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Rotating Magnetospheres: A Three Dimensional Approach

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ABSTRACT

Several authors have calculated the torque due to a stellar wind on a rotating body with an axisymmetric magnetic field. These discussions are restricted to the equatorial plane because the magnetic field structure is unknown elsewhere. Their treatment is here extended to the non-equatorial case. The axisymmetric magnetic field is described by a scalar function f. The various conservation equations allow the plasma flow to be described in terms of f. Although f has not been determined in the general case, it can be obtained in two special cases when the particle inertia is neglected. In the general case the Alfvén critical points are obtained in terms of f, and with an assumption of a "minimum torque" solution, conditions on the angular momentum flow are found.

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I. INTRODUCTION

With the discovery of pulsars and the proposal by Gold (1968) and Pacini (1968) that pulsars are rapidly rotating neutron stars with intense magnetic fields, interest in the properties of rapidly rotating magnetized objects has increases considerably. There has long been interest in the behavior of the plasma surrounding a star, primarily in connection with the solar corona and the solar wind. However, in the case of the sun, thermodynamical properties seem to dominate the electromagnetic aspects, at least in the Parker (1963) model and its subsequent revisions. In a pulsar, it is expected that the opposite is the case, that thermodynamical effects are completely dominated by electromagnetic effects and are thus negligible. One would like to have a model for a pulsar that is analytically tractable and is at the same time physically interesting in the sense that the pulsar emission mechanism is a direct result of the dominant electromagnetic effects. Unfortunately, no completely self-consistent model for a pulsar yet exists. Indeed, there is still considerable controversy over the emission mechanism. In any case, a preliminary step is to find the magnetic field structure and resultant plasma properties for a rotating magnetized star, regardless of the emission mechanism.

There is no generally accepted mechanism for generating a magnetic field of a star. This need not enter any discussions concerning pulsars except insofar as pulsars are thought to possess tremendous magnetic fields, on the order of 10^{12} gauss. The discussion presented by Ginzburg and Ozernoi (1965) indicate that if a frozen-in flux approximation is valid in a stellar collapse, then such strong magnetic fields are easy to achieve provided there was some small initial field present. A mechanism of this sort would probably be sufficient to create a large pulsar magnetic field.

With such a large magnetic field, the existence of a pulsar magnetosphere follows almost immediately [Goldreich and Julian, 1969; Michel, 1969b]. A simple comparison of the gravitational force on a particle at the surface to the electric force required by the high plasma conductivity shows that the electrical force is roughly eight orders of magnitude larger. This immediately precludes the possibility of a vacuum about a pulsar.

Ferraro (1937) pointed out that plasma around a rotating magnetized conducting body corotates with the body. If the plasma does not corotate, potential differences develop on the body and cause currents to flow, thus acting to slow the body and to accelerate the plasma until corotation is achieved. It is clear, however, that corotation cannot hold beyond the "light cylinder", where the corotation velocity exceeds the velocity of light. Inside the light cylinder, the magnetic field decreases with distance, while the centrifugal forces exerted by the plasma increase with distance. At some point, the magnetic field is overcome by the centrifugal force and the magnetic field lines are pulled open. For a magnetic dipole there will result two distinct regions, one containing corotating plasma on closed field lines and the other containing outflowing plasma on open field lines (see Figure 1).

Several interesting features arise from such a model that have not yet been considered in sufficient detail. Some features of interest are:

1. The exact structure of the magnetic field.
2. A neutral sheet resulting from the broken field lines, perhaps similar th that of the earth's magnetosphere. Michel and Tucker (1969) have suggested that such a sheet is relevant to the emission mechanism.
3. The torque due to the outflowing plasma, acting to slow the rotation rate.
4. A possible shock wave (Bertotti, Cavaliere, Pacini;1969) associated with the point in the equatorial plane where the magnetic field lines are broken first. Such a shock front may be a radiation source.
5. Particles of cosmic ray energies could result from the acceleration (Ostriker and Gunn; 1969).

In this paper, features (1) and (3) are considered. Both have already received considerable attention in the literature. Weber and Davis (1967) and Modisette (1967) have estimated the torque on the sun due to the solar wind, where the magnetic field is assumed to be a monopole and actual calculations are done only for the equatorial plane. Combining this with satellite observations, they found that the angular momentum loss rate is about one thousand times greater than the surface evaporation would provide. Dicke (1964) reached essentially the same conclusion by arguing that corotation is effective out to a point where shear signals can no longer be propagated back to the star (the Alfvén point). Michel (1969a) and Hendriksen and Rayburn (1970, 1971) have extended the treatment to include relativistic effects. Goldreich and Julian (1970) showed that Michel's treatment was a special case achieved by placing the magnetosonic point at infinity. Hendriksen and Rayburn's results appear to be valid only inside the light cylinder, apparently becoming time dependent outside it. Its physical acceptability is as yet unclear. In all cases only the equatorial plane is considered.

The other feature, the magnetic field structure, has been considered by several authors. The most common assumption is that the field is a monopole. Mestel (1968) assumed a dipole-like structure and then made various approximations. Ferraro and Bhatia (1967) and Bhatia (1968a,b) have considered the magnetic field structure of the sun, unfortunately neglecting the important electric field. Their results are therefore questionable. Cohen and Rosenblum (1972) and Michel (1973a) have presented models for a corotating magnetosphere, and Michel (1973b) has obtained a general solution for a monopole field structure in the case where plasma inertia may be neglected entirely.

In this paper, the magnetic field structure is examined in terms of a scalar function f. The equation describing the field structure has been solved only in two special cases, both of which have previously been considered by Michel. However, his results have not been obtained with the MHD approach used here. The angular momentum flux and the energy flux are also derived in terms of the field structure scalar, although they cannot be completely specified.

II. SIMPLIFYING ASSUMPTIONS

The treatment parallels that given by Weber and Davis (1967) and Michel (1969a). The major simplifying assumptions are the following:

1. The magnetic field is taken to be azimuthally symmetric at the stellar surface and hence everywhere external to the star. With this assumption, all derivatives with respect to the azimuthal angle vanish. As this is the most simple case possible and is still unsolved, there is as yet no point in considering the more complex case of a non-aligned field.
2. Changes in mass and rotation rate are ignored. The flow is therefore taken to be steady, and all derivatives with respect to time vanish. Goldreich (1969) has suggested that there are no steady state solutions, but stationary solutions must first be found before instabilities are to be found.
3. Gravitational and thermodynamical effects are ignored. In the case of the sun, rotation apparently has little effect. Although the acceleration is hydrodynamical with a sonic critical point at about three solar radii, at large distances the flow velocity should approach a constant along a streamline whereas the sound velocity falls monotonically. Thus at large distances the internal thermal energy becomes unimportant and the

flow can be accurately described with pressure neglected. Gravitational effects may be neglected once the flow velocity much exceeds the local escape velocity. A flat metric is also used.

5. Enough plasma is assumed to be present so that the conductivity is essentially infinite. The magnetic field is thus frozen into the plasma and they share the same streamline surfaces.

It is apparent that near the stellar surface some of these assumptions are not very good, and the application of any results obtained must be applied to this region with extreme caution.

