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THE EFFECT OF COHERENT SIGNALS ON THE CAPABILITY OF ARRAY PROCESSING ALGORITHMS TO RESOLVE SOURCE BEARINGS

by

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Abstract

In the ocean environment, where multipath propagation of sound
is common, the acoustic signals incident on a passive sonar array are
often coherent. Cox has demonstrated that the capability of minimum
energy adaptive beamforming to resolve the source bearings of
incoherent signals is superior to that of classical beamforming.
Seligson observed that when the signal wavefronts deviate from their
assumed shape, the adaptive beamformer can be inferior to the classi¬
cal beamformer in this regard.

We study the effect of coherent signals on the capability of
classical, adaptive, and linear predictive array processing algo¬
rithms to resolve source bearings. For a linear array of equally
spaced sensors, we demonstrate the superior resolution capability of
the linear predictive algorithm, and the significant effect of signal
coherence on all three processing algorithms. We demonstrate the
value of utilizing prediction elements in the center of the array to
resolve closely-spaced signal bearings. Finally, we investigate the
sensitivity of the processing algorithms to imperfections introduced
into the correlation matrix by finite averaging, and establish the
tradeoff that exists between this sensitivity and the capability to resolve closely-spaced signal bearings.
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CHAPTER 1

INTRODUCTION

The determination of the bearings of distant sources of acoustic energy in a noisy ocean environment is a primary objective of a passive sonar system. A group of acoustic sensors arranged in a known spatial pattern (an array) is deployed to record the acoustic field. By processing the data in various ways we can evaluate the bearings from which acoustic energy is incident on an array. A useful measure of the performance of an array and the processing algorithm is the signal to noise ratio required to resolve closely-spaced sources.

We model the acoustic environment as a discrete-time random field which is composed of stationary noise and sustained random signals. We assume that the $m^{th}$ sensor is located at position $z_m$. See figure 1.1. $x_m(n)$ represents the output of the $m^{th}$ sensor in an array of $M$ sensors at discrete-time index $n$, and $X_m(f)$ the corresponding discrete time Fourier transform defined as

$$X_m(f) = \frac{1}{N} \sum_{n=0}^{N-1} x_m(n) \cdot e^{-j2\pi fn}$$

$X(f)$ is a complex column vector of dimension $M$ with components $X_m(f)$.

There are two fundamentally different ways of deriving algorithms to evaluate the bearings from which acoustic energy is incident on an array. The first is a beamforming approach which leads to both classical and high resolution adaptive beamforming.
Figure 1.1

Plane wave impinging on an array of M sensors.

The bearing of incidence is along $k_1$, whose length has units of samples/distance. Sensor $m$ is located at position $k_m$. The signal seen at the origin of the array is denoted by $s(n)$. When there is no ambient noise, the output of sensor $m$ is a delayed version of this signal.
M sensors: \( m = 0, 1, \ldots, M-1 \)

No noise case: \( x_m(n) = s_1(n - k_1 z_m) \)
techniques. The second is a linear predictive approach.

Beamforming:

We form a quantity $y(n)$ called the beam as follows:

$$y(n,a_m\tau_m) = \sum_{m=0}^{M-1} a_m x(n-\tau_m). \quad (1.1)$$

The beam is a linear combination of the sensor outputs $x_m$, each of which has been delayed by $\tau_m$ and weighted by $a_m$. The parameters $\{a_m, \tau_m\}$ characterize the beam. Evaluating the discrete Fourier transform,

$$Y(f,a_m\tau_m) = \sum_{m=0}^{M-1} a_m e^{-j2\pi fc_m} X_m(f),$$

we can interpret beamforming as a type of multidimensional spectral analysis. In this context the delays do not have to be integral values. To simplify notation, define the steering vector $A$ to have components $A_m = a_m e^{j2\pi fc_m}$. Then

$$Y(f,\Delta) = \Delta^t X$$

where $\Delta'$ denotes the conjugate transpose of $\Delta$.

By Parseval's theorem, the energy in the beam is

$$E \left( \sum_{n=0}^{N-1} y^2(n) \right) = E \left( \frac{1}{2} \int_{-1/2}^{1/2} Y(f,\Delta)^* Y(f,\Delta) df \right)$$

where $E()$ denotes expected value. The quantity
\[ P_A(f) = E[Y(f, A)Y^*(f, A)] \]

is therefore the beam energy density, with \( P_A(f)df \) being the energy in the frequency band \([f, f+df)\). It is convenient to suppress the variable \( f \) and refer to \( P_A \) simply as the energy in the beam. Then

\[ P_A = E[Y(A)Y^*(A)] = E[A' \overline{XX} A] = A' \Sigma A \]

where \( \Sigma \) is the cross spectral correlation matrix of the acoustic field with elements

\[ \Sigma_{ij}(f) = E[X_i(f)X_j^*(f)] \]

\( \Sigma \) is positive definite and Hermitian.

Assume for the moment that \( x_m(n) = s_1(n-k_1 \cdot z_m) \), i.e. that a single plane wave propagating with bearing \( k_1 \) impinges on the array (see figure 1.1). In this case \( X = \mathcal{F}(f)S_1(f) \) where \( \mathcal{F}(f) \) is the discrete Fourier transform of the signal \( s_1(n) \), and \( S_1 \) is the signal direction vector with components \( S_{1m} = e^{-j2\pi f k_1 \cdot z_m} \). Its norm-squared is fixed at \( M \).

The classical, or Bartlett, approach to beamforming restricts \( A \) to be of the form \( A_m = e^{-j2\pi f k \cdot z_m} \). In other words, the beam is formed by delaying and adding the outputs of the sensors. Note that the norm-squared of \( A \) is fixed at \( M \). A beam characterized by a steering vector of this form is said to be "steered" on bearing \( k \). The beam-pattern is a plot of beam energy as a function of bearing \( k \). The beam energy will exhibit a global maximum when the beam is steered on target (along the bearing of incidence \( k = k_1 \)) because \( A \)
and \( S_1 \) are parallel. The on-target beam energy is

\[
P_{\text{BART}} = E[\mathbf{A}' \mathbf{XX}' \mathbf{A}] = M^2 E[\Phi(f)\Phi^*(f)]
\]

which is \( M^2 \) times the energy density of the propagating signal \( s(n) \) at frequency \( f \). When the beam is steered off target \( (k \neq k_1) \) the beam energy is not necessarily zero because \( \mathbf{A} \) may not be orthogonal to \( S_1 \). In addition to the global maximum on target, the beam-pattern will have local maxima off target; these maxima are known as sidelobes. Figure 1.2 shows the Bartlett beam-pattern due to a broadside signal impinging on a linear array of ten equally spaced sensors.

To suppress the level of these sidelobes and obviate the on-target peak to a greater degree, Capon[1] developed the technique of minimum energy adaptive beamforming. As with classical beamforming, we are interested in the beam energy. However, the steering vector is no longer restricted to be of the above form. We define a direction of look vector \( \mathbf{W} \) with components \( W_m = e^{-j2\pi f k \cdot z_m} \) and then seek to minimize the energy in the beam subject to the constraint \( \mathbf{A}' \mathbf{W} = 1 \). The constraint forces plane wave energy incident on the array with bearing \( k \) to be passed with unit gain. Minimization of the beam energy then has the effect of minimizing the energy due to sources and/or noise propagating from other bearings. The ME beam

---

1 This beamforming algorithm is often referred to as the maximum likelihood method in the literature. Actually, the beam does not maximize a likelihood function, so the customary terminology is misleading.
Figure 1.2
Bartlett beam-pattern for a linear array of equally spaced elements.

The array consists of 10 elements with a spacing of $\lambda/2$; the signal-to-noise ratio is one. The source is at array broadside.
pattern is a plot of the minimum beam energy consistent with the constraint as a function of the bearing of look $k$.

To minimize $\mathbf{A}' \mathbf{R}_{1} \mathbf{A}$ such that $\mathbf{A}' \mathbf{W} = \mathbf{W} \mathbf{A} = 1$ we utilize the method of Lagrange multipliers.

$$g(\mathbf{A}) = \mathbf{A}' \mathbf{R}_{1} \mathbf{A} - \lambda (\mathbf{W}' \mathbf{A} - 1)$$

$$\nabla g_{\mathbf{A}}|_{\mathbf{A}_{ME}} = \mathbf{A}_{ME}' \mathbf{R} - \lambda \mathbf{W}$$

where we have followed the standard practice [2] of assuming that $\mathbf{A}$ and $\mathbf{A}'$ are independent vectors. This gradient must be zero and the constraint must be satisfied. Thus

$$\mathbf{A}_{ME} = \lambda \mathbf{R}^{-1} \mathbf{W}$$

and

$$\lambda = \frac{1}{\mathbf{W}' \mathbf{R}^{-1} \mathbf{W}}$$

so

$$\mathbf{A}_{ME} = \frac{\mathbf{R}^{-1} \mathbf{W}}{\mathbf{W}' \mathbf{R}^{-1} \mathbf{W}} \cdot$$

The energy in the adaptive beam is

$$P_{ME}(k) = \frac{1}{\mathbf{W}' \mathbf{R}^{-1} \mathbf{W}} \cdot$$

The beam weightings and delays are now functions of the bearing of look as well as the observed data. It is in this sense that the beam is adaptive. When the beam looks on target ($k = k_{1}$) the adaptive
beam energy is the same as that of the conventional beamformer because the energy incident from this bearing passes into the beam with unit gain. The adaptive steering vector minimizes the energy passed to the beam from non-look bearings, thereby reducing the level of the adaptive sidelobes relative to the level of the classical beam sidelobes. Figure 1.3 shows the high resolution beam-pattern corresponding to the same array and acoustic environment as figure 1.2; the sidelobes of the ME beam-pattern are much smaller than those of the Bartlett beam-pattern.

**Linear Prediction:**

Linear predictive filtering is commonly used to model time series data, particularly speech; in this application future samples of a waveform are modelled as a linear combination of past samples. In the context of array processing, the output of one sensor is modelled as a linear combination of the remaining sensor outputs. Let us model the output of sensor \( q \) as

\[
\hat{X}_q(f) = - \sum_{m=0, m \neq q}^{M-1} a_m X_m(f).
\]

The observed prediction error is

\[
e_q(f) = X_q(f) - \hat{X}_q(f) = \sum_{m=0}^{M-1} a_m X_m(f)
\]

where \( a_q = 1 \). Let \( A \) be the vector of conjugate filter coefficients \( A_m = a_m^* \), and \( U_q \) be a vector with its \( q \)th component equal to one, and
Figure 1.3

ME beam-pattern for a linear array of equally spaced elements.

The array consists of 10 elements with a spacing of $\lambda/2$; the signal-to-noise ratio is one. The source is at array broadside.
Bearing of Look in Degrees

Beam Energy in dB re Maximum
all others zero. Then

$$e_q(f) = A' X \text{ and } A' U_q = 1.$$  

The observed prediction error energy is

$$P_{obs} = E[e_q e_q^*] = A' R A.$$  

The best set of filter coefficients is the one which minimizes the observed prediction error energy subject to the constraint $A' U_q = 1$. Again we utilize the method of Lagrange multipliers, and find that the prediction filter which is most consistent with the observed data is

$$A_{LP_q} = \frac{1}{[R^{-1}]_{qq}} R^{-1} U_q.$$  

Again let us define a direction of look vector $W$ with components $w_m = e^{-j2\pi f_k z_m}$. We refer to the quantity

$$P_{seen}(k) = A_{LP_q} W W' A_{LP_q}$$  

as the prediction error energy seen looking along bearing $k$; it is the prediction error which we would observe if the observed data were generated by a unit energy plane wave with bearing $k$. When the ratio of $P_{obs} / P_{seen}(k)$ is small, the data is unlikely to have been generated by such a plane wave. When it is maximized as a function of $k$ at $k = k_1$ then it is reasonable to assume that the data was generated by a unit energy plane wave propagating along bearing $k_1$. The degree to which the plane wave model of the observed data is accurate can be judged
by the closeness of the peak value to one, since if the model were
perfect \( P_{\text{obs}} = P_{\text{seen}}(k_1) \). We define the LP beam-pattern as a plot of

\[
P_{\text{LP}}(f,k) = \frac{A_{\text{LP}}}{\Delta_{\text{LP}}} \left( \frac{R_{\text{LP}}}{q} \right)^{\frac{1}{2}}
\]

as a function of the bearing of look \( k \). It will exhibit a global max-
imum on target where \( k = k_1 \). We emphasize that no beam as defined in
equation (1.1) exists for the linear predictive array processing
algorithm. Figure 1.4 shows the LP beam-pattern corresponding to
the same array and acoustic environment as figures 1.2 and 1.3. In
computing this beam-pattern, we have neglected the power of 2 in the
denominator of equation (1.2). The linear predictive peak is sharper
than either of the beamformer peaks. Notice that the ratio of on-
target to off-target values is the same in figures 1.2, 1.3, and 1.4.

