix} \qquad (80)$$

Now as is clear from (79) and (80), $R(i + 1, j)$ and $R(i, j + 1)$ can be obtained from $R(i, j)$ by one addition and subtraction. Thus, the matrix Q can be computed using approximately M^2 multiplications, instead of $M^2(K-M)$ in a straightforward manner. Also,

$$H_3^T h^1 = \begin{bmatrix} R(M-2,1) \\ R(M-3,2) \\ \\ R(0,M-1) \end{bmatrix} \triangleq c \tag{81}$$

3.2 Error Interpretation in Terms of Correlation

According to equation (22)

$$\varepsilon = H_3 b^* + h^1$$

For the best solution of b^*, should be orthogonal to all columns of H_3. If a crosscorrelation function is defined as

$$R_{h\varepsilon}(n) = \sum_{j=0}^{K-M} \varepsilon(j)h(j+n), \quad \text{for } N = M - 1 \tag{82}$$

where $R_{h\varepsilon}(n)$ denotes correlation between ε and columns of H_3. For the best solution of b^*

$$R_{h\varepsilon}(n) = 0, \quad \text{for } n = 0,1,\ldots,M-2 \tag{83}$$

Also $$||\varepsilon||^2 = \sum_{i=0}^{K-M} \varepsilon_i^2 = \underline{\varepsilon}^T\underline{\varepsilon}$$

$$= \varepsilon^T(H_3 b^* + h^1) = \varepsilon^T H_3 b + \varepsilon^T h^1$$

$$= 0 + \varepsilon^T h^1 = R_{h\varepsilon}(M-1) \tag{84}$$

This gives the norm of error as a correlation function.

3.3 Least Square Error Minimization

Also $$\begin{bmatrix} \underline{a} \\ \underline{0} \end{bmatrix} + [\underline{\varepsilon}] = [H][\underline{b}]$$

If $\hat{H}(z)$ is the actual impulse response, then

$$\hat{H}(z) = \frac{A(z) + \varepsilon(z)}{B(z)} = H(z) + \frac{\varepsilon(z)}{B(z)} = H(z) + e(z) \tag{85}$$

where H(z) is the approximated impulse response. Prony's method minimizes $\varepsilon(z)$, while the least square method will minimize e(z).

If X(z) is the input and Y(z) is the desired output to the system, then

$$Y(z) = X(z)\hat{H}(z) = X(z)H(z) + X(z)\,\frac{\varepsilon(z)}{B(z)} \tag{86}$$

Now if the least square error at the output is to be minimized, then if X(z) = B(z), Prony's method will minimize the least square error. This is obvious from equation (86).

3.4 To Excite a Particular Mode of the Impulse Response

Let $$H(z) = \sum_{i=0}^{P} H_i(z) = \sum_{i=0}^{P} \frac{A_i(z)}{B_i(z)}$$

Let $H_k(z)$ be the desired mode to be excited. Now if X(z) = $B_1(z)B_2(z) \cdot B_{K-1}(z) \cdot B_{K+1}(z) \ldots B_p(z)$, then

$$Y(z) = \hat{A}_1(z) + \hat{A}_2(z) + \ldots \frac{\hat{A}_K(z)}{B_K(z)} + \ldots + \hat{A}_p(z)$$

After a finite interval of time, the output will be due to $A_K(z)/B_K(z)$ only.

CHAPTER 4

SOME MODIFICATIONS OF PRONY'S METHOD

4.1 To Make All Poles/Zeros Lie on a Circle

If all the poles of a system are inside the unit circle, then the system is stable. Using this technique, all the poles can be made to lie on a circle inside the unit circle. This is done by putting restraints on Prony's method in such a way that the denominator of the impulse response turns out to be a symmetric polynomial, which is necessary but not sufficient condition for its roots to lie on a circle. For this the general procedure has to be modified slightly. This is shown next by considering an example. Let

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}$$

$$= h_0 + h_1 z^{-1} + \ldots + h_K z^{-K}$$

where h_i are given and a_i's and b_i's have to be found. For roots to lie on the unit circle, one should have $b_4 = b_0 = 1$ and $b_1 = b_3$. For roots to lie on a circle of radius r, where r is less than 1, the following should be satisfied, i.e.,

$$b_4 = b_0 r^{-4} = r^{-4}$$

$$b_3 r^{-3} = b_1 r^{-1}$$

Here only former case is considered which can always be generalized.