III. BASIC EQUATIONS

Due to axial symmetry, the cylindrical coordinates (x, ϕ, y) will be used. The metric is

$$ds^2 = c^2 dt^2 - dx^2 - d\phi^2 - dy^2. \tag{1}$$

The conductivity of the plasma requires that there be no electric field in the proper reference frame. The electric field is then determined by setting the proper force equal to zero:

$$F_{ik} \frac{dx^k}{ds} = 0 , \tag{2}$$

or (see Appendix I)

$$\vec{E} = - \frac{\vec{v}}{c} \times \vec{B} . \tag{3}$$

Magnetic fields can be described in terms of the (scalar) Euler potentials α and β, where

$$\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta . \tag{4}$$

In the case of axial symmetry this conveniently reduces to

$$\vec{B} = \vec{\nabla}\alpha \times \frac{\hat{n}}{x\hat{n} \cdot \hat{\phi}} \tag{5}$$

where $\hat{\phi}$ is the unit azimuthal vector and $\hat{n}$ is a unit vector perpendicular to $\vec{B}$ in a surface where α = constant. With this description, α is a constant along a given field line. Each field line could be described by a different constant, but the axial symmetry present will require all field lines at the stellar surface at a given latitude to have the same α . Thus surfaces of α = constant are actually formed (see Figures 2, 3, 4). The significance of α can be seen further if the expressions for B_x and B_y are written:

$$B_x = - (1/x)\ \alpha_y$$
$$B_y = + (1/x)\ \alpha_x \qquad (6)$$

where the subscripts on α denote partial derivatives. It is trivial to show that $\vec{\nabla}\cdot\vec{B} = 0$. Another very useful component of the magnetic field is the meridional component B_m , defined by

$$B_m = [B_x{}^2 + B_y{}^2]^{\frac{1}{2}}$$
$$= (1/x)\,.|\vec{\nabla}\alpha| \qquad (7)$$

Axial symmetry will result in the use of B_m often.

Two important results follow from equations (2) and (5). First, the electic field can be written as

$$\vec{E} = - \frac{\Omega}{c}\,\vec{\nabla}\alpha \quad , \qquad (8)$$

so it is seen that α acts like an electrostatic potential. For this reason, the surfaces of constant α are often referred to as equipotential surfaces. Also, from $\vec{\nabla} \times \vec{E} = 0$ there results

$$B_m x \Omega = B_m v_\phi - B_\phi v_m \ . \qquad (9)$$

This is a very standard result for radial field lines (cf. Mestel, 1968; Chandrasekhar, 1956) which generalizes to axially symmetric field lines. Equations (8) and (9) are derived in Appendix I.

The other equations needed are the various conservation equations. The simplest of these is the conservation of mass flux. For a steady state radial mass flow, the conservation equation is usually written

$$\rho v r^2 = \text{constant} \tag{10}$$

where ρ is the mass density and v is the magnitude of the radial velocity at a distance r from the origin. It is convenient in a relativistic treatment to define ρ to be the density as measured in the frame where the fluid is at rest. In the case of axial symmetry, this equation becomes

$$(\rho u_m)\frac{x}{|\vec{\nabla}\alpha|} = F(\alpha) \tag{11}$$

where $F(\alpha)$ is a constant along a given equipotential surface. Equation (7) allows this to be written instead as

$$\rho u_m = F(\alpha) B_m \,. \tag{12}$$

This is the integrated form of the mass flux conservation equation (all conservation equations are derived in detail in the appendices).

The conservation equations of angular momentum and energy can be obtained from the divergencelessness of the energy-momentum tensor. These two components are easily integrated, and are

$$\rho u_2 u_m - \frac{x}{4\pi} B_m B_\phi = L(\alpha) B_m \tag{13}$$

and

$$\rho u_o u_m + \frac{\Omega}{4\pi} x B_m B_\phi = -\chi(\alpha) B_m \ . \tag{14}$$

As is the case with $F(\alpha)$, $L(\alpha)$ and $\chi(\alpha)$ are constants along an equipotential surface.

The other two components of the energy-momentum tensor are non-integrable. These two equations are:

$$\rho u^1 \frac{\partial}{\partial x}(u_1) + \rho u^3 \frac{\partial}{\partial y}(u_1) - (1/x)\rho\, u_2 u^2 - \frac{E_x}{4\pi x}\frac{\partial}{\partial x}(xE_x) - \frac{E_x}{4\pi}\frac{\partial}{\partial y}(E_y) + \frac{B_\phi}{4\pi x}\frac{\partial}{\partial x}(B_\phi) + \frac{B_y}{4\pi}\frac{\partial}{\partial x}(B_y) - \frac{B_y}{4\pi}\frac{\partial}{\partial y}(B_x) = 0 \tag{15}$$

and

$$\rho u^1 \frac{\partial}{\partial x}(u_3) + \rho u^3 \frac{\partial}{\partial y}(u_3) - \frac{E_y}{4\pi x}(xE_x) - \frac{E_y}{4\pi}\frac{\partial}{\partial y}(E_y) - \frac{B_x}{4\pi}\frac{\partial}{\partial x}(B_y) + \frac{B_y}{4\pi}\frac{\partial}{\partial y}(B_x) + \frac{1}{8\pi}\frac{\partial}{\partial y}(B_\phi^2) = 0 \ . \tag{16}$$

From equations (13) and (14), or alternatively from equations (15) and (16), another important equation may be derived (see Appendix III):

$$\gamma - \frac{\Omega}{c^2} x\, u_\phi = 1 \ . \tag{17}$$

From the metric condition, equation (1), it follows that

$$1 = \gamma^2 - u^2 - v^2 \tag{18}$$

where u and v are defined in Appendix I.

All necessary equations have now been derived.

IV. CRITICAL POINTS

In doing any computations with the equations derived in the last section, it is first convenient to put them in a dimensionless form. First, a new scalar potential is defined by

$$f = \frac{\Omega}{c} \alpha. \tag{19}$$

The field line constants can then be redefined as $F(f)$, $L(f)$, and $\chi(f)$. With this definition,

$$\vec{E} = - \vec{\nabla} f \tag{20}$$

and

$$B_m = \frac{c}{\Omega} \frac{1}{x} |\vec{\nabla} f| . \tag{21}$$

New coordinates are then defined by the obvious notation

$$(\omega, z) = \frac{\Omega}{c} (x, y) . \tag{22}$$

Combining equations (12) and (14) yields

$$u_o F(f) B_m + \frac{\Omega}{4\pi} x B_m B_\phi = - \chi(f) B_m . \tag{23}$$

As $u_o = -c^2\gamma$, this becomes

$$\gamma - \frac{\Omega}{4\pi F(f) c^2} x B_\phi = \frac{\chi(f)}{c^2 F(f)} . \tag{24}$$

In the new coordinates this is

$$\gamma - \frac{\omega}{4\pi F(f) c} B_\phi = \mu(f) \tag{25}$$

where

$$\mu(f) = \frac{\chi(f)}{c^2 F(f)} . \tag{26}$$

Combining equations (12) and (13) yields

$$F(f)u_2 B_m - \frac{x}{4\pi} B_m B_\phi = L(f) B_m \tag{27}$$

or in the new coordinates,

$$\gamma\omega\beta_\phi - \frac{\omega}{4\pi F(f)c} B_\phi = \frac{\Omega L(f)}{c^2 F(f)} . \tag{28}$$

Defining $\lambda(f)$ and $\sigma(f)$ by

$$\lambda(f) = 4\pi L(f)$$

$$\sigma(f) = \frac{\Omega}{4\pi F(f) c^2} , \tag{29}$$

equations (25) and (28) become

$$\gamma - \frac{c}{\Omega} \sigma(f)\omega B_\phi = \mu(f) \tag{30}$$

and

$$\gamma\omega\beta_\phi - \frac{c}{\Omega} \sigma(f)\omega B_\phi = \sigma(f) \ \lambda(f) . \tag{31}$$

Last, define

$$\vec{e} = \frac{c}{\Omega} \vec{E}$$

$$\vec{b} = \frac{c}{\Omega} \vec{B} \tag{32}$$

so that

$$\vec{e} = - \vec{\nabla} f$$

$$b_m = \frac{1}{\omega} |\vec{\nabla} f| , \tag{33}$$

where the $\vec{\nabla}$ operator is now in terms of the coordinates (ω, ϕ, z). Equations (30) and (31) become

$$\gamma - \sigma(f)\omega b_\phi = \mu(f) \tag{34}$$

and

$$\gamma \omega \beta_{\phi} - \sigma(f) \omega \beta_{\phi} = \sigma(f) \ \lambda(f). \tag{35}$$

Equation (9) is now

$$\omega b_m = b_m \beta_{\phi} - b_{\phi} \beta_m \ . \tag{36}$$

Equation (17) is quite clearly

$$\gamma - \gamma \beta_{\phi} \omega = 1 \ . \tag{37}$$

Note that with this equation, equations (34) and (35) subtract to give

$$1 + \sigma(f) \lambda(f) = \mu(f) \ . \tag{38}$$

This is the same boundary condition that Michel (1969a) derived for the equatorial plane by considering the surface conditions. This condition is here seen to be a direct result of the equations without reference to the stellar surface.

