RICE UNIVERSITY

Relativistic Wave Equations and
Multiple Dirac Algebras

by

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ABSTRACT

Relativistic Wave Equations and
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A new formalism for finding and studying relativistic wave equations of the form

\[(\alpha^\mu \partial_\mu + \chi) \psi(x) = 0\]

has been developed. Using the algebra of matrices for the Dirac equation, higher order algebras, called multiple Dirac algebras, were created and a proof was given that the \(\alpha^\mu\) matrices for any such equation were essentially elements of these algebras. Further, the transformation properties of the elements of the algebras under the proper Lorentz group were determined in order to specify the elements \(\alpha^\mu\).

The classical analysis of this problem, due to Bhaba, was reproduced and compared to the analysis in the new formalism. The use of the multiple Dirac algebras in determining the algebraic structure of the matrices \(\alpha^\mu\) was demonstrated in a series of examples and followed by a discussion of problems that could be approached with the formalism.
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I. INTRODUCTION

In quantum mechanics, a "system" is described by a wavefunction $\psi(x)$, where $x$ is a point $(x^0, x^1, x^2, x^3)$ in Minkowski space-time. $\psi$ itself may be a column vector, in which case each entry is a function of $x$. The physics of a system takes the form of a set of differential equations which the wavefunction must satisfy. Ideally, this set of equations allows the state of the system at any time to be determined from knowledge of the state of the system at any previous time. The actual form of the equations involved is dictated by the dual considerations of physical "intuition" and simplicity.*

The former consideration manifests itself in several ways. The first is the concept of relativistic invariance. It is logical to assume that the actual physics involved in the development of a system does not depend on the frame of reference that the system is viewed from. Thus, the physical laws implied by a set of equations viewed from two different inertial reference frames should be identical. This requirement does not imply that the equations themselves are the same in both frames, only that their implications are. However, the consideration of simplicity leads to the investigation of equations which are themselves

*For a more complete development of the restrictions and the equations they imply, the reader is referred to Bhaba.1
relativistically invariant, a more limited class of equations.

The second main restriction on the type of equation comes from the requirement that the equation predict future states. That is, given the value of \( \psi \) on any space-like surface, it is possible to calculate its value on any later space-like surface. Once again, the equations satisfying this condition form a larger class than desirable. Thus, the criterion of simplicity is again invoked to limit the consideration to equations which embody the concept of "near action." Thus, the equation is required to be a first order differential equation in the variables \( x^0, x^1, x^2, x^3 \).

The most well-known of the relativistic wave equations is the Dirac equation

\[
(\sum_{\mu=0}^{3} \gamma^\mu \partial/\partial x^\mu + \chi) \psi(x) = 0
\]

where \( \psi(x) \) is a 4x1 column matrix, the \( \gamma^\mu \) are constant 4x4 matrices satisfying the Dirac anti-commutation rules, and \( \chi \) is a non-zero constant. This form of the equation is very appealing for many reasons. Since the partial derivatives \( \partial/\partial x^\mu \) are all on a similar footing in the equation, the requirement that (1.1.1) be relativistically invariant is a condition on the matrices \( \gamma^\mu \) alone. Further, the equation is obviously first order and the properties of the particle such as mass and spin can be determined
directly from the matrices $\gamma^\mu$ and $\chi$. Finally, the predictions of the Dirac equation for interacting spin $\frac{1}{2}$ particles have been extremely successful.

As a result, one line of approach to relativistic equations has been to simply generalize the degree of the linear equation with constant coefficients (1.1.1). That is, to investigate the equation

\[(1.1.2) \quad \left( \sum_{\mu} \alpha^\mu \partial / \partial x^\mu + \chi \right) \psi(x) = 0\]

for matrices of degree $nxn$. As with the Dirac equation itself, the requirement that this equation be relativistically invariant is a condition on the matrices $\alpha^\mu$. There are other physical conditions and mathematical restrictions which can be applied to Eq. (1.1.2) in the search for physically meaningful equations, but the first consideration is simply to find sets of matrices $\{\alpha^\mu\}$ such that (1.1.2) is a relativistically invariant equation.

The first such equation other than the Dirac equation to be studied was the Duffin-Kemmer equation.² This equation is formed with a set of four 16x16 matrices $\rho^\mu$ satisfying a certain set of algebraic relations. The equation can be separated into three separate equations: one 10x10 matrix equation for a spin 1 particle, one 5x5 matrix equation for a spin 0 particle, and a 1x1 trivial equation. The Duffin-Kemmer equation has been successful in the study of particle aspects of meson theory, despite
the fact that is not as mathematically "nice" as the Dirac equation.

The general solution for matrices $\gamma^{\mu}$ which make (1.1.2) a relativistic equation was developed by Bhaba.\(^3\) His work involved a direct mathematical decomposition of (1.1.2) into matrix equations for sub-matrices of the $\gamma^{\mu}$'s. As a result, the general solution of (1.1.2) is a general composition of the sub-matrix solutions.

However, the Bhaba solution does have its drawbacks. The physical properties of a solution to Eq. (1.1.2) such as mass and spin are expressed as algebraic properties of the $\gamma^{\mu}$ matrices and the constant $\chi$. But the sub-matrix composition of the Bhaba solution makes these properties rather difficult to determine, especially for higher degree solutions. As a result, the Bhaba solution is more easily applied in a case-by-case study of the solutions than in the search for a class of solution with certain properties.

One approach to this difficulty was given by the relation between the Dirac equation and the Duffin-Kemmer equation. It was shown that the four matrices defined by

$$\beta^{\mu} = \gamma^{\mu} \times 1 + 1 \times \gamma^{\mu}$$

where $1$ is the 4x4 unit matrix, were a form of the $\gamma^{\mu}$ matrices of the Duffin-Kemmer equation. This led to the concept of constructing solutions to (1.1.2) from already known solutions. The most well-known example of a general
method of constructing solutions to (1.1.2) is the de Broglie "méthode de fusion." In this method, the matrices $a^\mu$ are of the form

$$\mathcal{A}^\mu = \gamma^\mu_1 + \gamma^\mu_2 + \ldots + \gamma^\mu_n$$

where

$$\gamma^\mu_i = \mathbb{1} \times \mathbb{1} \times \ldots \times \mathbb{1} \times \gamma^\mu \times \mathbb{1} \times \ldots \times \mathbb{1}$$

This form, however, only picks out certain solutions to Eq. (1.1.2) and does not cover all solutions.

The purpose of this work is to develop a method of constructing solutions to Eq. (1.1.2) from matrices of known algebraic properties such that any solution to (1.1.2) is in the set of constructed solutions. The matrices used will be the familiar Dirac matrices and will proceed along the line of the de Broglie method, but in a more general manner.

Because much of the application of this work would be in a comparison of the constructed solutions and the Bhaba solutions, it is necessary to include the actual Bhaba analysis and solution to (1.1.2). With this in mind, the second chapter of the work develops the Lorentz and rotation groups, whose properties dictate the nature of relativistic invariance. Also in this chapter is the corresponding development of the spinor groups and notation which make the Bhaba analysis possible. The third chapter introduces the concept of group representation and
applies it to the transformation groups. Further, the Dirac $u$ and $v$ matrices are defined. These matrices, first introduced by Dirac$^5$ and treated by Fierz,$^6$ are the key to the Bhaba solution. These two chapters also develop the notation and approach for the later chapters.

The fourth chapter uses the material in the earlier parts to reproduce the Bhaba analysis in detail. The formalisms of both relativistic wave equations and algebraic representations are also done in detail here. In the fifth chapter, the construction technique for solutions to (1.1.2) is presented and compared with the Bhaba solutions to prove completeness. The sixth chapter further develops this technique by providing examples of the construction method. These examples are then related directly to the standard solutions to (1.1.2) provided by Kemmer, Fierz, Gupta, and others. The final chapter summarizes the conclusions of the thesis.

I have attempted to minimize the problem of notation by using the usual conventions whenever possible. Further, in order to facilitate the comparison between the theory and the examples, specific numerical representations are chosen at each stage of the development. The specific notation conventions used in this work and the ambiguities involved are explained in the Appendix.
II. THE TRANSFORMATION GROUPS

In order to perform a coherent treatment of relativistic wave equations, it is necessary to have at hand the properties of the groups involved, i.e., the rotation group, the Lorentz group, and their universal covering groups, $SU(2)$ and $SL(2, \mathbb{C})$, along with their representations. This section will develop the necessary properties and notation for these groups.*

2.1 The Rotation Group

The rotation group, $\mathcal{R}$, is the group of all rotations of 3-dimensional space about a fixed point. That is, $\mathcal{R}$ is the group of linear transformations $R: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which preserve the magnitude of the vectors in $\mathbb{R}^3$ and which have determinant $+1$. A linear transformation on $\mathbb{R}^3$ which preserves vector magnitudes must also preserve the inner product. So, if $\vec{x} = (x^i) = (x^1, x^2, x^3)$ and $\vec{y} = (y^i)$ are elements of $\mathbb{R}^3$ and $R \in \mathcal{R}$, then

$$\vec{x} \cdot \vec{y} = R(\vec{x}) \cdot R(\vec{y})$$

(2.1.)

Since $R$ is linear, it can be regarded as a real 3x3 matrix $(R^i_j)$, so that $(R(\vec{x}))^i = R^i_j x^j$. Equation (2.1.1) holds for all $\vec{x}, \vec{y} \in \mathbb{R}^3$, so it is equivalent to

$$\delta_{ij} = \sum_k R^k_i R^k_j$$

(2.1.2)

* More detail and development on these groups is available in Gelfand and Naimark.
For convenience, (2.1.2) can be written in matrix form as

\[(2.1.3) \quad \mathbf{1} = R^T R\]

\(\mathcal{R}\) consists of all real 3x3 matrices satisfying (2.1.3) and having \(\det R = 1\). Note that (2.1.3) implies that \((\det R)^2 = 1\) or \(\det R = \pm 1\). The matrices satisfying (2.1.3) with determinant \(-1\) correspond to linear transformations which both rotate and invert the axes of \(\mathbb{R}^3\). These are not pure rotations of \(\mathbb{R}^3\) and so are excluded from \(\mathcal{R}\) by the condition \(\det R = +1\).

As transformations on \(\mathbb{R}^3\), certain elements of \(\mathcal{R}\) have standard interpretations. These are*

\[
\begin{align*}
\mathbf{R}_i^j(\theta) &= \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} ; & \quad \mathbf{R}_{ij}^k(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \\
\mathbf{R}_i^j(\theta) &= \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}
\end{align*}
\]

where \(\mathbf{R}_i^j(\theta)\) is a rotation of the \(i-j\) plane by the angle \(\theta\).

The elements \(\mathbf{R}_i^j(\theta)\) of \(\mathcal{R}\) have inverses \(\mathbf{R}_i^j(-\theta)\), so a subgroup \(\mathcal{A}'\) of \(\mathcal{R}\) may be defined as the group generated by the set \(\{\mathbf{R}_i^j(\theta) : \forall i, j, \theta\}\). In fact, \(\mathcal{A}'\) and \(\mathcal{R}\) are identical by the following argument.

Suppose \(R \in \mathcal{A}\). The three entries \(R_{i1}\) of the matrix \(R\) can be considered as a three vector

* See the Appendix for an explanation of the notation \(R_{i1}^j(\theta)\).
\( \vec{v}_1 = (R_1^1, R_1^2, R_1^3) \). By defining \( \theta \) such that 
\[
\cos \theta = \frac{R_1^2}{s} \text{ and } \sin \theta = \frac{R_1^3}{s} \text{ where} 
\]
\[
s = \left[ (R_1^2)^2 + (R_1^3)^2 \right]^{1/2} \text{ and } \varphi \text{ such that} 
\]
\[
\cos \varphi = \frac{R_1^1}{s'} \sin \varphi = \frac{s}{s'} \text{ where} 
\]
\[
s' = \left[ (R_1^1)^2 + (s')^2 \right]^{1/2} = \left[ \sum_k (R_1^k)^2 \right]^{1/2} = 1 
\]
\( \vec{v}_1 \) can be written \( \vec{v}_1 = (\cos \varphi, \sin \varphi \cos \theta, \sin \varphi \sin \theta) \).
By definition, \( R_2^{-3}(+\theta) \) and \( R_1^{-2}(+\varphi) \) are elements
of \( \mathcal{R} \), so
\[
R' = R_1^{-3}(\varphi) R_2^{-3}(\theta) R
\]
\[
= R_1^{-3}(\varphi) R_2^{-3}(\theta) \begin{pmatrix} \cos \varphi & R_2' & R_3' \\ \sin \varphi \cos \theta & R_2' & R_3' \\ \sin \varphi \sin \theta & R_3' & R_3' \end{pmatrix}
\]
\[
= \begin{pmatrix} 1 & R_2' & R_3' \\ 0 & R_2' & R_3' \\ 0 & R_2' & R_3' \end{pmatrix}
\]
is also an element of \( \mathcal{R} \). Further, (2.1.3) 
implies that \( R'^T \in \mathcal{R} \), so
\[
1 = \sum_k (R_1^k)^T \begin{pmatrix} R_1^1 \\ R_2 \\ R_3 \end{pmatrix}^T = 1 + (R_1'^2)^2 + (R_1'^3)^2 \text{ and } R_1'^{1} = R_1'^{1} = 0. \text{ So } R'
\]
has the form
\[
R' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_2' & R_3' \\ 0 & R_2' & R_3' \end{pmatrix}
\]
implies that \( \det R' = 1 \) and that \((R')^{-1} = R'^T\). Therefore, the matrix

\[
\begin{pmatrix}
R'_{11} & R'_{12} \\
R'_{21} & R'_{22}
\end{pmatrix}
\]

has the same two properties and so is of the form

\[
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\]

for some angle \( \phi \). Thus, \( R' = R_{2}^{3}(\phi) \) and

\[
R = R_{2}^{3}(-\theta) \ R_{1}^{3}(-\pi) \ R_{2}^{3}(\phi)
\]

so \( R \in \mathcal{R}' \). Therefore, \( \mathcal{R}' = \mathcal{R} \).

A connected set in the space \( \mathbb{R}^n \) is a set of points such that, if \( p_1 \) and \( p_2 \) are in the set, then there is a continuous path from \( p_1 \) to \( p_2 \) with every point of the path in the set. Since an \( n \times n \) matrix is an array of \( n^2 \) numbers, it can be considered as an element of \( \mathbb{R}^{n^2} \), and a set of matrices can be defined as connected if it is connected as a set in \( \mathbb{R}^{n^2} \). Thus, the connectedness of a group of matrices such as \( \mathcal{R} \) can be determined.

In the set \( \mathcal{R} \), the elements \( R_{i}^{j}(\theta) \) for all \( i, j, \theta \) are connected to the identity, \( \mathbb{I} \), by the paths given by varying \( \theta \) to \( 0 \). Therefore any element of \( \mathcal{R} \) is connected to \( \mathbb{I} \) since the \( R_{i}^{j}(\theta) \) generate \( \mathcal{R} \), and so \( \mathcal{R} \) is connected.

2.2 The Lorentz Group

The Lorentz group, \( \mathcal{L} \), is the group of linear transformations which transform from one relativistic coordinate system to another. Explicitly, \( \mathcal{L} \) is the group of linear
transformations \( L : \mathbb{R}^4 \to \mathbb{R}^4 \) which preserve the metric

\[
g = (g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

That is, if \( x = (x^\mu) = (x^0, x^1, x^2, x^3) \) and \( y = (y^\mu) \) are elements of \( \mathbb{R}^4 \) and \( L \in \mathcal{L} \), then

\[
g_{\mu\nu} x^\alpha y^\nu = g_{\mu\nu} (L(x))^{\alpha} (L(y))^\nu
\]

Just as in the case of the rotation group, \( L \) can be regarded as a \( 4 \times 4 \) matrix \((L^\mu_\nu)\), so that \((L(x))^\mu = L^\mu_\nu x^\nu\), and (2.2.1) is equivalent to

\[
g_{\mu\nu} = g_{\rho\lambda} L^\rho_\mu L^\lambda_\nu
\]

or, in matrix form,

\[
g = L^T g L
\]

\( \mathcal{L} \) consists of all real \( 4 \times 4 \) matrices satisfying (2.2.3).

The elements of \( \mathcal{L} \) are transformations on Minkowski space, so certain elements of \( \mathcal{L} \) have a standard interpretation just as in the rotation group. These are

\[
L^\gamma_i(\theta) =\begin{pmatrix} 1 & 0 \\ 0 & R^1_i(\theta) \end{pmatrix} ;
L^\gamma_3(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & R^3_3(\theta) \end{pmatrix} ;
L^\gamma_3(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & R^3_3(\theta) \end{pmatrix}
\]

\[
L^\gamma_5(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} ;
L^\gamma_3(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}
\]

\[
L^\gamma_3(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} ;
I_3 = g ;
I_4 = -I_3 ;
I_{3t} = I_3 I_4 = -I_4
\]
where $L_{ij}^j(\theta)$ is a rotation of the i-j plane by the angle $\theta$, $L_0^k(\theta)$ is a boost along the k-axis by the velocity $\beta = \tanh \theta$, $I_s$ is space reflection, $I_t$ is time inversion, and $I_{st}$ is total inversion.

Because of the specific form of the $L_{ij}^j(\theta)$ and the fact that the $R_i^j(\theta)$ generate $\mathcal{R}$, $\mathcal{L}$ contains all elements of the form $(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix})$ where $R \in \mathcal{R}$. Therefore, $\mathcal{R}$ may be considered a subgroup of $\mathcal{L}$. Further, from here on any elements of $\mathcal{R}$ or $\mathbb{R}^3$ will automatically be considered as elements of $\mathcal{L}$ or $\mathbb{R}^4$ respectively.

Taking the determinant of both sides of (2.2.3) implies 
$$-1 = \det g = \det(L^T g L) = \det L^T \det g \det L = -(\det L)^2$$
so that

(2.2.4) \[ \det L = \pm 1 \]

Further, setting $\mu = \nu = 0$ in (2.2.2):

$$1 = g_{o\alpha} = g_{\rho \lambda} L^\rho_\alpha L^\lambda_\alpha = (L^\rho_\alpha)^2 - \Sigma_\Delta (L^\rho_\alpha) =$$

(2.2.5) \[ (L^\rho_\alpha)^2 = 1 + \Sigma_\Delta (L^\rho_\alpha)^2 \]

Therefore

(2.2.6) \[ |L^\rho_\alpha| \geq 1 \]

so can be separated into four subsets, defined by

\[ \mathcal{L}_{++}^+ \equiv \{ L \in \mathcal{L} : \det L = +1 ; \ L^\rho_\alpha \geq 1 \} \]
\[ \mathcal{L}_{+-}^+ \equiv \{ L \in \mathcal{L} : \det L = +1 ; \ L^\rho_\alpha \leq -1 \} \]
\[ \mathcal{L}_{-+}^- \equiv \{ L \in \mathcal{L} : \det L = -1 ; \ L^\rho_\alpha \leq -1 \} \]
\[ \mathcal{L}_{--}^- \equiv \{ L \in \mathcal{L} : \det L = -1 ; \ L^\rho_\alpha \geq 1 \} \]
From (2.2.3), \( L^{-1} = g L_T^T g \), so \( \mathbb{1} = L g L_T^T g \) and \( g = L g L_T = (L_T^T g L_T) \). Therefore, \( L \in \mathcal{L} \) implies \( L_T \in \mathcal{L} \). This fact, combined with (2.2.5), gives that

\[
(2.2.7) \quad L \in \mathcal{L} \Rightarrow \begin{cases} 
| L^o_o | > [ \Sigma \chi (L^\dagger_o)]^{1/2} \\
| L^o_o | > [ \Sigma \chi (L^\dagger_o)]^{1/2}
\end{cases}
\]

If \( L, L' \in \mathcal{L} \), then

\[
(2.2.8) \quad (L L')^o_o = L^o_o L'^\dagger_o = L^o_o L'^o_o - L^o_o L'^\dagger_o
\]

The Schwarz inequality implies

\[
| L^\dagger_o L'^\dagger_o | \leq | \Sigma \chi (L^\dagger_o)^2 |^{1/2} | \Sigma \chi (L'^\dagger_o)^2 |^{1/2}
\]

so (2.2.7) implies

\[
| L^o_o L'^o_o | \geq | L^\dagger_o L'^\dagger_o |
\]

and, from (2.2.8), the sign of \( (LL')^o_o \) is equal to the sign of \( L^o_o L'^o_o \).

This fact, together with the property \( \det(LL') = \det L \det L' \), proves that \( \mathcal{L}^+ \) is a group while \( \mathcal{L}^- \), \( \mathcal{L}^+ \), and \( \mathcal{L}^- \) are not. Further, it shows that, if \( L \in \mathcal{L}^+ \), then \( I_s L \in \mathcal{L}^+ \), \( I_t L \in \mathcal{L}^\nu \), and \( I_{st} L \in \mathcal{L}^\nu \). Therefore, \( \mathcal{L}^- \), \( \mathcal{L}^\nu \), and \( \mathcal{L}^\nu \) are exactly the cosets of \( \mathcal{L}^+ \) in \( \mathcal{L} \), and the factor group \( \mathcal{L} / \mathcal{L}^+ \) is isomorphic to the discrete group \( \mathbb{D} \) generated by the elements \( \{ \mathbb{1}, I_s, I_t, I_{st} \} \).

Just as in the case of the rotation group, the elements \( L_1 j (\theta) \) and \( L_0 k (\theta) \) generate a subgroup \( \mathcal{L}' \) of \( \mathcal{L} \) which is connected. All of these generators are in the subgroup \( \mathcal{L}^+ \), so \( \mathcal{L}' \) is also contained in \( \mathcal{L}^+ \). Further, it is
possible to show that $\mathcal{L}_+^t$ and $\mathcal{L}'$ are the same subgroup. The proof is below.

Suppose $L \in \mathcal{L}_+^t$. The three entries $L^i_i$ of the matrix $L$ can be considered as a three vector $\mathbf{v} = (L^1_0, L^2_0, L^3_0)$. As such, from the proof in Section 2.2, there exists a pair of angles $\theta$ and $\varphi$ such that $R_1^2(\varphi)R_2^3(\theta)\mathbf{v} = (|\mathbf{v}|, 0, 0)$. Thus

$$L_1^t(\gamma) L_2^t(\theta) L = \begin{pmatrix} L^0_0 & L^0_1 & L^0_2 & L^0_3 \\ |\mathbf{v}| & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \end{pmatrix}$$

From (2.2.5), $(L^0_0)^2 - |\mathbf{v}|^2 = 1$, so $|\mathbf{v}| = [(L^0_0)^2 - 1]^{\frac{1}{2}}$. $L \in \mathcal{L}_+^t$ implies that $L^0_0 \geq 1$, so there exists an angle $\theta'$ such that $\cosh \theta' = L^0_0$. Therefore,

$$\sinh \theta' = \sqrt{(\cosh \theta')^2 - 1} = |\mathbf{v}|$$

Multiplying $L_1^t(\varphi) L_2^t(\theta) L$ by $L_0^1(-\theta')$ gives

$$L_0^t(-\theta') L_1^t(\gamma) L_2^t(\theta) L = \begin{pmatrix} 1 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \end{pmatrix} \equiv L'$$

But, $L' \in \mathcal{L}$, so $1 = (L^0_0)^2 - \Sigma_i (L^i_0)^2 = 1 - \Sigma_i (L^i_0)^2$, which implies that $L^i_0 = 0$ for all $i$. Therefore, $L'$ has the form
Define $R'$ such that $L' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & L_1' & L_2' & L_3' \\ 0 & L_1'' & L_2'' & L_3'' \\ 0 & L_1''' & L_2''' & L_3''' \end{pmatrix}$. Since $L' \in \mathcal{L}$, $L'$ satisfies $g = (L')^T g(L)$. Therefore $R'$ satisfies $R = (R')^T R'$. Further, $L' \in \mathcal{L}_+$, so $\det L' = 1$ and $\det R' = 1$. Thus, $R' \in \mathcal{R}$ and there exist angles $\phi_1, \phi_2, \phi_3$, such that

$R' = R_2^{-1}(\theta_1) R_1^{-1}(\phi_1) R_2^{-1}(\varnothing)$. This implies that

$L' = L_2^{-1}(\theta_1) L_1^{-1}(\phi_1) L_2^{-1}(\varnothing)$ and that

$$(2.2.9) \quad L = L_2^{-1}(\theta) L_1^{-1}(\varnothing) L_2^{-1}(\theta') L_1^{-1}(\phi_1) L_2^{-1}(\varnothing)$$

Therefore, $L \in \mathcal{L}'$, which completes the proof that

$\mathcal{L}' = \mathcal{L}_+^\uparrow$.

Since $\mathcal{L}'$ and $\mathcal{L}_+^\uparrow$ are identical, $\mathcal{L}_+^\uparrow$ is a connected subgroup of $\mathcal{L}$. The functions $\det L$ and $L^0$ of $L \in \mathcal{L}$ are continuous functions on the set of 4x4 matrices. Since the image of $\mathcal{L}$ in $\mathbb{R}$ for each function has two disjoint components, the inverse images of these components are disjoint. Thus, the subsets $\mathcal{L}_+^\uparrow, \mathcal{L}_-^\uparrow, \mathcal{L}_+^\downarrow, \mathcal{L}_-^\downarrow$ are mutually disjoint. Further, the fact that $\mathcal{L}_+^\uparrow$ is connected implies that its cosets are connected.

Therefore, $\mathcal{L}$ is essentially the product of a discrete group $D$ and a connected group $\mathcal{L}_+^\uparrow$. $\mathcal{L}_+^\uparrow$ is called the proper Lorentz group.
2.3 The Spinor Groups

In a development of the Lorentz and rotation groups aimed towards the study of wave equations, it is necessary to introduce two other groups, \( SU(2) \) and \( SL(2, \mathbb{C}) \). The reason for this is that wave equations are allowed to transform under representations of the Lorentz group which are two-valued, i.e., elements of \( \mathcal{L} \) are represented by two elements of the representation rather than one. The groups \( SU(2) \) and \( SL(2, \mathbb{C}) \) are covering groups of \( \mathcal{R} \) and \( \mathcal{L} \) respectively such that the set of their single-valued representations is the same as the set of the two-valued representations of \( \mathcal{R} \) and \( \mathcal{L} \). Thus, in many cases, the wave equation formalism is more clearly developed with the \( SL(2, \mathbb{C}) \) structure than with the Lorentz group.*

Define \( \mathcal{H} \) to be the set of 2x2 Hermitian matrices. \( \mathcal{H} \) is an additive abelian group and a module over \( \mathbb{R} \) (but not over \( \mathbb{C} \)). Therefore, \( \mathcal{H} \) is a vector space over \( \mathbb{R} \). Further, any \( A \in \mathcal{H} \) is of the form

\[
A = \begin{pmatrix} a & b - i c \\ b + i c & d \end{pmatrix}
\]

where \( a, b, c, d \in \mathbb{R} \), so a possible basis for \( \mathcal{H} \) is the set of Pauli matrices

\[
(2.3.1) \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

* In addition to the previous references, the relation between the spinor and transformation groups is more explicitly explored in Corson.\(^9\)
and \( \mathcal{H} \) is a four-dimensional vector space.

With this basis, any \( A \in \mathcal{H} \) may be uniquely expressed as \( A = \alpha^\mu \sigma_\mu \) for a set \( \{ \alpha^\mu \} \subset \mathbb{R} \). This leads to a (basis-dependent) isomorphism \( I: \mathbb{R}^4 \rightarrow \mathcal{H} \) defined by
\[
I(\alpha^\mu e_\mu) = \alpha^\mu \sigma_\mu \text{ where } e_\mu \text{ is the } \mu^{th} \text{ basis vector of } \mathbb{R}^4,
\]
e.g., \( e_\circ = (1,0,0,0) \). Note that \( I^{-1}(\alpha^\mu \sigma_\mu) = \alpha^\mu e_\mu \).

Let \( T \in \text{Hom}(\mathbb{R}^4) \), the linear transformations from \( \mathbb{R}^4 \) to itself. \( T \) can be represented by a real 4x4 matrix \( T^\mu_\nu \) such that \( T(\alpha^\mu e_\mu) = \alpha^\nu T^\mu_\nu e_\mu \). Thus, \( I \) maps \( T \) to an element \( I(T) = ITI^{-1} \) of \( \text{Hom}(\mathcal{H}) \) by
\[
I(T)(A) = ITI^{-1}(A) = IT(\alpha^\mu e_\mu) = I(\alpha^\nu T^\mu_\nu e_\mu) = \alpha^\nu T^\mu_\nu \sigma_\mu \text{ for } A = \alpha^\mu \sigma_\mu \in \mathcal{H}.
\]

Let \( M \) be an arbitrary 2x2 complex matrix, so that \( M \in \text{Hom}(\mathbb{C}^2) \). \( M \) defines an element \( T_M \) of \( \text{Hom}(\mathcal{H}) \) by
\[
T_M(A) = MAM^+ \text{ for } A \in \mathcal{H}.* \text{ \( M \) also defines an element } I*(M) = I^{-1}T_MI \text{ of } \text{Hom}(\mathbb{R}^4) \text{ by }
\]
\[
I*(M)(\alpha^\mu e_\mu) = I^{-1}T_MI(\alpha^\mu e_\mu) = I^{-1}(M(\alpha^\mu \sigma_\mu) M^+).
\]
Thus, \( I*:\text{Hom}(\mathbb{C}^2) \rightarrow \text{Hom}(\mathbb{R}^4) \) and, since \( I*(MM^+) = I*(M)I*(M^+) \), \( I* \) preserves multiplication.