The preceding discussion indicates that when a single plane wave
propagates across an array, any of the beam-patterns

\[
P_{\text{BART}}(k) = W' W
\]

\[
P_{\text{ME}}(k) = \frac{1}{W' W^{-1} W}
\]
Figure 1.4

LP₀ beam-pattern for a linear array of equally spaced elements.

The array consists of 10 elements with a spacing of λ/2; the signal-to-noise ratio is one. The source is at array broadside.
\[
P_{LP, q}(k) = \frac{1}{\left| U_q R^{-1} \right|}
\]

(1.5)

will reveal the bearing of incidence as the location of the global maximum. We have eliminated the square from the denominator of the equation (1.2); this modification insures that the LP beam "energy" has units of energy and permits a direct comparison of the performance of the LP processing algorithm with that of the Bartlett and ME processing algorithms.

Unfortunately, the acoustic field present in an ocean environment can rarely be modelled satisfactorily by a single propagating plane wave. A more reasonable model is a superposition of plane waves propagating with different bearings and energies added to a background noise field.

\[
x_m(n) = n_m(n) + \sum_{p=1}^{P} s_p(n-k_p z_m)
\]

(1.6a)

\[
\mathbf{X}(f) = \sigma_0(f)N(f) + \sum_{p=1}^{P} \mathbf{\Phi}_p(f) \mathbf{S}_p(f)
\]

(1.6b)

where \( \sigma_0(f)N(f) \) is the vector of discrete Fourier transforms of the noise, \( \mathbf{S}_p(f) \) is the direction vector corresponding to signal \( p \) with components \( S_{pm} = e^{-j2\pi f k_p z_m} \), and \( \mathbf{\Phi}_p(f) \) is the discrete Fourier transform of \( s_p(n) \). \( \sigma_0(f) \) is the spatial variance of the noise and \( N(f) \) is normalized so that \( N' N = M \). In practice the number of incident signals \( P \) may not be known. It is no longer necessarily true that the peaks of the beam-patterns will be located exactly on target. More fundamentally, which maxima correspond to incident
plane waves and which do not? If we know \( P \) a priori we might assume that the \( P \) largest maxima correspond to those waves. Conversely, if there are \( P \) predominant maxima we might guess that there are \( P \) waves. Clearly both detection and estimation problems are important.

Of particular interest here is the study of linear arrays of equally spaced sensors. In this case, \( z_m = m \Delta z \), so the signal direction vectors and the direction of look vector have particularly simple forms (see figure 1.5):

\[
S_{pm}(f) = e^{-j\alpha_p}
\]

where \( \alpha_p = \pi \left( \frac{d}{\lambda/2} \right) \sin \theta_p \);

\[
W_m(f) = e^{-j\alpha_m}
\]

where \( \alpha = \pi \left( \frac{d}{\lambda/2} \right) \sin \theta \). \( \lambda = \frac{c}{f} \) is the wavelength associated with a sinusoidal signal of frequency \( f \) propagating with speed \( c \). (We have implicitly assumed that all of the narrowband signals share the common center frequency \( f \) and wavelength \( \lambda \).) Notice that we have chosen the zero phase reference point to be at element zero of the array (i.e. \( S_{00} = W_0 = 1 \)). The simple form of the direction of look vector allows us to compute the beam-patterns (1.3, 1.4, 1.5) efficiently via the FFT algorithm.

Consider the case \( P=2 \) in which the two equal energy plane waves propagate along nearly the same bearing. When a beam-pattern exhibits two distinct, predominant maxima we say that the targets are detected and resolved. Since both ME and Bartlett beams contain
Figure 1.5

Geometry of a linear array of M equally spaced sensors.

The element separation is d. Element 0 lies at the origin of the co-ordinate system. Signal $p$ impinges on the array along bearing $k_p$ and has wavelength $\lambda$. 
energy incident from non-look bearings, the beam-patterns may fail to exhibit two distinct maxima; instead they may display a broad maximum located at some intermediate bearing. Similarly the LP beam-pattern may exhibit only a single predominant maximum because the prediction error energy seen along an intermediate bearing may be smaller than that seen on either target. In this case we say that the targets are not resolved.

The foundation upon which any theoretical study of the detection or resolution capability of an array and processing algorithm rests is the assumed form of the cross spectral correlation matrix.

\[ R = E[XX'] \]

\[ \sigma_0^2 E[NN'] + E \left[ \sum_{p=1}^{P} \sigma_0 \left( \Psi_p \Psi_p^* \right) + \Psi_p \Psi_p^* \right] \]

\[ = \sum_{p=1}^{P} \sum_{p=m=1}^{P} E \left( \Psi_p \Psi_p^* \right) \]

\[ \sigma_0^2 E[NN'] + \sum_{p=1}^{P} \sigma_0 \left( \Psi_p \Psi_p^* \right) + E \left[ \Psi_p \Psi_p^* \right] \]

\[ = \sum_{p=1}^{P} \sum_{p=m=1}^{P} E \left( \Psi_p \Psi_p^* \right) \]

Cox[3] performed a detailed analysis of the capability of the Bartlett and ME array processing algorithms to resolve two equal energy signals. Cox implicitly assumed that \( \Psi_p \) and \( \Psi_m \) are zero mean uncorrelated (for \( m \neq p \)) random variables and that \( N \) is a zero mean vector that is independent of \( \Psi_p \). In this case
He demonstrated that the ME algorithm can resolve closely spaced signals at a lower signal to noise ratio than can the Bartlett algorithm. Studying the single signal case, Seligson[4] observed that the ME algorithm's performance is quite sensitive to deviations of the signal wavefront from its assumed shape (planar in this case). When a single source generates a signal which propagates along multiple bearings, the wavefronts incident on the array are obviously correlated; the superposition of correlated plane waves produces a net wavefront which is not planar. As multipath propagation is prevalent in the ocean [5], there is a need to understand how signal correlation affects the resolving capabilities of array processing algorithms. We shall pursue an analysis similar to Cox's to study the capabilities of Bartlett, ME, and LP algorithms to resolve coherent (i.e. correlated, narrowband) signals using linear arrays of equally spaced sensors. In addition we shall study the detection capabilities of Bartlett, ME, and LP processing algorithms. We will compare the algorithms on the basis of their abilities to simultaneously detect and resolve coherent signals. We shall also study the imperfections inherent in the empirical estimate of the correlation matrix and their effects on the various beam-patterns.

\[ R = \sigma_0^2 E[N^2] + \sum_{p=1}^{P} E(\hat{\Psi}_p^T \hat{\Psi}_p) S_p S_p^T. \]
CHAPTER 2

EFFECT OF SIGNAL COHERENCE ON ARRAY PROCESSING ALGORITHMS

We postulate the following simple narrowband model for our signals:

\[ s_p(n) = \sigma_p e^{j(2\pi f_p n + \phi_p)} \]

where \( \sigma_p \) is a real deterministic constant, \( f_p \) is the deterministic center frequency, and \( \phi_p \) is a random phase variable. The signal DFTs are

\[ \Phi_p(f) = \sigma_p e^{j[\phi_p + \pi(f_p - f)(N-1)]} \]

where

\[ \sigma_p = \frac{\sin\pi(f - f)N}{\sin\pi(f_p - f)} \quad \text{(2.1)} \]

The cross correlations of the signal DFTs are

\[ E[\Phi_p \Phi_m^*] = \sigma_p \sigma_m \left[ e^{j(\phi_p - \phi_m)} \right] e^{j\pi(f_p - f_m)(N-1)} \]

We define the signal coherence coefficients to be

\[ c_{pm} = \frac{E[\Phi_p \Phi_m^*]}{\sigma_p \sigma_m} \quad \text{(2.2)} \]

The coherence coefficient is analogous to a correlation coefficient. It follows that the magnitude of each coherence coefficient lies
between zero and one. When the signals \( s_p(n) \) and \( s_m(n) \) differ by only a constant deterministic phase and amplitude, \( |C_{pm}| = 1 \). This situation can arise, for example, when multiple propagation paths (with fixed differences in path length) exist between a single source and the array. While the magnitude of the inter-signal coherence is not a function of either the DFT length \( N \) or the center frequencies, the magnitude of the effective signal strengths \( \sigma_p \) (equation (2.1)) decrease \(^1\) (but not monotonically) as \( N \) increases when the analysis frequency \( f \) is offset from the center frequency \( f_p \). Consider the extreme situation when one signal, \( s_p(n) \), has a center frequency far removed from the analysis frequency \( f \). While this signal may be completely coherent with the remaining signals (i.e. \( |C_{pm}| = 1 \), \( \sigma_p = 0 \) so there are effectively \( P-1 \) signals rather than \( P \). Thus the fact that signals at differing frequencies can appear to be completely coherent is an artifact of the limited frequency resolution of the DFT.

We shall study the case where the noise (recall equation (1.6)) is spatially white \([\mathbb{E}[NN'] = I]\), has zero mean, and is independent of the signals. Subject to these assumptions, the cross spectral correlation matrix is

\[
R = \sigma_0^2 I + \sum_{p=1}^{P} \sum_{m=1}^{P} \sigma_p \sigma_m C_{pm} S_p S_m'. \tag{2.3}
\]

To simplify notation we introduce a coherence matrix \( C \) with elements \( C_{pm} \) and a matrix \( S \) whose \( p^{th} \) column is \( \sigma_p S_p \).

\(^1\) \(|f - f_p| \leq \frac{1}{2}\) because we assume that the data has been sampled at the Nyquist rate or higher.
Now we can express the correlation matrix of equation (2.3) in the convenient form

$$R = \sigma_0^2 I + SCS'$$

(2.4)

This is a form similar to that used by Cantoni and Godara[6] in their brief study of signal coherence.

We concentrate on the case where P=2 in order to study the effect of signal bearing, signal-to-noise ratio, and signal coherence on the detection and resolution capabilities of the various array processing algorithms. When two signals are detected and resolved, the beam-pattern exhibits two predominant peaks located nearly on-target. Figure 2.1 shows a classical beam-pattern in which two equal energy signals are both detected and resolved. The beam energy evaluated at either target bearing must be larger than the beam energy evaluated between the target bearings or off-target. Assume that the signal direction vectors $S_1$ and $S_2$ are known. The direction of look vector corresponding to midway between the targets is $W = W_0$ with components

$$W_{0m} = e^{-jm\left[\frac{a_1+a_2}{2}\right]}$$

A convenient mathematical description of looking off-target is $W S_1 = W S_2 = 0$. Our criterion for detection of the signals is that the ratio of on-target-to-largest-sidelobe beam energy exceed a threshold value of two. Our criterion for resolution of the signals
The array consists of 10 elements with a spacing of $\lambda/2$. The source bearings are $\pm 7^\circ$ about array broadside, the signal-to-noise ratio is one, and the signals are incoherent.
Bearing of Look in Degrees

Beam Energy in dB re Maximum

Source Separation: 14°
is that the ratio of on-target-to-between-target beam energy exceed a threshold value of \( \frac{2}{8} \). Using the assumed formulas for the correlation matrix and its inverse, we can obtain expressions for the on-to-off-target and on-to-between-target beam energy ratios. Once we have expressions for these ratios, we can evaluate the minimum signal-to-noise ratio required for each beam-pattern to detect and resolve the signals as a function of their bearing separation and coherence.