The critical point surface can now be found in terms of the magnetic field structure, that is, in terms of the scalar f. (As each equipotential surface will have a circle of critical points, a critical point surface should result when all field lines are considered.) From equation (36),

$$b_{\phi} = \frac{b_m}{\beta_m} (\beta_{\phi} - \omega). \tag{39}$$

Substituting this expression into equation (35) gives

$$\sigma(f) \lambda(f) = \gamma \omega \beta_{\phi} - \sigma(f) \omega \frac{b_m}{\beta_m} \beta_{\phi} + \sigma(f) \omega^2 \frac{b_m}{\beta_m} \ ,$$

which can be solved for β_{ϕ} :

$$\beta_\phi = \frac{\sigma(f)\lambda(f) - \sigma(f)\omega^2 \frac{b_m}{\beta_m}}{\gamma\omega - \sigma(f)\omega \frac{b_m}{\beta_m}} . \tag{40}$$

The denominator is

$$\gamma\omega - \sigma(f)\omega \frac{b_m}{\beta_m} = \gamma\omega\left[1 - \frac{\sigma(f)\, b_m}{\gamma \beta_m}\right]$$

$$= \frac{\gamma\omega}{M^2}\,(M^2 - 1) \tag{41}$$

where

$$M^2 = \frac{\gamma\beta_m}{\sigma(f)\, b_m} = \frac{c^3}{\Omega}\,\frac{4\pi\rho\, u^2}{b_m{}^2} \tag{42}$$

is the Alfvénic Mach number. The critical points are where $M^2 = 1$. β_ϕ can be rewritten as

$$\beta_\phi = \frac{\frac{M^2\sigma(f)\lambda(f)}{\gamma\omega} - \omega}{M^2 - 1} . \tag{43}$$

Thus v is

$$v = \gamma\beta_\phi = \frac{\frac{M^2\sigma(f)\lambda(f)}{\omega} - \gamma\omega}{M^2 - 1} . \tag{44}$$

From equation (36),

$$\frac{b_\phi}{b_m} = \frac{1}{\beta_m}\,(\beta_\phi - \omega)$$

$$= \frac{1}{\beta_m}\left[\frac{\frac{M^2\sigma(f)\lambda(f)}{\gamma\,\omega} - \omega}{M^2 - 1} - \omega\right]$$

$$= \frac{1}{M^2 - 1}\left[\frac{M^2\sigma(f)\lambda(f)}{\gamma\beta_m\omega} - \frac{\omega M^2}{\beta_m}\right]$$

$$= \frac{\frac{\lambda(f)}{|\vec{\nabla} f|} - \frac{\omega M^2}{\beta_m}}{M^2 - 1} . \tag{45}$$

Substituting this for b_ϕ into equation (35) gives

$$\gamma - \sigma(f)\omega\, b_m \left[\frac{\frac{\lambda(f)}{|\vec{\nabla} f|} - \frac{\omega M^2}{\beta_m}}{M^2 - 1}\right] = \mu(f) = 1 + \sigma(f)\lambda(f)$$

or

$$\gamma(M^2-1) - \sigma(f)\lambda(f) + \frac{M^2\omega\sigma(f)\,|\vec{\nabla} f|}{\beta_m} = M^2 + M^2\sigma(f)\lambda(f) - 1 - \sigma\lambda .$$

Now

$$\frac{M^2\omega\sigma|\vec{\nabla} f|}{\beta_m} = \frac{\gamma\beta_m}{\sigma b_m}\,\frac{\omega\sigma|\vec{\nabla} f|}{\beta_m} = \gamma\omega^2$$

so that

$$\gamma\,(1 - M^2 - \omega^2) = 1 - M^2(1 + \sigma\lambda) \;,$$

or

$$\gamma = \frac{1 - M^2(1 + \sigma\lambda)}{1 - M^2 - \omega^2}$$

$$= \frac{1 - \frac{\omega u}{\sigma|\vec{\nabla} f|}(1 + \sigma\lambda)}{1 - \omega^2\left(\frac{u}{\sigma\omega|\vec{\nabla} f|} + 1\right)} . \qquad (46)$$

Equations (18) and (46) may be combined to eliminate γ . Let $\gamma = A/B$, with obvious meaning for A and B. Using equation (44) for v,

$$1 + u^2 = A^2/B^2 - v^2 = A^2/B^2 - \frac{\omega^2\left(A/B - \frac{M^2\sigma\lambda}{\omega^2}\right)^2}{(1 - M^2)^2}$$

or

$$[B^2(1+u^2) - A^2][1-M^2]^2 = -\omega^2\left[A - \frac{Bu\lambda}{\omega|\vec{\nabla} f|}\right]^2 . \qquad (47)$$

Now

$$A - \frac{Bu\lambda}{\omega|\vec{\nabla} f|} = 1 - \frac{\omega u}{\sigma|\vec{\nabla} f|} - \frac{u\lambda}{\omega|\vec{\nabla} f|} + \frac{u^2\lambda}{\sigma|\vec{\nabla} f|^2}$$

$$= [1 - \frac{u\lambda}{\omega\,|\vec{\nabla} f|}]\ [1 - \frac{u\omega}{\sigma\,|\vec{\nabla} f|}]$$

$$= [1 - \frac{u\lambda}{\omega\,|\vec{\nabla} f|}]\ [1 - M^2].$$

Thus equation (47) becomes

$$B^2(1+u^2) - A^2 = -\ \omega^2(1 - \frac{u\lambda}{\omega\,|\vec{\nabla} f|})^2 .$$

Using the expressions for A and B, this becomes

$$(1+u^2)\ [1 - \frac{2\omega u}{\sigma\,|\vec{\nabla} f|} - 2\omega^2 + \omega^4(\frac{u}{\sigma\omega\,|\vec{\nabla} f|} + 1)^2] - 1 +$$

$$\frac{2u\omega^2}{\sigma\omega\,|\vec{\nabla} f|}(1+\sigma\lambda) - \frac{\omega^4 u^2}{\sigma^2\omega^2\,|\vec{\nabla} f|^2}(1+\sigma\lambda)^2 = -\ \omega^2 + \frac{2u\lambda\omega^2}{\omega\,|\vec{\nabla} f|} - \frac{\omega^2 u^2\lambda^2}{\omega^2\,|\vec{\nabla} f|^2}$$

or

$$\omega^4\ [\frac{u^2}{\sigma^2\omega^2\,|\vec{\nabla} f|^2}(1+\sigma\lambda)^2 - (1+u^2)(\frac{u}{\sigma\omega\,|\vec{\nabla} f|} + 1)^2\ ]$$

$$+\ \omega^2[1 + 2u^2(\frac{u}{\sigma\omega\,|\vec{\nabla} f|} + 1) - \frac{u^2\lambda^2}{\omega^2\,|\vec{\nabla} f|^2}\ ] - u^2 = 0 . \qquad (48)$$