Note that, if \( Q:\text{Hom}(\mathbb{C}^2) \rightarrow \text{Hom}(\mathcal{H}) \) is defined by \( Q(M) = T_M \), then the mappings \( Q, I^{-1}, I* \) form a commutative diagram, i.e.,

* The convention used here is \( M^+ \) for Hermitian conjugate, \( M^\top \) for transpose, and \( \bar{M} \) for complex conjugate.
\[ \text{Hom}(\mathcal{H}) \xrightarrow{I^*} \text{Hom}(\mathbb{R}^4) \]

\[ \uparrow Q \quad \text{C} \quad \downarrow I^* \]

\[ \text{Hom}(\mathbb{C}^2) \]

and \( I^{-1} \circ Q = I^* \). Further, \( Q \) is neither onto or one-to-one, so neither is \( I^* \).

Denote the \((a,b)\) entry of the matrix \( A \in \mathcal{H} \) by \( A^{ab} \).

The dotted and undotted indices of \( A \) specify the transformation properties of \( A \) under an element \( M \) of \( \text{Hom}(\mathbb{C}^2) \), i.e.,

\[ (T_M(A))^{\cdot d} = (MA^TM^+)^{\cdot d} = M^c_{\cdot a} \overline{M}^d_{\cdot b} A^{ab} \]

So, undotted indices transform by \( M \) and dotted indices by \( \overline{M} \).

There is a metric, \( g_{\mu\nu} \), defined on the space \( \mathbb{R}^4 \), and this metric, along with the isomorphism \( I \), can be used to define a metric \( \varepsilon \) on the space \( \mathcal{H} \) by defining

\[ \varepsilon(A, B) = g(\tilde{I}'(A), \tilde{I}'(B)) = g_{\mu\nu}(\tilde{I}'(A))^{\mu\nu}(\tilde{I}'(B))^{\nu} \]

Note that

\[ A^{\cdot b} = I(\alpha^\mu e_{\mu})^{\cdot b} = \alpha^\mu (\sigma^\mu)^{\cdot b} \]

so that

\[ A^{\cdot i} = \alpha^0 + \alpha^3 \quad ; \quad A^{\cdot \bar{i}} = \alpha^1 - i\alpha^2 \]

\[ A^{\bar{i}\bar{i}} = \alpha^0 - \alpha^3 \quad ; \quad A^{i\bar{i}} = \alpha^1 + i\alpha^2 \]

Therefore

\[ \alpha^0 = \frac{1}{2} (A^{\cdot i} + A^{\bar{i}\bar{i}}) \quad ; \quad \alpha^2 = \frac{i}{2} (A^{\bar{i}\bar{i}} - A^{i\bar{i}}) \]

\[ \alpha^1 = \frac{1}{2} (A^{i\bar{i}} + A^{\bar{i}\bar{i}}) \quad ; \quad \alpha^3 = \frac{1}{2} (A^{i\bar{i}} - A^{\bar{i}\bar{i}}) \]
and
\[ \epsilon(A, B) = \frac{1}{2} \sum g_{\alpha \beta} [A^{\mu}_{\beta} + A^{\alpha}_{\beta}][B^{\mu}_{\beta} + B^{\alpha}_{\beta}] + g_{\mu \nu} [A^{\mu}_{\nu} + A^{\nu}_{\mu}][B^{\mu}_{\nu} + B^{\nu}_{\mu}] \]
\[ - g_{\alpha \beta} [A^{\mu}_{\alpha} - A^{\mu}_{\beta}][B^{\mu}_{\alpha} - B^{\mu}_{\beta}] + g_{\mu \nu} [A^{\mu}_{\nu} - A^{\nu}_{\mu}][B^{\mu}_{\nu} - B^{\nu}_{\mu}] \]
\[ = \frac{1}{2} \left[ A^{\mu}_{\nu} B^{\mu}_{\nu} + A^{\nu}_{\mu} B^{\nu}_{\mu} - A^{\mu}_{\nu} B^{\nu}_{\mu} - A^{\nu}_{\mu} B^{\mu}_{\nu} \right] \]

Note that, by defining \( (\epsilon_{\alpha \beta}) = (\epsilon_{\beta \alpha}) = (0 1) \), the metric \( \epsilon \) may be written
\[ \epsilon(A, B) = \frac{1}{2} \epsilon_{\alpha \beta} \epsilon_{\beta \alpha} A^{\alpha}_{\beta} B^{\beta}_{\alpha} \]

This metric has the property \( \epsilon(A, A) = \det(A) \).

Just as the metric \( g_{\mu \nu} \) may be used to lower vector indices by the formula \( \alpha_{\mu} = g_{\mu \nu} \alpha^{\nu} \) and to raise indices by \( \alpha^{\mu} = g^{\mu \nu} \alpha_{\nu} \) where \( (g_{\mu \nu}) = (g^{\mu \nu}) \), the matrices \( (\epsilon_{\alpha \beta}) = (\epsilon^{\alpha \beta}) \) and \( (\epsilon_{\beta \alpha}) = (\epsilon^{\beta \alpha}) \) may be used to raise and lower "spinor" indices by
\[ A^{\alpha}_{\beta} = \epsilon_{\alpha \beta} A^{\beta}_{\epsilon}, \quad A^{\alpha}_{\beta} = \epsilon_{\alpha \beta} A^{\beta}_{\epsilon} \]

With these manipulations defined, important properties of the Pauli matrices can be developed:

By definition the mapping \( I \) preserves the metrics, so, \( \forall \alpha^{\mu}, \alpha^{' \nu} \):
\[ g_{\mu \nu} \alpha^{\mu} \alpha^{' \nu} = \frac{1}{2} \epsilon_{\epsilon \beta} \epsilon_{\gamma \delta} (\alpha^{\mu} \sigma_{\mu}^{\alpha \gamma})(\alpha^{' \nu} \sigma_{\nu}^{\beta \delta}) \]

Therefore
\[ g_{\mu \nu} = \frac{1}{2} \epsilon_{\epsilon \beta} \epsilon_{\gamma \delta} \sigma_{\mu}^{\alpha \gamma} \sigma_{\nu}^{\beta \delta} \]

Raising and lowering the appropriate indices:
(2.3.3) \[ \delta_{\mu \nu} = \frac{1}{2} \sigma_{\alpha \beta} \sigma_{\nu}^{\alpha \beta} \]
The sixteen numbers $\sigma_{\mu}^{ab}$ can be viewed as a 4x4 matrix by using $\mu$ as the row index and $ab$ as the column index. Call this matrix $S$. Similarly, call $P$ the matrix whose $(\mu,ab)$ entry is $\sigma_{\mu}^{ab}$. With this notation, (2.3.3) can be written as
$$1 = \frac{1}{2} P(S)^T.$$ Therefore, $P^{-1} = \frac{1}{2}(S)^T$ and $1 = \frac{1}{2}(S)^T P$, so
$$\delta_{d^c}^{a} = \frac{1}{2} \sigma_{\mu}^{ab} \sigma_{\mu}^{cd}$$
or
\begin{equation}
(2.3.4) \quad \sigma_{\mu}^{ab} \sigma_{\mu}^{cd} = 2 \delta_{d^c}^{a} \delta_{a^d}^{c}.
\end{equation}

In order to preserve the ordering of some matrix calculations, it is convenient to make the definitions $\sigma_{\mu}^{ab} = \sigma_{\mu}^{ba}$ and $\sigma_{\mu}^{ab} = \sigma_{\mu}^{ba}$. With these definitions it can be seen that, by inspection,
\begin{equation}
(2.3.5) \quad \sigma_{\mu}^{ab} = \sigma_{\mu}^{ba}
\end{equation}

Again, by inspection, the matrices $\{\sigma_{\mu}\}$ have the properties: $\sigma_{\mu}^{2} = 1$, $[\sigma_{\mu}, \sigma_{\nu}] = 0$, $[\sigma_{i}, \sigma_{j}] = 0$ ($i \neq j$). Using (2.3.5), these relations can be combined into the single property
\begin{equation}
(2.3.6) \quad \sigma_{\mu}^{ab} \sigma_{\nu}^{cd} + \sigma_{\nu}^{ab} \sigma_{\mu}^{cd} = 2 \delta_{d^c}^{a} \delta_{a^d}^{c}
\end{equation}

Also, $\sigma_{i} \sigma_{j} = i \epsilon_{i j} k \sigma_{k}$ and $\sigma_{o} \sigma_{k} = \sigma_{k}$ become
\begin{equation}
(2.3.7) \quad \sigma_{\mu}^{ab} \sigma_{\nu}^{cd} = -i/2 \epsilon_{\mu \nu \rho \lambda} \sigma_{\rho}^{ab} \sigma_{\lambda}^{cd}
\end{equation}
image of SL(2, C) under I* is a group contained in Hom( IR^4). If M SL(2, C) and I*(M) = T ε Hom( IR^4), then

\[ q \rho \lambda T^\gamma \nu = q \rho \lambda (\frac{1}{2} \sigma^e b \alpha M^c b d \sigma^c d) (\frac{1}{2} \sigma^d f M^f d \sigma^d) \]

\[ = \frac{1}{4} (2 \epsilon_{efg} \epsilon_{fgh} M^c b d M^f d g M^d c b \sigma^c d \sigma^d) \]

\[ = \frac{1}{2} (\epsilon_{efg} \epsilon_{fgh} M^c b d M^d c b) \sigma^c d \sigma^d \]

(Note that det M = 1 - \epsilon_{fa} M^f d g M^d c b = \epsilon_{gc})

\[ = \frac{1}{2} (\epsilon_{efg} \epsilon_{fgh} \sigma^c d \sigma^d) = \frac{1}{2} \sigma_{\mu \nu} \sigma_{\nu \mu} = q \mu \nu \]

Therefore: T ε L^*_\mu

Further, det T = det (\frac{1}{2} p M s^T) = det (M (\frac{1}{2} s^T p)) = det M

By definition, M = M × M so

\[ det T = det (M × M) = (det M)^2 (det M)^2 = 1 \]

Also

\[ T^0 = \frac{1}{2} \sigma^e b \alpha M^c b d \sigma^c d = \frac{1}{2} \sum_{\alpha, c} M^c b d M^c b d \]

\[ = \frac{1}{2} \sum_{\alpha, c} 1 M^c b d M^c b d \geq 0 \]

Therefore, T ε L^*_+ and I*: SL(2, C) → L^*_+

Note that, for θ real, SL(2, C) contains the matrices

\[ M_1 = \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}, \quad M_3 = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \]

Mapping these matrices into L^*_+ by I* yields

\[ I^* (M_1) = L^*_1(\theta) \]

\[ I^* (M_2) = L^*_2(\theta) \]

\[ I^* (M_3) = L^*_3(\theta) \]
From (2.2.9), these three elements of \( \mathbb{L}^+ \) for all \( \theta \) span \( \mathbb{L}^+ \), so \( I^* \) maps \( \text{SL}(2, \mathbb{C}) \) onto \( \mathbb{L}^+ \). Finally, if \( M \in \text{SL}(2, \mathbb{C}) \) and \( I^*(M) = 1 \), then

\[
1 = \frac{1}{2} s \mathbf{m} \mathbf{p}^T \Rightarrow \mathbf{m} = \frac{1}{2} \mathbf{p}^T \mathbf{s} = 1 \Rightarrow M^c \mathbf{m}^b d = \delta_c^a \delta^b_d \Rightarrow M = c \mathbf{1} \quad \text{where} \quad |c|^2 = 1
\]

But \( M \in \text{SL}(2, \mathbb{C}) \), so \( c = \pm 1 \). Thus, the kernel of \( I^* \) is \( \{ \pm 1 \} \).

So, the mapping \( I^* \) is a two-to-one isomorphism from the group \( \text{SL}(2, \mathbb{C}) \) to the group \( \mathbb{L}^+ \). By definition, this means that the two groups are locally isomorphic and that \( \text{SL}(2, \mathbb{C}) \) is a (two-sheeted) covering group of \( \mathbb{L}^+ \).

The group \( \text{SU}(2) \) is defined as the subgroup of all the unitary matrices of \( \text{SL}(2, \mathbb{C}) \). Thus, a general element \( M \) of \( \text{SU}(2) \) has the form

\[
M = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1 \quad ; \quad \alpha, \beta \in \mathbb{C}
\]

By defining \( \theta \) such that \( |\alpha| = \cos \theta \), \( |\beta| = \sin \theta \)

\( \theta \) such that \( \alpha = \cos \theta e^{i\varphi} \)

\( \theta \) such that \( \beta = i \sin \theta e^{i\varphi'} \)

\( M \) takes the form

\[
M = \begin{pmatrix} \cos \theta e^{i\varphi} & i \sin \theta e^{i\varphi'} \\ i \sin \theta e^{-i\varphi'} & \cos \theta e^{-i\varphi} \end{pmatrix}
\]

\[
= \begin{pmatrix} e^{i(\varphi + \varphi')} & \cos \theta \ i \sin \theta \\ i \sin \theta \ e^{-i(\varphi + \varphi')} & \cos \theta \ e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{i(\varphi - \varphi')} & 0 \\ 0 & e^{-i(\varphi - \varphi')} \end{pmatrix}
\]

Therefore

\[
I^*(M) = L_1^{\frac{3}{2}}(-2(\varphi + \varphi')) L_2^3(2\theta) L_1^{\frac{3}{2}}(-2(\varphi - \varphi'))
\]
This proves that $I^*$ maps $SU(2)$ onto $\mathcal{R}$ since $I^*(M)$ is an element of $\mathcal{R}$ and any element of $\mathcal{R}$ is of this form for some value of $\theta, \varphi, \varphi'$. As a result $I^*$ is a two-to-one isomorphism from $SU(2)$ to $\mathcal{R}$ and so $SU(2)$ is a (two-sheeted) covering group of $\mathcal{R}$.

The isomorphism $I: \mathbb{R}^4 \to \mathcal{H}$ can be restricted to elements of $\mathbb{R}^4$ of the form $(0, x^1, x^2, x^3)$ to give $I': \mathbb{R}^3 \to \mathcal{H}'$ where $\mathcal{H}'$ is the vector space of all traceless elements of $\mathcal{H}$. Since $I*: SU(2) \to \mathcal{R}$, a mapping $I'^*: SU(2) \to \mathcal{R}$ can be defined which differs from $I^*$ only in that it maps to 3x3 matrices rather than 4x4 matrices. Therefore, if $R \in \mathcal{R}$ and $M \in SU(2)$ such that $I'^*(M) = R$, then

$$ R^*_{ij} = \frac{1}{2} \sigma^i_{b\alpha} M^a_{c} \overline{M}^b_{d} \sigma^c_{j} \epsilon^{bd} $$

However, note that, since $M \in SU(2)$, $\overline{M}^1_1 = M^2_2$ and $\overline{M}^1_2 = -M^2_1$. Further, $M^b_a = \epsilon_{ac} M^c_d \epsilon^{db}$, so

$$M^1_1' = -M^2_2, \quad M^2_1 = M^2_1, \quad M^1_2 = M^1_2, \quad M^2_2 = -M^1_1$$

Therefore,

$$ \overline{M}^a_{b} = -M^a_{b} $$

and (2.3.11) can be written

$$ R^*_{ij} = -\frac{1}{2} \sum_{b,d} \sigma^i_{b\alpha} M^a_{c} M^d_{b} \sigma^c_{j} \epsilon^{cd} $$

This expression is clumsy because it sums dotted with undotted indices. This can be remedied by relabeling the entries in the $\sigma_i$ matrices so that
Note that the traceless property of the $\sigma_i$ implies that $\sigma_i^a b = \sigma_i b^a$, so that (2.3.14) does not conflict with the property $\sigma_i a b = \sigma_i b a$. Further, (2.3.14) implies that

$$
\sigma_{-i}^a b = \epsilon_{b^d} \epsilon_{a^c} \sigma^{-i}_{d^c} = \epsilon_{b^d} \epsilon_{a^c} \sigma^{-i c d}
$$

(2.3.15)

$$
= -\epsilon_{a^c} \sigma^{-i c} b \epsilon_{d^b} = -\sigma^{-i c} b
$$

Therefore, (2.3.13) becomes

$$
R^{-i}_j = \frac{1}{2} \sigma^{-i a b} M^{-a c} M^{-b d} \sigma^{-j c d}
$$

(2.3.16)

The main consequence of this development is that, for three-vectors, an upper dotted index is the same as a lower undotted index and a lower dotted index is equivalent to an upper undotted index with the addition of a minus sign.

The equivalence of the groups $SL(2, \mathbb{C})$ and $\mathcal{L}_+$ is useful in another way. If $A^\mu$ is a quantity which transforms by means of the 4-vector index $\mu$, then the equivalence allows a new set of quantities $A^{ab} = A^\mu \sigma_{-\mu}^a b$ to be defined such that the indices transform as spinor indices. The difference here is that the $A^\mu$ need not be real numbers but may also be matrices or linear operators. By means of the relations (2.3.3) and (2.3.4), this index transformation can be written as
where $\varepsilon_{\mu\nu\rho\lambda}$ is completely anti-symmetric with respect to the interchange of indices and $\varepsilon_{0123} = +1$.

With the properties of the Pauli matrices at hand, the mappings $I$ and $I^*$ can be explicitly developed. If $A \in \mathbb{R}$ such that $a_{ab} = \alpha^a \sigma_{\mu b}$, then

$$I(\lambda^\mu e_{\mu}) = \lambda^\mu \sigma_{\mu}$$

and

$$I^*(A) = \frac{1}{2} \sigma_{\mu} b_{ab} a_{ab} e_{\mu}$$

Let $M \in \text{Hom}(\mathbb{C}^2)$. If $I^*(M) = T \in \text{Hom}(\mathbb{R}^4)$ then

$$\lambda^\mu T_{\mu\nu} e_{\nu} = I(I^*(M) (\lambda^\mu e_{\mu})) = I^*(M) (\lambda^\mu e_{\mu}) = I^*(M (\lambda^\mu \sigma_{\mu}) M^*) = \frac{1}{2} e_{\mu} \sigma_{\mu} b_{ab} M^a \overline{M}^b \sigma_{\nu} c_{\nu d} \alpha^\gamma$$

which implies

$$T_{\mu\nu} = \frac{1}{2} \sigma_{\mu} b_{ab} M^a \overline{M}^b \sigma_{\nu} c_{\nu d}$$

Define the matrix $M$ by $(M)^{ab}_{cd} = M^a_c \overline{M}^b_d$. Thus, (2.3.9) becomes

$$T = \frac{1}{2} \rho M S^T$$

Define the group $SL(2, \mathbb{C})$ to be the group of all 2x2 complex matrices with determinant +1. Since $SL(2, \mathbb{C})$ is a group and the mapping $I^*$ preserves multiplication, the
The same development can be performed for 3-vectors. If \( A_i \) transforms as a 3-vector, then a new set of quantities \( A^a b \) can be defined by

\[
\begin{align*}
(2.3.18) \quad \left\{ \begin{array}{lcl}
A^a b &=& \Sigma_i A_i \sigma_i^a b \\
A_i &=& \frac{1}{2} \sigma_i^a b A^a b
\end{array} \right.
\end{align*}
\]

Relations (2.3.18) are derived from the 3-vector forms of (2.3.3) and (2.3.4), which are

\[
(2.3.19) \quad \frac{1}{2} \Sigma_i \sigma_i^a b \sigma_i^c d = \delta^a_c \delta^d_b - \frac{1}{2} \delta^a_b \delta^c d
\]

Note that, for any \( A^a b \) defined by (2.3.18), the relation \( A^a b = A_b^a \) holds, so \( A^{ab} = A^{ba} \).

These index transformations and manipulations provide the handle for the development of the representations of \( L_+ \) and of the wave equations by which they transform.

2.4 Infinitesimal Generators

The group \( L_+ \) is a continuous, connected matrix group. As such, a neighborhood of any point in \( L_+ \) is isomorphic, as a topological set, to a neighborhood about any other point of \( L_+ \). This can be shown by the construction of the isomorphism. If \( L_1 \) and \( L_2 \) are two points in \( L_+ \), then
the mapping \( I: \mathcal{L}_+^+ \to \mathcal{L}_+^+ \) given by \( I(L) = L_2 L_1^{-1} L \) is an isomorphism of topological spaces which maps \( L_1 \) to \( L_2 \).

Further, since \( \mathcal{L}_+^+ \) is a matrix group, each point of it has a neighborhood isomorphic to \( \mathbb{R}^n \) for some \( n \). Therefore, \( \mathcal{L}_+^+ \) is a manifold. A connected matrix group which is also a manifold is, by definition, a connected Lie group.

The Lie algebra of a connected Lie group is the tangent space of the manifold at the identity element.

The group \( \mathcal{L}_+^+ \) can be generated by the elements \( L_0^{-i}(\theta) \) and \( L_1^{j}(\theta) \), so the Lie algebra of \( \mathcal{L}_+^+ \) is spanned by the matrices \( I_o^i \) and \( I_1^j \) defined by

\[
\begin{align*}
I_o^i &= \frac{d}{d\theta} \left( L_0^{-i}(\theta) \right) \bigg|_{\theta=0} \\
I_1^j &= \frac{d}{d\theta} \left( L_1^{j}(\theta) \right) \bigg|_{\theta=0}
\end{align*}
\]

(2.4.1)

The specific forms of the \( I_o^i \) are

\[
\begin{align*}
I_1^1 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} ;
I_2^2 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} ;
I_3^3 &= \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
I_1^2 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} ;
I_2^3 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} ;
I_3^1 &= \begin{pmatrix} 0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

These forms can be extended to all values of \( \mu \) and \( \nu \) by requiring that \( I^{\mu\nu} = -I^{\nu\mu} \) and using the index raising and lowering properties of \( g_{\mu\nu} \). As a result, \( I_1^0 = I_0^1 \), \( I_1^j = -I_j^i \), and \( I^\mu_0 = 0 \). The \( I^{\mu\nu} \) are defined to be the infinitesimal generators of \( \mathcal{L}_+^+ \).
The Lie algebra of $\mathfrak{L}^+$ is spanned by the six independent matrices $\{I^{\mu \nu}\}$, so that any element of this algebra is of the form $w_{\mu \nu}I^{\mu \nu}$, where the $w_{\mu \nu}$ are real quantities such that $w_{\mu \nu} = -w_{\nu \mu}$. Thus, the Lie algebra is six-dimensional and $\mathfrak{L}^+$ is a six-dimensional real manifold.

The elements of the Lie algebra are, by definition, tangent vectors to the space $\mathfrak{L}^+$ at the identity element $I$. Therefore, for an element $w_{\mu \nu}I^{\mu \nu}$ of the algebra, there exists a one parameter subset $\{L(t)\}$ of $\mathfrak{L}^+$ such that $L(0) = I$ and $\frac{d}{dt}(L(t))|_{t=0} = w_{\mu \nu}I^{\mu \nu}$. In order for this subset to be a one parameter subgroup, it must have the property

$$L(t) \cdot L(t_o) = L(t + t_o)$$

This implies that

$$\frac{d}{dt} (L(t + t_o))|_{t=0} = \frac{d}{dt} (L(t) \cdot L(t_o))|_{t=0}$$

$$\Rightarrow$$

$$(2.4.2) \quad \frac{d}{dt} (L(t))|_{t=t_o} = (\omega_{\mu \nu}I^{\mu \nu}) \cdot L(t_o)$$

$$\Rightarrow$$

$$(2.4.3) \quad L(t_o) = \exp (t_o \omega_{\mu \nu}I^{\mu \nu})$$

where $\exp A \equiv \sum_{n=0}^{\infty} 1/n! (A)^n$ is well-defined for all matrices $A$. Equation (2.4.2) implies that, at any point $L(t_o)$ of the path $L(t)$, the tangent vector to the path at that point is an element of the tangent space of $\mathfrak{L}^+$ at that point. This fact, along with the fact that $L(0) = I$ is an element of $\mathfrak{L}^+$, proves that the path defined by (2.4.3)
is contained in $\mathfrak{L}_+^\uparrow$. Therefore, $\exp$ is a map from the Lie algebra of $\mathfrak{L}_+^\uparrow$ to $\mathfrak{L}_+^\uparrow$. The map $\exp$ is, by definition, a local isomorphism from the space of nxn matrices to the space of non-singular nxn matrices, so the restriction of $\exp$ to $\mathfrak{L}_+^\uparrow$ and its Lie algebra is also a local isomorphism.

The series expansions of the trigonometric and hyperbolic functions provide the specific results

$$\exp (\theta \mathbb{I}) = L_{\theta}^0$$

$$\exp (\theta \mathbb{J}) = L_{\theta}^0$$

By defining $L_{\theta}^0(\theta) = L_{\theta}^0(\theta)$ and $L_{\theta}^0(\theta) = L_{\theta}^0(-\theta)$, these relations become

$$(2.4.4) \quad \exp (\theta \mathbb{R}) = L_{\theta}^0$$

The $L_{\theta}^0$ generate $\mathfrak{L}_+^\uparrow$, so the map $\exp$ is onto.

So far, the Lie algebra of $\mathfrak{L}_+^\uparrow$ has been treated as a vector space tangent to the manifold $\mathfrak{L}_+^\uparrow$. However, the manifold $\mathfrak{L}_+^\uparrow$ has the additional structure of a group through the operation of matrix multiplication. This structure can be used to provide a multiplication on the Lie algebra. Let $I$ and $K$ be arbitrary elements of the Lie algebra. Then, for any real $t$, the quantity

$$L(t) = \exp (t \mathbb{I}) \exp (t \mathbb{K}) \exp (-t \mathbb{I}) \exp (-t \mathbb{K})$$

is an element of $\mathfrak{L}_+^\uparrow$. $L(t)$ is a path in $\mathfrak{L}_+^\uparrow$ and $L(0) = \mathbb{I}$, so, the tangent vector of $L(t)$ at $t = 0$ is an element of the Lie algebra. However,
so the tangent vector in this case is given by

\[ \frac{d^2}{dt^2} (L(t)) \big|_{t=0} = I + K - I - K = 0 \]

Therefore, the quantity \([I,K] = IK - KI\) is an element of the Lie algebra, and \([ , ]\) can be defined as the multiplication on the algebra.

The multiplication \([ , ]\) is distributive with respect to matrix addition and multiplication by a constant, so the action \([ , ]\) on any two elements of the Lie algebra is completely defined by the action on the basis set \(\{I^{\mu \nu}\}\). Therefore, given two basis vectors \(I^{\mu \nu}, I^{\rho \lambda}\), there exists a set of constants \(C^{\mu \nu \rho \lambda}_{\delta \sigma}\) such that

\[ [I^{\mu \nu}, I^{\rho \lambda}] = C^{\mu \nu \rho \lambda}_{\delta \sigma} I^{\delta \sigma} \]

This is required by the fact that \([I^{\mu \nu}, I^{\rho \lambda}]\) must be an element of the algebra. Further, the properties of \(I^{\mu \nu}\) and \([ , ]\) require that

\[ C^{\mu \nu \rho \lambda}_{\delta \sigma} = - C^{\mu \nu \rho \lambda}_{\sigma \delta} = - C^{\nu \mu \rho \lambda}_{\delta \sigma} = - C^{\mu \nu \rho \lambda}_{\delta \sigma} = - C^{\nu \mu \rho \lambda}_{\delta \sigma}\]

The \(C^{\mu \nu \rho \lambda}_{\delta \sigma}\) are the structure constants of the algebra and completely define the action \([ , ]\).
Any element of the group $\mathbb{L}^+$ can be written as a product of the elements $L^\mu_\mu(\theta)$. So, considered as an abstract group, the structure of $\mathbb{L}^+$ is determined by the commutation relations of these elements. In turn, the commutation relations of the $L^\mu_\mu(\theta)$ are determined by the commutation relations of their infinitesimal generators $I^\mu\nu$. Therefore, given the Lie algebra with the multiplication $[\ ,\ ]$ defined, the group $\mathbb{L}^+$ is completely specified.

The specific forms of the $I^\mu\nu$ are defined by (2.4.1). By direct calculation on these forms, the structure constants for the proper Lorentz group are given by

$$[I^\sigma\nu, I^\rho\lambda] = g^{\rho\sigma} I^\lambda\rho - g^{\rho\lambda} I^\sigma\rho + g^{\rho\lambda} I^\nu\sigma - g^{\rho\nu} I^\sigma\lambda$$

These equations are called the integrability conditions for $\mathbb{L}^+$. 

Any Lie group has a Lie algebra with a multiplication $[\ ,\ ]$ and a map exp associated with it, just as the proper Lorentz group does. The map exp and the multiplication $[\ ,\ ]$ have specific forms determined by the group structure. However, for a Lie group of matrices, the forms are always the same and $[\ ,\ ]$ is the commutator while exp has the specific expansion shown in (2.4.3).

As a connected subgroup of $\mathbb{L}^+$, $\mathcal{R}$ is also a connected Lie group. The infinitesimal generators of $\mathcal{R}$ are the $3 \times 3$ matrices $\{I^{ij}\}$, where $I^{ij}$ is equal to the matrix $I^{ij}$ with the first row and column deleted. Therefore, any element of $\mathcal{R}$ is of the form
(2.4.7) \[ R = \exp(\omega_{ij} I^{ij}) \]

for some set of real parameters \( \omega_{ij} = -\omega_{ji} \). Further, the generating elements \( R^i_j(\theta) \) have the specific form

(2.4.8) \[ R^i_j(\theta) = \exp(\theta I^{ij}) \]

in analogy to (2.4.4).

The integrability conditions for \( \mathcal{R} \) can be taken directly from (2.4.6).