To gain insight into the effect of coherence on the various beam-patterns, we shall make two simplifying assumptions. First, the signals impinge on the array from bearings which are symmetric about array broadside. Second, the signals are of equal energy. In mathematical terms, these assumptions dictate that:

\[
S_{1m} = e^{-j \alpha m} \quad S_{2m} = e^{j \alpha m} = S_{1m}^* \quad \mathbb{W}_m = 1
\]

\[
\sigma_1 = \sigma_2
\]

Referring to Figure 2.2 we see that

\[
\alpha = 2\pi \text{ when } dsin\left[\frac{\theta}{2}\right] = \lambda
\]

so

\[
\frac{\alpha}{\text{dsin}\left[\frac{\theta}{2}\right]} = \frac{2\pi}{\lambda}
\]

and

\[
\alpha = \pi \left[\frac{d}{\lambda/2}\right] \sin\left[\frac{\theta}{2}\right].
\]
Figure 2.2
Special geometry utilized in the resolution analysis.

Equal energy plane waves impinge on the linear array along bearings which are symmetric about array broadside. The bearing separation is $\theta$, and each plane wave has wavelength $\lambda$. 
In the course of our analysis, we shall require expressions for the inner products between the signal direction vectors and between \( \mathbf{W}_0 \) and the signal vectors.

\[
S_1^* S_2 = \sum_{m=0}^{M-1} e^{j2\pi m} \frac{\sin \alpha}{\sin \alpha} e^{j(M-1)\alpha} = [S_2^* S_1]^*
\]

\[
W_0^* S_1 = \sum_{m=0}^{M-1} e^{j\pi m} \frac{\sin \alpha}{\sin \alpha} e^{-j\frac{M-1}{2}\alpha} = [W_0^* S_2]^*
\]

To simplify notation we introduce generalized cosines defined as

\[
\cos[S_1^* S_2] = \frac{\sin \alpha}{M \sin \alpha} \quad \text{and} \quad \cos[W_0^* S_1] = \frac{\sin \alpha}{M \sin \alpha}.
\]

The motivation for this terminology comes from the familiar Euclidean inner product property

\[
A \cdot B = ||A|| ||B|| \cos(\theta)
\]

where \( \theta \) is the angle between the vectors \( A \) and \( B \). For notational convenience we define the variable

\[
\beta = \frac{M-1}{2} \quad \alpha = \frac{M-1}{2} \pi \left[ \frac{d}{\lambda/2} \right] \sin \frac{\pi}{2}.
\]

Then we can write

\[
S_1^* S_1 = S_2^* S_2 = M
\]

\[
S_1^* S_2 = M e^{j2\beta} \cos[S_1^* S_2] = [S_2^* S_1]^*
\]
\[
\begin{align*}
\mathbb{W}_0 S_1 &= \text{Me}^{-j\beta_c} \cos(\mathbb{W}_0, S_1) = [S_1' \mathbb{W}_0]^*
\\
\mathbb{W}_0 S_2 &= \text{Me}^{j\beta_c} \cos(\mathbb{W}_0, S_1) = [S_2' \mathbb{W}_0]^*
\end{align*}
\]

We express the signal coherence as

\[
c = C_{12} = |c| e^{j\phi} = C_{21}^*.
\]

Finally, let us define the **array signal-to-noise ratio** as

\[
A = \frac{\sigma_1^2}{\sigma_0^2}.
\]

We proceed to evaluate the various beam energies on-target, between-targets, and off-target. We shall see later that there is a simple relationship between the off-target beam energy and the energy in the largest sidelobes.

\[
R = \sigma_0^2 + \sigma_1^2 S_1 S_1' + \sigma_2^2 (c S_1 S_2' + c^* S_2 S_1') + \sigma_2^2 S_2 S_2'
\]

**Bartlett:**

**On target:** \( W = S_1 \)

\[
P_{\text{BART}}(\text{on}) = M_0^2 [1 + A [1 + \cos^2(S_1, S_2) + 2|c| \cos(S_1, S_2) \cos(2\beta - \phi)]]
\]

**Between targets:** \( W = W_0 \)

\[
P_{\text{BART}}(\text{between}) = M_0^2 [1 + 2A \cos^2(\mathbb{W}_0, S_1) [1 + |c| \cos^2(\mathbb{W}_0, S_1) \cos(2\beta - \phi)]]
\]
Off target: \( W'S_1 = W'S_2 = 0 \)

\[ P_{\text{BART}}(\text{off}) = \sigma_0^2 \]

\[
\frac{P_{\text{BART}}(\text{on})}{P_{\text{BART}}(\text{between})} = \frac{1 + A[1 + \cos^2[S_1,S_2] + 2|c|\cos[S_1,S_2]\cos(2\beta-\phi)]}{1 + 2A\cos^2[W_0,S_1][1 + |c|\cos^2[W_0,S_1]\cos(2\beta-\phi)]}
\]

(2.5)

\[
\frac{P_{\text{BART}}(\text{on})}{P_{\text{BART}}(\text{off})} = 1 + A[1 + \cos^2[S_1,S_2] + 2|c|\cos[S_1,S_2]\cos(2\beta-\phi)]
\]

(2.6)

To perform our detection and resolution analyses for the ME and LP beam-patterns we require an analytic expression for \( R^{-1} \). In general if \( R \) is of the form

\[ R = B + DFG' \]

then [7]

\[ R^{-1} = B^{-1} - B^{-1}DGH'B^{-1} \]

(2.7)

where

\[ H = [I + FG'B^{-1}D]^{-1}F \].

Letting \( B = \sigma_0^2 I, D = G = S, \) and \( F = C \) we find that
In the two signal case

\[ S = \begin{bmatrix} \sigma_1 S_1 & \sigma_2 S_2 \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Omitting the algebraic manipulations, we find the ME and LP beam energies on-target, between-targets, and off-target.

**ME:**

**On target:** \( W = S_1 \)

\[
P_{ME\,(on)} = \frac{\sigma_1^2}{A} \left[ 1 + 2A + A^2 \left[ 1 - \cos^2 [S_1, S_2] \right] \left[ 1 - |c|^2 \right] + 2A |c| \cos[S_1, S_2] \cos(2\beta - \phi) \right] \frac{1}{1 + A \left[ 1 - \cos^2 [S_1, S_2] \right]} \]
**Between targets:** $N = N_0$

$$P_{\text{NB (between)}} = \frac{\sigma^2}{\Lambda} \left[ \frac{1 + 2\Lambda + \Lambda^2 [1 - \cos^2 (S_1, S_2)] [1 - \cos^2 (S_1, S_2)] + 2\Lambda |\cos (S_1, S_2)| \cos (2\beta - \eta)}{1 + 2\Lambda + \Lambda^2 [1 - \cos^2 (S_1, S_2)] [1 - \cos^2 (S_1, S_2)] + 2\Lambda |\cos (S_1, S_2)| \cos (2\beta - \eta)} \right]$$

**Off targets:** $N S_1 = N S_2 = 0$

$$P_{\text{NB (off)}} = \frac{\sigma^2}{\Lambda}$$
Then the signals are incoherent (|c| = 0) this expression
reduces to Cox's formula.

\[
\frac{P_{ME}(on)}{P_{ME}(off)} = \frac{1 + 2A + A^2[1 - \cos^2(S_1, S_2)][1 - |c|^2] + 2A|c|\cos[S_1, S_2]\cos(2\beta - \varphi)}{1 + A[1 - \cos^2(S_1, S_2)]}
\]

(2.9)
LP:

**On target:** \( W = S_1 \)

\[
P_{LP}(\text{on}) = \frac{M_o^2}{A} \left[ \frac{1 + 2A + A^2 [1 - \cos^2 (S_1, S_2)] [1 - |\ell|^2] + 2A |\ell| \cos (S_1, S_2) \cos (2\beta - \varphi)}{1 + 2A + A^2 [1 - \cos^2 (S_1, S_2)] [1 - |\ell|^2] + 2A |\ell| \cos (S_1, S_2) \cos (2\beta - \varphi)} \right]^{1/2}
\]

**Between targets:** \( W = W_0 \)

\[
P_{LP}(\text{between}) = \frac{M_o^2}{A} \left[ \frac{1 + 2A + A^2 [1 - \cos^2 (S_1, S_2)] [1 - |\ell|^2] + 2A |\ell| \cos (S_1, S_2) \cos (2\beta - \varphi)}{1 + 2A + A^2 [1 - \cos^2 (S_1, S_2)] [1 - |\ell|^2] + 2A |\ell| \cos (S_1, S_2) \cos (2\beta - \varphi)} \right]^{1/2}
\]
**Off target:** \( y'_{S_1} = y'_{S_2} = 0 \)

\[
\frac{P_{LP}(on)}{P_{LP}(between)} = \left[ \begin{array}{c}
1 + 2A + A^2 [1 - \cos^2 (S_1, S_2)][1 - \cos(S_1, S_2)] + 2A|l|\cos[S_1, S_2]\cos(3\beta - \varphi) \\
- 2A[1 + A[1 - \cos(S_1, S_2)]\cos(S_1, S_2)]\cos(\beta - qa) \\
+ 2A^2 [1 - \cos(S_1, S_2)]\cos[S_1, S_2]\cos(S_1, S_2)\cos(\beta - qa) \\
- 2A|l|\cos[S_1, S_2]\cos(\varphi - qa - \beta)
\end{array} \right]^{1/2}
\]

\( (2.10) \)
\[
\frac{P_{\text{LP}}(\text{on})}{P_{\text{LP}}(\text{off})} = \frac{1 + 2A + A^2[\cos^2(S_1, S_2)][1-c^2] + 2|\cos[S_1, S_2]|\cos(2\beta) - A|c|\cos(q_\alpha)}{[\cos(q_\alpha - \phi) - A\cos[S_1, S_2]\cos(2\beta)]^2 + [\sin(q_\alpha - \phi) - A\sin[S_1, S_2]\sin(2\beta)]^2}^{1/2}
\]

(2.11)

Resolution:

The formulas we have developed (equations (2.5, 2.8, and 2.10)) for on-to-between-target beam energy ratios describe the exact effects of signal separation, coherence, signal-to-noise ratio, and array length on the resolving capacity of the array processing algorithms. Unfortunately, the complexity of the expressions prevents a clear understanding of these effects. To gain such an understanding we endeavor to find the minimum array signal-to-noise ratio which causes the ratio of on-to-between-target beam energies to exceed a threshold value which is greater than one. Implicit in this approach is the assumption that the ratio is a monotonically increasing function of \(A\) for each processing algorithm. For the beamformers, this assumption is always valid. For the linear predictive algorithm, the ratio is not necessarily monotonic; however, if one signal-to-noise ratio is sufficient to resolve the signals, then any larger signal-
to-noise ratio will yield resolved signals. The classical threshold value used by Cox[3] is \( \frac{\pi}{8} \). This is the value of \( \frac{P_{\text{BART (on)}}}{P_{\text{BART (between)}}} \) for an array composed of many elements when the signal-to-noise ratio is large and \( S_1 \) and \( S_2 \) are separated by the Rayleigh resolution limit defined by \( S_1 S_2 = 0 \) or \( \theta = 2\sin^{-1}\frac{\lambda}{2Md} \). The Rayleigh resolution limit is the bearing separation which places the second signal in the null of the classical beam-pattern arising from the first signal.

We can numerically solve for the minimum array signal-to-noise ratios required by Bartlett, ME, and LP array processing algorithms to resolve equal energy signals as a function of their separation. It is clear from the equations (2.5, 2.8, and 2.10) that this solvent array SNR is a function of the coherence magnitude and phase. We define the worst case coherence phase as the one which maximizes the minimum array signal-to-noise ratio required for signal resolution for a fixed coherence magnitude.