This is a quadratic equation in ω^2. Parker (1963) has shown that the critical points are at the intersection of two special solutions. In this quadratic, the roots are equal if and only if the discriminant is zero. The discriminant is

$$B^2 - 4AC = [1 + 2u^2(\frac{u}{\sigma\omega\,|\vec{\nabla} f|} + 1) - \frac{u^2\lambda^2}{\omega^2\,|\vec{\nabla} f|^2}\ ]^2$$

$$-\ 4(-u^2)[\frac{u^2}{\sigma^2\omega^2\,|\vec{\nabla} f|^2}(1+\sigma\lambda)^2 - (1+u^2)(\frac{u}{\sigma\omega|\vec{\nabla} f|} + 1)]$$

Expanding and simplifying, this becomes

$$1 - \frac{4u^3}{\sigma\omega\,|\vec{\nabla} f|} - \frac{4u^5\lambda^2}{\sigma\omega^3\,|\vec{\nabla} f|^3} - \frac{2u^2\lambda^2}{\omega^2\,|\vec{\nabla} f|^2} + \frac{u^4\lambda^4}{\omega^4\,|\vec{\nabla} f|^4} + \frac{8u^4\lambda}{\sigma\omega^2\,|\vec{\nabla} f|^2} = 0.$$

This factors to

$$[1 - \frac{u\lambda}{\omega|\vec{\nabla}f|}]^2[(1 + \frac{u\lambda}{\omega|\vec{\nabla}f|})^2 - \frac{4u^3}{\sigma\omega|\vec{\nabla}f|}] = 0 \ . \tag{49}$$

The last term of this equation does not have physical roots. The other term yields the (double) root

$$u_c = \frac{\omega|\vec{\nabla}f|}{\lambda(f)} \tag{50}$$

where the subscript denotes that this is at the critical point. The value of ω^2 at the critical point is

$$\omega_c^2 = \frac{-B}{2A} = \frac{-1-2u^2[\frac{u}{\sigma\omega|\vec{\nabla}f|}+1] + \frac{u^2\lambda^2}{\omega^2|\vec{\nabla}f|^2}}{\frac{2u^2}{\sigma^2\omega^2|\vec{\nabla}f|^2}(1+\sigma\lambda)^2 - 2(1+u^2)(\frac{u}{\sigma\omega|\vec{\nabla}f|}+1)^2}$$

$$= \frac{\frac{\omega^2|\vec{\nabla}f|^2}{\sigma\lambda^3}(1+\sigma\lambda)}{\frac{(1+\sigma\lambda)^2}{\sigma^2\lambda^2}[\frac{\omega^2|\vec{\nabla}f|^2}{\lambda(f)^2}]}$$

$$= \frac{\sigma\lambda}{1+\sigma\lambda} \ . \tag{51}$$

At this point, the critical points have been determined. As $\sigma(f)$ and $\lambda(f)$ are still unknown, this information is still incomplete. The determination of $\lambda(f)$ in terms of $\sigma(f)$, $|\vec{\nabla}f|$, and the coordinates is the next step.

V. MINIMUM TORQUE SOLUTION

As Michel (1969a) has pointed out, there is no a priori way to determine which solutions for different $\lambda(f)$ are stable and thus apply to the problem. Therefore some sort of assumption must be made, and following Michel, the minimum torque solution is chosen to be the proper one. That this is equivalent to putting the magnetosonic point at infinity has been demonstrated by Goldreich and Julian (1970). This should not be taken to imply that this is the only possible solution. The "near-zone" solution of Hendriksen and Rayburn (1970) illustrate another possibility. In any case, the minimum torque solution may be regarded as a special example.

Any solution extending to infinity must satisfy equation (48). The necessary condition for this is that as ω approaches infinity, the coefficient of ω^4 must approach zero. Thus the value of u at infinity, denoted by u_* , must be a root of the coefficient. The minimum torque solution, as the name implies, requires the smallest $\lambda(f)$ possible. If $\lambda(f)$ is decreased until there is no solution, then u_* must become a double root at the minimum $\lambda(f)$ allowing a solution. Calling the coefficient C, then the equation C = 0 is

$$\frac{u^2}{\sigma^2\omega^2 |\vec{\nabla} f|^2}(1+\sigma\lambda)^2 - (1+u^2)\left(\frac{u}{\sigma\omega|\vec{\nabla} f|} + 1\right)^2 = 0. \qquad (52)$$

Expanding terms, this is

$$\frac{u^2}{\sigma^2\omega^2|\vec{\nabla}f|^2}(1+\sigma\lambda)^2 - \frac{u^2}{\sigma^2\omega^2|\vec{\nabla}f|^2} - \frac{2u}{\sigma\omega|\vec{\nabla}f|} - 1 - \frac{u^4}{\sigma^2\omega^2|\vec{\nabla}f|^2}$$

$$-\frac{2u^3}{\sigma\omega|\vec{\nabla}f|} - u^2 = 0 ,$$

which simplifies to

$$u^4 + u^3[2\sigma\omega|\vec{\nabla}f|] + u^2[1 - (1+\sigma\lambda)^2 + \sigma^2\omega^2|\vec{\nabla}f|^2]$$
$$+ u[2\sigma\omega|\vec{\nabla}f|] + \sigma^2\omega^2|\vec{\nabla}f|^2 = 0. \tag{53}$$

This is a quartic equation of the form

$$u^4 + u^3D_3 + u^2D_2 + uD_1 + D_o = 0$$

with a double root $u = u_*$. If $(u - u_*)^2$ is divided into this quartic, the result is

$$\frac{u^4 + u^3D_3 + u^2D_2 + uD_1 + D_o}{(u - u_*)^2} = u^2 + u(D_3+2u_*) + D_2-u_*^2$$
$$+ 2u_*(D_3+2u_*) +$$
$$u \quad 2u_*[(D_2-u_*) + 2u_*(D_3+2u_*)] + D_1-u_*^2(D_3+2u_*)$$
$$+ u_*^2[(D_2-u_*^2)+2u_*(D_3+2u_*)] + D_o \quad .$$

For perfect factorization, the conditions which the coefficients must satisfy result:

$$2[(D_2-u_*^2)+2u_*(D_3+2u_*)]u_* + D_1 - u_*^2(D_3+2u_*) = 0 \tag{54}$$

$$D_o - u_*^2[(D_2 - u_*^2) + 2u_*(D_3+2u_*)] = 0 \tag{55}$$

Setting

$$K = D_2 + 2u_*D_3 + 3u_*^2 ,$$

the more compact forms are achieved:

$$2Ku_* + D_1 - u_* \ (D_3+2u_*) = 0 \tag{56}$$

$$D_o - u_* \ K = 0 \ . \tag{57}$$

Using the coefficients in equation (53), equation (57) gives

$$K = \frac{D_o}{u_*^2} = \frac{\sigma^2\omega^2 |\vec{\nabla} f|^2}{u_*^2} \tag{58}$$

Then from equation (56),

$$2\left(\frac{\sigma^2\omega^2 |\vec{\nabla} f|^2}{u_*^2}\right) u_* + 2\sigma\omega |\vec{\nabla} f| - u_*^2(2u_*+2\sigma\omega |\vec{\nabla} f|) = 0$$

or

$$2\sigma^2\omega^2 |\vec{\nabla} f|^2 + 2u_*\sigma \omega |\vec{\nabla} f| - 2u_*^4 - 2u_*^3\sigma\omega |\vec{\nabla} f| = 0.$$

This factors as

$$2[u_*+\sigma\omega |\vec{\nabla} f|][u_*^3 - \sigma\omega |\vec{\nabla} f|] = 0.$$