(2.4.9) \[ [I^{ij}, I^{nm}] = q^{jn} I^{im} - q^{jm} I^{in} + q^{im} I^{jn} - q^{in} I^{jm} \]

If the quantity \( \varepsilon_{ijk} \) is defined to be completely antisymmetric in its indices and to have the property \( \varepsilon_{123} = 1 \), then, for any \( R \in \mathcal{R} \),

\[ \varepsilon_{ijk} R^i_n R^j_m R^k_l = \varepsilon_{nml} \det R = \varepsilon_{nml} \]

Therefore the indices of \( \varepsilon_{ijk} \) and 3-vector indices. Further, \( \varepsilon_{ijk} \) has the property

(2.4.10) \[ \sum_k \varepsilon_{ijk} \varepsilon_{klm} = \delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk} \]

Defining \( I'_i = \frac{1}{2} i \varepsilon_{ijk} I^{jk} \) and using (2.4.9) and (2.4.10), the integrability conditions for \( \mathcal{R} \) can be set in a more familiar form.

\[ [I^{i'}, I^{j'}] = -\frac{1}{4} \varepsilon_{inm} \varepsilon_{jkl} \left[ I^{nm}, I^{kl} \right] = i \sum_k \varepsilon_{ijk} I^k_i \]
Therefore

\[(2.4.11) \quad [ \mathcal{I}_i, \mathcal{I}_j ] = \epsilon_{ij} \epsilon_k \mathcal{I}_k \]

The development of spinor indices in Section 2.3 provides a method for investigating the integrability conditions of $\mathcal{L}_+$. With the index transformations, the six matrices $\mathcal{I}^{\mu\nu}$ can be replaced by two spinors $K_a^b, L_{\dot{a}}^\dot{b}$ defined by

\[
(2.4.12) \begin{cases}
K_a^b = -\frac{1}{4} \mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{b}\dot{c}} \sigma_{\nu}^{\dot{c}\dot{a}} \\
L_{\dot{a}}^\dot{b} = \frac{1}{4} \mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{a}\dot{c}} \sigma_{\nu}^{\dot{c}\dot{b}}
\end{cases}
\]

This is an equivalent set because an inverse transformation exists.

\[
(2.4.13) \quad \mathcal{I}^{\mu\nu} = -\frac{1}{4\epsilon} \sigma_{\mu}^{\dot{c}\dot{a}} \sigma_{\nu}^{\dot{a}\dot{c}} \left( \epsilon^{\dot{d} \dot{e} \dot{f}} K^{\dot{d} \dot{e} \dot{f}} + \epsilon^{\dot{a} \dot{c} \dot{d}} L^{\dot{a} \dot{c} \dot{d}} \right)
\]

Note that $K_{ab} = K_{d}^{b} \epsilon^{da} = \frac{1}{4} (\mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{b}\dot{c}} \sigma_{\nu}^{\dot{c}\dot{a}}) \epsilon^{da}$

\[
= \frac{1}{4} (\mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{b}\dot{c}} \sigma_{\nu}^{\dot{c}\dot{a}}) = -\frac{1}{4} (\mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{b}\dot{a}} \sigma_{\nu}^{\dot{c}\dot{d}}) \epsilon^{db} = -\frac{1}{4} (\mathcal{I}^{\mu\nu} \sigma_{\mu}^{\dot{b}\dot{a}} \sigma_{\nu}^{\dot{c}\dot{d}}) \epsilon^{db} = K_{d}^{a} \epsilon^{da} = K^{ba}
\]

Similarly $L_{\dot{a}}^{\dot{b}} = L_{\dot{a}}^{\dot{b}}$.

Therefore, the integrability conditions for $\mathcal{L}_+$ can be restated as the set of commutation relations among the $K_{a}^{b}$ and $L_{\dot{a}}^{\dot{b}}$.

\[
[ K_a^b, K_c^d ] = \frac{1}{4\epsilon} \sigma_{\mu}^{\dot{b}\dot{c}} \sigma_{\nu}^{\dot{c}\dot{a}} \sigma_{\rho}^{\dot{d}\dot{f}} \sigma_{\lambda}^{\dot{f}\dot{c}} [ \mathcal{I}^{\mu\nu}, \mathcal{I}^{\rho\lambda} ]
\]
Similarly

\[
[ K^{ab}, L^{cd} ] = \frac{1}{2} [ \epsilon_{ad} K^{bc} + \epsilon_{bd} K^{ca} + \epsilon_{bc} K^{ad} + \epsilon_{ac} K^{bd} ]
\]

Further, the spinor \( K^{ab} \) is symmetric in the indices a and b, so these spinor indices may be replaced by a single 3-vector index. From (2.3.18):

\[
K \equiv \frac{1}{2} \sigma^{a b} K_{a b}
\]

so that (2.4.14) becomes

\[
[ K_{ij}, K_{j} ] = \frac{1}{2} \sigma_{i b} \sigma_{j c} [ K^{a b}, K^{c d} ] = i \sum_{k} \epsilon_{i j k} K_{k}
\]

The spinor \( L^{\hat{a} \hat{b}} \) is also symmetric in its indices and the 3-vector transformation can be applied to it also.

By changing the dotted indices to undotted indices, \( L^{\hat{a} \hat{b}} = L_{a b} \) and (2.4.15) takes the form

\[
[ L_{a b}, L_{c d} ] = \frac{1}{2} [ \epsilon_{a d} L^{b c} + \epsilon_{b d} L^{a c} + \epsilon_{b c} L^{a d} + \epsilon_{a c} L^{b d} ]
\]

which is equivalent to

\[
[ L^{a b}, L^{c d} ] = \frac{1}{2} [ \epsilon_{a d} L^{b c} + \epsilon_{b d} L^{c a} + \epsilon_{b c} L^{a d} + \epsilon_{a c} L^{b d} ]
\]
As before, the spinor indices are replaced by a three-vector index so that

\[(2.4.19) \quad L_i = \frac{1}{2} \sigma^a \sigma^b L_a b\]

and \((2.4.18)\) becomes

\[(2.4.20) \quad [L_i, L_j] = \epsilon_{ijk} L_k\]

Equation \((2.4.16)\) now takes the form

\[(2.4.21) \quad [K_i, L_i] = 0\]

Therefore, the \(\{I^\mu\nu\}\) have been replaced by two sets \(\{K_i\}\) and \(\{L_i\}\) of mutually commuting matrices, each of which satisfies the integrability conditions of \(R\).

With the preceding relations, the \(K^a b, L^a b, K_i,\) and \(L_i\) can be calculated explicitly in terms of the \(I^\mu\nu\). The forms are:

\[(2.4.22) \quad \begin{cases} 
K'' = -K_2^1 = -\frac{1}{2} \left( I^{01} - i I^{02} + i I^{23} + I^{31} \right) \\
K^{22} = K_1^2 = \frac{1}{2} \left( I^{01} + i I^{02} + i I^{23} - I^{31} \right) \\
K'^{12} = K^2_1 = K_1^1 = \frac{1}{2} \left( I^{03} + i I^{12} \right) 
\end{cases}\]

\[(2.4.23) \quad \begin{cases} 
L^{i i} = -L_2^i = -\frac{1}{2} \left( I^{01} + i I^{02} - i I^{23} + I^{31} \right) \\
L^{i 2} = L_1^i = \frac{1}{2} \left( I^{01} - i I^{02} - i I^{23} - I^{31} \right) \\
L^{i 1} = L^i_1 = L_1^i = \frac{1}{2} \left( I^{03} - i I^{12} \right) 
\end{cases}\]
\[ K_1 = \frac{\sqrt{2}}{2} (K_1^2 + K_2^1) = \frac{\sqrt{2}}{2} (I^0 + i I^{23}) \]
\[ K_2 = \frac{\sqrt{2}}{2} (i K_1^1 - i K_1^2) = \frac{\sqrt{2}}{2} (I^0 + i I^{31}) \]
\[ K_3 = K_1^1 = \frac{\sqrt{2}}{2} (I^0 + i I^{12}) \]

\[ L_1 = -\frac{\sqrt{2}}{2} (L_{11}^1 + L_{12}^2) = \frac{\sqrt{2}}{2} (-I^0 + i I^{23}) \]
\[ L_2 = -\frac{\sqrt{2}}{2} (i L_{11}^1 - i L_{12}^2) = \frac{\sqrt{2}}{2} (-I^0 + i I^{31}) \]
\[ L_3 = -L_{12}^1 = -L_{12}^2 = \frac{\sqrt{2}}{2} (-I^0 + i I^{12}) \]

Note that, if \( I_{1} \) is defined to be the 4x4 matrix corresponding to the 3x3 matrix \( I_{1} \), then (2.4.24) and (2.4.25) imply that

\[ K_{1} = \frac{\sqrt{2}}{2} (I_{1} + I_{0}^{1}) \]
\[ L_{1} = \frac{\sqrt{2}}{2} (I_{1} - I_{0}^{1}) \]

Therefore, the definition of the \( K_{1} \) and \( L_{1} \) in this section corresponds to the more familiar definition.
III. REPRESENTATIONS OF THE TRANSFORMATION GROUPS

Relativistic wave equations are characterized by the way they transform when the variables $x^\mu$ of $\mathbb{R}^4$ are transformed by elements of the Lorentz group. In order to be relativistic, a wave equation must transform by a representation of $\mathcal{L}$. This chapter will deal with the representations of $\mathcal{L}_+^r$, the maximal connected subgroup of $\mathcal{L}$, and with the representations of $\mathcal{R}$, which are closely related to them.* Further, the Dirac $u,v$ matrices, which are necessary in the Bhaba analysis of relativistic wave equations, will be developed.

This work will only deal with wave equations which are finite-dimensional matrix equations. As a result, all of the representations will be required to be continuous, finite-dimensional matrix representations.

3.1 Representations of $\mathcal{R}$

An $n$-dimensional matrix representation of a group $G$ is a mapping $T:G \to \text{Hom}(\mathbb{R}^n)$ such that

1) if $g_1, g_2, g_3 \in G$ such that $g_1 g_2 = g_3$, then
   \[ T(g_1) T(g_2) = T(g_3) \]

2) if $e$ is the identity element of $G$, then $T(e) = \mathbb{I}$.

Two representations of $G$ are "equivalent" if there exists a similarity transformation from one to the other. That is, * Once again, see Gelfand\textsuperscript{7} and Naimark\textsuperscript{8} for a more extensive development of these topics.
T and T' are equivalent if there exists a unitary matrix V such that $T'(g) = V^{-1}T(g)V$ for all $g \in G$. Two representations which are equivalent are considered to be essentially the same representation.

A subspace of $\mathbb{R}^n$ is said to be invariant with respect to T if each $T(g)$ maps the subspace to itself. If a subspace $M$ is invariant under $T$, then a new representation of $G$ can be defined by the action of $T$ on $M$ alone. A representation which has no invariant subspaces except $\{0\}$ and the entire space $\mathbb{R}^n$ is said to be irreducible.

A continuous representation of the group $G$ is one for which the mapping $T$ is continuous. As a result, if $G$ is a continuous Lie group of matrices, then the matrix group $\{T(g)\}$ is also. Further, the infinitesimals of $\{T(g)\}$ are well-defined, so the map $T$ maps the infinitesimal generators of $G$ to the infinitesimal generators of $\{T(g)\}$. Because a representation of $G$ preserves the group multiplication, the infinitesimal generators of $\{T(g)\}$ satisfy the same integrability conditions as those of $G$.

Let $T$ be a representation of $\mathcal{R}$ and $I_\xi$ be the infinitesimal generators of this representation. The $I_\xi$ satisfy the integrability conditions

$$[I_\xi, I_\eta] = i \sum_k \epsilon_{\xi \eta \kappa} I_k$$

Define the quantity $I^2$ by $I^2 = \sum_k (I_k)^2$ and note that
Thus, $I^2$ commutes with the entire Lie algebra since it commutes with all the $I_j$. As a result, the set of eigenvectors of $I^2$ corresponding to a particular eigenvalue form an invariant subspace of the representation. Every representation is a direct sum of such subspaces, so it is only necessary to deal with them in a study of the representations of $\mathcal{R}$. Further, it is possible to choose a representation which is equivalent to $T$ for which $I^2$ and $I_3$ are both diagonal matrices since they commute. Therefore, it suffices to deal with a representation $T$ of for which $I_3$ is diagonal and $I^2$ is of the form $I^2 = J I$ where $J \in \mathbb{C}$.

The calculation of exactly which values of $J$ are possible and what the exact forms of the corresponding representations are is common to most quantum mechanics texts and will not be done here. The results are that, for each non-negative integer or half-integer $k$, there exists a unique (up to equivalence) irreducible representation of $\mathcal{R}$ of dimension $2k+1$. This representation is called $\mathcal{R}_k$ and for it $J$ has the value $k(k+1)$. For the purposes of calculation, it is convenient to define a "standard" form for the elements of $\mathcal{R}_k$. This form is (where the $I_i(k)$ are the inf. gen. of $\mathcal{R}_k$)
\[ [I_1(k)]_{rs} = \frac{1}{2} \left[ r(2k-r+1) \right] \delta_{r+s,5} + \frac{1}{2} \left[ (r-1)(2k-r+2) \right] \delta_{r-s,5} \]

\[ [I_2(k)]_{rs} = -\frac{i}{2} \left[ r(2k-r+1) \right] \delta_{r+s,5} + \frac{i}{2} \left[ (r-1)(2k-r+2) \right] \delta_{r-s,5} \]

\[ [I_3(k)]_{rs} = (k-r+1) \delta_{r,s} \]

where \( r, s \in \{1, \ldots, 2k+1\} \)

or, more familiarly,

\[ [I_1(k)]_{m'm} = \frac{1}{2} \left[ (k-m)(k+m+1) \right] \delta_{m+m',m} + \frac{1}{2} \left[ (k+m)(k-m+1) \right] \delta_{m-m',m} \]

\[ [I_2(k)]_{m'm} = -\frac{i}{2} \left[ (k-m)(k+m+1) \right] \delta_{m+m',m} + \frac{i}{2} \left[ (k+m)(k-m+1) \right] \delta_{m-m',m} \]

\[ [I_3(k)]_{m'm} = m \delta_{m'm} \]

where \( m, m' \in \{k, \ldots, -k\} \)

Note that, with this form, \( I_j(k) \) is hermitian and \( I_j(\frac{1}{2}) = \frac{1}{2} \sigma_j \).

Strictly speaking, these representations are the irreducible representations of \( SU(2) \) because elements of the half-interval representations preserve the multiplication in \( \mathcal{R} \) only up to a sign. However, this distinction is unnecessary in wave mechanics since only the norm of a quantity is measurable.

3.2 Representations of \( \mathcal{L}^\uparrow_+ \)

Let \( T \) be a representation of \( \mathcal{L}^\uparrow_+ \) with infinitesimal generators \( I^{\mu\nu} \). The \( I^{\mu\nu} \) must satisfy (2.4.6), the integrability conditions for \( \mathcal{L}^\uparrow_+ \). From the development in Section 2.4, the two sets \( \{K_+\} \) and \( \{L_+\} \) can be defined so that each set satisfies the integrability conditions for
\( R \) and the sets commute. Therefore, each set defines a representation of \( R \). Without loss of generality, \( T \) can be chosen so that the representation defined by \( L_i \) is in block diagonal form, i.e.,

\[
L_i = \begin{pmatrix}
I_i(\ell_1) & 0 & \cdots & 0 \\
0 & I_i(\ell_2) & \cdots & 0 \\
& & \ddots & \vdots \\
0 & 0 & \cdots & I_i(\ell_n)
\end{pmatrix}
\]

where \( \ell_1 \geq \ell_2 \geq \ldots \geq \ell_n \) and \( I_i(\ell_a) \) is in standard form.

Let \( K_i \) be written in the form

\[
K_i = \begin{pmatrix}
(K_i)_{11} & (K_i)_{12} & \cdots & (K_i)_{1n} \\
(K_i)_{21} & (K_i)_{22} & \cdots & (K_i)_{2n} \\
& & \ddots & \vdots \\
(K_i)_{n1} & (K_i)_{n2} & \cdots & (K_i)_{nn}
\end{pmatrix}
\]

where \( (K_i)_{ab} \) is a \((2\ell_a + 1) \times (2\ell_b + 1)\) sub-matrix. The fact that \( K_i \) commutes with each \( L_j \) implies that

\[
(3.2.1) \quad (K_i)_{ab} I_j(\ell_b) = I_j(\ell_a) (K_i)_{ab} \quad \forall a,b,i,j
\]

Note that

\[
(3.2.2) \quad (K_i)_{ab} [I_j(\ell_b)]^2 = I_j(\ell_a) (K_i)_{ab} I_j(\ell_b)
\]

Summing (3.2.2) over \( j \) and using the fact that

\[
\Sigma_j [I_j(\ell_b)]^2 = \ell_b (\ell_b + 1) \quad , \quad (3.2.2) \text{ becomes}
\]
(3.2.3) \[ l_b (l_b + 1) (K_i)_{ab} = l_a (l_a + 1) (K_i)_{ab} \]
so that \((K_i)_{ib} = 0\) if \(l_a \neq l_b\). If \(l_a = l_b\), then, since
\(I_j(l_b)\) is irreducible, (3.2.1) implies that \((K_i)_{ab}\) commutes
with every \((2l_b + 1)\times(2l_b + 1)\) square matrix. Therefore,
\((K_i)_{ab}\) is a constant multiple of the identity matrix.

Let \(n'\) be the number of distinct values of \(l_a\) and
relabel them as \(l_1 > l_2 > \ldots > l_{n'}\). Further, let \(n_a\) be
the multiplicity of \(I_i(l_a)\) in \(L_i\). \(L_i\) can now be written
\[ L_i = \bigoplus_{a=1}^{n'} \left[ \mathbb{I}_{n_a} \times I_i(l_a) \right] \]
The analysis of \(K_i\) implies that there exist \(n_a\times n_a\) matrices
\((K_i)\) such that
\[ K_i = \bigoplus_{a=1}^{n'} \left[ (K_i)_{a} \times \mathbb{I}_{2l_a + 1} \right] \]
Since the \(K_i\) form a representation of \(A\), the \((K_i)\) for
each \(a\) must also. Therefore, \(T\) can be chosen so that each
\((K_i)\) is in block form. Thus, there exists an \(m\) and a
set of constants \(k_a, l_a\) for \(a = 1, \ldots, m\) such that
\[ K_i = \bigoplus_{a=1}^{m} \left[ I_i(k_a) \times \mathbb{I}_{2l_a + 1} \right] \]
(3.2.4) \[ L_i = \bigoplus_{a=1}^{m} \left[ \mathbb{I}_{2k_a + 1} \times I_i(l_a) \right] \]

Note that, for \(m \neq 1\), the set \(\{K_i, L_i\}\) is reducible
while, for \(m = 1\), it is irreducible. Thus the irreducible
representations of \(L^+_+\) are those for which there exists a
\(k\) and \(l\) such that
Define $I^\mu \nu(k, \ell)$ to be the infinitesimal generators of this irreducible representation and define the "standard form" of this representation to be the one for which (3.3.5) holds with $I_i^i(k)$ and $I_i^j(\ell)$ in the standard form for representations of $\mathcal{R}$. The representations generated by $I^\mu \nu(k, \ell)$ is called $\mathcal{O}(k, \ell)$.

Any (finite) representation of $\mathcal{Z}^\dagger_+$ is a direct sum of the $\mathcal{O}(k, \ell)$.

3.3 The Dirac u,v Matrices

This section will introduce the Dirac u and v matrices necessary in the Bhaba analysis performed in Section 4.2. However, it is necessary to show both the existence and uniqueness of these solutions along with specifying their defining equations. As a result, the development here starts with a situation which defines the matrices and guarantees their existence. From this beginning, the defining equations and uniqueness are then developed and proven. In addition, an additional property is gained from this method, the relationship between the u,v matrices and the reduction of product representations of $\mathcal{R}$.*

* The method of development in this section is a more detailed form of the one found in Corson.
A representation $M$ of $\mathcal{R}$ can be constructed from the two irreducible representations $\mathcal{D}_{\frac{1}{2}}$ and $\mathcal{D}_k$ by taking the direct product of the two. That is, $M$ is a $2(2k+1)$ dimensional representation of $\mathcal{R}$ with $M(R) = \mathcal{D}_{\frac{1}{2}}(R) \times \mathcal{D}_k(R)$ for all $R \in \mathcal{R}$. Every representation of $\mathcal{R}$ is a direct sum of representations $\mathcal{D}_k$, so $M$ is also. In fact, $M$ is equivalent to the representation $\mathcal{D}_{k+\frac{1}{2}} \oplus \mathcal{D}_{k-\frac{1}{2}}$. Therefore, a non-singular $2(2k+1) \times 2(2k+1)$ matrix $U$ exists such that

$$U M(R) U^{-1} = \begin{pmatrix} \mathcal{D}_{k+\frac{1}{2}}(R) & 0 \\ 0 & \mathcal{D}_{k-\frac{1}{2}}(R) \end{pmatrix} \quad \forall R \in \mathcal{R}$$

Define a matrix $A_k = \sum_i \sigma_i \times I_i(k)$. The infinitesimal generators of $\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_k$ are

$$\mathbf{I}_j' \equiv \frac{1}{2} \sigma_j \times 1 + 1 \times I_j^{(k)}$$

where $I_j^{(k)} = \frac{1}{2} \sigma_j$ is in standard form. Thus, $[[\mathbf{I}_j', A_k]] = 0$, i.e., $A_k$ commutes with all of the elements of $\frac{1}{2} \times k$. Therefore

$$U A_k U^{-1} = \begin{pmatrix} \lambda \mathbf{1}_{2(k+1)} & 0 \\ 0 & \beta \mathbf{1}_{2k} \end{pmatrix}$$

Further, one finds that $A_k^2 + A_k - k(k+1) = 0$, so that $A_k$ has eigenvalues $k$ and $-(k+1)$. Also, $\text{Tr}(A_k) = 0$ implies that the eigenvalue $k$ has multiplicity $2(k+1)$ and the eigenvalue $-(k+1)$ has multiplicity $2k$, so that

$$U A_k U^{-1} = \begin{pmatrix} k \mathbf{1}_{2(k+1)} & 0 \\ 0 & -(k+1) \mathbf{1}_{2k} \end{pmatrix}$$
Given \(V\) such that \(V\) diagonalizes \(A_k\) to this form, then \(VA_kV^{-1}\) commutes with \(UM(R)V^{-1}\) for all \(R \in \mathcal{R}\). Thus
\[
UM(R)V^{-1} = \begin{pmatrix} M_1(R) & 0 \\ 0 & M_2(R) \end{pmatrix}
\]
for all \(R\) and \(V\) reduces \(\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_k\). If \(V'\) also diagonalizes \(A_k\) to this form, then \(V'V'^{-1}\) commutes with \(VA_kV^{-1}\), so \(V' = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} V\) where \(T_1\) and \(T_2\) are non-singular.

Using the definition for \(\sigma_i\) (let \(I_j = I_j(k)\)) implies
\[
A_k = \begin{pmatrix} I_3 & I_1 - iI_2 \\ I_1 + iI_2 & -I_3 \end{pmatrix}
\]
From (2.3.18), the 3-vector index can be written as a pair of spinor indices by using \(I^a_b = \sum_i I_{i1} \sigma_i^a \). Therefore,
\[
A_k = \begin{pmatrix} I^1_{i1} & I^2_{i2} \\ I^2_{i1} & I^3_{i3} \end{pmatrix}
\]
Introduce \(u^a, v^a, u_a\) and \(v_a\) by
\[
V \equiv (2k+1)^{-\frac{1}{2}} \begin{pmatrix} u_1(k+\frac{v}{2}) & u_2(k+\frac{v}{2}) \\ \overline{v}_1(k) & \overline{v}_2(k) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2k} \]

\[
V^{-1} \equiv (-1)^{2k+1}(2k+1)^{-\frac{1}{2}} \begin{pmatrix} v_1'(k+\frac{v}{2}) & v_2'(k) \\ \overline{v}_1'(k+\frac{v}{2}) & \overline{v}_2'(k) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2k+1}
\]

Thus
\[
A_k = V^{-1} \begin{pmatrix} k & 0 \\ 0 & -(k+1) \end{pmatrix} V
\]
\[
= (-1)^{2k+1}(2k+1)^{-\frac{1}{2}} \begin{pmatrix} k v'u_1 - (k+1) u_vu_1 & k v'u_2 - (k+1) u_vu_2 \\ k v'u_1 - (k+1) u_vu_1 & k v'u_2 - (k+1) u_vu_2 \end{pmatrix}
\]
\[
\Rightarrow
\]
(3.3.1) \( k \nu^a \nu_b = (2k+1)(-1)^{2k+1} I^a_b + (k+1) u^a \nu_b \)

Further, \( UU^{-1} = U^{-1}U = 1 \) implies

(3.3.2) \( \nu^a \nu_b + u^a \nu_b = (2k+1)(-1)^{2k+1} \delta^a_b \)

(3.3.3) \( u^a \nu^a = \nu^a_a \nu^a = (2k+1)(-1)^{2k+1} \)

(3.3.4) \( u^a u^a = 0 = \nu^a \nu^a \)

However, (3.3.2) and (3.3.3) imply (3.3.4), so (3.3.4) is redundant. Also, (3.3.1) and (3.3.2) together can be rewritten as

\[
\begin{align*}
\nu^a \nu_b &= (I^a_b + (k+1) \delta^a_b)(-1)^{2k+1} \\
u^a \nu_b &= (-I^a_b + k \delta^a_b)(-1)^{2k+1}
\end{align*}
\]

The defining relations for the \( u^a \) and \( v^a \) matrices can now be written as

(3.3.5) \( u^a(k) \nu_b(k) = \left[ k \delta^a_b - I^a_b(k) \right] (-1)^{2k+1} \)

(3.3.6) \( \nu^a(k) u^a(k) = (2k+1)(-1)^{2k+1} \)

(3.3.7) \( \nu^a(k+1/2) \nu_b(k+1/2) = \left[ (k+1) \delta^a_b + I^a_b(k) \right] (-1)^{2k+1} \)
Note that (3.3.5) and (3.3.6) determine $u^a(k)$ and $v^b(k)$ up to an arbitrary $2k \times 2k$ non-singular matrix $T_1$. That is, $u'^a(k) = u^a(k)T_1^{-1}$ and also satisfy (3.3.5) and (3.3.6).

Further, $v'^b(k) = T_1v^b(k)$ also satisfy (3.3.7) and (3.3.8) for any arbitrary $(2k+2)\times(2k+2)$ non-singular matrix $T_2$.

Let $k' = k + \frac{1}{2}$ and define

$U^a(k') = u^a(k') \in b\omega$

$V^b(k') = V^b(c k')$

Using the relations (3.3.5)-(3.3.8), the following commutation relations can be established:

$[ U^1 V_1, U^1 V_2 ] = (-1)^{2k'+1} U^1 V_2$

$[ U^1 V_1, U^2 V_1 ] = -(-1)^{2k+1} U^2 V_1$

$[ U^2 V_1, U^1 V_1 ] = 2k' - (-1)^{2k'+1} U^1 V_1$

By defining

$I_1' = -\frac{1}{2} (-1)^{2k'+1} (U^1 V_2 + U^2 V_1)$

$I_2' = -\frac{1}{2} (-1)^{2k+1} (U^1 V_2 - U^2 V_1)$

$I_3' = k' - (-1)^{2k'+1} U^1 V_1$
these commutation relations become

$$[I_i', I_j'] = i \sum_j I^j I_k$$

Thus, $I_1'$, $I_2'$, $I_3'$ are the infinitesimal generators of a representation of $R$. Further, by direct calculation,

$$I_i' + I_j' + I_k' = i \sum_j I^j I_k$$

so the particular representation generated by the $I_i'$ is the irreducible representation $R_{k'} = k + \frac{1}{2}$. Introducing

$$I_i'(k') = I_i' \quad \text{AND} \quad I_{ab}^{\alpha}(k') = \sum_j I^j I_{ab}^\alpha_{\nu}$$

we find that $U^a(k')$ and $V_b(k')$ satisfy

$$U^a(k') V_b(k') = \left[ k' S_{ab} - I_{ab}^{\alpha}(k') \right] (-1)^{2k'+'1}$$

from the definition of $I_i'$. Also

$$V_b(k') U^b(k') = -U^a W_{\alpha} = (-1)^{2k'+1} (-1)^2 (k+1)$$

$\equiv (-1)^{2k'+1} (2k+1)$$$

Therefore:

Given $u_a(k+\frac{1}{2})$ and $v_b(k+\frac{1}{2})$ satisfying (3.3.7) and (3.3.8) for a representation of $B_k$, there exists a representation of $B_{k+\frac{1}{2}}$ such that $u_a(k+\frac{1}{2}) = u_b(k+\frac{1}{2}) e^{ba}$ and $v_b(k+\frac{1}{2}) = e_{ba} v^b(k+\frac{1}{2})$ satisfy (3.3.5) and (3.3.6) for $k' = k + \frac{1}{2}$.

Suppose that, for a particular $k$, $I_a^a(k)$ and $I^a_b(k-\frac{1}{2})$ are given. Then there exists a set of matrices $u_a(k)$ and $v^b(k)$ which satisfy (3.3.7) and (3.3.8) (with $k$ instead of
and there exists a representation $I^{\alpha}_{\beta}(k)$ such that $u^a(k)$ and $v^b(k)$ satisfy (3.3.5) and (3.3.6) (with $I'$ instead of $I$). Since $I^{\alpha}_{\beta}(k)$ and $I^{\alpha}_{\beta}(k)$ both generate the representation $\mathcal{D}_k$, there exists a non-singular matrix $T$ such that $I^{\alpha}_{\beta}(k) = T^{-1}I^\alpha_{\beta}(k)T$ for all $a, b$. Thus

$$u^a(k) v^b(k) = [k \delta^\alpha_{\beta} - T^{-1}I^\alpha_{\beta}(k)T] (-1)^{2k+1}$$

Define $u^a(k) = T u^a(k)$ and $v^b(k) = v^b(k)T'$. Note that this implies that $v^b(k)u^a(k) = v^b(k)u^a(k)$ and $v^b(k)u^a(k) = v^b(k)u^a(k)$. Further

$$u^a(k) v^b(k) = T u^a(k) v^a(k) T^{-1} = 2k (-1)^{2k}$$

Therefore:

Given $I^\alpha_{\beta}(k)$ and $I^\alpha_{\beta}(k-\frac{1}{2})$, there exists a set of matrices $u^a(k)$ and $v^b(k)$ such that $u^a$ and $v^b$ satisfy (3.3.7) and (3.3.8) for $k \rightarrow k-\frac{1}{2}$ and $u^a$ and $v^b$ satisfy (3.3.5) and (3.3.6).