For the above case, the following has to be solved, i.e.,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \underline{\varepsilon} \end{bmatrix} = \begin{bmatrix} h_0 & & & 0 \\ h_1 & h_0 & & \\ & & & \\ h_3 & h_2 & h_1 & h_0 \\ h_4 & & & \\ & & & \\ h_K & & & h_{K-5} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ b_1 \\ 1 \end{bmatrix} \tag{87}$$

Due to restrictions imposed, only b_1 and b_2 are unknown. Equation (87) can be written as

$$\begin{bmatrix} \underline{a} \\ \hline \underline{\varepsilon} \end{bmatrix} = \begin{bmatrix} & & H_1 & & \\ \hline h^1 & h^2 & h^3 & h^4 & h^5 \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ b_1 \\ 1 \end{bmatrix} \tag{88}$$

where h^1, h^2, h^3, h^4, h^5 are columns of the bottom part of partitioned matrix. The Solution of b_1 and b_2 is obtained by minimizing $\underline{\varepsilon}$ as before, i.e., by solving

$$h^1 + h^2 b_1 + h^3 b_2 + h^4 b_1 + h^5 = 0$$

or

$$(h^1 + h^5) + (h^2 + h^4) b_1 + h^3 b_2 = 0$$

or

$$\begin{bmatrix} (h^1 + h^5) & (h^2 + h^4) & h^3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \tag{89}$$

Example [8]:

This example is about fitting the logistic curve of growth. This curve is widely used in biological investigation. The equation of logistic curve may be written as

$$y_t = \frac{k}{1 + b\, e^{-at}} \tag{90}$$

where a, b, and k are constants or parameters to be estimated from observed data (t_i, y_i) for $i = 1,2,\ldots N$.

This problem has been solved in many different ways. But it can also be solved using Prony's method and fixing one of the poles. This is done by considering

$$y_t' = \frac{1}{y_r} = \frac{1}{k} + \frac{b\, e^{-at}}{k} \tag{91}$$

This implies that one pole of $y_t'(z)$ is at $Z = 1$. Taking the Z transform of equation (91)

$$y_t'(z) = \frac{\left(\frac{1 + b}{k}\right) - \left(\frac{e^{-a} + b}{k}\right)Z^{-1}}{1 - (e^{-a} + 1)Z^{-1} + e^{-a}Z^{-2}} \tag{92}$$

or

$$y_t'(z) = \frac{a_0 + a_1 Z^{-1}}{1 + b_1 Z^{-1} + b_2 Z^{-2}} \tag{93}$$

where $a_0 = (1 + b/k)$, $a_1 = -(e^{-a} + b/k)$, $b_1 = (e^{-9} + 1)$ and $b_2 = e^{-a}$. Now a_0, a_1, b_1 and b_2 are the parameters to be determined. This can be done using Prony's method. But as is clear from (92) and (93), the solution must be such that $-b_1 = b_2 + 1$. So the fact that one pole is at Z = 1 is transformed to a relationship between denominator coefficient.

So in this case

$$\begin{bmatrix} a_0 \\ a_1 \\ \underline{\varepsilon} \end{bmatrix} = \begin{bmatrix} y_0' & & \\ y_1' & y_0' & \\ & & \\ y_k' & & y_{k-2} \end{bmatrix} \begin{bmatrix} 1 \\ -b_2-1 \\ b_2 \end{bmatrix} \equiv \begin{bmatrix} \frac{H_1}{H_2} \end{bmatrix} \begin{bmatrix} 1 \\ b^* \end{bmatrix} \qquad (94)$$

Normally the solution is given by $H_2 b^* = 0$. Let h^1, h^2 and h^3 be columns of H_2. Then from equation (94)

$$h^1 + h^2(-b_2 - 1) + h^3 b_2 = 0$$

or

$$[h^3 - h^2]b_2 = [h^1 - h^2]$$

$$b_2 = \{[h^3 - h^2]^T[h^3 - h^2]\}^{-1}[h^3 - h^2]^T[h^1 - h^2] \qquad (95)$$

This problem was solved taking United States population as the data and using equation (95). The same problem can be generalized for any number of poles and is discussed in detail in the next chapter.