The physically meaningful root is thus

$$u_* = [\sigma\omega |\vec{\nabla} f|]^{1/3}. \tag{59}$$

This evaluates the asymptotic velocity in terms of $\sigma(f)$, ω, and $|\vec{\nabla} f|$. $\lambda(f)$ can also be evaluated in terms of these same quantities by combining equations (55) and (58):

$$D_2 + 3u_*^2 + 2u_*D_3 = \frac{D_o}{u_*^2}$$

$$3u_*^2 + 4u_*(\sigma\omega|\vec{\nabla} f|) + 1 + \sigma^2\omega^2 |\vec{\nabla} f|^2 - (1+\sigma\lambda)^2 = \frac{\sigma^2\omega^2 |\vec{\nabla} f|^2}{u_*^2}.$$

Using equation (59) this is

$$-(1+\sigma\lambda)^2 + 3u_*^2 + 4u_*^4 + 1 + u_*^6 = u_*^4$$

or

$$u_*^6 + 3u_*^4 + 3u_*^2 + 1 = (1 + \sigma\lambda)^2$$

which is just

$$\mu(f) = (1 + \sigma\lambda)^2 = (1 + u_*^2)^3. \tag{60}$$

Therefore the minimum torque $\lambda(f)$ and the energy per unit mass $\mu(f)$ have been evaluated entirely in terms of the coordinate ω and functions of the scalar f. As there is no exact method for determining $\sigma(f)$, $\lambda(f)$ cannot be evaluated further unless some choice for $\sigma(f)$ is made.

VI. MAGNETIC FIELD STRUCTURE

Assuming that some plausible choices for $\sigma(f)$ can be made, then the next step is to determine the scalar function f. This would complete the entire problem. For this purpose equation (16) can be used. This equation can be written (Appendix IV) as

$$\frac{1}{c\sigma(f)} b_\omega \frac{\partial}{\partial\omega} (u_z) + \frac{1}{c\sigma(f)} b_z \frac{\partial}{\partial z} (u_z)$$
$$- (e_z/\omega)\frac{\partial}{\partial\omega} (\omega e_\omega) - e_z \frac{\partial}{\partial z} (e_z) - b_\omega \frac{\partial}{\partial\omega}(b_z)$$
$$+ b_\omega \frac{\partial}{\partial z}(b_\omega) + \tfrac{1}{2} \frac{\partial}{\partial z}(b_\phi^2) = 0 \qquad (61)$$

Two special case can now be considered. The approximation of massless particles is made, and the first two terms disappear. In the first special case, strict corotation inside the light cylinder is assumed, and equation (61) reduces to

$$-(e_z/\omega)\frac{\partial}{\partial\omega}(\omega e_\omega) - e_z \frac{\partial}{\partial z}(e_z) - b_\omega \frac{\partial}{\partial\omega}(b_z)$$
$$+ b_\omega \frac{\partial}{\partial z}(b_\omega) = 0. \qquad (62)$$

Using the expressions for e_ω , e_z , b_ω, and b_z in terms of derivatives of f, this equation becomes

$$f_z[\frac{1}{\omega} \frac{\partial}{\partial\omega}(\omega f_\omega) + \frac{\partial}{\partial z}(f_z)] - \frac{f_z}{\omega}[\frac{\partial}{\partial\omega}(\frac{1}{\omega} f_\omega) + \frac{1}{\omega} \frac{\partial}{\partial z}(f_z) = 0.$$

Expanding this,

$$f_{\omega\omega}f_z + f_z f_\omega \frac{1}{\omega} + f_z f_{zz} + \frac{1}{\omega^3} f_\omega f_z - \frac{1}{\omega^2} f_z f_{\omega\omega} - \frac{1}{\omega^2} f_z f_{zz} = 0 ,$$

or

$$(1-\frac{1}{\omega^2})f_{\omega\omega}f_z + (1+\frac{1}{\omega^2})\frac{1}{\omega}f_\omega f_z + (1-\frac{1}{\omega^2})f_z f_{zz} = 0.$$

Dividing through by $(1-1/\omega^2)f_z$ yields

$$f_{\omega\omega} + f_{zz} - \frac{1}{\omega}\frac{1+\omega^2}{1-\omega^2}f_\omega = 0 . \tag{63}$$

This is the same equation that has been considered in great detail by Michel (1973a), although a different approach was used in its derivation.

The other special case is much more interesting. Again, the particles are assumed to be massless, but rigid corotation is not assumed. Equation (61) becomes just equation (62) but with an added term involving b_ϕ . Putting this in terms of f gives

$$-f_z[\frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega f_\omega) + \frac{\partial}{\partial z}(f_z)] + \frac{1}{\omega}f_z[\frac{\partial}{\partial\omega}(\frac{f_\omega}{\omega}) + \frac{1}{\omega}\frac{\partial}{\partial z}(f_z)]$$

$$+ \tfrac{1}{2}\frac{\partial}{\partial z}(b_\phi{}^2) = 0 . \tag{64}$$

Now from equation (13), with $\rho = 0$,

$$B_\phi = \frac{-4\pi L(\alpha)}{x}$$

or

$$b_\phi = -\frac{1}{\omega}\lambda(f).$$

Substituting this into equation (64) and expanding,

$$f_z[f_{\omega\omega} + f_{zz} - \frac{1}{\omega}\frac{1+\omega^2}{1-\omega^2}f_\omega] + \tfrac{1}{2}\frac{\omega^2}{1-\omega^2}\frac{\partial}{\partial z}[\frac{\lambda^2}{\omega^2}] = 0.$$

Now as the field lines for large ω are open, in this region the magnetic field will appear to be a distorted monopole. Of course the field lines in the northern and southern hemispheres will be oppositely directed. Near the equatorial plane the field lines should be wound up like an Archimedes spiral, with a distorted version of this on the other equipotential surfaces. It is therefore not too unreasonable to assume that the surface magnetic field is also a monopole and then to investigate the results. Then near the stellar surface,

$$f = f_o \frac{z}{r} \tag{65}$$

where f_o is a constant determined by the magnetic field strength and $r = (z^2+\omega^2)^{\frac{1}{2}}$ is the distance from the origin. Then

$$f_z = \frac{f_o\ \omega^2}{(\omega^2+z^2)^{3/2}}, \qquad f_\omega = \frac{-\ f_o \omega z}{(\omega^2+z^2)^{3/2}}$$

$$f_{\omega\omega} = \frac{-f_o z}{(\omega^2+z^2)^{3/2}} + \frac{3f_o\omega^2 z}{(\omega^2+z^2)^{5/2}}, \quad f_{zz} = \frac{-3f_o\omega^2 z}{(\omega^2+z^2)^{5/2}}$$

Equation (65) becomes

$$\frac{f_o\omega^2}{r^3}[-\frac{f_o z}{r^3}(1-\frac{1+\omega^2}{1-\omega^2})] = \tfrac{1}{2}\frac{1}{\omega^2}\frac{\omega^2}{1-\omega^2}\frac{\partial}{\partial z}(\lambda^2)$$

or

$$\frac{2f_o\omega^4 z}{r^6} = \tfrac{1}{2}\frac{\partial}{\partial z}(\lambda^2).$$

This can be integrated for give λ^2:

$$\lambda^2 = \frac{f_o^{\,2}\omega^2}{(\omega^2+z^2)^2} + \text{constant.}$$

As b_ϕ should be zero at the poles, the constant must be zero. Thus

$$\lambda = \frac{f_o\omega^2}{(\omega^2+z^2)}, \tag{66}$$

where the positive root is chosen to indicate a net outflow of angular momentum. At this point a very interesting observation can be made. $\lambda(f)$ has been determined such that the partial differential equation (64) is satisfied near the star. However, once this is done, it is readily apparent that with $\lambda(f)$ in this form the monopole solution is valid everywhere:

$$f = f_o \frac{z}{r} \tag{67}$$

For massless particles this should not be an entirely unexpected result, for the only forces are electromagnetic, which are all in balance at the surface and all behave the same way as one goes further from the star. It is also worthwhile to observe that for $v \to c$, equation (71) will reduce to equation (64) exactly. This suggests that $v=c$ is valid everywhere if the monopole solution is used. This can be readily verified: From $(\vec{\nabla} \times \vec{E})_\phi = 0$,

$$\frac{v_z}{b_z} = \frac{v_\omega}{b_\omega}$$

or

$$v_m = k\, b_m , \tag{68}$$

where k is some scalar function of position. Now

$$b_m = \frac{1}{\omega} |\vec{\nabla} f| = \frac{-f_o}{r^2} ,$$

and as $v \to c$,

$$k = \frac{-cr^2}{f_o}$$

is required. Then, however, it is found that $v_m = c$ everywhere. It thus appears that by assuming massless particles and a monopole magnetic field that the velocity of the particles is of necessity c. In a more realistic sense, as the plasma is initially accelerated by an electric field near the stellar surface, only a vanishingly small electric field is required to accelerate the particles to light speed.