In figure 2.3 we compare the minimum array SNR required for a ten element array to resolve incoherent (\(|c| = 0\)) signals using the Bartlett, ME, and LP\(_0\) processing algorithms. Of the three, the LP\(_0\) algorithm requires the least SNR to resolve the signals. The ME and LP\(_0\) algorithms are both capable of resolving arbitrarily close signals if the SNR is high enough. In contrast, the Bartlett algorithm is incapable of resolving signals spaced more closely than the Rayleigh limit for any SNR. Figures 2.4a, b, and c show Bartlett, ME, and LP\(_0\) beam-patterns, respectively, which just resolve the indicated incoherent signals. In each case, the array SNR is 10dB, and the
Figure 2.3

Minimum array SNR (in dB) necessary to resolve incoherent signals as a function of signal bearing separation using Bartlett, ME, and LP₀ algorithms.

The curves are for a 10 element array with λ/2 spacing.
Bearing Separation of Sources in Degrees

Minimum SNR Necessary to Resolve Signals
Figures 2.4a, b, and c

Beam-patterns which just resolve the indicated incoherent signals.

a) Bartlett
b) ME
c) $L_p$

The array consists of 10 elements with a spacing of $\lambda/2$; the array SNR is 10dB.
Beam Energy in dB re Maximum

Bartlett

Source Separation: 11.8°

ME

Source Separation: 8.4°

LP_0

Source Separation: 5.4°

Bearing of Look in Degrees
bearing separation is the smallest resolvable (from figure 2.2).

Figures 2.5a, b, and c show the minimum array SNR required for a ten element array to resolve signals whose coherence magnitude is .7 using Bartlett, ME, and LP₀ algorithms. For each algorithm twelve plots are shown, each corresponding to a different coherence phase. Clearly the relative phase of the coherent signals greatly influences the resolvability of the signals. A sensible way to compare the resolving capabilities of the algorithms is to select the worst case coherence phase at each signal separation for each algorithm, and compute the worst case minimum array SNR necessary for the algorithms to resolve the signals. Figure 2.5d compares the worst case minimum array SNR required for the Bartlett, ME, and LP₀ algorithms to resolve signals with coherence magnitude .7. As with the incoherent signals, the ME and LP₀ algorithms are capable of resolving signals which are arbitrarily close when the SNR is sufficiently high. Again, the LP₀ algorithm is capable of resolving closely spaced signals at a lower SNR than is the ME algorithm. Figures 2.6a, b, and c show Bartlett, ME, and LP₀ beam-patterns, respectively, which just resolve the indicated coherent signals. In each case, the coherence magnitude is .7, the coherence phase is zero, the array SNR is 10dB, and the bearing separation is the smallest resolvable (from figures 2.5a, b, and c).

Figures 2.7a, b, c, and d depict the effect of a coherence magnitude of .99. Figure 2.7c shows clearly that for particular coherence phases, the resolving array SNR required by the LP₀ algorithm
Figures 2.5a, b, and c

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .7 as a function of signal bearing separation using

a) Bartlett algorithm
b) ME algorithm
c) LP₀ algorithm.

Each figure depicts curves for coherence phases which are multiples of 30⁰ from 0⁰ to 330⁰. The plots are for a 10 element array with λ/2 spacing.
Figure 2.5d

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .7 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP₀ algorithms.

The curves are for a 10 element array with λ/2 spacing.
Figure 2.6a, b, and c

Beam-patterns which just resolve the indicated coherent signals; the coherence has magnitude .7 and zero phase.

a) Bartlett  
b) ME  
c) LP₀

The array consists of 10 elements with a spacing of λ/2; the array SNR is 10dB.
Beam Energy in dB re Maximum

Bartlett

Source Separation: 9.2°

ME

Source Separation: 8.4°

LP 0

Source Separation: 7°

Bearing of Look in Degrees
Figures 2.7a, b, and c

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .99 as a function of signal bearing separation using

a) Bartlett algorithm
b) ME algorithm
c) LP₀ algorithm.

Each figure depicts curves for coherence phases which are multiples of 30° from 0° to 330°. The plots are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Resolve Signals (dB)

Bearing Separation of Sources in Degrees
Figure 2.7d

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP\textsubscript{0} algorithms.

The curves are for a 10 element array with \( \lambda/2 \) spacing.
exhibits oscillatory behavior at large signal separations; this is an artifact of our particular approach to resolution analysis caused by the presence of a peak in the beam-pattern located between targets and/or a shift of the signal peaks off target. Figure 2.8 illustrates this phenomenon for a ten element array when the signal separation is $32^\circ$, the array SNR is ten decibels; the coherence magnitude is .99 and the phase is $90^\circ$. This phenomenon draws attention to a particular flaw of the LP beam-pattern: in the presence of multiple signals, the peaks are often biased slightly off target. The high array SNR required by the $LP_0$ algorithm to resolve "widely" separated signals reflects both biasing of the peaks and the appearance of intermediate peaks. Consequently, one could argue that resolution criterion that we have employed is of limited value when applied to the LP beam-pattern in the presence of widely separated coherent signals. However, the criterion accurately reflects the resolving performance of the LP algorithm in the presence of either incoherent or closely spaced coherent signals, and demonstrates the superior resolving capabilities of the linear predictive processing. In addition, we shall see that the array SNR required to detect widely separated signals is higher than that required to resolve them. Peak biasing can occur in the Bartlett and ME beam-patterns as well, but it is significant only for specific pathological coherence phases.

Sequential examination of figures 2.3, 2.5d, and 2.7d reveals that as the magnitude of the coherence increases, the minimum array
Figure 2.8

Example of peak bias produced by LP₀ algorithm.

The sources are located at ±16°. The coherence has magnitude .99 and phase 90°. The array consists of 10 elements with a spacing of λ/2; the array SNR is 10 dB.
SNR required by the ME and LP algorithms to guarantee resolution increases. The Bartlett algorithm exhibits a minimum signal separation below which signals cannot necessarily be resolved; this limit increases as the magnitude of the coherence increases.

Figure 2.9 compares the minimum array SNR required by the various LP\textsubscript{q} processing algorithms to resolve incoherent signals using a ten element array and prediction elements 0 to 4. The minimum array SNR as a function of prediction element \(q\) is symmetric about the center of the array. For closely spaced signals (\(0^\circ < 4^\circ\)) the center prediction elements (\(q=4,5\)) require the least array SNR to resolve the signals. Figures 2.10 a and b show the LP\textsubscript{0} and LP\textsubscript{4} beam-patterns, respectively, which result when incoherent signals separated by 2\(^\circ\) impinge on a ten element array; the array SNR is 21 dB. As expected (figure 2.9), the LP\textsubscript{4} algorithm resolves the signals whereas the LP\textsubscript{0} algorithm does not. Referring back to figure 2.9, it is clear that more widely separated signals (\(4^\circ < \theta < 22^\circ\)) the end prediction elements (\(q=0,9\)) require the least array SNR to resolve the signals. The sharp peaks in the plots for prediction elements 3–6 are interesting. Figure 2.11 shows the LP\textsubscript{3} beam-pattern for incoherent signals corresponding to a point just below the peak in figure 2.9 for prediction element 3. The signals are visibly resolved, but peak biasing and the large intermediate peak fools our resolution criterion into thinking that the signals are not resolved. Thus the peaks in figure 2.9 are located at signal separations for which utilizing prediction elements 3–6 results in spurious peaks in
Figure 2.9

Minimum array SNR (in dB) necessary to resolve incoherent signals as a function of signal bearing separation using prediction elements 0–4 in the LP algorithm.

The curves are for a 10 element array with λ/2 spacing.
Bearing Separation of Sources in Degrees

Minimum Array SNR Necessary to Resolve Signals

Prediction Elements 0 - 4

\( |\mathcal{C}| = 0 \)
Figures 2.10a and b

LP beam-patterns demonstrating the superior resolving capability of the center prediction elements.

a) LP\text{4}
b) LP\text{0}

The array consists of 10 elements with a spacing of $\lambda/2\delta$; the array SNR is 21 dB. The incoherent signals are separated by 2 degrees.
Beam Energy in dB re Maximum

Bearing of Look in Degrees

Source Separation: $2^\circ$

LP$_4$

LP$_0$
Example of spurious peak between targets produced by LP algorithm when a center prediction element is used.

The incoherent signals are separated by $14.023^\circ$. The array consists of 10 elements with a spacing of $\lambda/2$; the array SNR is 30 dB.
Figures 2.12a and b compare the minimum array SNR required by the various LP$^q$ algorithms to resolve signals with coherence magnitude .7 and zero phase using a ten element array. In this case, the minimum array SNR is not symmetric about the array center as a function of prediction element q. This is due to the asymmetric interference pattern created along the array because of the relative phase of the signals. Again, it makes little sense to compare the resolving performance of the LP$^q$ algorithms for particular coherence phases. Figure 2.12c compares the minimum array SNR required by the various LP$^q$ algorithms to resolve signals with coherence magnitude .7 and worst case phase using a ten element array. The worst case minimum array SNR is symmetric about the array center as a function of prediction element q. Figures 2.13a, b, and c depict the effect of a coherence magnitude of .99. As with the incoherent signals, the center prediction elements (q=4,5) require the least array SNR to resolve closely spaced signals while the end prediction elements (q=0,9) require the least array SNR to resolve more widely separated signals. For coherent signals the beam-patterns produced by prediction elements 3-6 continue to exhibit spurious peaks for some bearing separations.

Sequential examination of figures 2.9, 2.12c, and 2.13c reveals that as the magnitude of the coherence increases, the minimum array SNR required by the LP algorithms to guarantee resolution increases. When the signals become perfectly coherent ($|c| = 1$) LP processing
Figures 2.12a and b

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .7 zero phase as a function of signal bearing separation

a) using prediction elements 0-4 in the LP algorithm
b) using prediction elements 5-9 in the LP algorithm.

The plots are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Resolve Signals (dB)

Prediction Elements 0 - 4

Prediction Elements 5 - 9

Bearing Separation of Sources in Degrees
Figure 2.12c

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude 0.7 and worst case phase as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with λ/2 spacing.
Bearing Separation of Sources in Degrees

Minimum Array SNR Necessary to Resolve Signals

Prediction Elements 0 - 4

$|c| = .7$
Figures 2.13a and b

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .99 zero phase as a function of signal bearing separation

a) using prediction elements 0–4 in the LP algorithm
b) using prediction elements 5–9 in the LP algorithm.

The plots are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Resolve Signals (dB)

LPq

Prediction Elements 0 - 4

Prediction Elements 5 - 9

Bearing Separation of Sources in Degrees
Figure 2.13c

Minimum array SNR (in dB) necessary to resolve signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using prediction elements 0–4 in the LP algorithm.

The curves are for a 10 element array with $\lambda/2$ spacing.
cannot necessarily resolve signals at any SNR regardless of their separation. The reason is that particular coherence phases can cause an interference null to appear at any prediction element. Thus the output of the $q^\text{th}$ sensor may contain no signal component whatsoever.