CHAPTER 5

PRONY'S METHOD FOR A FIXED NUMBER OF PRESPECIFIED POLES

5.1 General Procedure

Out of, say, (M - 1) poles, let P poles be specified and the remaining M - 1 - P = L poles are to be found. Again for simplicity, it is assumed that N = M - 1. Let the given poles be the roots of the known polynomial

$$\hat{B}(z) = 1 + \hat{b}_1 Z^{-1} + \ldots + \hat{b}_p Z^{-p} \tag{96}$$

Then let the remaining poles to be found be the roots of the unknown polynomial

$$\tilde{B}(z) = 1 + \tilde{b}_1 Z^{-1} + \ldots + \tilde{b}_L Z^{-L} \tag{97}$$

Essentially, now $b_1, \ldots b_L$ have to be found. The overall polynomial, which is the denominator of the rational function, i.e., approximated impulse response, is given by

$$B(z) = \hat{B}(z) \cdot \tilde{B}(z) \tag{98}$$

or in terms of convolution, equation (95) can be written as

$$\underline{b} = \underline{\hat{b}} * \underline{\tilde{b}} \tag{99}$$

or

$$\begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{M-1} \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ \hat{b}_1 & \ddots & \\ & \ddots & 1 \\ \hat{b}_p & & \hat{b}_1 \\ & \ddots & \\ 0 & & \hat{b}_p \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_L \end{bmatrix}$$

or $$b_i = \sum_{j=0}^{i} b_j b_{i-j} \tag{100}$$

Now to get $\underline{b}$, usually

$$H_2\underline{b} \approx 0 \tag{101}$$

is solved as in equation (22) in Section 1.4. But here

$$\underline{b} = \hat{b} * \tilde{b} \tag{102}$$

From equations (101) and (102), it is implied that now

$$H_2\hat{B}\tilde{b} = 0 \tag{103}$$

has to be solved, where

$$\hat{B} = \begin{bmatrix} 1 & & 0 \\ \hat{b}_1 & & 1 \\ & & \hat{b}_1 \\ \hat{b}_p & & \\ & 0 & \hat{b}_p \end{bmatrix}$$

The procedure to find $\tilde{b}$ is obviously as before. Instead of the matrix H_2, $H_2\hat{B}$ is considered, i.e.,

$$H_2\hat{B} = \hat{H}_2 = \left[\hat{h}^1 \,\vdots\, \hat{h}^2 \,\vdots\, \cdots \,\vdots\, \hat{h}^\ell\right] \tag{104}$$

where $\hat{h}^1, \ldots \hat{h}^\ell$, etc., are columns of H_2. From equations (101), (102) and (103) it implies that $\underline{\tilde{b}}$ can be found by solving

$$\hat{H}_2\tilde{b} \approx 0 \tag{105}$$

or as before

$$\hat{h}^1 + \hat{H}_3\tilde{b} = \varepsilon \tag{106}$$

where

$$\hat{H}_2 = \left[h^1 \,\vdots\, H_3 \right]$$

From equation (106), $\tilde{b}^*$, which will minimize the norm of ε, is given by

$$\tilde{b}^* = -[\hat{H}_3^T\hat{H}_3]^{-1}\hat{H}_3^T\hat{h}^1 \tag{107}$$

5.2 Decoupling of Known and the Unknown Part

When the pole positions assumed are not the right ones, then the error will be in general more than obtained by unrestricted Prony's method. But preferably, the method should be such that if assumed pole positions are accurate, then the restrictions should not contribute to any additional error. The method described in Section 5.1 satisfies this requirement. This method isolates the known and the unknown parts. This is shown here in this section. Now

$$H_2 = \left[h^1 \,\vdots\, h^2 \,\vdots\, \cdots \,\vdots\, h^M \right]$$

Then

$$\begin{aligned}\hat{H}_2 &= H_2\hat{B} \\ &= \left[h^1 + \hat{b}_1 h^2 + \ldots + \hat{b}_p h^p \,\vdots\, h^2 + b_1\hat{h}_3 + \ldots + \hat{b}_p h^{p+1} \,\vdots\, \cdots\cdots \right]\end{aligned} \tag{108}$$