This last solution has also been found by Michel (1973b) with a different approach.

In the more general case, equation (61) must be solved. Every term is easily expressed in terms of f, and has in fact been done so, except for the terms involving u_z and b_ϕ . This situation can be remedied by the following considerations. From $(\vec{\nabla} \times \vec{e})_\phi = 0$ there results

$$b_\omega u_z = b_z u_\omega .$$

Thus

$$b_\omega{}^2 u_z{}^2 = b_z{}^2 u_\omega{}^2$$

$$(b_m{}^2 - b_z{}^2) u_z{}^2 = b_z{}^2 u_\omega{}^2$$

$$b_m{}^2 u_z{}^2 = b_z{}^2 (u_z{}^2 + u_\omega{}^2) = b_z{}^2 u_m{}^2$$

and thus

$$u_z = u_m \frac{b_z}{b_m} .$$

Similarly,

$$u_\omega = u_m \frac{b_\omega}{b_m} .$$

Thus, using the expressions for b_m, b_z , and b_ϕ ,

$$u_z = - \frac{f_z}{|\vec{\nabla} f|} u_m$$

$$u_\omega = \frac{f_\omega}{|\vec{\nabla} f|} u_m . \tag{69}$$

The problem now is that u_m is determined only by equation (48). b_ϕ is determined by equation (45), which also involves u_m, so b_ϕ and u_m are not readily useful. Some assumptions fortunately simplify things. One approach is to look only at the asymptotic expressions for u_m and b_ϕ, and then assume that the solution between the surface and large ω can be done numerically by a relaxation method.

For large ω, equation (59) may be used for u_m. The expression for b_ϕ also simplifies considerably:

$$b_\phi = \frac{\frac{\lambda}{|\vec{\nabla} f|} - M^2\omega/\beta_m}{M^2 - 1} b_m = \frac{\lambda b_m/|\vec{\nabla} f| - (b_m\omega/\beta_m)(\gamma\beta_m/\sigma b_m)}{M^2 - 1}$$

$$= \frac{1}{M^2-1} \frac{\lambda/\omega - \frac{\lambda u}{\sigma|\vec{\nabla} f|} + \lambda\omega - \omega/\sigma - \frac{\omega^2 u}{\sigma 2|\vec{\nabla} f|} - \frac{\omega^2 u\lambda}{\sigma|\vec{\nabla} f|}}{1 - \omega\left(\frac{u}{\sigma|\vec{\nabla} f|} + \omega\right)}$$

$$= \frac{1}{M^2-1} \frac{(M^2-1)(\omega/\sigma - \lambda/\omega + \omega\lambda)}{1 - \omega\left(\frac{u}{\sigma|\vec{\nabla} f|} + \omega\right)}$$

$$= \frac{\omega/\sigma + \omega\lambda - \lambda/\omega}{1 - \omega\left(\frac{u}{\sigma|\vec{\nabla} f|} + \omega\right)} .$$

Now let ω become very large and approximate b_ϕ by

$$b_\phi = -\frac{\omega/\sigma + \lambda/\omega}{\omega^2(\frac{u}{\sigma\omega|\vec\nabla f|}+1)} = \frac{-(\omega/\sigma)(1+\sigma\lambda)}{\omega^2(\frac{1}{u^2}+1)} = \frac{-u^2(1+u^2)^{3/2}}{\sigma\omega(1+u^2)}$$

$$= -\frac{u^2(1+u^2)^{\frac{1}{2}}}{\sigma\omega|\vec\nabla f|}\nabla f = -|\vec\nabla f|(1+u^2)^{-\frac{1}{2}}$$

and as this is at large ω,

$$b_\phi = -|\vec\nabla f|\ (1+u_*{}^2)^{-\frac{1}{2}}\ . \qquad (70)$$

Equation (61) then becomes (for large ω)

$$-\frac{1}{\omega\,\sigma(f)}\ f_z \frac{\partial}{\partial\omega}[\frac{f_\omega}{|\vec\nabla f|}\ (\sigma\omega|\vec\nabla f|)^{1/3}] + \frac{1}{\omega\sigma}f_\omega\ \frac{\partial}{\partial z}\ [\frac{f_\omega}{|\vec\nabla f|}(\sigma\omega|\vec\nabla f|)^{1/3}]$$

$$-\ f_z[\frac{1}{\omega}\ \frac{\partial}{\partial\omega}(\omega f_\omega) + \frac{\partial}{\partial z}(f_z)] + \frac{1}{2}\frac{\partial}{\partial z}\ (|\vec\nabla f|^2)$$

$$+\ \frac{1}{\omega}f_z\ [\ \frac{\partial}{\partial\omega}(\frac{f_\omega}{\omega}) + \frac{1}{\omega}\ \frac{\partial}{\partial z}(f_z)$$

$$+\ \frac{1}{2}\frac{\partial}{\partial z}\ [\ \frac{|\vec\nabla f|^2}{(\sigma\omega|\vec\nabla f|\ )^{2/3}}]\ = 0\ . \qquad (71)$$

Thus far, no physical solutions have been obtained for this equation, with various plausible choices for $\sigma(f)$.

VII. CONCLUSION

The equatorial treatment of the stellar wind torque has been extended out of the equatorial plane. The entire description has been carried out in terms of a scalar function which can be determined by solving a non-linear partial differential equation. To find a general solution describing a physically reasonable magnetic field is a difficult task and has not yet been done. Nevertheless the differential equation simplifies in certain special cases to a form that can be solved. The special cases in the text are found to agree with those obtained by Michel using a particle approach rather than an MHD approach. The critical points, the angular momentum flux, the energy flux, and the asymptotic velocity have all been found in terms of the scalar function. This turned out to be remarkably simple, and the only difficulty remaining is to solve the differential equation to determine the system completely.

APPENDIX I

A. Tensor components of the electromagnetic field

As a covariant formulation is used throughout, the tensor components and the more usual vector components of the electromagnetic field are listed for reference.

$$F_{12} = xB_y \qquad F^{12} = (1/x)B_y$$
$$F_{13} = -B_\phi \qquad F^{13} = -B_\phi$$
$$F_{23} = xB_x \qquad F^{23} = (1/x)B_x$$
$$F_{10} = cE_x \qquad F^{10} = -(1/c)E_x$$
$$F_{20} = 0 \qquad F^{20} = 0$$
$$F_{30} = cE_y \qquad F^{30} = -(1/c)E_y$$

B. Frozen - in flux

The plasma conductivity will require that the electric field vanish in the proper frame. This is done by setting the proper force equal to zero.