Further, $u^a(k)$ and $v^b(k)$ are unique up to the transformation $u^a \rightarrow cu^a$, $v^b \rightarrow c^{-1}v^b$ for $c \in \mathbb{C}$. For, if $u^a$ and $v^b$ also satisfy (3.3.7) and (3.3.8) for $k \rightarrow k-\frac{1}{2}$, then there exists a matrix $T$ such that $u^a = Tu^a$ and $v^b = v^bT^{-1}$. But if $u^a$ and $v^b$ satisfy (3.3.5) and (3.3.6) also, then $T$ commutes with $I^\alpha_{\beta}(k)$ for all $a, b$. Thus $T = c$. By raising all lower indices, (3.3.5)-(3.3.8) can now be written
Note that (3.3.9) implies (3.3.9') and (3.3.10) implies (3.3.10'), so \(u^a(k)\) and \(v^b(k)\) are entirely specified by (3.3.9) and (3.3.10).

By applying \(u^c(k)\) to the right in (3.3.9) and to the left in (3.3.10), the resulting two equations imply

\[
(3.3.11) \quad [ -k e^{ab} + I^{ab}(k) ] u^c(k) = u^a(k) [ (k+\frac{1}{2}) e^{bc} + I^{bc}(k-\frac{1}{2}) ]
\]

The indices in (3.3.11) can be permuted to give a set of equations. Combining these equations and using the properties of \(I^{ab}\) and \(e^{ab}\), (3.3.11) can be shown to be equivalent to

\[
(3.3.12) \quad u^a(k) I^{bc}(k-\frac{1}{2}) - I^{bc}(k) u^a(k) = \frac{1}{2} e^{ab} u^c(k) + \frac{1}{2} e^{ac} u^b(k)
\]

Similarly, the following equation can be derived for \(v^a(k)\).

\[
(3.3.13) \quad v^a(k) I^{bc}(k) - I^{bc}(k-\frac{1}{2}) v^a(k) = \frac{1}{2} e^{ab} v^c(k) + \frac{1}{2} e^{ac} v^b(k)
\]
The two equations (3.3.12) and (3.3.13) together imply

\[(3.3.14) \ [u^a v^d, I^{bc}(k)] = \frac{\gamma_2}{2} (\epsilon^{ac} u^b v^d + \epsilon^{ab} u^c v^d + \epsilon^{db} u^a v^c + \epsilon^{dc} u^a v^b)\]

Note that, if $I^{ad}(k)$ is substituted for $u^a v^d$ in (3.3.14), the result would be the spinor form of the integrability conditions for $\mathcal{R}$. Thus, the symmetric part of $u^a v^b$ is a multiple of $I^{ad}(k)$. The antisymmetric part, $N^{ad}$, of $u^a v^b$ must satisfy $[N^{ad}, I^{bc}(k)] = 0$ by inspection, so $N^{ab}$ is a multiple of the scalar $\epsilon^{ad}$. Therefore,

\[(3.3.15) \ u^a(k) v^d(k) = \alpha \ I^{ad}(k) + \beta \epsilon^{ad}\]

Similarly, (3.3.12) and (3.3.13) imply

\[(3.3.16) \ v^a(k) u^d(k) = \alpha' \ I^{ad}(k-\frac{\gamma_2}{2}) + \beta' \epsilon^{ad}\]

Manipulating (3.3.15) and (3.3.16) and using (3.3.12) and (3.3.13), it can be shown that $a$, $a'$, $b$ and $b'$ are not independent quantities and that

\[(3.3.17) \ u^a(k) v^b(k) = \alpha \ [-k \epsilon^{ab} + I^{ab}(k)]\]
\[(3.3.17) \ v^a(k) u^b(k) = \alpha' \ [(k+\frac{\gamma_2}{2}) \epsilon^{ab} + I^{ab}(k-\frac{\gamma_2}{2})]\]

Note that Eqs. (3.3.17) are the same as (3.3.9) and (3.3.10) up to a constant factor. Therefore, the following result has been shown:

Given the representations $I^{ab}(k)$ and $I^{ab}(k-\frac{\gamma_2}{2})$ for $\mathcal{R}$, there exist non-trivial solutions to the equations...
Further, \( u^a(k) \) and \( v^b(k) \) are solutions to (3.3.12) and (3.3.13) if and only if they satisfy Eqs. (3.3.17) for some value of the constant \( a \). Finally, \( u'^a(k) \) and \( v'^b(k) \) are another set of solutions of (3.3.12) and (3.3.13) if and only if there exist constants \( b \) and \( c \) such that \( u'^a(k) = bu^a(k) \) and \( v'^b(k) = cv^b(k) \).

The goal of this section has been to derive the existence and uniqueness of a set of solutions to Eqs. (3.3.12) and (3.3.13), equations which appear in a later section in the Bhaba analysis of relativistic wave equations. In addition, the connection between these solutions and the matrices which reduce product representations of \( \mathcal{A} \) has been derived. This connection will be developed more in Section 3.4 and in the later examples. The final step in this section is to obtain the exact forms of \( u^a \) and \( v^a \) for the "standard" representations of \( \mathcal{A} \).

By definition, the \( I^{ab} \) forms of the infinitesimal generators are related to the \( I_i \) by

\[
I'' = -I'_2 = -I_1 + iI_2
\]

\[
I^{22} = I^{31} = I_1 + iI_2
\]

\[
I^{11} = I'_1 = I_3 = -I^2_2 = I^{21}
\]

Therefore,

\[
\left[ I''(k) \right]_{r,s} = -\left[ (r-1) (2k-r+2) \right] \frac{\nu}{2} S_{r-1,s}
\]

\[
\left[ I^{22}(k) \right]_{r,s} = \left[ r (2k-r+1) \right] \frac{\nu}{2} S_{r+1,s}
\]

\[
\left[ I^{11}(k) \right]_{r,s} = \left[ I^{21}(k) \right]_{r,s} = (k-r+1) S_{r,s}
\]
Choose the standard form of the $u^a(k)$ and $v^a(k)$ to be

\begin{align}
U^l(k)_{\alpha,\beta} &= (-1)^{-\frac{l}{2}} \delta_{\alpha,\beta} \\
U^2(k)_{\alpha,\beta} &= (2k - r + 1)^{\frac{l}{2}} \delta_{\alpha,\beta} \\
V^l(k)_{\alpha,\beta} &= (2k - r + 1)^{\frac{l}{2}} \delta_{\alpha,\beta} \\
V^2(k)_{\alpha,\beta} &= (-1)^{k} \delta_{\alpha,\beta}
\end{align}

(3.3.18)

By inspection, these forms satisfy (3.3.9) and (3.3.10). Further, they have the property

$$\begin{bmatrix} u^a(k) & v^a(k) \end{bmatrix}^T = (-1)^{k} v^a(k)$$

since $v_1(k) = v^2(k)$ and $v_2(k) = -v^1(k)$.

### 3.4 Product Representations of $L^\uparrow$.

In order to study particular examples of the wave equation formalism to be developed in later sections, it will be useful to have a direct method for decomposing certain product representations of $L^\uparrow$. With the aid of the $u^a$ and $v^a$ matrices from the previous section, this can now be done. All of the representations in this section will be assumed to be in standard form.

The representation of $L^\uparrow$ defined by $\mathcal{B}(k,\ell) \otimes \mathcal{B}(k,\ell)$ is reducible and decomposes into the direct sum $\mathcal{B}(k+\frac{\ell}{2},\ell) \otimes \mathcal{B}(k-\frac{\ell}{2},\ell)$. Therefore there exists a matrix $U(k,\ell)$ such that

\begin{equation}
U(k,\ell) \left( I^{\otimes (k,\ell)} \otimes 1 + 1 \otimes I^{\otimes (k,\ell)} \right) U(k,\ell)^{-1}
\end{equation}

(3.4.1)
By transforming the vector indices to spinor indices, (3.4.1) is equivalent to the two equations

\[(3.4.2) \quad \mathcal{U}(k, l) \left( K^{ab}_{(k \cdot 0)} \times 1 + 1 \times K^{ab}_{(k, l)} \right) \mathcal{U}^{-1}(k, l) = \begin{pmatrix} K^{ab}_{(k + \frac{1}{2}, l)} & 0 \\ 0 & K^{ab}_{(k - \frac{1}{2}, l)} \end{pmatrix} \]

\[(3.4.3) \quad \mathcal{U}(k, l) \left( L^{ab}_{(k \cdot 0)} \times 1 + 1 \times L^{ab}_{(k, l)} \right) \mathcal{U}^{-1}(k, l) = \begin{pmatrix} L^{ab}_{(k + \frac{1}{2}, l)} & 0 \\ 0 & L^{ab}_{(k - \frac{1}{2}, l)} \end{pmatrix} \]

The $L^{ab}$ are in standard form, so (3.4.3) can be written

\[(3.4.4) \quad \mathcal{U}(k, l) \left( 1_{2(2k+1)} \times L^{ab}_{(l)} \right) \mathcal{U}^{-1}(k, l) = 1_{2(2k+1)} \times L^{ab}_{(l)} \]

since $L^{ab}(k, 0) = 0$. This implies that $\mathcal{U}(k, l)$ has the form $\mathcal{U}(k, l) = \mathcal{U}(k) \times 1_{2l+1}$. Thus, (3.4.2) has the form

\[(3.4.5) \quad \mathcal{U}(k) \left( K^{ab}_{(k \cdot 0)} \times 1 + 1 \times K^{ab}_{(k)} \right) \mathcal{U}^{-1}(k) = K^{ab}_{(k + \frac{1}{2})} \]

If the spinor indices here are transformed to a single vector index, (3.4.5) becomes

\[
\begin{align*}
\mathcal{U}(k) \left( K_{a}(k \cdot 0) \times 1 + 1 \times K_{a}(k) \right) \mathcal{U}^{-1}(k) &= \begin{pmatrix} K_{a}(k + \frac{1}{2}) & 0 \\ 0 & K_{a}(k - \frac{1}{2}) \end{pmatrix} \\
\mathcal{U}(k) &= \begin{pmatrix} u_{1}(k + \frac{1}{2}) & u_{2}(k + \frac{1}{2}) \\ u_{1}(k) & u_{2}(k) \end{pmatrix} \\
\mathcal{U}^{-1}(k) &= \begin{pmatrix} u_{1}^{1}(k + \frac{1}{2}) & u_{1}^{1}(k) \\ u_{2}^{1}(k + \frac{1}{2}) & u_{2}^{1}(k) \end{pmatrix} \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
(3.4.6) \quad \left\{ \begin{array}{l}
\mathcal{U}(k) = (-1)^{2k+1} \begin{pmatrix} u_{1}(k + \frac{1}{2}) & u_{2}(k + \frac{1}{2}) \\ u_{1}(k) & u_{2}(k) \end{pmatrix} \\
\mathcal{U}^{-1}(k) = (-1)^{2k+1} \begin{pmatrix} u_{1}^{1}(k + \frac{1}{2}) & u_{1}^{1}(k) \\ u_{2}^{1}(k + \frac{1}{2}) & u_{2}^{1}(k) \end{pmatrix}
\end{array} \right. 
\end{align*}
\]
where \( u_a \) and \( v_a \) are in standard form.

Note that \( \mathcal{I}^{\mu\nu}(\frac{1}{2}, 0) \times \mathcal{I}^{\mu\nu}(0, \lambda) = \mathcal{I}^{\mu\nu}(\frac{1}{2}, \lambda) \) so that \( \mathcal{U}(0, \lambda) = I \).

Similarly, \( \mathcal{D}(0, \frac{1}{2}) \times \mathcal{D}(k, \lambda) \) reduces to \( \mathcal{D}(k, \lambda+\frac{1}{2}) \oplus \mathcal{D}(k, \lambda-\frac{1}{2}) \). If \( \mathcal{V}(k, \lambda) \) is the matrix that performs the reduction, then

\[
(3.4.7) \quad \mathcal{V}(k, \lambda) \left( K^{ab}(0, \frac{1}{2}) \times I + I \times K^{ab}(k, \lambda) \right) V^{-1}(k, \lambda) = 
\begin{pmatrix}
K^{ab}(k, \lambda+\frac{1}{2}) & 0 \\
0 & K^{ab}(k, \lambda-\frac{1}{2})
\end{pmatrix}
\]

\[
(3.4.8) \quad \mathcal{V}(k, \lambda) \left( L^{ab}(0, \frac{1}{2}) \times I + I \times L^{ab}(k, \lambda) \right) V^{-1}(k, \lambda) = 
\begin{pmatrix}
L^{ab}(k, \lambda+\frac{1}{2}) & 0 \\
0 & L^{ab}(k, \lambda-\frac{1}{2})
\end{pmatrix}
\]

(3.4.7) implies

\[
(3.4.9) \quad \mathcal{V}(k, \lambda) \left( I \times [K^{ab}(k) \times I] \right) V^{-1}(k, \lambda) = 
\begin{pmatrix}
K^{ab}(k) \times I_{2\lambda+2} & 0 \\
0 & K^{ab}(k) \times I_{2\lambda}
\end{pmatrix}
\]

Define \( O(k, \lambda) \) to be the matrix such that

\[
O(k, \lambda) [ A \times B ] O'(k, \lambda) = B \times A
\]

where \( A \) is a \((2k+1)\times(2k+1)\) matrix and \( B \) is a \((2\lambda+1)\times(2\lambda+1)\) matrix. Note that \( O^{-1}(k, \lambda) = O(\lambda, \kappa) \). Therefore, if

\[
O_1 = O(k, \lambda+\frac{1}{2}) \oplus O(k, \lambda-\frac{1}{2}) \quad \text{and} \quad O_2 = I_{2\lambda} \times O(k, \lambda),
\]

(3.4.9) has the form

\[
(3.4.10) \quad (O_1 \mathcal{V}(k, \lambda) O_2^{-1}) \left( I_{2\lambda+2} \times K^{ab}(k) \right) (O_2^{-1} \mathcal{V}(k, \lambda) O_1') = I_{2\lambda+2} \times K^{ab}(k)
\]
Thus, $0\text{v}(k, l)O_2^{-1}$ has the form $v(l) \times 1_{2k+1}$ and (3.4.8) becomes

$$v(l) (L^\alpha_\beta(\frac{l}{2}) \times 1 + 1 \times L^\alpha_\beta(l)) V^{-i}(l)$$

(3.4.11)$$
= \begin{pmatrix} L^\alpha_\beta(l+\frac{1}{2}) & 0 \\ 0 & L^\alpha_\beta(l-\frac{1}{2}) \end{pmatrix}
$$

Once again, spinor indices can be transformed into 3-vector indices, implying

$$v(l) (L^\alpha_\beta(\frac{l}{2}) \times 1 + 1 \times L^\alpha_\beta(l)) V^{-i}(l)$$

(3.4.12)$$
= \begin{pmatrix} L^\alpha_\beta(l+\frac{1}{2}) & 0 \\ 0 & L^\alpha_\beta(l-\frac{1}{2}) \end{pmatrix}
$$

Therefore, $v(l) = \mathcal{U}(l)$.

Finally, look at the representation of $\mathcal{L}^\dagger$ defined by

$$[\mathcal{B}(\frac{k}{2}, 0) \oplus \mathcal{B}(\frac{k}{2}, \frac{1}{2})] \times [\mathcal{B}(k_1, l_1) \oplus \ldots \oplus \mathcal{B}(k_n, l_n)]$$

This representation is already in the form

$$[\mathcal{B}(\frac{k}{2}, 0) \times (\mathcal{B}(k_1, l_1) \oplus \ldots \oplus \mathcal{B}(k_n, l_n))]

\oplus [\mathcal{B}(\frac{k}{2}, 0) \times (\mathcal{B}(k_1, l_1) \oplus \ldots \oplus \mathcal{B}(k_n, l_n))]$$

By applying $0(\frac{k}{2}, N) \oplus 0(\frac{k}{2}, N)$ where $N = \frac{1}{2} \left[ \sum_{i=1}^{n} (2k_i+1)(2l_i+1)-1 \right]$ to this representation, it becomes

$$[\mathcal{B}(k_1, l_1) \oplus \ldots \oplus \mathcal{B}(k_n, l_n) \times \mathcal{B}(\frac{k}{2}, 0)]

\oplus [(\mathcal{B}(k_1, l_1) \oplus \ldots \oplus \mathcal{B}(k_n, l_n)) \times \mathcal{B}(\frac{k}{2}, 0)]$$

which is in the form

$$[\mathcal{B}(k_1, l_1) \times \mathcal{B}(\frac{k}{2}, 0)] \oplus \ldots \oplus [\mathcal{B}(k_n, l_n) \times \mathcal{B}(\frac{k}{2}, 0)]

\oplus [(\mathcal{B}(k_1, l_1) \times \mathcal{B}(\frac{k}{2}, 0)] \oplus \ldots \oplus [(\mathcal{B}(k_n, l_n) \times \mathcal{B}(\frac{k}{2}, 0)]$$

Define $N_1 = \frac{1}{2}[ (2k_1+1)(2l_1+1)-1 ]$ and apply
\[ \bigoplus (N_1, \frac{1}{2}) \oplus \ldots \oplus \bigoplus (N_n, \frac{1}{2}) \oplus \bigoplus (N_1, \frac{1}{2}) \oplus \ldots \oplus \bigoplus (N_n, \frac{1}{2}) \]
to the representation. Now, the representation has the form
\[
\bigoplus [ \bigoplus (\frac{1}{2}, 0) \times \bigoplus (k_1, l_1) ] \oplus \ldots \oplus [ \bigoplus (\frac{1}{2}, 0) \times \bigoplus (k_n, l_n) ]
\oplus [ \bigoplus (0, \frac{1}{2}) \times \bigoplus (k_1, l_1) ] \oplus \ldots \oplus [ \bigoplus (0, \frac{1}{2}) \times \bigoplus (k_n, l_n) ]
\]
and can be fully reduced by the matrix
\[ R = \bigoplus (k_1, l_1) \oplus \ldots \oplus \bigoplus (k_n, l_n) \oplus \bigoplus (k_1, l_1) \oplus \ldots \oplus \bigoplus (k_n, l_n) \]
IV. RELATIVISTIC WAVE EQUATIONS

4.1 Wave Equation Formalism*

The standard form for a wave equation of a free particle is

\[(4.1.1) \quad \left( \kappa \partial_\mu + \chi \right) \psi(x) = 0 \]

where \( \psi(x) \) is an nx1 column matrix whose entires are functions of \( x \), \( \partial_\mu \) is the first-order partial derivative \( \partial / \partial x^\mu \), each \( \alpha^\mu \) is a constant nxn matrix, and \( \chi \) is a constant. The constant \( \chi \) determines the mass of the particle and is assumed to be non-zero in this development. Zero mass particles require a somewhat different formalism than the one here.

In order for a wavefunction \( \psi(x) \) to describe a relativistic particle, there must exist a representation \( T \) of \( L^\uparrow \) such that, if \( L \in L^\uparrow \), then

\[(4.1.2) \quad \psi(L(x)) = T(L) \psi(x) \]

Note that the definition of relativistic here is restricted to the proper Lorentz group. Changes in this theory as a result of using \( L \) or \( L_+^\uparrow \cup L_-^\uparrow \) instead of \( L_+^\uparrow \) are in the nature of restrictions to the original development and may be developed separately.

* The reader is referred to Corson, Harish-Chandra, and Takahashi for more extensive studies of this formalism.
If Eq. (4.1.1) is to be relativistically (or Lorentz) invariant, then, if \( x' = L(x) \) for \( L \in \mathcal{L}_+ \), the equation

\[
\left( \alpha^\mu \partial^\prime_\mu + \chi \right) \psi(x') = 0
\]

must be essentially the same as (4.1.1). Note that

\[
\partial^\prime_\mu = \partial/\partial x'^\mu = \left( \partial x''_\mu / \partial x^\mu \right) \partial/\partial x''_\nu = L^\nu_\mu \partial^\prime_\nu
\]

Therefore, (4.1.1) is equal to

\[
\left( \alpha^\mu L^\nu_\mu \partial^\prime_\nu + \chi \right) T(L)^{-1} \psi(x') = 0
\]

In order for (4.1.3) and (4.1.4) to be essentially the same, there must exist a representation \( T'(L) \) such that

\[
T'(L) \left( \alpha^\mu \partial^\prime_\mu + \chi \right) = \left( \alpha^\mu L^\nu_\mu \partial^\prime_\nu + \chi \right) T(L)^{-1}
\]

The fact that \( \chi \) is non-zero implies that \( T'(L) = T(L)^{-1} \), so that (4.1.5) holds if

\[
\alpha^\nu = T(L) \left( L^\nu_\mu \alpha^\mu \right) T(L)^{-1}
\]

Therefore, the problem of finding all possible Lorentz invariant equations of the form (4.1.1) is equivalent to finding sets \( \{ \alpha^\mu, T \} \) such that

\[
T(L)^{-1} \alpha^\nu T(L) = L^\nu_\mu \alpha^\mu \quad \forall L \in \mathcal{L}_+
\]

Since this development is only considering Lorentz invariance for the proper Lorentz group, (4.1.6) may be replaced by an equation involving the infinitesimal generators \( \Gamma^\mu_\nu \) of \( T \). (4.1.6) implies
\[ [L_\rho^\lambda(\theta)]^\gamma_\mu A^\mu = T(L_\rho^\lambda(\theta))^{-1} A^\gamma T(L_\rho^\lambda(\theta)) \]
\[ = T(L_\rho^\lambda(-\theta)) A^\gamma T(L_\rho^\lambda(\theta)) \]

Taking the derivative with respect to \( \theta \) at \( \theta = 0 \) gives

\[ [I_\rho^\lambda]_\mu A^\mu = -I_\rho^\lambda A^\gamma + A^\gamma I_\rho^\lambda \]

From Section 2.4, by inspection,

\[ [I_\rho^\lambda]_\mu A^\mu = \delta^\mu_\rho \delta^\lambda_\mu - q^\mu_\rho q^\lambda_\nu \]

Therefore, the infinitesimal equivalent of (4.1.6) is

\[ [\alpha^\nu, I_\rho^\lambda] = q^{\mu\nu} \alpha^\lambda - q^{\lambda\nu} \alpha^\rho \]

Any set of ten matrices \( \{\alpha^\mu, I_\rho^\lambda\} \) which satisfy (4.1.9) and the integrability conditions for the proper Lorentz group determine a Lorentz invariant equation of the form (4.1.1).

4.2 The Bhaba Analysis

A general analysis of the solutions to Eq. (4.1.9) has been made by Bhaba. This analysis is reproduced here so that the solutions can be compared to the constructions in later sections.

If the set \( \{\alpha^\mu, I_\rho^\lambda\} \) is a solution to Eqs. (4.1.9) and (2.4.6), it is possible to put the \( I_\rho^\lambda \) in standard form without loss of generality. Therefore, let \( I_\rho^\lambda \) have the form

\[ I_\rho^\lambda = \begin{pmatrix} I_\rho^\lambda(k_1, l_1) & 0 & \ldots & 0 \\
0 & I_\rho^\lambda(k_2, l_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I_\rho^\lambda(k_m, l_m) \end{pmatrix} \]
where each $I^{\rho \lambda}(k_i, l_i)$ is in standard form. The vector index on the $a^\mu$ can be transformed to a pair of spinor indices by defining $A^{ab} = a^\mu \sigma_\mu^{ab}$. With this definition, (4.1.9) can be put into spinor form by changing the $I^{\rho \lambda}$ to $K^{cd}$ and $L^{\cd}$, $\cd$. Thus, (4.1.9) is equivalent to the two equations

\begin{align}
(4.2.1) \quad [ A^{ab} \cdot \cdot K^{cd} ] &= \frac{1}{2} \left[ \epsilon^{ac} A^{db} + \epsilon^{ad} A^{bc} \right]
\end{align}

\begin{align}
(4.2.2) \quad [ A^{ab} \cdot L^{\cd} ] &= \frac{1}{2} \left[ \epsilon^{bc} A^{ad} + \epsilon^{bd} A^{ac} \right]
\end{align}

By subdividing the $A^{ab}$ into blocks $(A^{ab})_{ij}$ corresponding to the blocks of $I^{\rho \lambda}$, (4.2.1) and (4.2.2) become

\begin{align}
(4.2.3) \quad (A^{ab})_{ij} \cdot K^{cd}(k_i, l_i) - K^{cd}(k_i, l_i) (A^{ab})_{ij} = \frac{1}{2} \left[ \epsilon^{ac} (A^{db})_{ij} + \epsilon^{ad} (A^{bc})_{ij} \right]
\end{align}

\begin{align}
(4.2.4) \quad (A^{ab})_{ij} \cdot L^{\cd}(k_j, l_j) - L^{\cd}(k_j, l_j) (A^{ab})_{ij} = \frac{1}{2} \left[ \epsilon^{bc} (A^{ad})_{ij} + \epsilon^{bd} (A^{ac})_{ij} \right]
\end{align}

Note that $(A^{ab})_{ij}$ is a $(2k_i+1) \times (2k_i+1)$ matrix and $K^{cd}(c_i, l_i)$ and $L^{\cd}(c_j, l_j)$ are in the form

\begin{align}
K^{cd}(k_i, l_i) &= K^{cd}(k_i) x \mathbb{1}_{2k_i+1} \\
L^{\cd}(k_j, l_j) &= \mathbb{1}_{2k_j+1} x L^{\cd}(l_j)
\end{align}

Further, the form of $L^{\cd}(k_i, l_i)$ implies that $(A^{ab})_{ij}$ can be subdivided into subblocks $[ (A^{ab})_{ij} ]_{rs}$ of size $(2k_{ij}+1) x (2l_{ij}+1)$, all of which satisfy the equation
\[ [A^{ab}]_{ij} \]_{rs} L^{cd}(l_j) - L^{cd}(l_i) [A^{ab}]_{ij} \]_{rs} \\
\text{(4.2.6)}
\[ = \frac{1}{2} [e^{bc} (A^{ab})_{ij}]_{rs} + e^{bd} (A^{bc})_{ij} \]_{rs} \]

For the moment, let \([A^{ab}]_{ij} \]_{rs} = B^{ab} to simplify the notation.

Since the \(L^{cd}(l_j)\) are in standard form:

\[ (L^{i}(l_j))_{m'm} = (I^{is}l_{j})_{m'm} = \frac{1}{\sqrt{z}} S_{m+1,m'} \]
\[ (L^{i}(l_j))_{m'm} = (I^{is}l_{j})_{m'm} = -\frac{1}{\sqrt{z}} S_{m-1,m'} \]
\[ (L^{i}(l_j))_{m'm} = (L^{i}(l_j))_{m'm} = -m \delta_{m'm} \]

Let \(C = 1\) and \(D = 2\) in (4.2.6)

\[-(B^{ab})_{m'm}(m) + (m') (B^{ab})_{m'm} = \frac{1}{2} [e^{bi} (B^{ai})_{m'm} + e^{b'i} (B^{ai})_{m'm}] \]
\[ \Rightarrow \]
\[ (B^{ab})_{m'm} [m'-m \neq \frac{1}{2}] = 0 \quad (- \text{for } b = 1, + \text{ for } b = 2) \]
\[ \Rightarrow \]
\[ \begin{cases} 
B^{ab} = 0 & \text{if } l_j \neq l_i + \frac{n}{2} \text{ for some odd integer } n \\
(B^{ab})_{m'm} = 0 & \text{if } l_j = l_i + \frac{n}{2} \text{ and } m' \neq m \pm \frac{1}{2}
\end{cases} \]

Let \(C = D = B = 2\), \(l_j = l_i + n/2\) in (4.2.6):

\[-(B^{ai})_{m'm} [(l_j+m+1)(l_j-m)]^{1/2} [(l_i+m+1)(l_i-m')]^{1/2} (B^{ai})_{m'1,m+1} \]
\[ = 0 \]

Both matrix elements are zero unless \(m' = m - \frac{1}{2}\), so

\[ (B)^{ai}_{m'-\frac{1}{2},m} [(l_i+\frac{n}{2}+m+1)(l_i+\frac{n}{2}-m)]^{1/2} \]
\[ \text{(4.2.7)} \]
\[ = [(l_i+m+\frac{n}{2})(l_i-m+\frac{n}{2})]^{1/2} (B^{ai})_{m'+\frac{n}{2},m+1} \]

Let \(n \geq 1\): \(m = l_i + \frac{n}{2} \Rightarrow (B^{ai})_{l_i, l_i + \frac{n}{2}} [(l_i+\frac{n}{2}+m+1)(l_i+\frac{n}{2}-m)]^{1/2} = 0 \]
\[ \Rightarrow \text{either } n = 1 \text{ or } (B^{ai})_{l_i, l_i + \frac{n}{2}} = 0 \]

But \([(l_i+\frac{n}{2}+m+1)(l_i+\frac{n}{2}-m)]^{1/2} = 0 \text{ only for } n = 1 \text{ and} \]
m = \ell_1 + \frac{3}{2} if n \geq 1. Therefore, \((B^{a,i}_1)\ell_1, \ell_1 + \frac{3}{2} = 0\) implies \(B^{a,i}_1 = 0\) from (4.2.7).

Let \(C = D = 1, B = 2\) in (4.2.6):

\[
(B^{a,i}_1) L^{ii}_i (l_j) - L^{ii}_i (l_{\ell_1}) (B^{a,i}_1) = -B^{a,i}_1
\]

Thus \(B^{a,i}_1 = 0 \implies B^{a,i}_1 = 0\)

Therefore:

\[n \geq 1 \implies \text{either } n = 1 \text{ or } B^{ab} = 0.\]

Let \(C = D = B = 1, \ell_1 = l_j + n'/2\) in (4.2.6):

\[
(B^{a,i}_1) m' m' \left[ (l_j - m + 1) (l_j + m) \right]^{\frac{v}{2}} - \left[ (l_{\ell_1} - m' + 1) (l_{\ell_1} + m') \right]^{\frac{v}{2}} (B^{a,i}_1) m_{-1} m_{-1} = 0
\]

Both matrix elements are zero unless \(m = m' - \frac{3}{2}\), so

\[
(B^{a,i}_1) m' m' \left[ (l_j - m' + \frac{3}{2}) (l_j + m' - \frac{3}{2}) \right]^{\frac{v}{2}} = 0
\]

(4.2.8)

Let \(n' \geq 1, m' = l_j + \frac{3}{2} \implies \left\{ (2l_j + \frac{3}{2}) (2l_j - \frac{3}{2}) \right\}^{\frac{v}{2}} (B^{a,i}_1) m_{-1} m_{-3/2} = 0\]

\[\implies \text{either } n' = 1 \text{ or } (B^{a,i}_1) m_{-1}, m_{-3/2} = 0.\]

But \(\left\{ (l_j + \frac{3}{2}) (l_j + \frac{3}{2} + m') \right\}^{\frac{v}{2}} = 0\) only for \(n' = 1\) and \(m' = l_j + 3/2\) if \(n' \geq 1\). Therefore, \((B^{a,i}_1) m_{-1}, m_{-3/2} = 0\) implies \(B^{a,i}_1 = 0\) from (4.2.8).