The first column of $\hat{H}_2$ in equation (108), i.e., $\bar{h}^1$

$$\bar{h}^1 = h^1 + \hat{b}_1 h^2 + \ldots + \hat{b}_p h^p \tag{109}$$

Now let $h(k) = \tilde{h}(k) + \hat{h}(k)$, where $\tilde{h}(k)$ is the part of the impulse response due to unknown poles and $\hat{h}(k)$ is due to known poles.

Then from equation (109)

$$\begin{aligned} \bar{h}^1 &= (\hat{h}^1 + \tilde{h}^1) + \hat{b}_1(\hat{h}^2 + \tilde{h}^2) + \ldots + \hat{b}_p(\hat{h}^p + \tilde{h}^p) \\ &= (\hat{h}^1 + \hat{b}_1 \hat{h}^2 + \ldots + \hat{b}_p \hat{h}^p) + (\tilde{h}^1 + \hat{b}_1 \tilde{h}_2 + \ldots + \hat{b}_p \tilde{h}^p) \end{aligned} \tag{110}$$

Now according to equation (32) in Section 1.4

$$\hat{h}^k = \sum_{i=0}^{p-1} (c_i \lambda_i^{\,p-k}) \Lambda_i \tag{111}$$

where λ_i's are the P assumed eigenvalues of the system. From equations (110) and (111)

$$\begin{aligned} \bar{h}^1 = {} & (\tilde{h}_1 + \hat{b}_1 \tilde{h}_2 + \ldots + \hat{b}_p \tilde{h}_p) + \\ & \sum_{i=0}^{p-1} c_i (\lambda_i^{\,p} + \hat{b}_1 \,_i^{\,p-1} + \ldots + \hat{b}_p) \end{aligned} \tag{112}$$

Now the second part of (112), which involves contributions from the known poles, is given by

$$\begin{aligned} X &= \sum_{i=0}^{p-1} c_i (\lambda_i^{\,p} + \hat{b}_1 \lambda_i^{\,p-1} + \ldots + \hat{b}_p) \\ &= \sum_{i=0}^{p-1} c_i \lambda_i^{\,p} (1 + \hat{b}_1 \,_i^{-1} + \ldots + \hat{b}_p \,_i^{-p}) \end{aligned} \tag{113}$$

Now if the assumed eigenvalues λ_i, $i = 1, 2, \ldots p$ are also exact eigenvalues of the system, then equation (113) is equal to zero. This implies that effect of known poles is totally removed in finding the other unknown poles, but only if the assumption is exact. Otherwise, some additional error contribution may be there.

5.3 Z-domain Interpretation and Error Analysis

Again consider

$$H(z)B(z) = A(z) + \varepsilon(z) \tag{114}$$

as in equation (85) of Section 3.3. The solution is such that $\varepsilon(z)$ is minimized, where $\varepsilon(z)$ is the error in the coefficients. Now

$$B(z) = \hat{B}(z)\tilde{B}(z)$$

Therefore, from equation (114)

$$H(z)\hat{B}(z)\tilde{B}(z) = A(z) + \varepsilon(z) \tag{115}$$

$H(z)\hat{B}(z)$ can be considered together. This represents the convolution, $h * \hat{b}$, and will be of length K + P. K and P are as given before in previous chapters. Let $\bar{H}(z)$ represent first K terms of $H(z)\hat{B}(z)$. Then equation (115) becomes