$$F_{ik} \frac{dx^k}{ds} = 0 ,$$

or

$$F_{i0} \frac{dx^o}{ds} = - F_{ia} \frac{dx^a}{ds} , \quad a \neq 0.$$

The following equations result:

i = 1: $$F_{10} \frac{dx^o}{ds} = - F_{12} \frac{dx^2}{ds} - F_{13} \frac{dx^3}{ds}$$

or

$$cE_x = B_\phi v_y - B_y v_\phi \tag{A1}$$

i = 2: $$F_{20} \frac{dx^o}{ds} = - F_{21} \frac{dx^1}{ds} - F_{23} \frac{dx^3}{ds}$$

or

$$cE_\phi = 0 = B_y v_x - B_x v_y \tag{A2}$$

i = 3:

$$F_{30}\frac{dx^o}{ds} = -F_{31}\frac{dx^1}{ds} - F_{32}\frac{dx^2}{ds}$$

or

$$cE_y = B_\phi v_x - B_x v_\phi \tag{A3}$$

where the velocity components are defined by

$$\begin{aligned} v_x &= dx/dt \\ v_\phi &= x\, d\phi/dt \\ v_y &= dy/dt \ . \end{aligned} \tag{A4}$$

Equations (A1)-(A3) are seen to be the components in the usual frozen-in flux condition

$$\vec{E} = -(\vec{v}/c) \times \vec{B} \ .$$

C. Magnetic field line description

The magnetic field is described as stated in the main text. The electric field can be written as

$$\vec{E} = -\vec{\nabla}m\mu \ ,$$

where $m\mu$ is the scalar electric potential. From the frozen-in flux condition and equation (5) of the text,

$$-\vec{\nabla}m\mu = -\frac{\vec{v}}{c} \times [\vec{\nabla}\alpha \times \frac{\hat{n}}{x\,\hat{n}\cdot\hat{\phi}}]$$

$$= \frac{1}{cx}\left[\frac{\hat{n}}{\hat{n}\cdot\hat{\phi}}(\vec{v}\cdot\vec{\nabla}\alpha) - \vec{\nabla}\alpha(\vec{v}\cdot\frac{\hat{n}}{\hat{n}\cdot\hat{\phi}})\right] .$$

As the plasma flow is along the surfaces α = constant, and $\vec{\nabla}\alpha$ is of course perpendicular to these surfaces, $\vec{v}\cdot\vec{\nabla}\alpha = 0$ and

$$\vec{\nabla}m\mu = \frac{\vec{v}\cdot\hat{n}}{c x \hat{n}\cdot\hat{\phi}} \vec{\nabla}\alpha \, .$$

The coefficient of $\vec{\nabla}\alpha$ can be interpreted by referring to the stellar surface. There the particle velocity is just the corotation velocity, $v = \Omega x \hat{\phi}$, where Ω is the angular rotation rate. there,

$$\frac{\vec{v}\cdot\hat{n}}{c x \hat{n}\cdot\hat{\phi}} = \frac{\Omega x \hat{n}\cdot\hat{\phi}}{c x \hat{n}\cdot\hat{\phi}} = \frac{\Omega}{c} \, . \qquad \text{(A5)}$$

Thus

$$\vec{\nabla}m\mu = \frac{\Omega}{c} \vec{\nabla}\alpha \, .$$

This result is equation (8) in the main text. Equation (9) can also be obtained: Consider equation (A5) above, and refer to the diagram in Figure 3b.

First,

$$\hat{n}\cdot\vec{v} = n_\phi v_\phi + n_m v_m \, .$$

Then

$$\frac{\hat{n}\cdot\vec{v}}{\hat{n}\cdot\hat{\phi}} = v_\phi + v_m \frac{n_m}{n_\phi} \, .$$

As

$$n_m = \sin\chi, \quad n_\phi = \cos\chi$$

$$B_m = \cos\chi, \quad B_\phi = -\sin\chi \, ,$$

it follows that

$$-\frac{n_m}{n_\phi} = \frac{B_\phi}{B_m} \, .$$

Then equation (A5) becomes

$$\Omega = \frac{\vec{v}\cdot\hat{n}}{x\hat{n}\cdot\hat{\phi}} = \frac{1}{x}\left(v_\phi - v_m \frac{B_\phi}{B_m}\right)$$

or

$$B_m x\Omega = B_m v_\phi - B_\phi v_m \ .$$

This is equation (9) in the text.

The following velocities are defined for reference:

$$u^o = dt/ds$$

$$u^1 = dx/ds$$

$$u^2 = d\phi/ds$$

$$u^3 = dy/ds$$

$$u_o = g_{oo}dt/ds = -c^2u^o$$

$$u_1 = g_{11}dx/ds = u^1$$

$$u_2 = g_{22}d\phi/ds = x^2u^2$$

$$u_3 = g_{33}dy/ds = u^3$$

$$u_x = (g_{11})^{\frac{1}{2}}dx/ds = u^1$$

$$u_\phi = (g_{22})^{\frac{1}{2}}d\phi/ds = xu^2$$

$$u_y = (g_{33})^{\frac{1}{2}}dy/ds = u^3$$

$$u_m = (u_x{}^2 + u_y{}^2)^{\frac{1}{2}}$$

$$u = \gamma v_m/c = u_m/c$$

$$v = \gamma v_\phi/c$$

$$\beta_i = v_i/c$$

APPENDIX II

A. Conservation of mass

The conserved particle current is

$$s^i = \rho \frac{dx^i}{ds}$$

and

$$s^i_{\ ;i} = \frac{1}{(-g)^{\frac{1}{2}}} \frac{\partial}{\partial x^i} [(-g)^{\frac{1}{2}} s^i] = 0$$

is the covariant divergencelessness of that current. For the metric in equation (1),

$$(-g)^{\frac{1}{2}} = x ,$$

so that

$$s^i_{\ ;i} = \frac{1}{x} \frac{\partial}{\partial x}(x\rho u^1) + \frac{1}{x} \frac{\partial}{\partial y}(x\rho u^3) = 0. \qquad \text{(A6)}$$

The terms involving derivatives with respect to ϕ and t are of course zero and have accordingly not been written. Equation (A6) may be written as

$$\frac{1}{x} \frac{\partial}{\partial x}(\rho x u_x) + \frac{\partial}{\partial y} (\rho u_y) = 0 ,$$

or

$$\vec{\nabla} \cdot (\rho \vec{u}_m) = 0 . \qquad \text{(A7)}$$

Equation (A7) is very simple in appearance and can be readily integrated. As the method used in this integration is the same as that used in the integration of the other conservation equations to follow, the process is now illustrated in detail.

Refering to figure (4), it is readily apparent that with the use of the divergence theorem that

$$(\rho u_m)(2\pi x\, ds) = \text{constant } (\alpha) \ ,$$

where ds is an infinitesimal element of length and the constant is a function of field line position. As

$$ds \sim \frac{1}{|\nabla\alpha|},$$

this becomes

$$(\rho u_m) \frac{x}{|\nabla\alpha|} = \text{constant } (\alpha) \ .$$

This is equation (11) in the text.