**Detection:**

We seek the minimum array SNR necessary to achieve a ratio of on-target to highest sidelobe energy greater than or equal to two. Consequently, we must develop the relationship between the off-target beam energy and the energy in the highest sidelobe. We shall derive this relation for the single signal case. Generally, the sidelobes present in the two signal case are smaller than in the single signal case. For one signal and independent, spatially-white noise

$$R = \sigma_0^2 I + \sigma_1^2 S_1 S_1'$$

and

$$R^{-1} = \frac{1}{\sigma_0^2} \left[ \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2 S_1 S_1'} \right]$$

Without loss of generality, we shall assume that the signal impinges on the array from broadside so that $S_{1m} = 1$. The direction of look vector has components $\mathbf{w}_m = e^{-jma}$. 
Bartlett:

\[
\mathcal{W}_\mathcal{W} = \sigma_0^2 M 1 + A \left[ \frac{\sin \frac{M \alpha}{2}}{M \sin \frac{\alpha}{2}} \right]^2
\]

\[
= \frac{P_{\text{BART}}(\text{off})}{1 + A \left[ \frac{\sin \frac{M \alpha}{2}}{M \sin \frac{\alpha}{2}} \right]^2} > P_{\text{BART}}(\text{off})
\]

The largest sidelobe occurs roughly where \( \frac{M \alpha}{2} = \frac{3\pi}{2} \) or \( \alpha = \frac{3\pi}{M} \). Thus the ratio of the largest-sidelobe energy to the off-target beam energy is:

\[
\frac{P_{\text{BART}}(\text{highest sidelobe})}{P_{\text{BART}}(\text{off})} = 1 + A \left[ \frac{\sin \frac{M \alpha}{2}}{M \sin \frac{\alpha}{2}} \right]^2
\]

When \( 2M \gg 3\pi \) or \( M \gg 5 \) we can approximate \( \sin \frac{3\pi}{2M} = \frac{3\pi}{2M} \) so

\[
\frac{P_{\text{BART}}(\text{highest sidelobe})}{P_{\text{BART}}(\text{off})} = 1 + \frac{4A}{9\pi^2} \quad (2.12)
\]

Notice that the ripple in the beam-pattern off target increases as the array SNR increases. Figures 2.14a and b demonstrate this effect.
Figures 2.14a and b

Dependence of Bartlett sidelobe height (relative to nulls) on array SNR.

a) Array SNR = 10dB
b) Array SNR = 30dB

The array consists of 10 elements with a spacing of $\lambda/2$. 
The largest sidelobe occurs roughly where \( \frac{Ma}{2} = \frac{3\pi}{2} \) or \( \alpha = \frac{3\pi}{M} \). Thus the ratio of the largest-sidelobe energy to the off-target beam energy is:

\[
\frac{P_{ME}^{\text{(highest sidelobe)}}}{P_{ME}^{\text{(off)}}} = \frac{1}{2} \left( 1 - \frac{A}{1 + A} \left[ \frac{1}{M \sin^2 \frac{3\pi}{2M}} \right] \right)
\]

When \( M \gg 5 \) we can approximate this expression as:

\[
\frac{P_{ME}^{\text{(highest sidelobe)}}}{P_{ME}^{\text{(off)}}} = \frac{1}{1 - \frac{A}{1 + A} \left[ \frac{2}{3\pi} \right]^2} \tag{2.13}
\]

\[
= \frac{1 + A}{1 + A \left[ 1 - \frac{4}{9\pi^2} \right]} \leq \frac{1}{1 - \frac{4}{9\pi^2}} = 1.05
\]

The ripple in the ME beam-pattern off target asymptotically approaches 5 percent as the array SNR increases.
This expression yields considerable insight into the workings of the LP beam-pattern. The denominator is the magnitude of one plus a phasor having magnitude between zero and one. On target ($\alpha = 0$) the phasor is a negative real number which minimizes the denominator leading to a global maximum of the beam-pattern on target. As the direction of look begins to move off target, there are two effects which cause the beam energy to decrease. First, the magnitude of the phasor decreases, and second, the phasor rotates so that the magnitude of the complex quantity increases. It should be clear that if the phasor rotates quickly as $\alpha$ moves away from zero the on-target peak will be sharp. If the phasor rotates slowly as $\alpha$ moves away from zero the on-target peak will be broader. Thus we see that the choosing the prediction element at either end of the array will result in a sharper on-target peak than will choosing the prediction.
element in the center of the array. Simulations confirm this phenomenon. The LP sidelobes are generated by the rotating phasor. The sidelobes can extend both above and below the off-target value. The magnitude of the rotating phasor has several local maxima as a function of $\alpha$, the first occurring on-target. The next largest maximum occurs roughly where $\frac{Ma}{2} = \frac{3\pi}{2}$ or $\alpha = \frac{3\pi}{M}$. At this point the potential deviation of the beam energy from its off-target value is greatest. Therefore the maximum ratio of sidelobe to off-target energy is:

\[
\frac{P_{LP}(\text{highest sidelobe})}{P_{LP}(\text{off})} = \frac{1}{1 - \frac{A}{1+A}\frac{1}{M\sin \frac{3\pi}{2M}}}
\]

When $M \gg 5$ we can approximate this expression as:

\[
\frac{P_{LP}(\text{highest sidelobe})}{P_{LP}(\text{off})} = \frac{1}{1 - \frac{A}{1+A}\frac{2}{3\pi}}
\]

(2.14)

\[
= \frac{1 + A}{1 + A\left[1 - \frac{2}{3\pi}\right]} < \frac{1}{1 - \frac{2}{3\pi}} = 1.27
\]

The ripple in the LP beam-patterns off target asymptotically approaches about 27 percent as the array SNR increases.

Using equations (2.6, 2.9, and 2.11) in conjunction with equations (2.12, 2.13, and 2.14) we can numerically solve for the minimum array signal-to-noise ratios required by Bartlett, ME, and LP array processing algorithms to detect equal energy signals as a function of
their separation. It is clear from the equations (2.6, 2.9, and 2.11) that this detectable array SNR is a function of the coherence magnitude and phase. We define the worst case coherence phase as the one which maximizes the minimum array signal-to-noise ratio required for signal detection for a fixed coherence magnitude.

Figure 2.15 compares the minimum array SNR required for a ten element array to detect incoherent (|c| = 0) signals using the Bartlett, ME, and LP₀ processing algorithms. Of the three, the LP₀ algorithm requires the highest array SNR to detect the signals. The array SNR necessary for the Bartlett and ME algorithms to detect the signals is nearly identical.

Figures 2.16a, b, and c show the minimum array SNR required for a ten element array to detect signals whose coherence magnitude is .7 using Bartlett, ME, and LP₀ algorithms. For each algorithm twelve plots are shown, each corresponding to a different coherence phase. Clearly the relative phase of the coherent signals greatly influences the detectability of the signals. A sensible way to compare the detection capabilities of the algorithms is to select the worst case coherence phase at each signal separation for each algorithm, and compute the worst case minimum array SNR necessary for the algorithms to detect the signals. Figure 2.16d compares the worst case minimum array SNR required for the Bartlett, ME, and LP₀ algorithms to detect signals with coherence magnitude .7. As with the incoherent signals, the LP₀ algorithm requires the greatest array SNR to detect the signals. Except for closely spaced signals, the Bartlett algorithm is
Minimum array SNR (in dB) necessary to detect incoherent signals as a function of signal bearing separation using Bartlett, ME, and $LP_0$ algorithms.

The curves are for a 10 element array with a spacing of $\lambda/2$. 
Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .7 as a function of signal bearing separation using

a) Bartlett algorithm
b) ME algorithm
c) LP₀ algorithm.

Each figure depicts curves for coherence phases which are multiples of 30° from 0° to 330°. The plots are for a 10 element array with a spacing of λ/2.
Figure 2.16d

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .7 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP₀ algorithms.

The curves are for a 10 element array with a spacing of λ/2.
Bearing Separation of Sources in Degrees

Minimum Array SNR Necessary to Detect Signals

LP₀
ME
Bartlett
capable of detecting signals at a lower array SNR than the ME algorithm. Figures 2.17a, b, c, and d depict the effect of a coherence magnitude of .99. In this case, the Bartlett algorithm is incapable of detecting signals separated by less than about 4° for some coherence phases. This result is neither intuitively satisfying nor correct, and is an artifact of the simple relationship we assumed between the sidelobe and off-target beam energies. This relationship holds rigorously only in the single signal case. However, when the signals are either incoherent or coherent but widely separated (i.e. resolvable), the detection results based on the simple relationship seem adequate.

Sequential examination of figures 2.15, 2.16d, and 2.17d reveals that as the magnitude of the coherence increases, the minimum array SNR required by the ME and LP algorithms to guarantee detection increases.

Figure 2.18 compares the minimum array SNR required by the various LP processing algorithms to detect incoherent signals using a ten element array and prediction elements 0 to 4. The minimum array SNR as a function of prediction element q is symmetric about the center of the array. For closely spaced signals (θ<4°) the center prediction elements (q=4,5) require the least array SNR to detect and resolve the signals.

Figures 2.19a and b compare the minimum array SNR required by the various LP algorithms to detect signals with coherence magnitude .7 and zero phase using a ten element array. In this case, the
Figures 2.17a, b, and c

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .99 as a function of signal bearing separation using

a) Bartlett algorithm
b) ME algorithm
c) LP₀ algorithm.

Each figure depicts curves for coherence phases which are multiples of 30° from 0° to 330°. The plots are for a 10 element array with a spacing of λ/2.
Minimum Array SNR Necessary to Detect Signals (dB)

Bartlett

ME

LP₀

Bearing Separation of Sources in Degrees
Figure 2.17d

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP₀ algorithms.

The curves are for a 10 element array with a spacing of λ/2.
Figure 2.18

Minimum array SNR (in dB) necessary to detect incoherent signals as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with a spacing of λ/2.
Bearing Separation of Sources in Degrees

LPq
Prediction Elements 0-4

Minimum Array SNR Necessary to Detect Signals
Figures 2.19a and b

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .7 zero phase as a function of signal bearing separation

a) using prediction elements 0-4 in the LP algorithm
b) using prediction elements 5-9 in the LP algorithm.

The plots are for a 10 element array with $\lambda/2$ spacing.
Minimum Array SNR Necessary to Detect Signals (dB)

Bearing Separation of Sources in Degrees

Prediction Elements 0 - 4

Prediction Elements 5 - 9
minimum array SNR is not symmetric about the array center as a function of prediction element q. This is due to the asymmetric interference pattern created along the array because of the relative phase of the signals. Again, it makes little sense to compare the detection performance of the LPq algorithms for particular coherence phases. Figure 2.19c compares the minimum array SNR required by the various LPq algorithms to detect signals with coherence magnitude .7 and worst case phase using a ten element array. The worst case minimum array SNR is symmetric about the array center as a function of prediction element q. As with the incoherent signals, the center prediction elements (q=4,5) require the least array SNR to detect and resolve closely spaced signals.

Figures 2.20a, b, and c depict the effect of a coherence magnitude of .99.

Sequential examination of figures 2.18, 2.19c, and 2.20c reveals that as the magnitude of the coherence increases, the minimum array SNR required by the LP algorithms to guarantee detection increases.

**Combined Detection and Resolution:**

It is unreasonable to discuss the resolution of two undetectable signals or the detection of two unresolvable signals. Instead, we must consider what conditions are necessary to both detect and resolve two signals. All of the algorithms require a higher array SNR to detect widely separated signals than to resolve them. Figure
Minimum array SNR (in dB) necessary to detect signals with coherence magnitude 0.7 and worst case phase as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with $\lambda/2$ spacing.
Figures 2.20a and b

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .99 zero phase as a function of signal bearing separation

a) using prediction elements 0–4 in the LP algorithm
b) using prediction elements 5–9 in the LP algorithm.

The plots are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Detect Signals (dB)

Bearing Separation of Sources in Degrees

LPq
Prediction Elements 0 - 4

LPq
Prediction Elements 5 - 9
Figure 2.20c

Minimum array SNR (in dB) necessary to detect signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with $\lambda/2$ spacing.
2.21 shows an example of two signals which are resolved but not detected, as our theory predicts. Conversely, all of the algorithms require a higher array SNR to resolve closely spaced signals than to detect them.

Figures 2.22, 2.23 and 2.24 show the minimum array SNR required by the Bartlett, ME, and LP₀ processing algorithms to guarantee both detection and resolution of equal energy signals when the coherence magnitude is 0, .7, and .99, respectively. As the magnitude of the coherence increases, so does the array SNR necessary to guarantee the detection and resolution of the signals.