$$H(z)\tilde{B}(z) = A(z) + \varepsilon(z) \tag{116}$$

Comparing equations (114) and (116) shows that the procedure for solving $B(z)$ in (114) is exactly similar to that required for solving $\tilde{B}(z)$ in equation (116). In both cases, obviously, the pseudoinverse is taken and the error $\varepsilon(z)$ is minimized. Thus, this method will minimize the same error $\varepsilon(z)$, as in the original Prony method. Thus, this method is optimum in Prony's sense, i.e., the norm of the error in coefficient a's is minimized. Although error obtained by this procedure will be more in this case because of the imposed constraints. But as said before, if the assumptions are correct, no additional error contribution is made. This can also be shown using Z-domain interpretation. Since h(k) can be written as

$$h(k) = \hat{h}(k) + \tilde{h}(k)$$

Then H(z) can be written as

$$H(z) = \hat{H}(z) = \tilde{H}(z) = \frac{\hat{A}(z)}{\hat{B}(z)} + \tilde{H}(z) \tag{117}$$

where $\hat{A}(z)$ is a polynomial of degree P-1. Now $\bar{H}(z) = H(z)\hat{B}(z)$. Using equation (117)

$$\bar{H}(z) = \hat{H}(z)\hat{B}(z) + \tilde{H}(z)\hat{B}(z) = \hat{A}(z) + \tilde{H}(z)\hat{B}(z) \tag{118}$$

Thus, only first (P-1) terms of $\bar{H}(z)$ and hence $\bar{h}(k)$ contain contributions from $\hat{h}(k)$. Equation (116) can be written in matrix form as

$$\begin{bmatrix} a_0 \\ a_1 \\ \\ a_{M-2} \\ a_{M-1} \\ \\ a_{K-1} \end{bmatrix} = \begin{bmatrix} \bar{h}_0 & \\ \bar{h}_1 & \\ \bar{\bar{h}}_{M-2} & \\ \bar{\bar{h}}_{M-1} & h_p \\ \bar{h}_{K-1} & \bar{h}_{K+P-M} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \\ b_\ell \end{bmatrix} \tag{119}$$

Note that P + L = M - 1

Also, $\bar{h}(k)$ can be considered to be prefiltered data. From equation (119), it can be seen that the computation of $\tilde{b}$ requires values of $\bar{h}$ frop $\bar{h}_p$ onwards, which do not have any contribution from $\hat{H}(z)$. Thus, for the computation of $\tilde{b}$ with $H(z) = \tilde{H}(z)\hat{B}(z)$, that part of the impulse response which contains the response due to $\hat{B}(z)$ gets completely eliminated in the process. It should be noted that here in equation (118) it is assumed that $\hat{H}(z)$ is really equal to $\hat{A}(z)/\hat{B}(z)$. This result is the same as obtained in Section 5.2.

Again in this case optimum $\tilde{b}$ is given by

$$\tilde{b}^* = [\bar{H}_3^T\bar{H}_3]^{-1}\bar{H}_3^T\bar{h}^1 \tag{120}$$

where

$$\bar{H}_3^T = \begin{bmatrix} \bar{h}_{M-2} \cdots \bar{h}_p \\ \vdots \\ \bar{h}_{K-2} \cdots \bar{h}_{K+P-M} \end{bmatrix} \quad \text{and} \quad \bar{h}^1 = \begin{bmatrix} \bar{h}_{M-1} \\ \vdots \\ \bar{h}_{K-1} \end{bmatrix} \tag{121}$$

5.4 <u>Error in Terms of the Autocorrelation Function</u>

Here in this section results similar to those in Section 3.2 are derived. For this case also the optimum $\tilde{b}$ would be such that the error ε is orthogonal to the columns of matrix $\bar{H}_3$, where $\bar{H}_3$ is as shown in equation (121).

Again if the autocorrelation function, showing autocorrelation between the error ε and the columns of $\bar{H}_3$, is defined as

$$R_{\bar{h}\varepsilon}(n) = \sum_{j=0}^{K-M} \varepsilon(j)\bar{h}(j + n) \tag{122}$$

Then $R_{\bar{h}\varepsilon}(n) = 0$ for $n = p, p + 1, \ldots, M - 2$. Again

$$||\varepsilon||^2 = \sum_{i=0}^{K-M} \varepsilon_i^{\,2} = \varepsilon^T\varepsilon = \varepsilon^T[\bar{H}_3\tilde{b}^* + \bar{h}^1] = R_{\bar{h}\varepsilon}(M - 1) \tag{123}$$

This result is identical to the result derived in equation (84), Section 3.2. The new error is again ε, i.e., error when some of the poles are fixed. Now $||\varepsilon||^2$ would certainly be more than the $||\varepsilon||^2$ for the straightforward Prony method of the same order. This is true because some degree of freedom is lost.