Let \(C = D = 2, B = 1\) in (4.2.6):

\[
(B^{a,i}_2) L^{ii}_i (l_j) - L^{ii}_i (l_{\ell_1}) (B^{a,i}_2) = B^{a,i}_2
\]

Thus \(B^{a,i}_2 = 0 \implies B^{a,i}_2 = 0\).

Therefore

\[n' \geq 1 \implies \text{either } n' = 1 \text{ or } B^{ab} = 0.\]
Thus $B^{\hat{a}\hat{b}} = 0$ unless $\xi^i = \xi^j \pm \frac{1}{2}$.

If $\xi^i = \xi^j \pm \frac{1}{2}$, then Eq. (4.2.6) becomes

$$B^{\hat{a}\hat{b}} \tilde{l}^\dagger(l_j) - \tilde{l}^\dagger(l_j \pm \frac{1}{2}) B^{\hat{a}\hat{b}} = \frac{1}{2} [\epsilon^{\hat{a}\hat{c}} B^{\hat{a}\hat{d}} + \epsilon^{\hat{a}\hat{d}} B^{\hat{a}\hat{c}}]$$

Since $L^{\hat{c}\hat{d}}(k)$ $I_{\hat{c}\hat{d}}(k)$, the solution to (4.2.9) can be derived directly from the solutions to (3.3.12) and (3.3.13). Therefore, $(u_b$ and $v_b$ are in standard form)

$$B^{\hat{a}\hat{b}} = \begin{cases} c \ u_b(x_j + \frac{1}{2}) & \text{IF } \xi^i = \xi^j + \frac{1}{2} \\ c \ v_b(x_j) & \text{IF } \xi^i = \xi^j - \frac{1}{2} \end{cases}$$

where $c$ is an arbitrary constant. In order to regularize the notation, define $u^\hat{a}(k) = u_a(k)$ and $v^\hat{a}(k) = v_a(k)$.

By returning the deleted subscripts, these results can all be written as

$$[(A^{\hat{a}\hat{b}})_{ij}]_{rs} = \begin{cases} (c_{\hat{a}i})_{rs} u^\hat{b}(x_j + \frac{1}{2}) & \text{IF } \xi^i = \xi^j + \frac{1}{2} \\ (c_{\hat{a}i})_{rs} v^\hat{b}(x_j) & \text{IF } \xi^i = \xi^j - \frac{1}{2} \\ 0 & \text{OTHERWISE} \end{cases}$$

Therefore, if $C^a_{ij}$ is the matrix with elements $(C^a_{ij})_{rs}$,

$$A^{\hat{a}\hat{b}} = \begin{cases} c_{\hat{a}i} x u^\hat{b}(x_j + \frac{1}{2}) & \text{IF } \xi^i = \xi^j + \frac{1}{2} \\ c_{\hat{a}i} x v^\hat{b}(x_j) & \text{IF } \xi^i = \xi^j - \frac{1}{2} \\ 0 & \text{OTHERWISE} \end{cases}$$

Using the notation of (4.2.11), Eq. (4.2.3) now takes the form

$$C_{\hat{a}i}^a \cdot K_{\hat{c}j}^{cd}(k_j) - K_{\hat{c}j}^{cd}(k_i) C_{\hat{a}i}^c = \frac{1}{2} [\epsilon^{ac} C_{\hat{a}i}^d + \epsilon^{ad} C_{\hat{a}i}^c]$$
By similar reasoning to that for Eq. (4.2.6), (4.2.12) implies that \( C_{ij}^a = 0 \) unless \( k_i = k_j \pm \frac{1}{2} \), and, further, that

\[
C_{ij}^a = \begin{cases} 
  d_{ij} \ u^a(k_j + \frac{1}{2}) & \text{if } k_i = k_j + \frac{1}{2} \\
  d_{ij} \ u^a(k_j) & \text{if } k_i = k_j - \frac{1}{2}
\end{cases}
\]

where \( d_{ij} \in \mathbb{C} \).

Therefore, the solution to Eqs. (4.2.1) and (4.2.2) is that

\[
(A_{abc})_{ij} = d_{ij} \begin{cases} 
  u^a(k_j + \frac{1}{2}) \times u^b(l_j + \frac{1}{2}) & k_i = k_j + \frac{1}{2}, \ l_i = l_j + \frac{1}{2} \\
  u^a(k_j + \frac{1}{2}) \times u^b(l_j) & k_i = k_j + \frac{1}{2}, \ l_i = l_j - \frac{1}{2} \\
  u^a(k_j + \frac{1}{2}) \times u^b(l_j) & k_i = k_j - \frac{1}{2}, \ l_i = l_j + \frac{1}{2} \\
  u^a(k_j) \times u^b(l_j + \frac{1}{2}) & k_i = k_j - \frac{1}{2}, \ l_i = l_j + \frac{1}{2} \\
  u^a(k_j) \times u^b(l_j) & \text{otherwise}
\end{cases}
\]

where \( u^a, v^a, t^a = u_a, v^a = v_a \) are in standard form if \( K^{ab}(k) \) and \( L^{cd}(l) \) are, and \( d_{ij} \in \mathbb{C} \).

4.3 Algebras and Representations

An "algebra" \( A \) over \( \mathbb{C} \) is a set of elements with the operations addition, multiplication and multiplication by a constant (\( \in \mathbb{C} \)) defined such that \( A \) is a vector space under addition and multiplication by a constant and is a ring under addition and multiplication. \( A \) is finite if it has a finite basis as a vector space. An algebra can be generated from a finite set of elements by taking all possible products of these elements as the basis for the algebra and formally defining sums of constant multiples of
these products as the elements of the algebra. A finite matrix representation of order n of an algebra \( \mathfrak{a} \) is a mapping from \( \mathfrak{a} \) to a set of \( nxn \) matrices which preserves the operations of the algebra.*

In Section 4.1, it was shown that an equation of the form (4.1.1) is determined by a set of ten matrices \( \{ \alpha^\mu, \lambda^\rho \} \) which satisfy (4.1.9) and (2.4.6), the integrability conditions for \( \mathcal{X}_+ \). Another way of looking at this is to use the ten elements \( \{ \alpha^\mu, \lambda^\rho \} \), with the relations (4.1.9) and (2.4.6), to generate an algebra \( \mathfrak{a} \). Then any finite representation of \( \mathfrak{a} \) would determine an equation of the form (4.1.1) and any such equation would determine a finite representation of \( \mathfrak{a} \). Further, two equivalent representations determine essentially the same equation. At this point, it can be seen that the reducibility of representations of \( \mathfrak{a} \) plays a major role in determining the interpretation of equations of the form (4.1.1). Therefore, it is desirable to develop the concept of reduction for representations of general algebras.

Just as in the case of group representations, an \( nxn \) representation \( T \) of an algebra \( \mathfrak{a} \) is irreducible if the space \( \mathbb{R}^n \) has no invariant subspaces under the action of the set of matrices \( T(a) \equiv \{ T(a): a \in \mathfrak{a} \} \). If there exists an invariant subspace, then \( T \) is reducible, and the form of \( T \) may be chosen such that, for all \( a \in \mathfrak{a} \), \( T(a) \) has the form

* An excellent study of these topics and their relations to group representations can be found in Burrow.12
\[ T(a) = \begin{pmatrix} T_1(a) & N(a) \\ 0 & T_2(a) \end{pmatrix} \]

where 0 is a sub-matrix with all entries 0. Note that, in this form, both \( T_1 \) and \( T_2 \) are representations of \( \mathcal{A} \) since

\[
T(a_1) T(a_2) = \begin{pmatrix} T_1(a_1) T_1(a_2) & T_1(a_1) N(a_2) + N(a_1) T_2(a_2) \\ 0 & T_2(a_1) T_2(a_2) \end{pmatrix}
\]

for \( a_1, a_2 \in \mathcal{A} \). It is also possible to choose the form of \( T \) so that \( T_1 \) and \( T_2 \) are reduced if they are reducible. Therefore, \( T \) can be fully reduced, i.e., put in the form

\[
T(a) = \begin{pmatrix} T_1(a) & N_{12}(a) & \ldots & N_{1m}(a) \\ 0 & T_2(a) & \ldots & N_{2m}(a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & T_m(a) \end{pmatrix}
\]

(4.3.1)

where each \( T_1(a) \) is an irreducible representation of \( \mathcal{A} \).

One important result for representations of algebras is given by the Generalized Burnside Theorem. When applied to algebras over \( \mathbb{C} \), it states that an irreducible algebra of \( nxn \) matrices over \( \mathbb{C} \) has dimension \( n^2 \) as a vector space over \( \mathbb{C} \) and so contains all \( nxn \) complex matrices. The proof of this particular theorem is rather lengthy, so the reader is referred to Chapter 2 of Burrow\(^{13}\) for more detail.

Suppose \( T_1 \) is an irreducible representation of \( \mathcal{A} \). This implies that the mapping \( T_1 : \mathcal{A} \rightarrow T_1(\mathcal{A}) \) is a homomorphism of algebras, so the kernel \( \mathcal{K}_1 \) of \( T_1 \) is a (two-sided)
ideal of \( \mathcal{A} \) and \( T_1(\mathcal{A}) \) is isomorphic to the factor group \( \mathcal{A}/\mathcal{K}_1 \). Further, since \( T_1 \) is irreducible, it is a complete matrix algebra (i.e., all complex \( n_1 \times n_1 \) matrices) and has no two-sided ideals except for itself and \{0\}. Therefore, \( \mathcal{K}_1 \) is a maximal ideal of \( \mathcal{A} \). Thus any irreducible representation of \( \mathcal{A} \) corresponds to a maximal ideal of \( \mathcal{A} \).

(All ideals from here on are two-sided.)

An algebra is called "simple" if it has no two-sided ideals other than itself and \{0\}. Therefore, it has only one non-trivial irreducible representation. This representation is an isomorphism since the kernel is \{0\}, so the algebra is isomorphic to some complete matrix algebra. A representation which is also an isomorphism is called "faithful." Thus, any irreducible representation of a simple algebra is faithful.

The radical, \( \mathcal{N} \), of an algebra \( \mathcal{A} \) is defined to be the set of all elements \( n \) of \( \mathcal{A} \) such that \( T_1(n) = 0 \) for any irreducible representation \( T_1 \). \( \mathcal{N} \) is an ideal of by definition. Note that, if \( T \) is a representation of in the form (4.3.1), then any \( n \in \mathcal{N} \) has only zero blocks on the diagonal. If \( T_1, \ldots, T_m \) are the distinct irreducible representations of \( \mathcal{A} \) with kernels \( \mathcal{K}_1, \ldots, \mathcal{K}_m \), then \( \mathcal{N} = \mathcal{K}_1 \cap \mathcal{K}_2 \cap \ldots \cap \mathcal{K}_m \) by definition.

An algebra is called "semi-simple" if its radical is zero. A simple algebra has a faithful irreducible representation, so the only element which is zero in the repre-
sentation is zero itself. Thus a simple algebra is semi-simple. If \( T \) is a representation of a semi-simple algebra in the form (4.3.1), then the mapping from this representation to a representation \( T' \) which replaces all the \( N_{ij} \) matrices with zero blocks is a homomorphism of algebras. Further, if \( a_1, a_2 \in A \) such that \( T'(a_1) = T'(a_2) \), then \( T(a_1) = T(a_2) \) since \( T(a_1 - a_2) \) would be a non-zero element of \( N \) otherwise. Thus, \( T' \) and \( T \) are isomorphic. Therefore, any representation \( T \) of a semi-simple algebra can be put in the form

\[
T(a) = \begin{pmatrix}
T_1(a) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & T_m(a)
\end{pmatrix}
\]

(4.3.2)

where each \( T_i \) is an irreducible representation of \( A \).

Any representation of an algebra which can be put into this form is called completely reducible.

It is worthwhile to comment here that the concepts "simple" and "semi-simple" and the difference between "reducible" and "completely reducible" have their analogs in the theory of group representations. However, these topics have not been developed here because the groups studied, i.e., \( \mathfrak{L}_+ \) and \( \mathcal{R} \), are semi-simple by virtue of the complete reducibility of their representations.

Suppose \( T_1, \ldots, T_m \) are irreducible representations of \( A \) with kernels \( \mathcal{K}_1, \ldots, \mathcal{K}_m \) where \( \mathcal{K}_i \neq \mathcal{K}_j \) for all \( i, j \) (i.e., the representations are distinct). Define the ideal
$B_i$ of $a$ to be $\cap_{j \neq i} \mathcal{K}_j$. If $B_i$ is contained in $K_i$, then the representation $T = \bigoplus_{j \neq i} T_j$ has its kernel contained in $\mathcal{K}_i$. This implies that $T_i$ can be considered as a representation of $T(a)$ and is, therefore, not distinct from the $T_j$. Thus, $B_i$ contains elements not in $\mathcal{K}_i$. As a result, the set $T_i(B_i)$ is a non-zero ideal of $T_i(a)$. $T_i$ is irreducible, so $T_i(B_i) = T_i(a)$. Suppose the representation $T_i$ is of order $n_i$. Given any complex $n_i \times n_i$ matrix $M_i$, there exists an $a_i \in L_i$ such that $T_i(a_i) = M_i$ and $T_j(a_i) = 0$ for $j \neq i$. For each $i$, let $M_i$ be an $n_i \times n_i$ matrix and $a_i \in B_i$ such that $T_i(a_i) = M_i$ and $T_j(a_i) = 0$ for $j \neq i$. Therefore, if $a = \sum a_i$, then an element of $\mathcal{A}$ is defined such that $T_i(a) = M_i$ for each $i$.

Suppose $T$ is a representation of $\mathcal{A}$ in the form (4.3.2), i.e., $T(\mathcal{A})$ is a finite, semi-simple algebra. It is not true that the irreducible components $T_1', \ldots, T_m'$ must all be distinct, so let $T_1', \ldots, T_m'$ be the distinct irreducible representations and $n_1', \ldots, n_m'$ their respective orders. From the discussion in the previous paragraph, the matrix $T_i'(a)$ can be any $n_i \times n_i$ matrix, independently of the values of the $T_j'(a)$ for $j \neq i$. Since these are the only values the $T_i'(a)$ can take, $T(\mathcal{A})$ must have dimension $n_1^2 + \ldots + n_m^2$, as a complex vector space. Therefore, if $\mathcal{A}$ is a finite, semi-simple algebra with distinct irreducible representations $T_1, \ldots, T_m$, then $\mathcal{A}$ has dimension $n_1^2 + \ldots + n_m^2$, as a complex vector space. Further, if $\mathcal{A}$ is
finite but not necessarily semi-simple, then the dimension is greater than or equal to \(n_1^2 + \ldots + n_m^2\).

If \(A\) is a complete matrix algebra of dimension \(n^2\), then there exists a basis for \(A\) such that each element of the basis has a multiplicative inverse. Thus, if \(A\) is semi-simple, such a basis can be chosen by taking direct products of the invertible basis elements of each irreducible representation. Note, however, that any element of the radical must be zero in each irreducible representation and is, therefore, singular in any representation, even a faithful one. Therefore, an invertible basis for \(A\) exists if and only if \(A\) is semi-simple.

The center \(Z(a)\) of \(A\) is defined to be the subspace of elements of \(A\) which commute with all elements of \(A\). If \(z \in Z(a)\), and \(T_i\) is an irreducible representation of \(A\), then \(T_i(z)\) must commute with every element of \(T_i(a)\). \(T_i(a)\) is a complete matrix algebra, so \(T_i(z)\) must be a multiple of the identity matrix in \(T_i(a)\). If \(A\) is finite and semi-simple, then any \(z \in A\) which has the property \(T_i(z) = c_i I\) for every distinct irreducible representation is an element of \(Z(a)\). Therefore, the dimension of \(Z(a)\) as a vector space over \(\mathbb{C}\) is \(m\), where \(m\) is the number of distinct, irreducible representations of \(A\).

4.4 Decomposition of the Bhaba Solutions

Let \(I_{-}^{\mu \nu}\) be the representation of \(Z^+\) which has the form
\[ I^{\mu \nu} = \begin{pmatrix} I^{\mu \nu}(\gamma_2, o) & o \\ o & I^{\mu \nu}(o, \gamma_2) \end{pmatrix} \]

From the analysis in Section 4.2, the only possible non-trivial solutions to (4.1.9) with \( I^{\mu \nu} \) in this form have the form

\[ A^{\alpha \beta} = \begin{pmatrix} o & c_{12} u^\alpha(\gamma_2) \times v^\beta(\gamma_2) \\ c_{21} u^\beta(\gamma_2) \times u^\alpha(\gamma_2) & o \end{pmatrix} \]

Suppose \( c_{21} = 0 \) and \( c_{12} = 1 \), so that

\[ A^{\alpha \beta} = \begin{pmatrix} o & u^\alpha(\gamma_2) \times v^\beta(\gamma_2) \\ o & o \end{pmatrix} \]

In standard form, the matrices \( u^\alpha(\gamma_2) \times v^\beta(\gamma_2) \) are

\[ u^\alpha(\gamma_2) \times v^\beta(\gamma_2) = \begin{pmatrix} o & o \\ o & 1 \end{pmatrix} \]

\[ u^\alpha(\gamma_2) \times v^\beta(\gamma_2) = - \begin{pmatrix} o & o \\ 1 & o \end{pmatrix} \]

\[ u^\alpha(\gamma_2) \times v^\beta(\gamma_2) = - \begin{pmatrix} o & o \\ o & o \end{pmatrix} \]

\[ u^\alpha(\gamma_2) \times v^\beta(\gamma_2) = \begin{pmatrix} o & o \\ o & o \end{pmatrix} \]

Therefore, the algebra generated by the matrices \( A^{\alpha \beta} \) consists of all matrices of the form \( \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \) where \( c \) is any 2x2 complex matrix. By adding the \( I^{\mu \nu} \) to the lists of generating elements, the algebra becomes all matrices of the form \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) where \( A, B, C \) are any 2x2 complex matrices. This representation is definitely not completely reducible. Therefore the algebra \( \mathcal{A} \) generated by the general solutions \( \{ \alpha^\mu, I^{\rho \lambda} \} \) to (4.1.9) and (2.4.6) is itself not semi-simple.
This implies that it is not enough to limit this development to irreducible representations of \( a \).

However, it is possible to limit the solutions considered somewhat. A representation of an algebra which is reducible but not completely reducible is called "indecomposable." By completely reducing a representation as far as possible, any representation \( T \) can be put in the form \( T = \oplus T_i \), where each \( T_i \) is an indecomposable representation. Therefore, it is only necessary to study indecomposable representations of an algebra.

The degree of a representation \( \mathcal{D}(k,l) \) of \( \mathbb{Z}_+ \) is defined to be \( k + l \). Suppose \( \{a^\mu, i^\rho \} \) is a solution to (4.1.9) and (2.4.6). Let \( i^\rho \) be put in standard form,

\[
I^{\rho \lambda} = \begin{pmatrix}
I^{\rho \lambda}(k_i, l_i) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & I^{\rho \lambda}(k_m, l_m)
\end{pmatrix}
\]

with the added requirement that \( I^{\rho \lambda}(k_i, l_i) \) have integral degree for \( 1 \leq i \leq m_0 \) and half-integral degree for \( m_0 < i \leq m \). If the \( a^\mu \) are put into the corresponding form, let \( (a^\mu)_{ij} \) be the subblock of \( a^\mu \) in the \((i,j)\) position. The Bhaba analysis implies that the only non-trivial blocks \( (a^\mu)_{ij} \) are those such that \( k_i = k_j \pm \frac{1}{2} \) and \( l_i = l_j \pm \frac{1}{2} \).

If \( t_i \) is the degree of \( \mathcal{D}(k_i, l_i) \), then \( t_i = t_j \) or \( t_i = t_j \pm 1 \) for a non-trivial block. Therefore, blocks connecting representations of integral degree with those of half-
integral degree are zero, and $\alpha^\mu$ is in the completely reduced form

$$\lambda^\mu = \begin{pmatrix}
(\alpha^\mu)_1 & \ldots & (\alpha^\mu)_m & 0 & \ldots & 0 \\
\vdots & & \vdots & \ddots & & \vdots \\
(\alpha^\mu)_{m_1} & \ldots & (\alpha^\mu)_{m_2} & 0 & \ldots & 0 \\
0 & \ldots & 0 & (\alpha^\mu)_{m_1+1, m_1+1} & \ldots & (\alpha^\mu)_{m_2+1, m_2+1} \\
\vdots & & \vdots & \ddots & & \vdots \\
0 & \ldots & 0 & (\alpha^\mu)_{m_1+1, m_2+1} & \ldots & (\alpha^\mu)_{m_2+1, m_2+1}
\end{pmatrix}$$

Thus, the only indecomposable representations of $\alpha$ are those with only integral or only half-integral degree representations of $\mathcal{L}^\uparrow$. 
V. MULTIPLE DIRAC ALGEBRAS

The Bhaba analysis of Eq. (4.1.1) has provided a complete description of the solutions to the equation. However, this description has its disadvantages. The algebraic properties of the $\sigma^\mu$ matrices are not readily apparent from this form. These properties determine the characteristics of the particle described such as mass and spin and provide an insight to the description of the particle as "fundamental." The purpose of this section, and of this work as a whole, is to give a method for constructing solutions to (4.1.1) which covers all solutions given in the Bhaba analysis and which gives a handle on the algebraic structures of the solutions. The starting point for this construction is the Dirac Algebra.

5.1 The Dirac Algebra

A Dirac Algebra $\mathcal{D}$ is the algebra generated over by the four elements $\{\gamma^\mu: \mu = 0,1,2,3\}$ with the relations $[\gamma^\mu, \gamma^\nu] = 2g^\mu\nu$. The linearly independent elements of are, therefore:

\[
\begin{align*}
1 \quad (1 \text{ element}) & \quad \gamma^\mu \gamma^\nu (\mu < \nu) \quad (6 \text{ elements}) & \quad \gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad (1 \text{ element}) \\
\gamma^\mu \quad (4 \text{ elements}) & \quad \gamma^\mu \gamma^\nu (\mu > \nu) \quad (4 \text{ elements}) & \quad (4 \text{ elements})
\end{align*}
\]

So $\mathcal{D}$ is a complex vector space of dimension 16. Define a basis for $\mathcal{D}$ to be the 16 elements

\[
\left\{ 1, \gamma^\mu, \sigma^{\mu\nu} \equiv \gamma^\mu \gamma^\nu (\mu < \nu), \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \gamma^\mu \gamma_5 \right\}
\]
Let these elements be represented by \( \{ \Gamma_a : A = 0, \ldots, 16 \} \).

Note that, since \((\gamma^\mu)^2 = \pm 1\) for all \( \mu \), \( \Gamma^2_a = \pm 1 \) for all \( A \).

Therefore, each basis element of \( \mathcal{O} \) has an inverse in \( \mathcal{O} \), which implies that \( \mathcal{O} \) is semi-simple.

Suppose that \( y \) is an element of the center of \( \mathcal{O} \), i.e., that \( y \in \mathcal{O} \) and \([y, \Gamma] = 0\) for all \( \Gamma \in \mathcal{O} \). Because \( y \in \mathcal{O} \) and the set \( \{ \Gamma_a \} \) is a basis for \( \mathcal{O} \), there exist constants \( a_a \in \mathbb{C} \) such that \( y = \sum a_a \Gamma_a \). Since \([\gamma^\mu, \gamma_5] = 0\) for all \( \mu \),

\[
[\gamma_5, y] = 2 \sum \alpha \mu \gamma_5 \gamma^\mu - 2 \sum \alpha \mu_5 \gamma^\mu
\]

Thus, \([\gamma_5, y] = 0\) implies that \( a_\mu = 0 = a_\mu_5 \) for all \( \mu \).

Therefore,

\[
y = a_1 + \sum \alpha \mu \gamma_5 \gamma^\mu + a_5 \gamma_5
\]

But,

\[
[\gamma^\rho, y] = 2 \sum (\gamma^\rho)_{\lambda \mu} \alpha \mu \gamma^\rho \gamma^\lambda - 2 \sum (\gamma^\rho)_{\lambda \mu_5} \alpha \mu \gamma^\rho + 2 a_5 \gamma^\rho \gamma_5
\]

So, \([\gamma^\rho, y] = 0\) implies \( a_\rho = 0 = a_\mu_5 = 0 = a_5 = 0 \). Therefore \( y = a_1 \). Since the center of \( \mathcal{O} \) is one-dimensional and is semi-simple, \( \mathcal{O} \) must be simple. Therefore, \( \mathcal{O} \) has only one irreducible representation, of dimension 4 since \( 4^2 = 16 \), and this representation is faithful since \( \mathcal{O} \) is simple.

From here on, let \( \gamma_\mu \) and \( \Gamma_a \) stand for elements of either the algebra or its representation.

Define \( \mathcal{I}^{\gamma_\mu} = \gamma_\mu [\gamma^\mu, \gamma_\nu] = \gamma_2 (\gamma^\mu \gamma_\nu - \gamma_\mu \gamma^\nu) \)

By inspection, the \( \mathcal{I}^{\gamma_\mu} \) have the commutation relations of the proper Lorentz group \( \mathbb{L}_+ \), so the \( \mathcal{I}^{\gamma_\mu} \) generate a repre-
sentation of $L^\uparrow$. The $I^{\mu\nu}$ are elements of $\mathcal{D}$ and can be used to generate a subalgebra $\mathcal{D}'$ of $\mathcal{D}$. Note that

\[
\begin{align*}
I^{\mu\nu}I^{\rho\lambda} &= \mathbb{1} \\
I^{\mu\nu}I^{\rho\mu} &= \sigma^{\nu\rho} \\
I^{\mu\nu}I^{\rho\lambda} &= \gamma_5
\end{align*}
\]

Thus, $\{1, \sigma^{\nu\rho}, \gamma_5\}$ is a basis for $\mathcal{D}'$, so $\mathcal{D}'$ has dimension 8 over $\mathbb{C}$ and $\mathcal{D}'$ is semi-simple. Further, by a similar calculation to that for $\mathcal{D}$, it can be shown that the center of $\mathcal{D}'$ consists of all elements of the form $y = a\mathbb{1} + a_5\gamma_5$. The center of $\mathcal{D}'$ is therefore two-dimensional and $\mathcal{D}'$ has two irreducible representations. The $I^{\mu\nu}$ generate a representation of $L^\uparrow$, so $I^{\mu\nu}$ is of the form $I^{\mu\nu} = I^{\mu\nu}(k_1, l_1)$ $I^{\mu\nu}(k_2, l_2)$. The fact that $\mathcal{D}'$ has dimension 8 implies that

\[
(2k_1 + 1)^2 (2l_1 + 1)^2 + (2k_2 + 1)^2 (2l_2 + 1)^2 = 8
\]

and that $(k_1, l_1) \neq (k_2, l_2)$. Therefore,

\[
I^{\mu\nu} = I^{\mu\nu}(1/2, 0) \oplus I^{\mu\nu}(1/2, 0)
\]

Further,

\[
[\gamma^\mu, I^{\nu\lambda}] = \frac{1}{2} [\gamma^\mu, \gamma^\nu \gamma^\lambda] = q^{\mu\nu} \gamma^\lambda - q^{\mu\lambda} \gamma^\nu
\]

So, if $T = \mathcal{D}(1/2, 0) \oplus \mathcal{D}(0, 1/2)$ is the representation of $L^\uparrow$ generated by the $I^{\mu\nu}$, then

\[
T(\ell)^\dagger \gamma^\mu T(\ell) = L^{\mu\nu} \gamma^\nu
\]

from the development in Section 4.1. If $T'$ is any other representation of $L^\uparrow$ such that

\[
T'(\ell)^\dagger \gamma^\mu T(\ell) = L^{\mu\nu} \gamma^\nu
\]
then \[ T'(L) T(L)^{-1} \gamma^\nu T(L) T'(L)^{-1} = \gamma^\nu \quad \forall \mu, L \in \mathcal{L}_+ \]

Therefore, \( T'(L) = c(L) T(L) \) where \( c(L) \in \mathbb{C} \). This implies that \( c(L) \) is a one-dimensional representation of \( \mathcal{L}_+^\mu \), so \( c(L) = 1 \) and \( T'(L) = T(L) \).