Figures 2.25, 2.26, and 2.27 compare the minimum array SNR required by the various LPₚ algorithms to guarantee both detection and resolution of equal energy signals when the coherence magnitude is 0, .7, and .99, respectively. As the magnitude of the coherence increases, so does the array SNR necessary to guarantee the detection and resolution of the signals. Notice that for closely spaced signals the performance of the center prediction elements is superior to that of the end prediction elements. This superiority is independent of the coherence magnitude.
Figure 2.21

Example of a ME beam-pattern which resolves but does not detect the signals.

The sources are located at ±5°. The coherence has magnitude 0.99 and zero phase. The array consists of 10 elements with a spacing of \( \lambda/2 \); the array SNR is 0 dB.
Minimum array SNR (in dB) necessary to resolve and detect incoherent signals as a function of signal bearing separation using Bartlett, ME, and LP_0 algorithms.

The curves are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Detect and Resolve Signals

Bearing Separation of Sources in Degrees

|c| = 0

Bartlett

ME

LP₀

0 5 10 15 20 25 30 35 40 45

0 10 20 30 40 50 60 70 80

-20 -10 0
Minimum array SNR (in dB) necessary to resolve and detect signals with coherence magnitude .7 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP₀ algorithms.

The curves are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Detect and Resolve Signals

Bearing Separation of Sources in Degrees

|c| = 0.7

Bartlett

ME

LP₀
Figure 2.24

Minimum array SNR (in dB) necessary to resolve and detect signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using Bartlett, ME, and LP_0 algorithms.

The curves are for a 10 element array with \( \lambda/2 \) spacing.
Bearing Separation of Sources in Degrees

Minimum Array SNR Necessary to Detect and Resolve Signals

Bartlett

|c| = .99

LP0

ME
Figure 2.25

Minimum array SNR (in dB) necessary to resolve and detect incoherent signals as a function of signal bearing separation using prediction elements 0–4 in the LP algorithm.

The curves are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Detect and Resolve Signals

L_{Pq}

Prediction Elements 0 - 4

|c| = 0

Bearing Separation of Sources in Degrees
Figure 2.26

Minimum array SNR (in dB) necessary to resolve and detect signals with coherence magnitude .7 and worst case phase as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with λ/2 spacing.
Minimum Array SNR Necessary to Detect and Resolve Signals

Bearing Separation of Sources in Degrees

LPₚ

Prediction Elements 0 - 4

|c| = .7
Figure 2.27

Minimum array SNR (in dB) necessary to resolve and detect signals with coherence magnitude .99 and worst case phase as a function of signal bearing separation using prediction elements 0-4 in the LP algorithm.

The curves are for a 10 element array with $\lambda/2$ spacing.
Bearing Separation of Sources in Degrees

LP\(q\)

Prediction Elements 0 - 4

|c| = .99

Minimum Array SNR Necessary to Detect and Resolve Signals

Bearing Separation of Sources in Degrees
CHAPTER 3
EFFECTS OF CORRELATION MATRIX COMPUTATION

In the previous chapter our theoretical analysis was based on use of the true correlation matrix. Now we consider the effect of using an empirical computation of the correlation matrix by averaging the available sensor output data. We shall see that the previous definition of signal coherence must be modified slightly. While signal coherence is a deterministic quantity when infinite averaging is employed, in realistic situations where the averaging is finite, signal coherence is a random quantity. We would like to understand how signal coherence depends on the amount of averaging used to compute the correlation matrix. In addition, there will be non-zero cross terms between signals and noise present in the correlation matrix when the averaging is finite. Recall that in Chapter 2 we assumed that these terms were zero (equation (2.3)). By studying the single signal case, we will probe the effect of signal-noise cross terms on the on-target and off-target beam energies. Our goal is to determine the amount of averaging required by each processing algorithm to justify the assumption of zero cross terms in the analysis in Chapter 2.

Assume that T=NK samples of the sensor outputs are available. We break the data into K consecutive non-overlapping segments each of length N as follows (see figure 3.1):

---

1 We refer to K as the time-bandwidth product.
Figure 3.1

Division of data into $K$ segments of $N$ samples.
\[ x_m^{(k)}(n) = x_m(n + kN) \quad 0 \leq n < N \]

Assuming that the sensor outputs are given by equation (1.6a),

\[ x_m^{(k)}(n) = n_m(n + kN) + \sum_{p=1}^{P} s_p(n-k_p \cdot z_m + kN) \]

\[ = n_m^{(k)}(n) + \sum_{p=1}^{P} s_p^{(k)}(n-k_p \cdot z_m) . \]

The N point DFT of the kth output vector is

\[ X^{(k)}(f) = \sigma_0(n)N^{(k)}(f) + \sum_{p=1}^{P} \phi_p^{(k)}(f)S_p(f) . \]

Following common practice[8,9], we utilize the correlation matrix estimate defined by

\[ R = \frac{1}{K} \sum_{k=0}^{K-1} X^{(k)}(f)X^{(k)\prime}(f) \]

\[ = \frac{1}{K} \sum_{k=0}^{K-1} \sigma_0 N^{(k)}(k)^{\prime} + \sum_{p=1}^{P} 1 \sum_{k=0}^{K-1} \left[ \phi_p^{(k)\ast}N_p^{(k)}S_p + \phi_p^{(k)}S_p N_p^{(k)\ast} \right] \]

\[ + \sum_{p=1}^{P} \sum_{p=1}^{P} S_p S_p^{\prime} \sum_{k=0}^{K-1} \phi_p^{(k)}\phi_p^{(k)\ast} \]

The statistics of this random matrix have been studied by Goodman[9]. It is important to note that R will not be positive definite or invertible if K < M because it will have M-K zero eigenvalues.

We consider a somewhat more general narrowband signal model than in the preceding chapter.

\[ s_p(n) = \sigma_p e^{-j(2\pi f n + \phi_p)} \]

\[ s_p(n) = \sigma_p e^{-j(2\pi f n + \phi_p(n))} \]
where $f_p$ is the center frequency of signal $p$ and $\phi_p(n)$ is a slowly varying random phase. We assume that $\phi_p(n)$ varies slowly enough that we can consider it to be essentially constant over a length $N$ DFT. Thus:

$$s_p(n) = \sigma_p e_p, k e_{p, k} j2\pi f n, \quad kN \leq n < (k+1)N$$

$$s_p^{(k)}(n) = \sigma_p e_p, k e_{p, k} j2\pi f n, \quad 0 \leq n < N$$

Evaluating the DFT of $s_p^{(k)}(n)$ yields

$$q_p^{(k)}(k) = \sum_{n=0}^{N-1} \sigma_p e_p, k e_{p, k} j2\pi(f - f) n$$

$$= \sigma_p e_p, k + k2\pi f N + \pi(f - f)(N - 1)$$

(3.1)

where

$$\sigma_p = \frac{\sin(f - f)N}{\sin(f - f) N}$$

(2.1)

as in the last chapter.

We define the $p^{th}$ average noise vector $\overline{N}_p$ as

$$\overline{N}_p = \frac{1}{K} \sum_{k=0}^{K-1} q_p^{(k)}(k) e^{-j[\phi_p, k + k2\pi f N + \pi(f - f)(N - 1)]}$$

and the coherence coefficient between signals $p$ and $m$ as

$$C_{pm} = \frac{\frac{1}{K} \sum_{k=0}^{K-1} q_p^{(k)} q_m^{(k)*}}{\sigma_p \sigma_m}.$$  

(3.2)
Since $\mathcal{P}_p^{(k)}$ and $\mathcal{P}_m^{(k)}$ are random variables (equation (3.1)) so are the coherence coefficients. Comparing equations (3.2) and (2.2) we see that the expectation operator has been replaced by a time average of the sample functions. Notice that the coherence coefficients given by equation (3.2) are not necessarily zero even when those given by equation (2.2) are. We can therefore express the correlation matrix as

$$R = \sum_{k=0}^{K-1} \mathcal{N}^{(k)} \mathcal{N}^{(k)'} + \sum_{p=1}^{P} \sigma_0 \sigma_0 \left[ \sum_{p} \mathcal{N}_{p} \mathcal{S}_{p}^* + \sum_{p} \mathcal{S}_{p} \mathcal{N}_{p}^* \right] + \sum_{p=1}^{P} \sum_{m=1}^{P} \sigma_{pm} \sigma_{pm} \mathcal{C}_{pm} \mathcal{S}_{p} \mathcal{S}_{m}$$

Comparison of this formula with equation (2.3) illustrates the need to understand how the coherence terms and signal-noise cross terms depend on the time-bandwidth product $K$.

One important mechanism which generates signal coherence in the ocean is specular reflection. To see how the coherence due to specular reflection depends on the time-bandwidth product, we consider the two signal case. The coherence between signals 1 and 2 is $c = |c| e^{j\phi} = C_{12}$. We model the effect of multipath propagation by writing

$$\phi_{1,k} - \phi_{2,k} = \phi + \phi_k$$

where $\phi$ is a constant random phase shift due to the average difference in path lengths, and $\phi_k$ is a zero mean random phase variable which accounts for a time varying difference in path lengths due to platform motion, source motion, or motion of the scattering surface (see figure 3.2). For mathematical convenience we shall assume that
Multipath propagation in the ocean due to specular reflection.

The difference in path lengths varies as a function of time because of the dynamic ocean surface.
\( \phi_i \) and \( \phi_k \) are statistically independent when \( i \neq k \), and that the probability density of \( \phi_k \) is

\[
p_{\phi_k}(\phi) = \begin{cases} 
\frac{1}{\gamma 2\pi} & -\gamma \pi < \phi < \gamma \pi \\
0 & \text{otherwise}
\end{cases}
\]

The coherence is

\[
c = e^{j[\phi + \pi(f_1 - f_2)(N-1)]} \sum_{k=0}^{K-1} e^{j[\phi_k - k2\pi(f_1 - f_2)N]}
\]

and its conditional expected value given \( \phi \) is

\[
E(c|\phi) = \text{sinc} \gamma \pi \frac{\sin\pi(f_1 - f_2)NK}{\pi \sin\pi(f_1 - f_2)N} e^{j[\phi + \pi(f_1 - f_2)(NK-1)]}
\]

The magnitude of the conditional expected value of \( c \) is always less than or equal to one. It tends toward zero, but not monotonically, as the time bandwidth product and/or the inter-signal frequency difference and/or the randomness of the inter-signal phase increase. The coherence between signals which have nearly the same center frequencies or phases is reduced by increasing the amount of averaging (i.e. increasing \( K \)). If the signals share the same center frequencies and have a constant phase relationship then the coherence magnitude will be one regardless of the amount of averaging performed.

It is worth noting that even when the inter-signal phase sequence is completely random (\( \gamma = 1 \) and \( E(c|\phi) = 0 \)), the expected value of the squared magnitude of the coherence is \( \frac{1}{K} \), not zero.
Effect of noise on beam-patterns:

In the previous chapter we stated that the correlation cross terms between signals and noise were zero. Since we perform finite averaging to obtain our correlation estimate, these cross terms will not be zero even when the assumption of statistically independent of signals and noise is correct. Capon and Goodman\[8\] have studied the statistics of the Bartlett and ME beam-patterns when finite averaging is performed to obtain the correlation matrix estimate, and Baggeroer\[10\] has studied the statistics of the LP spectral estimator for time series data. These analyses are rigorous mathematically, but fail to explain the sensitivity of the array processing algorithms to finite averaging.