5.5 Iterative Scheme for This Method

Again as in Section 2.3, $e(z) = \varepsilon(z)/B(z)$ is to be minimized. For this iterative procedure similar to that used in Section 2.3 is used. First consider to minimize $e(z) = \varepsilon(z)/B_i(z)$ for a given $A_i(z)$. Again as in equation (68) in Section 2.3

$$\frac{H(z)}{B_i(z)}\, B_{i+1}(z) = \frac{A_i(z)}{B_i(z)} + \frac{\varepsilon(z)}{B_i(z)} \tag{124}$$

Now in this case

$$B_i(z) = \hat{B}_i(z)\tilde{B}_i(z)$$
$$B_{i+1}(z) = \hat{B}_{i+1}(z)\tilde{B}_{i+1}(z)$$
$$\hat{B}_i(z) = \hat{B}_{i+1}(z) \tag{125}$$

Using equation (125), equation (124) can be written as

$$\frac{H(z)}{\tilde{B}_i(z)}\, \tilde{B}_{i+1}(z) = \frac{A_i(z)}{B_i(z)} + e(z) \tag{126}$$

or

$$(H * \tilde{\beta}_i) * \tilde{b}_{i+1} = a_i * \beta_i + e(z) \tag{127}$$

where $\beta_i(z) = 1/B_i(z)$ and $\beta_i = B^{-1}$. Thus, equation (127) can be formulated as

$$\beta_i \underline{a} + \underline{e} = (\tilde{\beta}_i h)\tilde{b}_{i+1} \tag{128}$$

Thus, by taking pseudoinverse optimum value of $\tilde{b}_{i+1}$ can be found to minimize $||e||$. Using equation (128) for given b_{i+1}, a_{i+1} can be found. Also a weighted function $W(z)$ can be used to minimize $||W(z)e(z)||$. For this both sides of equation (126) should be multiplied by $W(z)$.

The procedure for iteration is again as before. First an initial estimate of $A(z)$ and $B(z)$ is made. Then for the same $A(z)$, the next set of $B_1(z)$ is found, so as to minimize the norm of $\varepsilon(z)/B(z)$. Then $A_1(z)$ is found using equation (128). The process is repeated till at the i^{th} iteration $B_i(z) = B_{i+1}(z)$. As said in Section 2.1, then actual $e(z)$ is minimized.

CONCLUSIONS

In this thesis, the problem of the design of recursive digital filters with a fixed number of prespecified poles is considered. The method proposed is a modified version of Prony's method, but the results obtained are identical in both cases. The expressions for finding the filter coefficients are the same in both cases, except that in the modified case prefiltered data is used. Just as in Prony's original method, only linear equations have to be solved.

Again the type of error minimized is the error in coefficients. The relationship between this error and the actual error in the impulse response has been found and is the same as for Prony's method. So this method is optimum in Prony's sense, which also minimizes the error in the filter coefficients. This method is also optimum in the sense that when assumption made about the poles is correct, no additional error contribution is made and the known and the unknown parts are decoupled.

In the second chapter of this thesis, Steiglitz's iterative scheme applied to Kalman's method for the estimation of filter parameters is given. Also, it is shown that Steiglitz's method is the same as Burrus's method, when instead of the impulse response, the input and the output of the filter are given. In this chapter an iterative scheme for Burrus's method is given. The same iterative procedure is applied to this new modified Prony method. The results are again similar to the original iterated Prony method. The computations from one iteration to another are identical except for the prefiltering of the data at each step. For this efficient use of digital computer can be made.

Furthermore, some interesting interpretations are given. But in most of the cases, the proposed modified method and the original Prony method give similar results.

An example of the modified scheme is given in Section 4.2.

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