Under the action of \( T(L) \), the basis elements of transform as

\[
\begin{align*}
T(L)^{-1} 1 T(L) &= 1 \\
T(L)^{-1} \gamma^\nu T(L) &= L^\nu_\mu \gamma^\mu \\
T(L)^{-1} \sigma^{\mu\nu} T(L) &= L^\nu_\rho L^\rho_\lambda \sigma^{\mu\lambda} \\
T(L)^{-1} \gamma_5 T(L) &= (\det L) \gamma_5 = \gamma_5 \\
T(L)^{-1} \gamma^\mu \gamma_5 T(L) &= (\det L) L^\mu_\nu \gamma^\nu \gamma_5 = L^\mu_\nu \gamma^\nu \gamma_5
\end{align*}
\]

The notation for the basis can be changed in order to express these transformation properties explicitly. Let \( i \in \{0,1,2,3,4\} \) and

\[
\eta_i = \begin{cases} 
0 & i = 0,4 \\
1 & i = 1,3 \\
2 & i = 2 
\end{cases}
\]

and let a general basis element of \( \mathcal{S} \) be

\[
(\Gamma_i)^{\mu_1 \ldots \mu_n} \quad \text{where} \quad (\Gamma_0) = 1 \\
(\Gamma_1)^{\mu_1} = \gamma^{\mu_1} \\
(\Gamma_2)^{\mu_1 \mu_2} = \gamma^{\mu_1} \gamma^{\mu_2} - \epsilon^{\mu_1 \mu_2} \\
(\Gamma_3)^{\mu_1} = \gamma^{\mu_1} \gamma^5 \\
(\Gamma_4) = \gamma_5
\]

Therefore, a general element of \( \mathcal{S} \) has the form

\[
\Gamma = \sum_i c_{i \mu_1 \ldots \mu_n} (\Gamma_i)^{\mu_1 \ldots \mu_n}
\]
where \( c_{1\mu_1 \mu_2} = -c_{1\mu_2 \mu_1} \) so that the \( c_i \)'s are unique. Given this form, the transformation properties of \( \Gamma \in \mathcal{D} \) are

\[
\Gamma(L)^{-1} \Gamma \Gamma(L) = \sum_\lambda c_{i_1 \mu_1 \cdots \mu_n} L^{\lambda_1 \nu_1} \cdots L^{\lambda_n \nu_n} \left( \Gamma \right)^{\gamma_1 \cdots \gamma_n}
\]

The fact that the \( \gamma^\mu \) and \( I^\mu \) satisfy (4.1.1) implies that the \( \gamma^\mu \) correspond to a particular solution from the Bhaba analysis. The representation of the \( I^\mu \) can be chosen so that

\[
I^\mu = \begin{pmatrix}
I^{\mu \nu}(\frac{1}{2}, 0) & 0 \\
0 & I^{\mu \nu}(0, \frac{1}{2})
\end{pmatrix}
\]

where the \( I^{\mu \nu}(k, \ell) \) are in standard form. As a result, the \( \gamma^\mu \) must have the form

\[
\gamma^\mu = i^\frac{1}{2} \sigma^\mu_{ab} \begin{pmatrix}
0 & c_{12} u^a(\frac{1}{2}) \times u^b(\frac{1}{2}) \\
c_{21} u^a(\frac{1}{2}) \times u^b(\frac{1}{2}) & 0
\end{pmatrix}
\]

where \( c_{12} \) and \( c_{21} \) are constants. In the standard form

\[
u^i(\frac{1}{2}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -u^i(\frac{1}{2}) \quad ; \quad \nu^i(\frac{1}{2}) = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -v^i(\frac{1}{2})
\]

\[
u^i(\frac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u^i(\frac{1}{2}) \quad ; \quad \nu^i(\frac{1}{2}) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = v^i(\frac{1}{2})
\]

Therefore

\[
u^i(\frac{1}{2}) \times u^i(\frac{1}{2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -v^i(\frac{1}{2}) \times u^i(\frac{1}{2})
\]

\[
u^i(\frac{1}{2}) \times u^i(\frac{1}{2}) = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \nu^i(\frac{1}{2}) \times u^i(\frac{1}{2})
\]

\[
u^i(\frac{1}{2}) \times u^i(\frac{1}{2}) = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = u^i(\frac{1}{2}) \times u^i(\frac{1}{2})
\]

\[
u^i(\frac{1}{2}) \times v^i(\frac{1}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -\nu^i(\frac{1}{2}) \times u^i(\frac{1}{2})
\]

and
Thus,
\[ \gamma^\mu = \begin{pmatrix} 0 & \frac{1}{2} \, c_{12} \, \sigma^\mu \\ -\frac{1}{2} \, c_{21} \, \sigma^\mu & 0 \end{pmatrix} \]

Note that \((\gamma^\mu)^2 = g^\mu\nu\) and
\[
(\gamma^\mu)^2 = \begin{pmatrix} -\frac{1}{4} \, c_{21} \, c_{12} \, \sigma^\mu \sigma^\nu \, o \\ o & -\frac{1}{4} \, c_{21} \, c_{12} \, \sigma^\nu \sigma^\mu \end{pmatrix} = -\frac{1}{4} \, c_{21} \, c_{12} \, g^\mu\nu \, \mathbb{1}
\]

Therefore, \(c_{12} \, c_{21} = -4\). Let \(c_{12} = -c_{21} = 2\) so that
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \]

Let this representation of the \(\gamma^\mu\) be the "standard form" of the Dirac Algebra. Note that, in this form
\[
\gamma_5 = i \, \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} i \, (g_{00}) \, (g_{22}) \, \sigma_0 \, \sigma_1 \, \sigma_2 \, \sigma_3 \\ o & o & i \, (g_{11}) \, (g_{33}) \, \sigma_0 \, \sigma_1 \, \sigma_2 \, \sigma_3 \end{pmatrix} \\
= \begin{pmatrix} \mathbb{1}_2 & o \\ o & -\mathbb{1}_2 \end{pmatrix}
\]

5.2 Multiple Dirac Algebras

Let \(\mathcal{B}_{(i)}\) be a set of \(n\) Dirac Algebras, where
\(\mathcal{B}_{(i)}\) is generated by \(\{\gamma^\mu_{(i)}, \gamma^\nu_{(i)}\} = 2g^\mu\nu_{(i)}\). Define the algebra \(\mathcal{A}_{(i)}^n\) by
\[ a^n = \mathcal{D}(\gamma_1) \times \mathcal{D}(\gamma_2) \times \ldots \times \mathcal{D}(\gamma_n) \]

This algebra is generated by the 4n elements \( \{\gamma_1^\mu\} \) where

\[ \gamma_{\mu}^\alpha = 1 \times \ldots \times 1 \times \gamma_{\mu}^\alpha \times 1 \times \ldots \times 1 \]

and is a complex vector space of dimension \( 16^n \), where the basis elements are \( \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\} \) and

\[ \Gamma_\alpha^i = 1 \times \ldots \times \gamma_{\alpha}^i \times \ldots \times 1 \]

where \( \Gamma_\alpha^i \) is a basis element of \( \mathcal{D}(\gamma_\alpha) \).

From now on, let \( \Gamma^{(i)} \) be an element of \( \mathcal{D}(\gamma^{(i)}) \) and \( \Gamma^{i} \) its embedding in \( a^n \). Therefore, \( [\Gamma^i, \Gamma^j] = 0 \) if \( i \neq j \).

As in the case of a single Dirac Algebra, each element of the basis of \( a^n \) has an inverse in \( a^n \), so \( a^n \) is semi-simple. Further, if \( \gamma \in \text{center of } a^n \), then \( [\Gamma_\alpha^i, \gamma] = 0 \) for all \( i, \alpha \). Therefore, just as \( \mathcal{D} \) is the complete algebra of all 4x4 complex matrices, \( a^n \) is the complete algebra of \( 4^n \times 4^n \) complex matrices.

The transformation properties of elements of \( \mathcal{D}(\gamma^{(i)}) \) under the action of elements of \( \mathcal{L}^\uparrow_+ \) are uniquely defined by the criteria \( L(\gamma_\alpha^{(i)}) = L^\mu_\nu \gamma_\alpha^{(i)} \) for \( L \in \mathcal{L}^\uparrow_+ \). This leads to the unique representation \( T^{(i)}(L) \) such that \( T^{(i)}(L)^{-1} \gamma_\alpha^{(i)} T^{(i)}(L) = L^\mu_\nu \gamma_\alpha^{(i)} \), so that \( \Gamma^{(i)} \in \mathcal{D}(\gamma^{(i)}) \) implies \( L(\Gamma^{(i)}) = T^{(i)}(L)^{-1} \Gamma^{(i)} T^{(i)}(L) \). So, in order to define the transformation properties of the elements of \( a^n \) under the action of elements of \( \mathcal{L}^\uparrow_+ \), the definition must be consistent with the relations \( L(\gamma_\alpha^{(i)}) = L^\mu_\nu \gamma_\alpha^{(i)} \) in order to
preserve the vector characteristics of $\gamma^\mu_i$. Therefore, 
$L(\Gamma^i) = T^i(L)^{-1} \Gamma^i T^i(L)$ where $T^i(L)$ is the embedding of $T^{(i)}(L)$ into $\alpha^n$. (This is possible since $T^{(i)}(L) \in \mathcal{B}_{(i)}$ for all $i, L \in \mathcal{L}^\uparrow_+$.)

Therefore

$L(\Gamma^{r^1} \Gamma^{r^2} \ldots \Gamma^{r^n}) = [T^{(r^1)}(L) \Gamma^{(r^1)} \Gamma^{(r^2)}(L)] \ldots [T^{(r^n)}(L) \Gamma^{(r^n)}(L)]$

So the elements of $\alpha^n$ transform by a representation $\mathcal{T}^n$ of $\mathcal{L}^\uparrow_+$ which is defined by

$\mathcal{T}^n(L) \equiv T^{(r^1)}(L) \times T^{(r^2)}(L) \times \ldots \times T^{(r^n)}(L)$

$= T^{(r^1)}(L) T^{(r^2)}(L) \ldots T^{(r^n)}(L)$

This representation is unique by the same reasoning as in the Dirac case.

Each $T^{(i)}(L)$ is generated by the infinitesimal generators $I^{\mu\nu}_{(i)} \equiv i\gamma^{\mu}_{(i)} \gamma^{\nu}_{(i)}$. This implies that the infinitesimal generators of $\mathcal{T}^n$ are

$\mathcal{L}^{\uparrow\downarrow}_{\mu\nu} = I^{\uparrow\downarrow}_{\mu\nu} \times \ldots \times I^{\uparrow\downarrow}_{\mu\nu}$

(5.2.1)

$= I^{\uparrow\downarrow}_{1} + I^{\uparrow\downarrow}_{2} + \ldots + I^{\uparrow\downarrow}_{n}$

$= \gamma^\alpha (\gamma^\alpha, \gamma^\alpha) + \ldots + (\gamma^\alpha, \gamma^\alpha)$

Each $T^{(i)}(L)$ is the representation $\mathcal{D}(\frac{1}{2}, 0) \oplus \mathcal{D}(0, \frac{1}{2})$ of $\mathcal{L}^\uparrow_+$, so $\mathcal{T}^n(L)$ is the representation $\hat{\otimes} (\mathcal{D}(\frac{1}{2}, 0) \oplus \mathcal{D}(0, \frac{1}{2}))$. This is also a representation of $\mathcal{L}^\uparrow_+$ and may be written in the reduced form

$\mathcal{D}(\kappa_1, \ell_1) \oplus \mathcal{D}(\kappa_2, \ell_2) \oplus \ldots \oplus \mathcal{D}(\kappa_n, \ell_n)$

Up to similarity, the order of the irreducible representations in this representation does not matter. Therefore,
the representation may be characterized by a set of parameters \{c(k, l)\} such that c(k, l) is the number of times the representation \( \mathcal{D}(k, l) \) occurs in the representation. Let the set \{c^n(k, l)\} be the parameters for \( \mathcal{F}^n \).

The reduction formulas for representations of \( \mathcal{L}^+ \) imply that

\[
\mathcal{D}(k, l) = \left[ \mathcal{D}(\frac{k}{2}, 0) \oplus \mathcal{D}(0, \frac{l}{2}) \right] = \mathcal{D}(k+\frac{l}{2}, l) \oplus \mathcal{D}(k-\frac{l}{2}, l) \oplus \mathcal{D}(k, l+\frac{l}{2}) \oplus \mathcal{D}(k, l-\frac{l}{2})
\]

(ters with \( k - \frac{l}{2} < 0 \) or \( l - \frac{l}{2} < 0 \) are deleted)

Therefore, by defining \( c(k, l) = 0 \) if \( k < 0 \) or \( l < 0 \), the following relation among the \( c^n(k, l) \) can be derived

\[
(5.2.2) \quad c^n(k, l) = c^{n-1}(k-\frac{l}{2}, l) + c^{n-1}(k+\frac{l}{2}, l) + c^{n-1}(k, l-\frac{l}{2}) + c^{n-1}(k, l+\frac{l}{2})
\]

Equation (5.2.2) and the initial conditions \( \{c^1(\frac{k}{2}, 0) = c^1(0, \frac{l}{2}) = 1, c^1(k, l) = 0 \text{ otherwise}\} \) are enough to completely determine the \( c^n(k, l) \).

Iterating (5.2.2) gives the second-order equation

\[
(5.2.3) \quad c^n(k, l) = 4c^{n-2}(k, l) + 2c^{n-2}(k+\frac{l}{2}, l-\frac{l}{2}) + 2c^{n-2}(k-\frac{l}{2}, l+\frac{l}{2})
\]

+ \( 2c^{n-2}(k+\frac{l}{2}, l+\frac{l}{2}) \) + \( 2c^{n-2}(k+\frac{l}{2}, l-\frac{l}{2}) \) + \( c^{n-2}(k-1, l) \) + \( c^{n-2}(k+1, l) \) + \( c^{n-2}(k, l-1) \) + \( c^{n-2}(k, l+1) \)

Since \( c^n(k, l) \geq 0 \) for all \( n, k, l \), (5.2.3) implies that:

\[
(5.2.4) \quad c^n(k, l) > c^{n-2}(k, l) \quad \text{unless} \quad c^n(k, l) = c^{n-2}(k, l) = 0
\]
The degree of $\mathfrak{S}(k, \ell)$ or $c(k, \ell)$ is $k + \ell$. From (5.2.2), a coefficient in $\mathfrak{T}^n$ of degree $p$ depends only on the coefficients in $\mathfrak{T}^{n-1}$ of degree $p-\frac{1}{2}$ or $p+\frac{1}{2}$. Thus implies that, if $\mathfrak{T}^{n-1}$ has only non-zero coefficients of integral (half-integral) degree, then $\mathfrak{T}^n$ has only non-zero coefficients of half-integral (integral) degree. Further, if the highest degree in $\mathfrak{T}^{n-1}$ is $p$, then (5.2.2) implies that the highest degree in $\mathfrak{T}^n$ is $p+\frac{1}{2}$. $\mathfrak{T}$ has only non-zero coefficients of half-integral degree $\frac{1}{2}$. Therefore, $\mathfrak{T}^n$ can only have non-zero coefficients of degree $n/2-p$ where $p$ is a non-negative integer.

Note that the possible coefficient of degree $n-1/2$ in $\mathfrak{T}^{n-1}$ are $c^{n-1}(n-1/2, 0), c^{n-1}(n-2/2, 1/2), \ldots, c^{n-1}(1/2, n-2/2), c^{n-1}(0, n-1/2)$. If these coefficients are non-zero, then, again from (5.2.2), the coefficients $c^n(n/2, 0), c^n(n-1/2, 1/2), \ldots, c^n(0, n/2)$ in $\mathfrak{T}^n$, i.e., all the coefficients of degree $n/2$ in $\mathfrak{T}^n$, are non-zero. This fact, together with (5.2.4), implies that any coefficient of degree $n/2-p$ in $\mathfrak{T}^n$ is non-zero for any non-negative integer $p$.

The result of these conclusions is that: A coefficient $c^n(k, \ell)$ is non-zero if and only if $k+\ell = n/2-p$ for some non-negative integer $p$. Further, given an integer $m$ and a representation $(k, \ell)$, it is possible to find an $n$ such that $c^n(k, \ell) \geq m$. These conclusions can be illustrated by a table of the coefficients of the first few $\mathfrak{T}^n$. 
<table>
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<th>degree</th>
<th>( n \rightarrow )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>0</td>
<td>( c^n(0,0) )</td>
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<tr>
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<tr>
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<td>5</td>
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<tr>
<td>1</td>
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<td></td>
<td>9</td>
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<td></td>
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<tr>
<td>1</td>
<td>( c^n(1/2,1/2) )</td>
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<td></td>
<td>16</td>
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<tr>
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<tr>
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Suppose that the set \( \{ \alpha^\mu, I^\rho \} \) is a solution set to Eqs. (4.1.9) and (2.4.6) and, further, that the algebra generated by this set is indecomposable. From Section 4.4, the representation of \( \mathcal{L}_+^\rho \) generated by the \( I^\rho \) can contain irreducible representations of only integral or only half-integral degree. Let \( d(k, \ell) \) be the coefficient of the representation \( \mathcal{O}(k, \ell) \) in \( I^\rho \). Since the representation generated by \( I^\rho \) is finite there must exist a minimum degree \( p_0 \) such that \( d(k, \ell) = 0 \) for \( k-\ell > p_0 \). Therefore, the only possible non-zero coefficients in \( I^\rho \) are the \( d(k, \ell) \) for which \( k+\ell = p_0-p \) for some non-negative integer \( p \).

Given a non-zero \( d(k, \ell) \), it is possible to find an \( n \) such that \( c^n(k, \ell) \geq d(k, \ell) \). Further, the choice can be made so that \( n \) is the minimum integer with this property. By choosing such an \( n(k, \ell) \) for each non-zero \( d(k, \ell) \), a finite set of integers is chosen, all of which are either even or odd. If \( n_o \) is the maximum integer in this set, then (5.2.4) implies that \( c^{n_o}(k, \ell) \geq d(k, \ell) \) for all \( (k, \ell) \), and that \( n_o \) is the minimum integer with this property. Bearing in mind the definitions of \( c^n(k, \ell) \) and \( d(k, \ell) \), this development implies that there exists an \( n_o \) and a representation \( I_o^\mu \nu \) such that \( c^{n_o} \) can be put in the form

\[
(5.2.5) \quad c^{n_o} = \left( \begin{array}{cc} I^\mu \nu & \circ \\ \circ & I_o^\nu \mu \end{array} \right)
\]
The method of choosing \( n_0 \) described above also implies that \( n_0 \) is the least such integer for which (5.2.5) is true.

The group \( A^{n_0} \) contains all \( 4^{n_0} \times 4^{n_0} \) complex matrices, so it contains the four matrices \( \alpha \) defined by

\[
\alpha' = \begin{pmatrix}
\alpha & 0 \\
0 & 0
\end{pmatrix}
\]

Since the set \( \{\alpha, I_0^{\rho \lambda}\} \) satisfies (4.1.9), the set \( \{\alpha', \mu \nu\} \) does also. The set of matrices \( \{\alpha\} \) and the set \( \{\alpha'\} \) can each be said to generate an algebra. The fact that these algebras are isomorphic by the mapping \( \alpha = \alpha' \) is clear. However, the algebras generated by \( \{\alpha, I_0^{\rho \lambda}\} \) and \( \{\alpha', \mu \nu\} \) are not necessarily isomorphic because \( I_0^{\mu \nu} \) is not necessarily the trivial representation.

The basic conclusion of this development is:

Given a finite representation \( T \) of \( L_+ \) and a set of matrices \( \{\alpha\} \) such that \( T(L)^{-1}\alpha T(L) = L_{\mu \nu} \alpha_{\mu \nu} \) for all \( \mu \) and \( L \in L_+ \) and such that the set \( \{\alpha, T(L)\} \) is indecomposable, then there exists an \( n_0 \) and a set \( \{\alpha'\} \) of elements of \( A^{n_0} \) such that

\[
\mathcal{J}^{n}(L)^{-1} \alpha' \mathcal{J}^{n}(L) = L_{\mu \nu} \alpha'_{\mu \nu}
\]

for all \( \mu \) and \( L \in L_+ \) and such that the algebra generated by the set \( \{\alpha\} \) is isomorphic to the algebra generated by the set \( \{\alpha'\} \) by the mapping \( \alpha = \alpha' \).

Therefore, in order to study the algebraic properties of the solutions to (4.1.9), it is enough to study the properties of elements \( \alpha \) of \( A^n \) which transform like vectors under \( \mathcal{J}^{n} \).
5.3 Vectors in $\alpha^n$

The basis set for the algebra $\alpha^n$ is the set of $16^n$ elements $\{\Gamma_{a_1}^{1} \Gamma_{b_2}^{2} \ldots \Gamma_{x_n}^{n}\}$. In Section 5.1 an alternate basis $\{(\Gamma_i)^{\mu_1 \ldots \mu_n}_{i}\}$ was defined for $\alpha$. This definition may be extended to the basis set for $\alpha^n$ by introducing the notation

$$(\Gamma_{i_1}^{1} \Gamma_{i_2}^{2} \ldots \Gamma_{i_n}^{n})^{\mu_1 \ldots \mu_n} = \prod_{j=1}^{n} (\Gamma_{i_j}^{j})^{\mu_{j_1} \ldots \mu_{j_{n,j-1}}},$$

where $m = \sum_{j=1}^{n} n_j$ and $m_j = 1 + \sum_{k=1}^{j-1} n_k$. With this notation, the basis set for $\alpha^n$ may be written $\{(\Gamma_{i_1}^{1} \ldots \Gamma_{i_n}^{n})^{\mu_1 \ldots \mu_m}\}$ and an arbitrary element of $\alpha^n$ has the form

$$\Gamma = \sum_{\iota_1, \ldots, \iota_n} (c_{\iota_1}^{1} \ldots c_{\iota_n}^{n})_{\mu_1 \ldots \mu_m} (\Gamma_{\iota_1}^{1} \ldots \Gamma_{\iota_n}^{n})^{\mu_1 \ldots \mu_m}$$

From the definition of the basis elements, requiring antisymmetry in any pair of vector indices corresponding to a $j$ such that $i_j = 2$ implies that the coefficients are unique.

Suppose $\{\alpha^{\mu}\}$ is a set of four elements of $\alpha^n$ which transform as a vector under the action of $\mathcal{J}^n$, i.e.,

$$\mathcal{J}^n(L)^{1} \alpha^{\mu} \mathcal{J}^n(L) = L^{\gamma \gamma} \alpha^{\nu} \quad \forall \mu, \nu \in \mathcal{L}^+_n$$

Since $\alpha^{\mu} \in \alpha^n$, there exist unique coefficients $(c_{i_1}^{\mu} \ldots i_n^{\mu})_{\mu_1 \ldots \mu_m}$ such that

$$\alpha^{\mu} = \sum_{\iota_1, \ldots, \iota_n} (c_{\iota_1}^{\mu} \ldots c_{\iota_n}^{n})_{\mu_1 \ldots \mu_m} (\Gamma_{\iota_1}^{1} \ldots \Gamma_{\iota_n}^{n})^{\mu_1 \ldots \mu_m}$$
Therefore, (5.3.1) implies that

\[ \mathcal{F}(L)^{-1} \mathcal{A} \mathcal{F}(L) = \sum_{i_1, \ldots, i_n} (c_{i_1, \ldots, i_n})_{\mu_1 \ldots \mu_m} L^{i_1} v_1 \ldots L^{i_n} v_n (\Gamma_{\alpha_1}^1 \ldots \Gamma_{\alpha_n}^n)^{\nu_1 \ldots \nu_m} \]

\[ = \sum_{i_1, \ldots, i_n} L^{i} v (c_{i_1, \ldots, i_n})_{\mu_1 \ldots \mu_m} (\Gamma_{\alpha_1}^1 \ldots \Gamma_{\alpha_n}^n)^{\nu_1 \ldots \nu_m} \]

The coefficients of such an expansion are unique, so

\[ (5.3.2) \quad L^{i} v (c_{i_1, \ldots, i_n})_{\mu_1 \ldots \mu_m} = (c_{i_1, \ldots, i_n})_{\nu_1 \ldots \nu_m} L^{i} v \]

Suppose \( m \) is even. By defining the matrix \( C^\mu \) so that

\[ (C^\mu)^{(\mu_1 \ldots \mu_m)}_{(\mu'_1 \ldots \mu'_{m'})} = (c_{i_1, \ldots, i_n})_{\mu_1 \ldots \mu_m} \]

where \( m' = m/2 \) and using the fact that \( (L^{-1})^\mu_v = L^\mu_v \), Eq. (5.3.2) can be written as

\[ (5.3.3) \quad L^{i} v C^\nu = \left( L \times L \times \ldots \times L \right)^{\text{m factors}} \left( L \times L \times \ldots \times L \right)^{\text{m' factors}} \]

The manipulations at the end of Section 2.4 imply that

\[ \sum_{\lambda} \mathcal{K}^2_{\lambda} = \sum_{\lambda} L^{\lambda} = \frac{\mathcal{P}}{4} \mathcal{I} \]

so that the identity representation of \( \mathcal{L}_+^\uparrow \) is the representation \( \mathcal{B}(\frac{1}{2}, \frac{1}{2}) \). Thus, Eq. (5.3.3) is just (4.1.6) with the representation

\[ T^{m'} = \mathcal{B}(\frac{1}{2}, \frac{1}{2}) \times \ldots \times \mathcal{B}(\frac{1}{2}, \frac{1}{2}) \]

This representation can be reduced to a direct product of irreducible representations just as any representation of \( \mathcal{L}_+^\uparrow \) can. Note that...
\[ \mathcal{O}(\nu_1, \nu_2) \times \mathcal{O}(k, l) = \mathcal{O}(k+\nu_2, l+\nu_2) \oplus \mathcal{O}(k+\nu_2, l-\nu_2) \]
\[ \oplus \mathcal{O}(k-\nu_2, l+\nu_2) \oplus \mathcal{O}(k-\nu_2, l-\nu_2) \]

Thus, \( T_m \) contains only representations \( \mathcal{O}(k, l) \) with \( k \) and \( l \) both integral if \( m' \) is even and both half-integral if \( m' \) is odd. Solutions to (4.1.6) exist only in the sub-blocks for which \( k' = k \pm \frac{1}{2} \) and \( l' = l \pm \frac{1}{2} \) and these cannot occur in \( T_m' \). Therefore, there are no non-trivial solutions for \( m \) even.

For \( m \) odd it is possible to construct solutions to Eq. (5.3.2) by using the defining relation for \( \mathcal{L} \), Eq. (2.2.2). This equation implies that the coefficients defined by

\[ (c^{a_1...a_n})_{\mu_1...\mu_m} = C \delta^{\mu_1}_{\mu_2} g_{\mu_2 \mu_3} g_{\mu_3 \mu_4} ... g_{\mu_{m-1} \mu_m} \]

where \( C \) is a constant, form a solution to (5.3.2). Further, any permutation of the indices \( \{\mu, \mu_1, ... , \mu_m\} \) on the left side of Eq. (5.3.4) also results in a solution to (5.3.2). This implies that there are \( m(m-2)(m-4)...(3)(1) \) independent solutions of this form.

This development has dealt with only elements of \( \mathcal{L}_+^\top \), the proper Lorentz transformations. As a result, the fact that \( \det L = 1 \) for all \( L \in \mathcal{L}_+^\top \) implies that

\[ \epsilon_{\mu_1 \nu_\rho \lambda} \ L^{\mu_1}_{\nu} \ L^{\nu}_{\rho} \ L^{\rho}_{\lambda} \ L^{\lambda} = \epsilon_{\mu_1 \nu_\rho \lambda} \]

and that

\[ (c^{a_1...a_n})_{\mu_1...\mu_m} = \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} (c^{a_1})_{\mu_5 \mu_6 ... \mu_m} \]
where \( d \) is a solution to (5.3.2) for \( m - m - 4 \), is also a solution to (5.3.2). Once again, any permutation of indices in (5.3.5) also leads to a solution of (5.3.2). Note, however, that these solutions aren't always independent of the ones in (5.3.4). For example,

\[
\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} = - \text{det} \begin{pmatrix}
\varphi_{\mu_1 \nu_1} & \varphi_{\mu_1 \nu_2} & \varphi_{\mu_1 \nu_3} & \varphi_{\mu_1 \nu_4} \\
\varphi_{\mu_2 \nu_1} & \varphi_{\mu_2 \nu_2} & \varphi_{\mu_2 \nu_3} & \varphi_{\mu_2 \nu_4} \\
\varphi_{\mu_3 \nu_1} & \varphi_{\mu_3 \nu_2} & \varphi_{\mu_3 \nu_3} & \varphi_{\mu_3 \nu_4} \\
\varphi_{\mu_4 \nu_1} & \varphi_{\mu_4 \nu_2} & \varphi_{\mu_4 \nu_3} & \varphi_{\mu_4 \nu_4}
\end{pmatrix}
\]

In many cases these \( \epsilon \) solutions appear as products of \( \gamma_5 \)'s.

For a specific \( m \), the number of independent solutions to Eq. (5.3.2) can be calculated. By defining a matrix \( C \) such that (m is odd)

\[
C_{(\mu_1 \ldots \mu_{m'})}^{(\mu_1 \ldots \mu_m)} = (C_{1 \ldots 4})^{\mu_1 \ldots \mu_{m'}}
\]

for \( m' = (m+1)/2 \), it is possible to put (5.3.2) in the form

\[
C' = T_{m'}(L)^{-1} C T_{m'}(L)
\]

Thus, the number of independent solutions to (5.3.2) is equal to the number of independent matrices which commute with \( T_{m'}(L) \) for all \( L \).

Suppose that \( T_m \) has been fully reduced and is described by the parameters \( C^{m'}(k, L) \) as defined in Section 5.2. Since any matrix which commutes with all the \( T_{m'}(L) \) must have zero sub-blocks connecting different representations
and multiples of the unit matrix in the other sub-blocks, 
the number of independent matrices is \( x_{m'} \), where

\[
X_{m'} = \sum_{(k,l)} \left[ C^{m'}(k,l) \right]^2
\]

For example, \( m' = 1 \) implies that \( T_{m'} = \mathcal{S}(1/2,1/2) \), so \( x_1 = 1 \).
If \( m' = 2 \), then

\[
T_{m'} = \mathcal{S}(1,1) \oplus \mathcal{S}(1,0) \oplus \mathcal{S}(0,1) \oplus \mathcal{S}(0,0)
\]
and \( x_2 = 1 + 1 + 1 + 1 = 4 \). Finally, if \( m' = 3 \), then

\[
T_{m'} = \mathcal{S}(1/2,3/2) \oplus 2 \mathcal{S}(3/2,1/2) \oplus 2 \mathcal{S}(1/2,3/2) \oplus 4 \mathcal{S}(1/2,1/2)
\]
and \( x_3 = 1 + 4 + 4 + 16 = 25 \).