Due to finite averaging, the noise alone portion of the correlation matrix
\[ \frac{\sigma_0^2}{K-1} \sum_{k=0}^{N-1} \mathbf{N}_k \mathbf{N}_k^T \]
is generally not proportional to the identity matrix. Nevertheless, we shall continue to assume that the noise alone portion of the correlation matrix is \( \sigma_0^2 \mathbf{I} \). Thus we isolate the effect of the finitely averaged signal-noise cross terms. For the single signal case, the correlation matrix is:

\[ \mathbf{R} = \sigma_0^2 \mathbf{I} + \sigma_1 \sigma_0 \left[ \mathbf{N}_1 \mathbf{S}_1^T + \mathbf{S}_1 \mathbf{N}_1^T \right] + \sigma_1^2 \mathbf{S}_1 \mathbf{S}_1^T \]

We now analyze the effect of the signal-noise cross terms on the various on and off-target beam energies. For convenience let us define the random variables \( a = \mathbf{S}_1 \mathbf{N}_1 \) and \( b = \mathbf{N}_1 \mathbf{N}_1 \).
Bartlett:

On target: $W = S_1$.

$$P_{\text{BART}}(\text{on}) = \sigma_0^2 M + 2\sigma_1 \sigma_0 \text{Re}(a) + \sigma_1^2 M^2 \quad (3.3)$$

Off target: $W S_1 = 0$.

$$P_{\text{BART}}(\text{off}) = \sigma_0^2 M \quad (3.4)$$

To analyze the effect of the signal-noise cross terms on the ME and LP beam-patterns we require an analytic expression for $R^{-1}$. We can write $R$ in the form

$$R = \sigma_0^2 I + \left[ \sigma_0 \overline{N_1} \sigma_1 S_1 \right] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_0 \overline{N_1} \\ \sigma_1 ^2 S_1 \end{bmatrix}.$$ 

Let $S = \left[ \sigma_0 \overline{N_1} \sigma_1 S_1 \right]$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then the correlation matrix has the familiar form

$$R = \sigma_0^2 I + SCS'.$$

Utilizing equation (2.7) we find that
\[ R^{-1} = \begin{bmatrix} \frac{1}{\sigma_0^2} & 0 \\ 0 & 1 \end{bmatrix} \left( I - \frac{\begin{bmatrix} \sigma_0^2 \\ -M_1 + a + \frac{a}{\sigma_0} \\ \frac{1}{\sigma_0} + 1 \end{bmatrix}}{\sigma_0^2 + 2\sigma_1\sigma_0\text{Re}a + M_1\left[1 - b + \frac{|a|^2}{M}\right]} \right) \]

**ME:**

**On target:** \( W = S_1 \)

\[ P_{\text{ME (on)}} = \frac{\sigma_0^2}{M} \left[ 1 + 2\frac{\sigma_1}{\sigma_0}\text{Re}a + \frac{\sigma_0^2}{\sigma_0^2} \left[1 - b + \frac{|a|^2}{M}\right]\right] \quad (3.5) \]

**Off target:** \( W' S_1 = 0 \).

Define \( a_0 = \frac{W}{N_1} \).

\[ P_{\text{ME (off)}} = \frac{\sigma_0^2}{M} \left[ 1 + 2\frac{\sigma_1}{\sigma_0}\text{Re}(a) + \frac{\sigma_0^2}{\sigma_0^2} \left[1 - b + \frac{|a|^2}{M} + \frac{|a_0|^2}{M}\right]\right] \quad (3.6) \]
LP:

**On target:** \( W = S_1 \)

\[
P_{LP}(on) = \frac{\sigma_0^2}{1 + 2 \frac{1}{\sigma_0} \text{Re} + \frac{\sigma_1}{\sigma_0} \left[ 1 - \frac{b + |a|^2}{M} \right]} \]

(3.7)

Notice that the relative phase of \( S_{1q} \) and \( N_{1q} \) can have a significant effect on the on-target beam energy.

**Off target:** \( W S_1 = 0 \)

\[
P_{LP}(off) = \frac{\sigma_0^2}{W q \left[ 1 + 2 \frac{1}{\sigma_0} \text{Re} + \frac{\sigma_1}{\sigma_0} \left[ 1 - \frac{b + |a|^2}{M} \right] \right]} \]

\[\left(1 + 2 \frac{1}{\sigma_0} \text{Re} + \frac{\sigma_1}{\sigma_0} \left[ 1 - \frac{b + |a|^2}{M} \right]\right) \]

\[- S_{1q} \left[ \frac{\sigma_1}{\sigma_0} a^0 + \frac{\sigma_2}{\sigma_0} a_s \right]^* \]

\[+ N_{1q} \frac{\sigma_1}{\sigma_0} a^0 \]

(3.8)
The complexity of equations (3.5 to 3.8) precludes direct evaluation of the first or second moments of the ME and LP beam energies. Instead we propose to study the bias and variability of the beam energies induced by the noise by substituting the expected values of the real random variables Re(a) and b and the expected values of the magnitudes and squared magnitudes of the complex random variables a and \( a_0 \) into equations (3.3 to 3.8). Although generally \( E(f(x)) \neq f(E(x)) \), the expressions which result do yield insight into the variability of on and off-target beam energies that is the consequence of finite averaging.

We proceed to obtain expressions for the expected values of Re(a), \( |a|^2 \), and \( |a_0|^2 \). We assume that the noise is spatially white, zero mean and Gaussian. Thus \( N^{(k)}(f) \) is a complex Gaussian zero mean random vector with independent components. We also assume that the real and imaginary portions of each component are independent. Finally, assume that \( N^{(i)}(f) \) and \( N^{(k)}(f) \) are independent vectors when \( i \neq k \). These assumptions imply that

\[
E \left[ \sum_{k=0}^{K-1} N^{(k)} N^{(k)\prime} \right] = I
\]

or

\[
E \left[ \sum_{k=0}^{K-1} N^{(k)} m N^{(k)\prime} m \right] = 1.
\]

The stationarity of the noise field implies that
Thus the real and imaginary parts of $N_m^{(k)}$ are independent zero mean Gaussian random variables with variance $1/2$. $\overline{N}_1$ is a linear combination of Gaussian random vectors having independent components; consequently, it too, is a Gaussian random vector with independent components. Each real and imaginary component of $\overline{N}_1$ has zero mean and variance $\frac{1}{2K}$.

$$a = \frac{\overline{N}_1}{\Sigma_1}$$

$$E(Re(a)) = 0 \text{ since } E(\overline{N}_1) = 0.$$ 

$$|a| = \left| \sum_{m=0}^{M-1} e^{-jm\alpha_1} N_{1m} \right|$$

$$= \sqrt{x^2 + y^2}$$

where

$$x = \sum_{m=0}^{M-1} \left[ \cos \alpha_1 \text{ Re}[\overline{N}_{1m}] + \sin \alpha_1 \text{ Im}[\overline{N}_{1m}] \right]$$

and

$$y = \sum_{m=0}^{M-1} \left[ \cos \alpha_1 \text{ Im}[\overline{N}_{1m}] - \sin \alpha_1 \text{ Re}[\overline{N}_{1m}] \right]$$

are independent, real Gaussian random variables with zero mean and variance $\sigma^2 = \frac{M}{2K}$. The distribution of $|a|$ is $\text{[11]}$. 

$$E[N_m^{(k)}N_m^{(k)*}] = 1 = E\left[\text{Re}[N_m^{(k)}]^2\right] + E\left[\text{Im}[N_m^{(k)}]^2\right] = 2E\left[\text{Re}[N_m^{(k)}]^2\right]$$
\[ P(a|r) = \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \] for \( r \geq 0 \).

so that

\[ E(|a|) = \frac{1}{\sigma^2} \int_0^\infty r^2 e^{-\frac{r^2}{2\sigma^2}} \, dr \]

\[ = \frac{1}{\sqrt{4\pi}} E(|a_0|), \]

and

\[ |a|^2 = \left[ \sum_{m=0}^{M-1} e^{-j\alpha_1 N_{1m}} \right] \left[ \sum_{n=0}^{M-1} e^{j\alpha_1 N_{1n}} \right] \]

\[ E[|a|^2] = \sum_{m=0}^{M-1} E[|N_{1m}|^2] = \frac{M}{K} \]

\[ E[|a|^2] = \text{var}(a) = \frac{M}{K} = \text{var}(a_0). \]

\[ b = \sum_{m=0}^{M-1} |N_{1m}|^2 \]

\[ E(b) = \sum_{m=0}^{M-1} E[|N_{1m}|^2] = \frac{M}{K} \]

Substituting these expected values into the expressions for on and off-target beam energies yields:
Bartlett:

\[ P_{\text{BART}}(\text{on}) = M\sigma_0^2 + M^2\sigma_1^2 \] (3.9)

\[ P_{\text{BART}}(\text{off}) = M\sigma_0^2 \] (3.10)

Finite averaging causes no bias of the beam energies away from their correct (infinite averaging) values. However, finite averaging leads to variability of the on-target beam energy. We can crudely estimate the degree of this variability by substituting \( \pm \text{var}(a) \) in place of \( \text{Re}(a) \) in equation (3.3). This yields:

\[ \sigma_0^2M - 2\sigma_1\sigma_0\sqrt{\frac{M}{K}} + \sigma_1^2M^2 \leq P_{\text{BART}}(\text{on}) \leq \sigma_0^2M + 2\sigma_1\sigma_0\sqrt{\frac{M}{K}} + \sigma_1^2M^2. \]

The variability is small when \( K \gg M \). When the time bandwidth product satisfies this constraint, we are well justified in neglecting the signal-noise cross terms of the correlation matrix.

For a large array with an array signal to noise ratio of one, this reduces to \( K \gg M \). Figures 3.3a, b, and c show several Bartlett beam-patterns for a ten element array when \( K=10, 100, \) and \( 1000, \) respectively; the array SNR is one. The variability of the patterns decreases as \( K \) increases. The variability of both the \( K=100 \) and \( K=1000 \) beam-patterns is small, as expected, since \( M=10. \)
Variability of Bartlett beam-patterns induced by finite averaging.

Each plot shows 4 Bartlett beam-patterns corresponding to 4 distinct estimates of the correlation matrix. As the amount of averaging increases, the variability of the beam-patterns diminishes. The array consists of 10 elements with a spacing of \( \lambda/2 \); the signal-to-noise ratio is one. The source is at array broadside.
ME:

\[
P_{\text{ME}}(\text{on}) = \sigma_0^2 \left[ \frac{1 + \frac{\sigma_1^2}{\sigma_0^2} \left( 1 - \frac{M-1}{K} \right)}{M} \right]
\]

which approaches \( \frac{\sigma_0^2}{M} + \sigma_1^2 \) as \( K \) becomes large.

\[
P_{\text{ME}}(\text{off}) = \frac{\sigma_0^2}{M} \left[ 1 + \frac{\sigma_1^2}{\sigma_0^2} \left( 1 - \frac{M-1}{K} \right) \right]
\]

which approaches \( \frac{\sigma_0^2}{M} \) as \( K \) becomes large.

The ME beam energy can be slightly sensitive to finite \( K \) and the resulting signal-noise cross terms when evaluated both on and off-target. Specifically, finite averaging causes the beam energies to be biased away from their correct (infinite averaging) values. In addition, finite averaging leads to variability in the beam energies. As the time bandwidth product increases and the beam energies tend toward their asymptotic values, the degree of variability decreases. When \( K \gg M \) the on and off-target beam energies are close to their asymptotic deterministic values. Consequently, the assumption that the signal-noise cross terms are negligible is well justified whenever \( K \gg M \). Figures 3.4a, b, and c show several ME beam-patterns for a ten element array when \( K=10, 100, \) and 1000, respectively; the array
Figures 3.4a, b, and c

Variability of ME beam-patterns induced by finite averaging.

a) \( K=10 \)
b) \( K=100 \)
c) \( K=1000 \)

Each plot shows 4 ME beam-patterns corresponding to 4 distinct estimates of the correlation matrix. As the amount of averaging increases, the variability of the beam-patterns diminishes. The array consists of 10 elements with a spacing of \( \lambda/2 \); the signal-to-noise ratio is one. The source is at array broadside.
SNR is one. The variability of the patterns decreases as \( K \) increases. The variability of both the \( K=100 \) and \( K=1000 \) beam-patterns is small, as we expect since \( M=10 \).

LP:

To be consistent with our use of \( E(b) = \frac{M}{K} \), we set the magnitude of \( N_{1q} \) equal to \( \frac{1}{\sqrt{K}} \). We define \( \gamma_1 \) as the random phase of \( a \), \( \gamma_2 \) as the random phase of \( N_{1q} \), and \( \gamma_3 \) as the random phase of \( a_0 \).

\[
\begin{align*}
\frac{P_{LP} (on)}{\sigma_0^2} &= e^{-jq_1} \frac{1 + \frac{\sigma_1^2}{\sigma_0^2} \left[1 - \frac{M-1}{K}\right]}{\left[1 + \left|\frac{M\pi}{4K} \frac{\sigma_1}{\sigma_0} e^{j\gamma_1}\right| - \frac{M}{K} \frac{\sigma_1}{\sigma_0} e^{j\gamma_2}\right]} \\
&\quad \text{which approaches } \sigma_0^2 + M\sigma_1^2 \text{ as } K \text{ becomes large.}
\end{align*}
\]
which approaches \( \sigma_0^2 \) as \( K \) becomes large.

The LP beam energy can be highly sensitive to finite \( K \) and the resulting signal-noise cross terms when evaluated both on and off-target. Specifically, finite averaging causes the beam energies to be biased away from their correct (infinite averaging) values. The variability that finite averaging can cause in the LP beam energies is extreme because of the complex interaction of the phasors in the denominators of equations (3.13 and 3.14). The random phases in equation (3.13) can cause reinforcement or cancellation in the denominator. Thus a range of on-target beam energies is possible:
For particular time-bandwidth products where \( \frac{M^2}{\pi} + \frac{M}{\left| K \right|} \sigma_0 > \frac{\sigma_0}{\sigma_1} \), the possibility of infinite on-target beam energy exists.

As the time bandwidth product increases and the beam energies tend toward their asymptotic values, the degree of variability decreases. When \( K \gg M \) and \( K \gg \frac{M^2}{\pi^2} \sigma_0^2 \), the on and off-target beam energies are close to their asymptotic deterministic values. Consequently, the assumption that the signal-noise cross terms are negligible is well justified only when both \( K \gg M \) and \( K \gg \frac{M^2}{\pi^2} \). The second constraint is interesting. It indicates that as the signal to noise ratio increases, more averaging must be performed to maintain a constant level of noise induced variability in the beam energies. For large arrays, the second constraint is dominant because of its proportionality to \( M^2 \). Equation (3.14) demonstrates the possible existence of spurious noise induced peaks off-target in the beam-pattern due to the complex interaction of the random phasors. Figures 3.5a, b, and c show several LP\(_0\) beam-patterns for a ten element array when \( K = 10, 100, \) and 1000, respectively; the array SNR is 10dB. The variability of the patterns decreases as the time-bandwidth product increases. Only the \( K = 1000 \) has small variability as expected.
Figure 3.6

LP$_0$ beam-patterns corresponding to 4 distinct estimates of the correlation matrix when the array SNR is 20 dB and $K=1000$.

The variability of the on-target beam energy increases as the SNR increases. The array consists of 10 elements with a spacing of $\lambda/2$. The source is at array broadside.
Beam Energy in dB re 4

K = 10

K = 100

K = 1000

Bearing of Look in Degrees
since $M^2=100$. Figure 3.6 shows several LP beam-patterns for a ten element array when $K=1000$; the array SNR is 20dB. Comparison of figures 3.5c and 3.6 confirms that the variability of the on-target beam energy increases as the SNR increases.

The analysis we have performed indicates how the on and off-target beam energies are likely to depend on the time bandwidth product. It does not, however, indicate how the degree of off target ripple varies with the amount of averaging performed. Empirically, the ripple in the LP beam-patterns increases rapidly as the averaging decreases; the ripple in the ME beam-pattern increases much more slowly as the averaging decreases. The ripple in the Bartlett beam pattern is the least sensitive to finite averaging.

---

2 The off-target beam energies are computed under the assumption that $\bar{W}S_1 = \bar{W}S_2 = 0$, which is only true at specific off-target bearings.
Figures 3.5a, b, and c

Variability of $LP_0^0$ beam-patterns induced by finite averaging.

a) $K=10$
b) $K=100$
c) $K=1000$

Each plot shows 4 $LP_0^0$ beam-patterns corresponding to 4 distinct estimates of the correlation matrix. As the amount of averaging increases, the variability of the beam-patterns diminishes. The array consists of 10 elements with a spacing of $\lambda/2$; The signal-to-noise ratio is one. The source is at array broadside.
CHAPTER 4

CONCLUSIONS

The resolution and detection performances of the various beamforming and linear predictive array processing algorithms described here are adversely affected by signal coherence and by limited averaging in the computation of the correlation matrix. Furthermore, the performance of each algorithm is affected to a different degree by these conditions. No one algorithm proves superior to the others in all situations. Consequently, the coherence and amount of averaging possible in particular applications greatly influence the choice of an "optimal" array processing algorithm. Table 4.1 summarizes the results of Chapters 2 and 3.

As the magnitude of the coherence increases from zero to one, the resolving capability of all of the algorithms diminishes. The incremental effect of a change in coherence magnitude on the resolving capabilities of the ME and LP algorithms is greatest when the coherence magnitude is large (near one); for the Bartlett algorithm, the incremental effect is greatest when the coherence magnitude is small. Except for perfectly coherent signals, the LP processing algorithm is uniformly the most capable of resolving closely-spaced signals, followed by ME and Bartlett processing. Only in the case of extremely coherent signals (\(|c| \gg .99\)) is the resolution capability of the Bartlett algorithm superior to the others.
Table 4.1

Summary of the effect of signal coherence on the processing algorithms' resolution and detection capabilities.
| Array processing algorithm | $|c|$ | Closely spaced signals | Widely separated signals | \(\theta=1^\circ\) | \(\theta=10^\circ\) | Averaging necessary to minimize variability |
|---------------------------|------|------------------------|------------------------|----------------|----------------|-----------------------------------------------|
| Bartlett                  | .7   | 4                      | 5                      | 0              | 1              | \(K \gg \frac{M}{\left[\frac{\sigma_0}{\sigma_1} + \frac{\sigma_0}{M\sigma_0}\right]^2}\) |
|                           | .99  | 5                      | 5                      | 1              | 1              |                                               |
| ME                        | 0    | 50                     | 5                      | 0              |                |                                               |
|                           | .7   | 54                     | 12                     | 2              |                |                                               |
|                           | .99  | 69                     | 27                     | 11             |                |                                               |
| LP_{end}                  | 0    | 34                     | 2                      | 1              |                |                                               |
|                           | .7   | 40                     | 8                      | 7              |                |                                               |
|                           | .99  | 55                     | 22                     | 22             |                |                                               |
| LP_{center}               | 0    | 28                     | *                      | 1              |                |                                               |
|                           | .7   | 34                     | *                      | 7              |                |                                               |
|                           | .99  | 49                     | *                      | 22             |                |                                               |

† Rayleigh resolution limit = 11°
‡ Rayleigh resolution limit = 15°
+ Rayleigh resolution limit = 16°
* Use of the center prediction elements for signal separations greater than 4° yields spurious peaks (see text).
The capability of the three algorithms to detect incoherent signals is basically the same. Coherence has negligible effect on the detection capabilities of Bartlett processing is negligible. In contrast, as signal coherence increases, the detection capabilities of ME and LP processing decrease, with ME processing always having a greater capability. This corroborates Seligson's[4] work and extends it to include the LP processing algorithms.

In Chapter 3 we established constraints on the time-bandwidth product to ensure minimal sensitivity of the beam-patterns to imperfections in the correlation matrix that result from finite averaging. These constraints are listed in table 4.1. For many practical sonar applications it is reasonable to assume that \( \frac{\sigma_0^2}{\sigma_1^2} \gg 1 \). Under these conditions the constraints reduce to

\[
\begin{align*}
\text{Bartlett: } & \ K \gtrsim 1 \\
\text{ME: } & \ K \gg M \\
\text{LP: } & \ K \gg M \frac{\sigma_1^2}{\sigma_0^2} \gg M.
\end{align*}
\]

Consequently, the Bartlett beam-pattern is least sensitive to finite averaging; the LP beam-patterns are most sensitive in this regard. The sensitivity of the ME beam-pattern to finite averaging lies between these extremes.

A tradeoff exists between resolving capability and sensitivity to finite averaging. The LP algorithms are most capable of resolving
closely-spaced signals; however, their high sensitivity to finite averaging restricts their application to environments where large amounts of averaging are possible. In addition, the LP algorithms require the highest SNR to detect widely-separated signals when the coherence is large. The Bartlett algorithm is least sensitive to finite averaging and requires the least SNR to detect widely-separated signals; however, its resolving capability is very poor.

The ME algorithm lies between the LP and Bartlett algorithms in terms of resolution and detection capabilities and sensitivity to noise. The ME algorithm seems best suited to applications where the amount of averaging possible is small and the capability to resolve closely-spaced signal bearings is requisite. The LP algorithms are of little value in these situations because of their extreme sensitivity to noise (recall figure 3.5). The Bartlett algorithm is also of little value here because of its extremely poor resolving capability. Table (4.2) crudely suggests which array processing technique is best suited to each of four combinations of signal separation and time-bandwidth product.

The novel variation to the LP algorithm of utilizing different prediction elements to produce different LP beam-patterns proves to be of some value. Moving the prediction element from the end to the center of a linear array of equally spaced sensors enhances the capability of the algorithm to resolve extremely closely-spaced signals. However, the beam-patterns produced by the center prediction elements are more likely to exhibit spurious peaks when the sources are less
Array processing algorithms best suited to each of four combinations of time-bandwidth product and signal bearing separation.
<table>
<thead>
<tr>
<th>Time-bandwidth product</th>
<th>Bearing separation of signals</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Closely spaced</td>
<td>Widely separated</td>
</tr>
<tr>
<td>Small</td>
<td>ME for all coherences.</td>
<td>Bartlett for all coherences</td>
</tr>
<tr>
<td>Large</td>
<td>LP end for all end coherences.</td>
<td>Bartlett for all coherences</td>
</tr>
<tr>
<td></td>
<td>LP center for all coherences.</td>
<td>LP or ME for small coherences</td>
</tr>
<tr>
<td></td>
<td>all coherences when signals are very close.</td>
<td></td>
</tr>
</tbody>
</table>
closely-spaced. Choice of prediction element has no effect on the capability of the LP algorithm to detect widely-separated signals. The effect of increasing signal coherence is the same for all prediction elements. For linear arrays of unequally spaced sensors and for non-linear array geometries, it is not clear that these observations hold. Variation of the prediction element for different array geometries might enhance the capability of the LP algorithm to resolve closely-spaced signals using these arrays.

There are three issues which need to be investigated. First, even in the infinite averaging case, the peaks in the beam-patterns are biased away from the source locations. Since passive sonar systems are often used to localize targets at great ranges, small biases in bearing estimates can lead to large errors in position estimates. An analysis of the peak biasing would therefore be of great value. Second, extreme sensitivity to finite averaging compromises the high resolution capability of the LP algorithms. The linear prediction algorithm is quite flexible and lends itself to modifications beyond those discussed here (i.e., choice of prediction element). Some work[12] has been done to decrease the sensitivity of linear prediction to noise in time series problems without significantly reducing the resolving capability of the algorithm. Such work needs to be extended to array processing applications. Finally, coherence continues to be a significant obstacle to array processing algorithms. Cantoni and Godara[6] discuss an array processing algorithm based on a simple eigenvalue analysis of the correlation matrix. In the case
of white sensor noise and infinite averaging, signal coherence does not affect the capability of the algorithm to resolve source bearings. However, their algorithm's performance deteriorates rapidly as the noise becomes colored and as the amount of averaging is reduced. Work needs to be done to find a robust processing algorithm which is insensitive to signal coherence. Alternatively, techniques to detect and suppress coherent terms in the correlation matrix would eliminate the need for an insensitive processing algorithm.
References