However, the independence of solutions to (5.3.2) 
does not imply that they are independent when applied to 
form elements of \( \alpha^n \). For example, the class of basis 
elements in \( \alpha^2 \) denoted by \( i_1 = 1, i_2 = 2 \) has elements

\[
\left( \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right)^{\mu_1 \mu_2 \mu_3} = \gamma_1^{\mu_1} \gamma_2^{\mu_2} \gamma_3^{\mu_3} = \gamma_1^{\mu_1} \left( \gamma_2^{\mu_2} \gamma_2^{\mu_3} - \gamma_2^{\mu_2} \gamma_2^{\mu_3} \right)
\]

By defining

\[
\left( C'_{i2} \right)^{\mu_1 \mu_2 \mu_3} = \delta^{\mu_1} \gamma_{\mu_2 \mu_3}
\]

it is possible to form a vector in \( \alpha^2 \). But

\[
\left( C'_{i2} \right)^{\mu_1 \mu_2 \mu_3} \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right)^{\mu_1 \mu_2 \mu_3} = \gamma_1^{\mu_1} \left( \gamma_{\mu_2 \mu_3} \gamma_2^{\mu_2} \gamma_2^{\mu_3} - \gamma_{\mu_2 \mu_3} \gamma_2^{\mu_2} \gamma_2^{\mu_3} \right)
\]

\[
= \gamma_1^{\mu_1} \left( 0 \right) = 0
\]
so this is not an independent vector. Thus, it is diffi-
cult to give a general approach to finding all solutions
of (5.3.1) for $a^n$ which also specifies which solutions are independent. Most such approaches have to be applied to specific cases in order to gain information about linear dependence.
VI. EXAMPLES

The purpose of this chapter is to present examples which will demonstrate several approaches to the use of the formalism developed in Chapter V. The first example is simply $\alpha^1$, the Dirac algebra itself, and is included primarily to state certain useful properties used in the later examples. The next example is a detailed presentation of the relation between $\alpha^2$ and the Bhaba solutions. The purpose of this example is to show how solutions from the Bhaba analysis may be translated directly into the multiple Dirac algebra formalism. It was specifically for this example that the "standard" representations were presented in the earlier chapters.

The third example is concerned with vectors in $\alpha^3$. In it, certain vectors are chosen and their algebras developed without relevance to the corresponding Bhaba solutions. This illustrates the use of the formalism in finding solutions to (4.1.1) and determining the algebraic properties of those solutions.

6.1 Vectors in the Dirac Algebra

In Section 5.1, the Dirac algebra and its "standard" representation were defined. As a result, the $\gamma^\mu$, $I^{\mu\nu}$, and $\gamma^5$ have the form

$$\gamma^\mu = \sigma^{\mu\nu\rho} \begin{pmatrix} \sigma & u^\nu(v_2) \times u^\rho(v_2) \\ -u^\nu(v_2) \times u^\rho(v_2) & \sigma \end{pmatrix} = \begin{pmatrix} \sigma & \sigma^\mu \\ \sigma^\mu & \sigma \end{pmatrix}$$
\[ I^{\mu\nu} = \left( \begin{array}{cc} I^{\mu\nu}(\gamma_5, \sigma) & 0 \\ 0 & I^{\mu\nu}(\sigma, \gamma_5) \end{array} \right) \]

\[ \gamma_5 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \]

The Bhaba solution implies that any vector in this algebra must have the form

\[ \alpha^\mu = \frac{1}{2} \sigma_{ab} \left( \begin{array}{cccc} 0 & c_{12} u^a(\gamma_5) x u^b(\gamma_5) & c_{12} u^a(\gamma_5) x u^b(\gamma_5) \\ c_{12} u^a(\gamma_5) x u^b(\gamma_5) & 0 \end{array} \right) \]

Comparing this form to that of the \( \gamma^\mu \) and \( \gamma_5 \) matrices leads to the relation

\[ \alpha^\mu = \frac{1}{2} \left[ (c_{12} - c_{21}) I + (c_{12} + c_{21}) \gamma_5 \right] \gamma^\mu \]

Therefore, any vector in \( a^\dagger \) has the form

\[ \alpha^\mu = \left( a + b \gamma_5 \right) \gamma^\mu \]

If \( c \neq \pm d \), then the matrix \( s = c + d \gamma_5 \) is a non-singular matrix which commutes with the \( I^{\mu\nu} \). Thus, the mapping \( T_s: a^\dagger \rightarrow a^\dagger \) defined by

\[ T_s = s \Gamma s^{-1} \quad \forall \Gamma \in a^\dagger \]

is an isomorphism which maps \( I^{\mu\nu} \) to itself, and, consequently, vectors to vectors. The inverse of \( s \) is

\[ s^{-1} = \left( c - d \gamma_5 \right)^{-1} \left[ c + d \gamma_5 \right] \]

Therefore,
\[ s \alpha^\mu s^{-1} = \left[ (c + d) \eta_5 \right] (a + b) \eta_5 \gamma^\mu (c - d) \eta_5 \right] 2 (c^2 - d^2)^{-1} \]

\[ = 2 (c^2 - d^2)^{-1} \left[ (ac^2 + ad^2 + 2c) \eta_5 + (bc^2 + bd^2 + 2b) \eta_5 \right] \gamma^\mu \]

If \( c/d \) has the property that

\[(6.1.1) \quad d (c/d)^2 + 2a (c/d) + a = 0 \]

then

\[ s s^{-1} = 2 d^2 (c^2 - d^2)^{-1} \left[ a (c/d)^2 + 2d (c/d) + a \right] \gamma^\mu \]

\[ \equiv a' \gamma^\mu \]

Note that, if \( a \neq \pm b \), then (6.1.1) implies that \( c \neq \pm d \), \( a \neq 0 \), and \( a' \neq 0 \). The result is that, if \( a \neq \pm b \), then there is an isomorphism from \( a' \) to itself which maps \( (a + b) \gamma_5 \gamma^\mu \) to a multiple of \( \gamma^\mu \). Thus, the algebra generated by the matrices \( (a + b) \gamma_5 \gamma^\mu \) is isomorphic to the Dirac algebra by an isomorphism which preserves the transformations \( J^{-1}(L) \) if \( a \neq \pm b \).

If \( a = \pm b \), then \( \alpha^\mu = a (1 \pm \gamma_5) \gamma^\mu \). Since

\[ \alpha^\mu \gamma^\nu = a^2 (1 \pm \gamma_5) \gamma^\mu (1 \pm \gamma_5) \gamma^\nu = a^2 (1 \pm \gamma_5) (1 \neq \gamma_5) \gamma^\mu \gamma^\nu = 0 \]

the algebra generated by the \( \alpha^\mu \) in either case consists only of constant multiples of the four elements \( \alpha^\mu \). The specific forms of these elements are

\[ a (1 + \gamma_5) \gamma^\mu = a \begin{pmatrix} 0 & 0 \\ o & o \end{pmatrix} \]

\[ a (1 - \gamma_5) \gamma^\mu = a \begin{pmatrix} 0 & 0 \\ \sigma^\mu & o \end{pmatrix} \]
As a result, the vectors in $\mathfrak{a}^1$ generate only two different algebras, the 16-dimensional Dirac algebra and a 4-dimensional nilpotent algebra. The nilpotent algebra here is the same as the example of a non-semi-simple algebra used in Section 4.4.

6.2 Vectors in $\mathfrak{a}^2$

The algebra $\mathfrak{a}^2 = \mathcal{B} \times \mathcal{B}$ is generated by the eight elements $\gamma_1^\mu = \gamma^\mu \times 1_4$, $\gamma_2^\mu = 1_4 \times \gamma^\mu$. If the $\gamma^\mu$ are in standard form, the representation of $\mathfrak{a}^2$ is generated by

$$\gamma_1^{\mu\nu} = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \overline{\sigma^\mu} & 0 \end{array} \right) \times 1_4 = \left( \begin{array}{cc} 0 & \sigma^\mu \times 1_4 \\ \sigma^\mu \times 1_4 & 0 \end{array} \right)$$

$$\gamma_2^{\mu\nu} = \left( \begin{array}{cc} 1_4 \times \gamma^\mu & 0 \\ 0 & 1_4 \times \gamma^\mu \end{array} \right)$$

Thus, the infinitesimal generators of $\mathfrak{g}^2$ are

$$\mathcal{L}_1^{\mu\nu} = I_1^{\mu\nu} + I_2^{\mu\nu} = I_1^{\mu\nu} \times 1_4 + 1_4 \times I_2^{\mu\nu}$$

where $I_1^{\mu\nu} = \frac{1}{4} \{ \gamma^{\mu}, \gamma^{\nu} \} = \frac{1}{4} \left( \begin{array}{ccc} \sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu & 0 \\ 0 & \sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu \end{array} \right)$

The irreducible components of $\mathfrak{g}^2$ were determined in Section 5.2, so the standard form of $\mathcal{L}_2^{\mu\nu}$ can be defined as
From Section 3.4, the representation in (6.2.1) can be reduced to the one in (6.2.2) by successive application of the three matrices

\[ O_1 \equiv O^{(\frac{1}{2}, \frac{3}{2})} \oplus O^{(\frac{1}{2}, \frac{1}{2})} \]
\[ O_2 \equiv O^{(\frac{1}{2}, \frac{1}{2})} \oplus O^{(\frac{1}{2}, \frac{3}{2})} \oplus O^{(\frac{3}{2}, \frac{3}{2})} \oplus O^{(\frac{3}{2}, \frac{1}{2})} \]
\[ R \equiv U(\frac{1}{2}, 0) \oplus U(0, \frac{1}{2}) \oplus U(\frac{1}{2}, 0) \oplus U(0, \frac{1}{2}) \]

Note that

\[ U(\frac{1}{2}, 0) = U(\frac{3}{2}) = (\sqrt{2})^{-\frac{1}{2}} \begin{pmatrix} u_{1,0} \ u_{2,0} \\ u_{1,\frac{1}{2}} \ u_{2,\frac{1}{2}} \end{pmatrix} = X \]
\[ U(0, \frac{1}{2}) = 1 \]
\[ V(\frac{1}{2}, 0) = O(\frac{1}{2}, \frac{1}{2}) = 0 \]
\[ V(0, \frac{1}{2}) = V(\frac{1}{2}) = U(\frac{1}{2}) = X \]

Let \( \tilde{\gamma}_1^{\mu} \) be the matrix form of \( \gamma_1^{\mu} \) in the reduced representation of (6.2.2). Therefore

\[ \tilde{\gamma}_1^{\mu} = R O_2 O_1 \gamma_1^{\mu} O_1^{-1} O_2^{-1} R^{-1} = \begin{pmatrix} o & o & X(\sigma^+ x 1_z) O & o \\ o & o & o & X(\sigma^+ x 1_z) X^{-1} \\ O(\sigma^+ \times 1_z) X^{-1} & o & o & o \\ o & X(\sigma^+ \times 1_z) & o & o \end{pmatrix} \]
Performing the explicit calculations and using the properties of the \( u, v \) matrices yields the specific forms for these vectors.

\[
\tilde{\gamma}_1^\mu = \gamma_2^\mu \sigma_{\alpha \beta} G_1^{\alpha \beta} \quad \text{WHERE}
\]

\[
G_1^{\alpha \beta} \equiv (2, 4) \left[ \begin{array}{cccccc}
0 & 0 & 0 & u^a(1) \times u^c(1) & 0 & 0 \\
0 & 0 & 0 & u^c(1) \times u^b(1) & 0 & 0 \\
0 & 0 & 0 & u^b(2) \times u^c(1) & 0 & 0 \\
0 & 0 & 0 & u^c(2) \times u^b(1) & 0 & 0 \\
3 & 1 & 4 & 4 & 3 & 1
\end{array} \right]
\]

\[
\tilde{\gamma}_2^\mu = \gamma_2^\mu \sigma_{\alpha \beta} G_2^{\alpha \beta} \quad \text{WHERE}
\]

\[
G_2^{\alpha \beta} \equiv (2, 4) \left[ \begin{array}{cccccc}
0 & 0 & 0 & u^a(1) \times u^b(1) & 0 & 0 \\
0 & 0 & 0 & -u^c(2) \times u^b(1) & 0 & 0 \\
0 & 0 & 0 & u^b(2) \times u^b(1) & 0 & 0 \\
0 & 0 & 0 & u^b(2) \times u^c(1) & 0 & 0 \\
0 & 0 & 0 & -u^c(2) \times u^c(1) & 0 & 0 \\
0 & 0 & 0 & u^c(2) \times u^c(1) & 0 & 0
\end{array} \right]
\]
A similar manipulation provides the representation of certain scalars in $\mathcal{A}^2$.

$$\tilde{\gamma}^1_5 = \gamma^1_5 = \begin{pmatrix} \mathbf{1} \mathbf{g} & 0 \\ 0 & -\mathbf{1} \mathbf{g} \end{pmatrix}; \quad \tilde{\gamma}^2_5 = \begin{pmatrix} \mathbf{1} \mathbf{g} & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{1} \mathbf{g} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \mathbf{g} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \mathbf{g} & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{1} \mathbf{g} \end{pmatrix}$$

$$g_{\mu \nu} \tilde{\gamma}^\mu \tilde{\gamma}^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{1} \mathbf{4} & 0 & 0 \\ 0 & 0 & \mathbf{1} \mathbf{4} & 0 & 0 & 0 \\ 0 & \mathbf{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 2 \tilde{\eta}$$

$$\tilde{\eta}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \mathbf{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \mathbf{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Bhaba analysis yields the result that the only possible vectors in $\mathcal{A}^2$ have the form (in the representation given by (6.2.2)) $\tilde{\alpha}^\mu = \frac{1}{2} \sigma^\mu_{ab} A^a_b$ where
By manipulating the scalars \( \tilde{\gamma}_3^1, \tilde{\gamma}_5^2 \) and \( \tilde{\eta} \) and the vectors \( \tilde{\gamma}_3^1 \) and \( \tilde{\gamma}_5^2 \), it is possible to write \( a^\mu \) as

\[
\tilde{a}^\mu = \left[ a_1 \mathbb{1} + a_2 \tilde{\gamma}_3^1 + a_3 \tilde{\gamma}_5^2 + a_4 \tilde{\gamma}_3^1 \tilde{\gamma}_5^2 + a_5 (1 - \tilde{\gamma}_3^1 \tilde{\gamma}_5^2) \tilde{\eta} + a_6 (\tilde{\gamma}_3^1 - \tilde{\gamma}_5^2) \tilde{\eta} + a_7 (1 + \tilde{\gamma}_3^1 \tilde{\gamma}_5^2) \tilde{\eta}^2 + a_8 (\tilde{\gamma}_3^1 + \tilde{\gamma}_5^2) \tilde{\eta}^2 \right] \tilde{\gamma}_3^1
\]

\[
+ \left[ b_1 \mathbb{1} + b_2 \tilde{\gamma}_3^1 + b_3 \tilde{\gamma}_5^2 + b_4 \tilde{\gamma}_3^1 \tilde{\gamma}_5^2 + b_5 (1 - \tilde{\gamma}_3^1 \tilde{\gamma}_5^2) \tilde{\eta} + b_6 (\tilde{\gamma}_3^1 - \tilde{\gamma}_5^2) \tilde{\eta} + b_7 (1 + \tilde{\gamma}_3^1 \tilde{\gamma}_5^2) \tilde{\eta}^2 + b_8 (\tilde{\gamma}_3^1 + \tilde{\gamma}_5^2) \tilde{\eta}^2 \right] \tilde{\gamma}_5^2
\]
where

\[ a_1 = \frac{1}{\sqrt{8}} (2 d_2 + 2 c_6 + c_2 + c_4 + d_5 + d_6) Z^{-1/2} \]
\[ a_2 = \frac{1}{\sqrt{8}} (2 d_2 - 2 c_6 - c_2 - c_4 + d_5 + d_6) Z^{-1/2} \]
\[ a_3 = \frac{1}{\sqrt{8}} (2 d_2 - 2 c_6 + c_2 + c_4 - d_5 - d_6) Z^{-1/2} \]
\[ a_4 = \frac{1}{\sqrt{8}} (2 d_2 + 2 c_6 - c_2 - c_4 - d_5 - d_6) Z^{-1/2} \]
\[ a_5 = \frac{1}{\sqrt{8}} (c_1 + c_3 + d_7 + d_8) Z^{-1/2} \]
\[ a_6 = \frac{1}{\sqrt{8}} (c_1 + c_3 - d_7 - d_8) Z^{-1/2} \]
\[ a_7 = \frac{1}{\sqrt{8}} (-c_6 + c_5 + d_4 - d_2) Z^{-1/2} \]
\[ a_8 = \frac{1}{\sqrt{8}} (-c_5 + c_6 + d_4 - d_2) Z^{-1/2} \]

\[ b_1 = \frac{1}{\sqrt{8}} (2 d_1 - 2 c_8 + c_1 - c_3 + d_7 - d_9) Z^{-1/2} \]
\[ b_2 = \frac{1}{\sqrt{8}} (2 d_1 + 2 c_8 + c_1 - c_3 - d_7 + d_9) Z^{-1/2} \]
\[ b_3 = \frac{1}{\sqrt{8}} (2 d_1 + 2 c_8 - c_1 + c_3 + d_7 - d_8) Z^{-1/2} \]
\[ b_4 = \frac{1}{\sqrt{8}} (2 d_1 - 2 c_8 - c_1 + c_3 - d_7 + d_8) Z^{-1/2} \]
\[ b_5 = \frac{1}{\sqrt{8}} (c_2 - c_4 + d_5 - d_6) Z^{-1/2} \]
\[ b_6 = \frac{1}{\sqrt{8}} (c_4 - c_2 + d_5 - d_6) Z^{-1/2} \]
\[ b_7 = \frac{1}{\sqrt{8}} (+c_7 + c_9 - d_1 - d_3) Z^{-1/2} \]
\[ b_8 = \frac{1}{\sqrt{8}} (-c_7 - c_9 - d_1 - d_3) Z^{-1/2} \]

Note that Eq. (6.2.4) is representation independent. Therefore, if \( \mathcal{L} \) is the scalar subspace of \( \mathcal{A}^2 \) spanned by the eight elements

\[ \{ 1, \gamma_5^1, \gamma_5^2, \gamma_5^3, (1 - \gamma_5^1 \gamma_5^2) \eta, (\gamma_5^1 - \gamma_5^2) \eta, (1 + \gamma_5^1 \gamma_5^2) \eta, (\gamma_5^1 + \gamma_5^2) \eta \} \]

then any vector in \( \mathcal{A}^2 \) has the form

(6.2.5) \[ \lambda^\mu = s_1 \gamma_1^\mu + s_2 \gamma_2^\mu \] where \( s_1, s_2 \in \mathcal{L} \)
\( \mathcal{L} \) is not the space of all scalars in \( \mathbb{R}^2 \) since it does not contain the scalars \((1 + \gamma_5^1 \gamma_5^2) \eta \) and \((\gamma_5^1 + \gamma_5^2) \eta \). But applying these scalars in \((6.2.5)\) does not yield any new vectors, so they are not included in \( \mathcal{L} \).

The correspondence given by Eqs. \((6.2.3)\) and \((6.2.4)\) allows solutions of the form \((6.2.3)\) to be transformed into algebraic elements of the form \((6.2.4)\). Since the commutation relations of the \(\gamma_i^\mu\) elements are known, the properties of elements such as those of \((6.2.4)\) can be determined. As a direct illustration of this point, the correspondence will be applied to the Duffin-Kemmer matrices.

The Duffin-Kemmer matrices are a set of four elements \(\beta^\mu\) which satisfy the commutation relations

\[
(6.2.6) \quad [\beta^\mu, \beta^\nu] = \delta^\nu_{\lambda} \beta^\mu + \delta^\mu_{\lambda} \beta^\nu
\]

The algebra generated by these matrices is semi-simple and has dimension 126 as a complex vector space.* The center of the algebra contains three linearly independent elements, so it has three irreducible representations, of dimensions 10, 5, and 1. The representations of \(\mathcal{L}_+\) by which each irreducible representation transforms and the form of the elements \(\beta^\mu\) as Bhaba solutions are well-determined.** They are

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* For a complete development of this algebra and its properties, see Kemmer\textsuperscript{2} or Heitler.\textsuperscript{13}

** See Corson\textsuperscript{9} or Shamaly and Capri.\textsuperscript{14}
(10-dimensional): \[ T = \mathcal{B}(1,0) \oplus \mathcal{B}(\frac{1}{2},\frac{1}{2}) \oplus \mathcal{B}(0,1) \]

\[ \beta^\mu = \frac{1}{2} \sigma^\mu_{\lambda\kappa} \begin{pmatrix}
0 & u^\lambda(1) \times u^\kappa(1) & 0 \\
0 & u^\lambda(\frac{1}{2}) \times u^\kappa(\frac{1}{2}) & 0 \\
0 & u^\lambda(0) \times u^\kappa(0) & 0
\end{pmatrix} \]

(5-dimensional): \[ T = \mathcal{B}(\frac{1}{2},\frac{1}{2}) \oplus \mathcal{B}(0,0) \]

\[ \beta^\mu = \frac{1}{2} \sigma^\mu_{\lambda\kappa} \begin{pmatrix}
0 & \sqrt{2} u^\lambda(\frac{1}{2}) \times u^\kappa(\frac{1}{2}) \\
\sqrt{2} u^\lambda(\frac{1}{2}) \times u^\kappa(\frac{1}{2}) & 0
\end{pmatrix} \]

(1-dimensional): \[ T = \mathcal{B}(0,0) \quad ; \quad \beta^\mu = 0 \]

By taking the direct product of these three representations, a faithful matrix representation of the \( \beta \) algebra is obtained. The representation of \( \mathcal{L}^+ \) which is created by this process is exactly \( \mathcal{T}^2 \), so the \( \beta^\mu \)'s have the form (6.2.3). In this form, the coefficients are

\[ c_1 = d_1 = c_5 = d_5 = \frac{1}{2} \]
\[ c_4 = d_4 = \sqrt{2} \]
\[ c_j = d_j = 0 \quad \text{for} \quad j \neq 1,4,5 \]

In order to simplify the calculation, let \( \beta'^\mu = S \beta^\mu S^{-1} \) where

\[ S \equiv \begin{pmatrix}
1_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2^{\frac{1}{2}} & 0 & 0 & 2^{-\frac{1}{2}} & 0 \\
0 & 0 & 2^{\frac{1}{2}} 1_4 & -2^{\frac{1}{2}} 1_4 & 0 & 0 \\
0 & 0 & 2^{-\frac{1}{2}} 1_4 & 2^{\frac{1}{2}} 1_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_3 & 0 \\
0 & -2^{\frac{1}{2}} & 0 & 0 & 0 & 2^{\frac{1}{2}}
\end{pmatrix} \]
This transformation implies that the coefficients of $\beta^\mu$ are

$$c_i = d_i = \lambda^{-\frac{1}{2}}$$  \hspace{1cm} \text{for} \hspace{1cm} i = 1, 2, 4, 5, 6, 7$$

$$c_j = d_j = -\lambda^{-\frac{1}{2}}$$  \hspace{1cm} \text{for} \hspace{1cm} i = 3, 8$$

Substituting these coefficients into (6.2.4) yields

$$a_i = b_i = 0 \hspace{1cm} \text{for} \hspace{1cm} i \neq 1$$

$$a_1 = b_1 = \frac{1}{2}$$

Therefore,

(6.2.7) \hspace{1cm} $\beta^\mu = \frac{1}{2} (\gamma_1^\mu + \gamma_2^\mu)$

Both the transformations $\beta^\mu \rightarrow \beta'^\mu$ and $\gamma_i^\mu \rightarrow \gamma_i^\mu$ are similarity transformations, so the mapping of algebras defined by $\beta^\mu \rightarrow \frac{1}{2}(\gamma_1^\mu + \gamma_2^\mu)$ is an isomorphism.

Of course, the isomorphism in this case is not as significant as it could be because the algebra of $\beta$ matrices is already well defined and its properties determined. It is in the case of having a known Bhaba solution with no information about its algebraic structure that the technique is valuable.

Before proceeding to the algebras developed in the next section, it is useful to look at one other vector of the form (6.2.5). This vector is

(6.2.8) \hspace{1cm} $\alpha^\mu = \lambda^{-\frac{1}{2}} (\gamma_1^\mu + \gamma_5^1 \gamma_2^\mu)$

Since the matrices $\gamma_5^1$ and $\gamma_2^\mu$ commute and $(\gamma_5^1)^2 = \mathbb{1}$, the four matrices $\gamma_5^1 \gamma_2^\mu$ generate a Dirac algebra. Further, the
\( \gamma_{1}^{\mu} \) matrices anti-commute with the \( \gamma_{5}^{1} \gamma_{2}^{\mu} \) matrices, so (6.2.8) is essentially the same form as (6.2.7) with the \( \tilde{\gamma}_{1}^{\mu} \) matrices anti-commuting. Note that

\[
\{ \gamma_{\mu}, \gamma_{\nu} \} = \frac{1}{2} \left[ \{ \gamma_{1}^{\mu}, \gamma_{1}^{\nu} \} + \{ \gamma_{2}^{\mu}, \gamma_{2}^{\nu} \} + \{ \gamma_{3}^{\mu}, \gamma_{3}^{\nu} \} + \{ \gamma_{5}^{\mu}, \gamma_{5}^{\nu} \} \right]
\]

Therefore, the \( \gamma_{\mu} \) matrices generate the Dirac algebra.

This example illustrates an important point. If a vector \( \alpha^{\mu} \) is in \( \mathcal{A}^{n} \), then it transforms by the representation \( \mathcal{J}^{n} \). However, it is possible that \( \mathcal{J}^{n} \) can be put in the form \( \mathcal{J}^{n} = T_{1} \oplus T_{2} \) or \( \mathcal{J}^{n} = T_{1}' \times T_{2}' \) such that \( \alpha^{\mu} \) has the form \( \alpha_{1}^{\mu} \otimes 1 \) or \( \alpha_{1}^{\mu} \times 1 \). In either case, there is a component or factor of \( \mathcal{J}^{n} \) which has no effect on \( \alpha^{\mu} \).

A physical property of an equation determined by the \( \alpha^{\mu} \) such as spin is determined only by the component of \( \mathcal{J}^{n} \) under which \( \alpha^{\mu} \) transforms. The point here is that this is not necessarily \( \mathcal{J}^{n} \). In the example of (6.2.8), the fact that the \( \alpha^{\mu} \) matrices generate a Dirac algebra implies that they transform by the representation \( \mathcal{J}^{1} \).

6.3 Single Mass Equations in \( \mathcal{A}^{3} \)

The purpose of this section is to illustrate the use of the multiple Dirac algebra formalism in the search for relativistic wave equations with certain properties. The example chosen here is the one of finding equations of the form (4.1.1) where the \( \alpha^{\mu} \) are in \( \mathcal{A}^{3} \) and the equation describes a particle with a single mass.
The single mass restriction on (4.1.1) can be stated as a relation on the $\alpha^\mu$ matrices of the form

$$ (6.3.1) \quad \sum (\alpha^\mu \alpha^\nu - g^\mu\nu) \alpha^\rho \alpha^\sigma \ldots \alpha^\tau = 0 $$

where the sum is over all permutations of the indices $\mu_1, \ldots, \mu_n$. This restriction comes from a relativistic generalization of the minimal equation for $\alpha^0$, which is

$$ (\alpha^0)^n - (\alpha^0)^{n-2} = 0 $$

This minimal equation is derived from the requirement that $\alpha^0$ have only the eigenvalues $\pm 1$ and $0$, a condition implied by the requirement of single mass.* Note that a solution to (6.3.1) for some $n = n_0$ also satisfies (6.3.1) for $n > n_0$.

The Dirac matrices are elements of $\alpha^1$ and solve (6.3.1) for $n = 2$. The Bhaba matrices $\beta^\mu$ can be considered as elements of $\alpha^2$ and solve (6.3.1) for $n = 3$. Therefore, it is useful to look at elements of $\alpha^3$ which satisfy (6.3.1) for $n = 4$. Further, the task of determining all vectors in $\alpha^3$ for which this is true is beyond the scope of this work. In order to limit the vectors considered, it can be noted that the most obvious elements of $\alpha^2$ satisfying (6.3.1) are

$$ \beta^\mu = \gamma_2 \left( \gamma_1^\mu + \gamma_2^\mu \right) $$

$$ \alpha^\mu = \sqrt{2} \left( \gamma_1^\mu + \gamma_5^\mu \gamma_2^\mu \right) $$

* The theory of single mass equations is completely developed in Bhaba, Harish-Chandra, and Glass.
Therefore, it is logical to investigate vectors of the form

\[ \lambda^\mu = a_1 s_1 \gamma_1^\mu + a_2 s_2 \gamma_2^\mu + a_3 s_3 \gamma_3^\mu \]

where \( a_1 \in \mathbb{C} \) and the \( s_i \) are scalars of the forms

\[ s_i = \Gamma_i^1 \Gamma_i^2 \Gamma_i^3 \]

have \( \Gamma_i^j = 1 \) or \( \gamma_5^j \). Note that the quantities \( i \gamma_5^i \gamma_i^\mu \) generate a Dirac algebra, so including \( \gamma_5^i \) in \( s_i \) does not affect the algebra generated by the \( \alpha^\mu \).

Thus, let \( \Gamma_i^i = 1 \) for all \( i \).

Before seeking vectors of the form (6.3.2) which satisfy (6.3.1), it is useful to note which of these vectors are significantly different. Let \( n_i \) be the number of \( \gamma_5^i \)'s in \( s_i \) so that \( s_i = 0,1,2 \). It is always possible to permute the indices 1,2,3 in (6.3.2) since the \( \gamma_i^\mu \) are all Dirac matrices. Therefore, the only significant sets of values for the \( n_i \) are

\[ (n_1, n_2, n_3) = \begin{cases} (0,0,0) & ; (0,0,1) & ; (0,0,2) ; \\ (0,1,1) & ; (0,1,2) & ; (0,2,2) ; \\ (1,1,1) & ; (1,1,2) & ; (1,2,2) ; \\ (2,2,2) \end{cases} \]

However, the element \( \gamma_5^1 \gamma_5^2 \gamma_5^3 \) anti-commutes with the \( \alpha^\mu \) of (6.3.2), so the algebra generated by \( i \gamma_5^1 \gamma_5^2 \gamma_5^3 \alpha^\mu \) is isomorphic to the one generated by the \( \alpha^\mu \). Since \( \gamma_5^i \) does not affect \( s_i \), multiplying \( \alpha^\mu \) by \( i \gamma_5^1 \gamma_5^2 \gamma_5^3 \) changes

\( (n_1, n_2, n_3) \) to \( (2-n_1, 2-n_2, 2-n_3) \). Thus, it is only necessary to consider the values
(6.3.4) \((n_1, n_2, n_3) = (0, 0, 0); (0, 0, 1); (0, 0, 2); (0, 1, 1); (1, 1, 1); (0, 1, 2)\)

from the set of values in (6.3.3).

The next step is to determine the specific values of \(s_1, s_2, \text{and} \ s_3\) which go with the values of \((n_1, n_2, n_3)\) in (6.3.4). The corresponding vectors are

\[
\begin{align*}
(0, 0, 0) : & \quad \gamma^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_3^\mu \\
(0, 0, 1) : & \quad \gamma^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \\
(0, 0, 2) : & \quad \gamma^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_5^\mu \\
(0, 1, 1) : & \quad \gamma^\mu = a_1 \gamma_1^\mu + a_2 \gamma_5^\mu \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \\
(1, 1, 1) : & \quad \gamma^\mu = a_1 \gamma_5^\mu \gamma_1^\mu + a_2 \gamma_5^\mu \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \\
(0, 1, 2) : & \quad \gamma^\mu = a_1 \gamma_1^\mu + a_2 \gamma_5^\mu \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \\
(1, 1, 2) : & \quad \gamma^\mu = a_1 \gamma_5^\mu \gamma_1^\mu + a_2 \gamma_5^\mu \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \\
\end{align*}
\]

Note that, if \(i \neq j\), the elements \(\gamma_5^i \gamma_j^\mu\) generate a Dirac algebra. The only consideration that prevents replacing \(\gamma_5^i \gamma_j^\mu\) with \(\gamma_j^\mu\) is that the \(\gamma_5^i\) affects the commutation relations of \(\gamma_j^\mu\) with \(\gamma_i^\mu\). However, if the simultaneous substitutions \(\gamma_5^i \gamma_j^\mu \rightarrow \gamma_j^\mu\) and \(\gamma_i^\mu \rightarrow \gamma_5^j \gamma_i^\mu\) are made, these relations are unaffected and the algebra is not changed. Therefore, the only significantly different vectors in (6.3.5) are
(6.3.6) \((0,0,0)\) \[ \lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_3^\mu \]

(6.3.7) \((0,0,1)\) \[ \lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^1 \gamma_3^\mu \]

(6.3.8) \((0,0,2)\) \[ \lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^1 \gamma_5^2 \gamma_3^\mu \]

(6.3.9) \((0,1,2)\) \[ \lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_5^1 \gamma_2^\mu + a_3 \gamma_5^1 \gamma_5^2 \gamma_3^\mu \]

Now, it is necessary to determine whether there are values of \(a_1, a_2,\) and \(a_3\) such that the vectors of (6.3.6) to (6.3.9) satisfy (6.3.1) for \(n = 4\).

**Case 1:** \[ \lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_5^1 \gamma_2^\mu + a_3 \gamma_5^1 \gamma_5^2 \gamma_3^\mu \]

\[ \Sigma \lambda^\mu, \lambda^\nu = 2 (a_1^2 + a_2^2 + a_3^2) \gamma^\mu \gamma^\nu \equiv 2x \gamma^\mu \gamma^\nu \]

\[ \Sigma (\lambda^\mu \lambda^\nu - \gamma^\mu \gamma^\nu) \lambda^\rho \lambda^\lambda = 8x(x-1) \left[ \gamma^\mu \gamma^\rho \gamma^\lambda + \gamma^\mu \gamma^\lambda \gamma^\rho + \gamma^\rho \gamma^\lambda \gamma^\mu \right] \]

\[ \equiv 8x(x-1) G^{\mu \nu \rho \lambda} \]

The tensor \(G^{\mu \nu \rho \lambda}\) is not identically zero, so this expression is zero only when \(x = 0\) or \(x = 1\).
Case 2: \( \alpha^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \)

\[ \xi \alpha^\mu, \xi \gamma^3 = 2x q^\mu + 2a_1 a_2 [ \gamma_1^\mu \gamma_2^\nu + \gamma_2^\mu \gamma_1^\nu ] \]

\[ = 2x q^\mu + 2s [ \gamma_3^\mu \nu ] \]

\[ \Sigma (\alpha^\mu \gamma^\nu - q^\mu) \alpha^\gamma \lambda = 8 (x^2 - x + 4s^2) G^{\mu \nu \rho \lambda} \]

\[ + 4s (2x - 1) [ q^\mu \nu \gamma_3^\rho \lambda + q^\rho \lambda \gamma_3^\mu \nu + q^\mu \rho \gamma_3^\nu \lambda + q^\nu \lambda \gamma_3^\mu \rho + q^\nu \rho \gamma_3^\mu \lambda ] \]

In order for this expression to equal zero, each of the two coefficients must be zero. Therefore, either \( x = \frac{1}{2} \) and \( s^2 = 1/16 \) or \( s = 0 \) and \( x = 0 \) or 1.

Case 3: \( \alpha^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_5^2 \gamma_3^\mu \)

\[ \xi \alpha^\mu, \xi \gamma^3 = 2x q^\mu + 2s [ \gamma_3^\mu \nu + 2a_1 a_3 \gamma_5^2 \gamma_1^\mu + \gamma_1^\mu \gamma_3^\nu ] \]

\[ = 2x q^\mu + 2s [ \gamma_3^\mu \nu + 2t \gamma_5^2 \gamma_2^\nu ] \]

\[ \Sigma (\alpha^\mu \gamma^\nu - q^\mu) \alpha^\gamma \lambda = 8 (x^2 - x + 4s^2 + 4t^2) G^{\mu \nu \rho \lambda} \]

\[ + 4s (2x - 1) [ q^\mu \nu \gamma_3^\rho \lambda + q^\rho \lambda \gamma_3^\mu \nu + \ldots ] \]

\[ + 4t (2x - 1) \gamma_5^2 [ q^\nu \rho \gamma_2^\mu \lambda + \gamma_2^\mu \rho \gamma_3^\nu \lambda + \ldots ] \]

This expression is identically zero only when each coefficient is zero. Thus, either \( k = \frac{1}{2} \) and \( s^2 + t^2 = 1/16 \) or \( s = t = 0 \) and \( x = 0 \) or 1.
Case 4: $\lambda^\mu = a_1 \gamma_1^\mu + a_2 \gamma_2^\mu + a_3 \gamma_3^\mu$

$$\Sigma \lambda^\mu = 2 x \bar{q}^\mu + 2 s \bar{r}^\mu + 2 t \bar{r}^\mu + 2 r \bar{r}^\mu$$

where $r = \alpha_1 \alpha_3$ and $\bar{r}^\mu = [\gamma_2^\mu \gamma_3^\mu + \gamma_3^\mu \gamma_2^\mu]$

$$\Sigma (\lambda^\mu \gamma^\nu - \bar{q}^\mu \gamma^\nu) \lambda^\rho \lambda^\lambda = \bar{g} (x^2 - x + 4 s^2 + 4 t^2 + 4 r^2) G^{\mu \nu \rho \lambda}$$

$$+ 4 \left[ s (2 x - 1) + 4 t r \right] \left[ \bar{q}^\mu \gamma^\nu \gamma^\rho \gamma^\lambda + \ldots \right]$$

$$+ 4 \left[ t (2 x - 1) + 4 s r \right] \left[ \bar{q}^\mu \gamma^\nu \gamma^\rho \gamma^\lambda + \ldots \right]$$

$$+ 4 \left[ r (2 x - 1) + 4 s t \right] \left[ \bar{q}^\mu \gamma^\nu \gamma^\rho \gamma^\lambda + \ldots \right]$$

In order for this expression to equal zero, all of the following quantities must be zero.

$$x^2 - x + 4 s^2 + 4 r^2 = 0 \quad s (2 x - 1) + 4 t r = 0 \quad t (2 x - 1) + 4 s r = 0 \quad r (2 x - 1) + 4 s t = 0$$

This implies that

If $x = \frac{1}{2}$; then $t r = s r = s t = 0$ so two of $r, s, t = 0$.

One of $r, s, t$ must be non-zero since $x$ is. Without loss of generality, say $r = t = 0$, $s \neq 0$. Thus, $s^2 = 1/16$.

If $x \neq \frac{1}{2}$; then $s^2 = t^2 = r^2$. Further,

$$s (2 x - 1)^2 = -4 rt (2 x - 1) = 16 s r^2 = 16 s^3.$$ 

Therefore, either $s = t = r = 0$ and $x = 0$ or $1$ or $s^2 = t^2 = r^2 = (2 x - 1)^2 / 16$ and $x = \frac{1}{4}$ or $3/4$.

Thus $a_1^2 a_2^2 = a_2^2 a_3^2 = a_3^2 a_1^2 = 1/64$ and $a_1^2 = a_2^2 = a_3^2 = \pm 1/8$ and $x = \pm 3/8$, which is a contradiction. Therefore, $s = t = r = 0$ and $x = 0$ or $1$. 
With these possibilities, for each case it is possible to restrict the ones considered by identifying the algebras generated.

**Case 1:** If \( x = 1 \), then \( \{ \alpha^\mu, \alpha^\nu \} = 2g^{\mu\nu} \) and the algebra generated by the \( \alpha^\mu \) is the Dirac algebra \( \mathcal{D} \). If \( x = 0 \), then \( \{ \alpha^\mu, \alpha^\nu \} = 0 \). Call the algebra \( \mathcal{D} \) generated by these \( \alpha^\mu \).

**Case 2:** If \( s = 0 \), then either \( a_1 \) or \( a_2 \) is zero. Let \( a_2 = 0 \). Therefore, both the cases \( x = 0 \) and \( x = 1 \) are merely instances of Case 1 with \( a_2 = 0 \).

If \( x = \frac{1}{2} \) and \( s^2 = \frac{1}{16} \), call the algebra \( C_1 \) generated by these \( \alpha^\mu \).

**Case 3:** If \( s = t = 0 \), then, either \( a_1 = 0 \), in which case the algebras for \( x = 0 \) and \( 1 \) are again \( \mathcal{D} \) and \( \mathcal{D} \) from Case 1, or \( a_2 = a_3 = 0 \), in which case the algebras generated are \( \mathcal{D} \) for \( x = 1 \) (i.e., \( \alpha^\mu = \pm \gamma_1^\mu \)) or \( \{0\} \) for \( x = 0 \) (i.e., \( \alpha^\mu = 0 \)).

If \( x = \frac{1}{2} \) and \( t^2 + s^2 = \frac{1}{16} \), call the algebra generated in this case \( C_2 \).

**Case 4:** If \( x \neq \frac{1}{2} \), then \( s = t = r = 0 \) and so two of \( a_1, a_2, a_3 \) are zero. Thus, the algebras for \( x = 0 \) or \( 1 \) are \( \{0\} \) and \( \mathcal{D} \) respectively.

If \( x = \frac{1}{2} \), \( t = r = 0 \), and \( s^2 = \frac{1}{16} \), then \( a_3 = 0 \), \( a_1 = \pm \frac{1}{2} \), and \( a_2 = \pm \frac{1}{2} \). Therefore, \( \alpha^\mu = \pm \frac{1}{2} (\gamma_1^\mu \pm \gamma_2^\mu) \). Thus, the \( \alpha^\mu \) in this case generate the Duffin-Kemmer algebra \( \mathcal{B} \).
As a result of these calculations, the possible new algebra from vectors of the form (6.3.2) have been restricted to three possible cases: \( C_1, C_2 \) and \( \mathcal{O} \). The next step is to consider each of these separately.

**\( C_1 \):** The equations \( x = \frac{1}{2} \) and \( s^2 = 1/16 \) imply the general relations

\[
\alpha_1 = i \frac{c}{z}, \quad \alpha_2 = i \frac{\delta}{2c}, \quad \alpha_3 = \epsilon \frac{(c + \sqrt{c})}{2}
\]

where \( c \) is a general non-zero complex constant and \( \delta, \epsilon = \pm 1 \). Therefore,

\[
(6.3.10) \quad \alpha^\mu = \frac{i}{2} \left( c \gamma_1^\mu + (\frac{\delta}{c}) \gamma_2^\mu \right) + \frac{\epsilon}{2} \left( c + \sqrt{c} \right) \gamma_3^1 \gamma_5^2 \gamma_3^\mu
\]

Define \( \eta^\mu = g^{\mu\mu} - 2 (\alpha^\mu)^2 = \gamma_1^\mu \gamma_2^\mu \).

If \( \mu \neq \nu \), then

\[
\alpha^\mu + \eta^\nu \alpha^\nu \eta^\nu = \epsilon \left( c + \sqrt{c} \right) \gamma_3^1 \gamma_3^2 \gamma_3^\nu
\]

Thus, if \( c \neq \pm i \), then \( \gamma_3^1 \gamma_3^2 \gamma_3^\mu \in C_1 \)

If \( c = \pm i \), then

\[
\alpha^\mu = \mp \frac{i}{2} \left( \gamma_1^\mu - \delta \gamma_2^\mu \right)
\]

which implies that \( C_1 \cong \mathcal{B} \), the Duffin-Kemmer algebra, so this case can be ignored. Therefore, \( C_1 \) can be generated by the eight elements

\[
\xi \gamma_3^1 \gamma_5^2 \gamma_3^\mu, \quad \frac{i}{2} \left( c \gamma_1^\mu + (\frac{\delta}{c}) \gamma_2^\mu \right) \equiv \rho^\mu \eta
\]

Note that

\[
(\rho^\mu)^2 = -\frac{i}{4} \left[ (c^2 + \sqrt{c}) \gamma_{\mu\mu} + 2 \delta \gamma_1^\mu \gamma_2^\mu \right]
\]

\[
(\rho^\mu)^3 = -\frac{i}{8} g_{\mu\nu} \left[ (c^2 + \sqrt{c}) c \gamma_1^\mu + (3c^2 + \sqrt{c}) \frac{\delta}{c} \gamma_2^\mu \right]
\]

\[
(\rho^\mu)^4 = -\frac{i}{64} g_{\mu\nu} \left[ (c^2 + \sqrt{c}) c \gamma_1^\mu + (3c^2 + \sqrt{c}) \frac{\delta}{c} \gamma_2^\mu \right]
\]
If \((c^2 + 3/c^2) \neq (3c^2 + 1/c^2)\), then \(\rho^\mu\) and \((\rho^\mu)^3\) are linearly independent combinations of \(\gamma_1^\mu\) and \(\gamma_2^\mu\). Therefore, both \(\gamma_1^\mu\) and \(\gamma_2^\mu\) are elements of \(C_1\).

Thus \(\{\gamma_1^\mu, \gamma_2^\mu, \gamma_3^\mu\} \in C_1\) and \(C_1 \cong \mathcal{A}^3\).

If \((c^2 + 3/c^2) = (3c^2 + 1/c^2)\), then \(c = \pm 1, \pm i\). The case of \(c = \pm i\) has been dealt with. For \(c = \pm 1\),

\[
\rho^\mu = i \left[ \pm \frac{1}{\sqrt{2}} (\gamma_1^\mu + \delta \gamma_2^\mu) \right]
\]

This implies that the \(\rho^\mu\) are generators of a Duffin-Kemmer algebra. Thus, let \(\rho^\mu = \beta^\mu\) and write

\[
(6.3.11) \quad \alpha^\mu = \gamma_3^\mu + i \gamma_5^3 \beta^\mu
\]

(For simplicity, \(\alpha^\mu\) has been multiplied by \(\gamma_5^1 \gamma_5^2 \gamma_5^3\) here.) Call the algebra \(C_1'\) generated by these \(\alpha^\mu\).

\(C_2\): If \(x = \frac{1}{2}\) and \(t^2 + s^2 = 1/16\), then \(a_1 = \pm \frac{1}{2}\) and \(a_2^2 + a_3^2 = \frac{1}{4}\). However, note that, if

\[
\delta^\mu = 2 \left[ a_2 \gamma_2^\mu + a_3 \gamma_3^2 \gamma_3^\mu \right]
\]

then

\[
\varepsilon \delta^\mu, \delta^\nu \gamma_3 = 2 \gamma_9^\mu
\]

so that the \(\delta^\mu\) generate a Dirac algebra. Therefore

\[
\alpha^\mu = \pm \frac{1}{2} \gamma_1^\mu + \frac{1}{2} \delta^\mu
\]

and \(C_2\) is just the Duffin-Kemmer algebra \(\mathcal{C}\).

\(\mathcal{S}\): If \(a_1^2 + a_2^2 + a_3^2 = 0\), suppose that \(a_1 \neq 0\). From the analysis of \(C_2\), the elements
\[ \Sigma^\mu \equiv (a_2^+ + a_3^+) - \frac{i}{\sqrt{2}} \left[ a_2 \gamma_1^\mu + a_3 \gamma_5^\mu \gamma_3^\mu \right] \]

generate a Dirac algebra. Therefore,

\[ \lambda^\mu = a_1 \left( \gamma_1^\mu + i \gamma_5^\mu \gamma_3^\mu \right) \]

Since \((\alpha^\mu)^2 = 0\), the only eigenvalue of \(a^0\) is zero and the \(\alpha^\mu\) do not describe a particle of non-zero mass.

Thus, there are only two significant simple-mass solutions to be gained from the vectors of the form (6.3.2).

The algebra \(C_1\) generated by \(\alpha^\mu\) of the form (6.3.10) with \(c \neq \pm i, \pm 1, 0\) is the complete algebra \(\alpha^3\) and is therefore simple. As a result, \(C_1\) has only one irreducible representation, of dimension \(4^3 = 64\). The algebra \(C_1'\), generated by (6.3.11), has been investigated before. It was incorrectly identified by Takahashi\(^{11}\) as the Fierz-Pauli-Gupta solution of spin \(3/2\). This is not the case since the Gupta\(^{16}\) solution is an irreducible set of \(16 \times 16\) matrices and \(C_1'\) has no such representation. \(C_1'\) was first studied by Harish-Chandra,\(^{10}\) where it was created as a generalization of the Dirac equation. It is a semi-simple algebra with three independent elements in its center. These elements are exactly the independent elements of the center of \(\mathcal{B}\). This leads to the conclusion that the three irreducible representations of \(C_1'\) have dimensions 40, 20, and 4 and correspond to the three representations of the Duffin-Kemmer algebra crossed with the single representation.
of the Dirac algebra. $C_1'$ (also called the Chandra ring) has a combination of spins $3/2$ and $\frac{1}{2}$ and was used by Harish-Chandra to illustrate the existence of single-mass equations other than the Dirac and Duffin-Kemmer equations.
VII. SUMMARY AND DISCUSSION

The purpose of this thesis has been to create an algebraic formalism for constructing all possible relativistic wave equations of the form

\begin{equation}
(\mathcal{L} - \chi) \psi(x) = 0
\end{equation}

Using the Dirac matrices and the algebra they generate, higher degree algebras were created by taking products of this algebra with itself. These higher degree algebras, called multiple Dirac algebras, were shown to be simple and, therefore, complete matrix algebras. The nature of relativistic wave equations of the form (7.1.1) implied that any set of matrices \( \{a^\mu\} \) for such an equation must essentially be contained in one of these higher degree algebras.

By definition, the fact that an equation such as (7.1.1) is relativistic means that there is a representation of the proper Lorentz group by which the \( a^\mu \) transform. The transformation properties of the \( \gamma_i^\mu \) matrices in the multiple Dirac algebras define the transformation properties of all elements in the algebras. The method of construction used to find the \( a^\mu \) matrices in the multiple Dirac algebras implied that the transformation associated with them is exactly the transformation defined by the \( \gamma_i^\mu \). Therefore, it was shown that any set of \( a^\mu \) matrices
defining a relativistic wave equation of the form (7.1.1) is essentially represented by a vector in some multiple Dirac algebra.

The motivation behind this construction has been shown to be the lack of algebraic structure or completeness in earlier solutions. The only other complete method of finding equations such as (7.1.1) has been the Bhaba solution. This solution is characterized by its technique of building $\gamma^\mu$ matrices from submatrix solutions. As a result, the algebraic properties of the $\gamma^\mu$ must be studied for individual solutions and the properties of the solutions of large degree are fairly obscure. Other methods of finding such equations have restricted the type of solution studied so that the techniques do not apply to all solutions. One common method of this type is due to Bhaba$^3$ and only considers equations with the relation

$$(7.1.2) \quad \gamma^{\nu} = c \ [\gamma^\alpha, \gamma^\nu]$$

between the $\gamma^\mu$ matrices and the infinitesimal generators of Lorentz transformations. Another method is the de Broglie method mentioned in the introduction, which suggested the construction method in this thesis. These methods provide a handle on the algebraic properties but are not complete.

The examples included in this thesis have illustrated the two basic uses of the technique developed. The first is that, by transforming a Bhaba solution into the multiple
Dirac algebra formalism, the algebraic properties of a known solution may be determined. However, the translation from one form to another is complex even in the second order case, so that in higher order cases where the algebraic structure is unknown, there remain practical problems to be solved in order to apply the technique. It is the latter example that illustrated the main point of the technique. In this example, it is shown that solutions may be found to give equations of certain characteristics by translating these characteristics into algebraic restrictions and looking for vectors in the multiple Dirac algebras which satisfy these restrictions. It is at this point that the completeness of the technique is important since, if a solution exists, then it can be found in this form.

The approach to relativistic wave equations in this thesis has been very general. In fact, aside from certain mathematical restrictions such as finiteness and the simplicity of form, the only real restriction has been that of invariance under proper Lorentz transformations. No mention has been made of such restrictions as space-inversion invariance. This has been deliberate and has been done for several reasons. The first is that it is customary to approach such topics as added restrictions on the more general solution. For example, invariance under space inversion manifests itself in the Bhaba solution as
a restriction on which representations of $\mathbb{L}_+^*$ can be used and on the coefficients of the submatrix solutions. Thus, the Bhaba analysis remains basically unchanged and is still valid. The other consideration is that there may be cases, such as mass zero, in which it is useful to investigate solutions which are not invariant under space inversion. Therefore, the proof in this thesis showed that the multiple Dirac algebra technique included all solutions, not just the inversion invariant ones.

One useful characteristic of this technique is that inversion characteristics of many of the solutions can be easily investigated. In the development here, the fact that only proper Lorentz transformations were used led to the identification of both $\mathbb{I}$ and $\gamma_5$ as scalars in the Dirac algebra. With space inversion, $\gamma_5$ becomes a pseudo-scalar, and it can be seen that it would retain this character in the generalization of the multiple Dirac algebras. Thus, the position of the $\gamma_5^i$ elements in any vector of $\mathcal{A}^n$ would indicate the scalar or pseudo-scalar nature of the vector. The final example of Chapter VI involved certain manipulations with these $\gamma_5^i$ elements in order to show which combinations generated distinct algebras. Once a particular algebra was chosen, it is feasible that the reverse manipulations could be performed to give the generating vector a particular vector or pseudo-vector nature. This points out another useful
feature of the formalism. The basic generating elements of the formalism, the $\gamma^\mu$ matrices of the Dirac algebra, have well-known space inversion characteristics. Thus, these characteristics of the elements of the $a^n$ follow from those of the Dirac algebra.

The next step in the development of the topic of this thesis would be to investigate the nature of the restrictions on the possible vectors in the $a^n$ algebras. The example $a^2$ in Chapter VI, which found all vectors in $a^2$, showed that the number of vectors in $a^n$ increases extremely rapidly as $n$ increases. In fact it is not feasible to look at all possible combinations of vectors in each $a^n$. It is true that many vectors generate the same algebra, but, as seen in the last example, even a slight change in coefficients can change the generated algebra. Therefore, it is logical to restrict the number of vectors considered.

As noted, the requirement of invariance under space inversion limits the allowed solutions. Other considerations which would impose restrictions are the concepts of mass and spin. The example of single mass was used to limit the vectors considered in the third example of Chapter VI. This particular restriction is motivated by the search for fundamental particle equation. It seems somehow wrong to have a fundamental particle equation which allows a number of mass eigenvalues. As a result,
certain classical theories, such as the one presented by Bhaba with Eq. (7.1.2), have been considered unacceptable with this characteristic. A direct extension of the topics in this thesis would be to consider which forms of vectors in the $\alpha^n$ are eliminated in general.

The restrictions on spin are less direct. It is desirable to find equations which predict a particle with a unique spin. However, the exact conditions which this imposes can be complicated. For example, the Duffin-Kemmer equation itself does not imply a unique spin. It is only the irreducible representations of this algebra that provide the spin one and spin zero solutions. Associated with this topic is the question of the relationship between the generators of Lorentz transformations, the $I^{\mu\nu}$, and the $\alpha^\mu$ matrices themselves. It is not necessarily true that the $I^{\mu\nu}$ are elements of the algebra generated by the $\alpha^\mu$, so restrictions implied by a unique spin do not have to apply to this algebra alone. The relationship between the vectors of $\alpha^n$ and the spin they imply is probably the next topic that should be approached with this formalism.

Another topic that can be dealt with this technique is that of Hamiltonian form. The Dirac equation can be put into Hamiltonian form easily since the $\gamma^\mu$ matrices are invertible. However, this is not true of the Duffin-Kemmer equation. The $\beta^\mu$ have zero eigenvalues and are not
invertible. As a result, the transition to Hamiltonian form involves projecting the equation onto the subspace of non-zero eigenvalues of $\gamma^0$ and then inverting. This process gives the Sakata-Taketani equation\textsuperscript{17} for spin one and spin zero particles. The process itself is an algebraic one, so it is well-suited to the multiple Dirac algebra formalism.

Finally, it can be noted that much of the search for relativistic wave equations relies on the observed patterns and similarities of the equations that have proven successful in dealing with the considerations stated above. One major advantage of a new formalism is that it provides a new way of looking at a problem. As a result, it is possible that new patterns will appear, or that old ones will be seen in a new way. This, then, is the primary purpose behind the formalism in this thesis. It takes a concept that has been touched on before, the concept for constructing new equations by algebraically combining older ones, and generalizes it so that it includes all possible solutions of a type in its framework.
APPENDIX: NOTATION

The problem of notation is an acute one in a work such as this. I have attempted to use the usual notation wherever possible so that there will be a minimum of confusion. The index conventions are the standard ones, i.e.,

\[
\begin{cases}
\mu, \nu, \rho, \lambda \ldots & \in \{0, 1, 2, 3\} \\
\alpha, \beta, \gamma, \delta \ldots & \in \{1, 2\}
\end{cases}
\]

These types of indices occur both as upper (superscript) and lower (subscript) indices and the Einstein summation convention is used when an upper index is summed with a lower one. That is,

\[
\begin{align*}
X^\mu Y_\mu &= X^0 Y_0 + X^1 Y_1 + X^2 Y_2 + X^3 Y_3 \\
X^\lambda Y_\lambda &= X^0 Y_0 + X^1 Y_1 + X^2 Y_2 + X^3 Y_3 \\
C^\alpha D_\alpha &= C^1 D_1 + C^2 D_2
\end{align*}
\]

Further, upper indices and lower indices are related by the use of a metric, such as \(g\) or \(\varepsilon\). These are matrices whose entries are denoted by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

and which are used to lower indices by the formalas
Lower indices are raised similarly, with the matrices

\[ g^{\mu \nu} = g_{\mu \nu} \quad \text{and} \quad \epsilon^{ab} = \epsilon_{ba}, \quad \text{i.e.} \]

\[ x^\alpha = g^{\mu \nu} x_\nu \]
\[ x^i = g^{ij} x_j = -x^i \]
\[ C^a = C_b \epsilon^{ba} \]

Note that the standard form is used for matrices here. That is, the matrix \( g \) has entries \( g_{\mu \nu} \), where \( \mu \) is a row index and \( \nu \) is a column index. Except for these three types of indices, the upper or lower position of an index will have no significance. Also, the summation convention does not apply for any other set of indices. Occasionally, it is necessary to use the indices \( i, j, k \) for other purposes, but the differences should be clear in context, and the summation convention will not be used in such cases.

In Section 2, the mathematical notation for matrix algebras is used, so that \( \text{Hom}(\mathbb{R}^n) \) is the set of all \( n \times n \) real matrices and \( \text{Hom}(\mathbb{C}^n) \) is the set of all \( n \times n \) complex algebras. In general, \( \text{Hom}(\mathcal{A}) \) is the set of all transformations from \( \mathcal{A} \) to itself which preserve the structure of \( \mathcal{A} \). Also in Section 2 there are certain transformations which are indexed such as \( R^J_i(\theta) \) and
The bar over the indices denotes that they are not to be treated as the indices in (A.1) even though they have the same range. This distinction is made in order to differentiate between these transformations and the $R^i_j$ and $L^\mu_\nu$, which are matrix elements.

The direct product, $\times$, of a pair of matrices $A$ and $B$, where $A$ is an $nxn$ matrix and $B$ is an $mxm$ matrix, is denoted by the matrix $AXB$, which is an $nmxn$ matrix. The convention for the form of this matrix is that

$$A \times B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \ldots & a_{nm}B \end{pmatrix}$$

where $a_{ij}$ is the $(i,j)$ entry of $A$.

Finally, the symbol $\mathbb{I}_n$ is used to denote the $nxn$ unit matrix, with the $n$ left out in cases where the context makes the dimension clear. Also, the convention for conjugates used here is $M^+$ for Hermitian conjugate, $M^T$ for transpose, and $\bar{M}$ for complex conjugate.
BIBLIOGRAPHY