B. Conservation of angular momentum

The other conservation equations are derived from the divergencelessness of the energy-momentum tensor,

$$T^k_{i;k} = \frac{1}{(-g)^{\frac{1}{2}}} \frac{\partial}{\partial x^k} [(-g)^{\frac{1}{2}} T^k_i] - \frac{1}{2} \frac{\partial g_{mn}}{\partial x^i} T^{mn} = 0 \ .$$

The energy-momentum tensor consists of two parts, one part describing the fluid and the other part describing the electromagnetic field :

$$T^k_i \text{ (fluid)} = (p + \rho)u_i u^k - p\delta^k_i$$

$$= \rho u_i u^k$$

as the pressure p is assumed to be zero, and

$$T^k_i \text{ (field)} = \frac{1}{4\pi}(F_{il}F^{kl} - \frac{1}{4}F_{lm}F^{lm}\delta^k_i) \ .$$

The equation for i = 2 represents the angular momentum conservation equation. From the definition of $T^k_{2;k}$,

the terms for k = 0 and k = 2 vanish as there are no non-zero derivatives with respect to t and ϕ. Also, $\frac{\partial g_{mn}}{\partial x^i}$ is zero unless i = 1. Hence only T^1_2 and T^3_2 need be considered.

$$T^1_2 = \rho u_2 u^1 + \frac{1}{4\pi}(F_{20}F^{10} + F_{23}F^{13})$$

$$= \rho u_2 u^1 + \frac{1}{4\pi}[0 + (xB_x)(-B_\phi)]$$

$$= \rho u_2 u^1 - \frac{x}{4\pi} B_x B_\phi$$

and

$$T^3_2 = \rho u_2 u^3 + \frac{1}{4\pi}(F_{20}F^{30} + F_{21}F^{31})$$

$$= \rho u_2 u^3 + \frac{1}{4\pi}[0 + (-xB_y)(B_\phi)]$$

$$= \rho u_2 u^3 - \frac{x}{4\pi} B_y B_\phi$$

Thus

$$T^k_{2;k} = \frac{1}{x}\frac{\partial}{\partial x}[x(\rho u_2 u^1 - \frac{x}{4\pi} B_x B_\phi)] + \frac{1}{x}\frac{\partial}{\partial y}[x(\rho u_2 u^3 - \frac{x}{4\pi} B_\phi B_y)]$$

$$= 0$$

which may be written

$$\vec{\nabla}\cdot[\rho u_2 \vec{u}_m - \frac{x}{4\pi}\vec{B}_m B_\phi] = 0$$

As the plasma is carried away along the equipotential surfaces, the angular momentum, which is carried in both the plasma and the electromagnetic field, is also carried along the equipotential surfaces. This equation can therefore immediately be integrated to

$$\rho u_2 u_m - \frac{x}{4\pi} B_m B_\phi = L(\alpha) B_m \ .$$

C. Conservation of energy

The case i = 0 for $T^k_{i;k} = 0$ represents the conservation of energy. The only non-zero terms in $T^k_{0;k} = 0$ are those involving T^1_o and T^3_0 .

$$T^1_0 = \rho u_o u^1 + \frac{1}{4\pi} (F_{02}F^{12} + F_{03}F^{13})$$

$$= \rho u_o u^1 + \frac{1}{4\pi} [\ 0 + (-cE_y)(-B_\phi)]$$

$$= \rho u_o u^1 + \frac{c}{4\pi} E_y B_\phi$$

and

$$T^3_o = \rho u_o u^3 + \frac{1}{4\pi}(F_{01}F^{31} + F_{02}F^{32})$$

$$= \rho u_o u^3 + \frac{1}{4\pi} [(-cE_x)(B_\phi\ ) + 0]$$

$$= \rho u_o u^3 - \frac{c}{4\pi} E_x B_\phi \ .$$

Then

$$T^k_{0;k} = (1/x) \frac{\partial}{\partial x} [x(\rho u_o u^1 + \frac{c}{4\pi} E_y B_\phi)]$$

$$+ (1/x) \frac{\partial}{\partial y} [x(\rho u_0 u^3 - \frac{c}{4\pi} E_x B_\phi)] = 0 \ ,$$

which may be written as

$$\vec{\nabla} \cdot (\rho u_o \vec{u}_m - \frac{c}{4\pi} \vec{E} \times \vec{B}_\phi) = 0 \ .$$

As $\vec{E}$ is always perpendicular to $\vec{B}$, from equations (7) and (8) this may be written as

$$\vec{\nabla}\cdot\left(\rho u_o \vec{u}_m + \frac{\Omega}{4\pi} \times \vec{B}_m B_\phi\right) = 0 \ .$$

As the energy flux is along the equipotential surfaces, this integrates to

$$\rho u_o u_m + \frac{\Omega}{4\pi} \times B_m B_\phi = -\chi(\alpha) B_m \ ,$$

where $\chi(\alpha)$ is a constant along a given field line.

D. Non-integrable components of the energy-momentum tensor

The other two equations, $T^k_{1;k} = 0$ and $T^k_{3;k} = 0$, cannot be integrated. They are necessary , and thus derived.

$$T^1_1 = \rho u_1 u^1 + \frac{1}{8\pi}\left(F_{10}F^{10} + F_{12}F^{12} + F_{13}F^{13} - F_{03}F^{03} - F_{23}F^{23}\right)$$

$$= \rho u^1 u_1 + \frac{1}{8\pi}\left(E_y^{\ 2} + B_\phi^{\ 2} + B_y^{\ 2} - E_x^{\ 2} - B_x^{\ 2}\right)$$

$$T^3_1 = \rho u_1 u^3 + \frac{1}{4\pi}\left(F_{10}F^{30} + F_{12}F^{32}\right)$$

$$= \rho u_1 u^3 - \frac{1}{4\pi}\left(E_x E_y + B_x B_y\right)$$

$$T^{22} = g^{22} T^2_2 = (1/x^2)\left[\rho u_2 u^2 + \frac{1}{8\pi}\left(F_{21}F^{21} + F_{23}F^{23} - F_{10}F^{10} - F_{13}F^{13} - F_{30}F^{30}\right)\right]$$

$$= \frac{1}{x^2}\left[\rho u_2 u^2 + \frac{1}{8\pi}\left(B_x^{\ 2} + B_y^{\ 2} + E_x^{\ 2} + E_y^{\ 2} - B_\phi^{\ 2}\right)\right] .$$

Then, as $(\partial g_{22}/\partial x) = 2x$, $T^k_{1;k} = 0$ becomes

$$\frac{1}{x}\frac{\partial}{\partial x} \; x[\rho u_1 u^1 + \frac{1}{8\pi}(E_y^2 + B_\phi^2 + B_y^2 - E_x^2 - B_x^2)]$$

$$+ \frac{\partial}{\partial y}[\rho u_1 u^3 - \frac{1}{4\pi}(E_x E_y + B_x B_Y)] - \frac{1}{x}\rho u_2 u^2$$

$$- \frac{1}{8\pi x}(B_x{}^2 + B_y{}^2 - B_\phi{}^2 + E_x{}^2 + E_y{}^2) = 0 .$$

With a considerable amount of algebra this equation can be reduced to the somewhat more familiar form

$$\rho u^1 \frac{\partial}{\partial x}(u_1) + \rho u^3 \frac{\partial}{\partial y}(u_1) - \frac{1}{x}\rho u_2 u^2$$

$$- \frac{E_x}{4\pi x}\frac{\partial}{\partial x}(xE_x) - \frac{E_x}{4\pi}\frac{\partial}{\partial y}(E_y) + \frac{B_\phi}{4\pi x}\frac{\partial}{\partial x}(xB_\phi)$$

$$+ \frac{B_y}{4\pi}\frac{\partial}{\partial x}(B_y) - \frac{B_y}{4\pi}\frac{\partial}{\partial y}(B_x) = 0 . \qquad (A8)$$

For the case i = 3,

$$T^1_3 = \rho u_3 u^1 + \frac{1}{4\pi}(F_{30}F^{31} + F_{32}F^{12})$$

$$= \rho u_3 u^1 - \frac{1}{4\pi}(E_x E_y + B_x B_y)$$

$$T^3_3 = \rho u_3 u^3 + \frac{1}{8\pi}(F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32}$$

$$- F_{10}F^{10} - F_{12}F^{12})$$

$$= \rho u_3 u^3 + \frac{1}{8\pi}(E_x^2 - E_y^2 + B_\phi^2 + B_x^2 - B_y^2).$$

Then

$$T^k_{3;k} = \frac{1}{x}\frac{\partial}{\partial x} \; x[\rho u_3 u^1 - \frac{1}{4\pi}(E_x E_y + B_x B_y)]$$

$$+ \frac{1}{x}\frac{\partial}{\partial y} \; x[\rho u_3 u^3 + \frac{1}{8\pi}(E_x^2 - E_y^2 + B_\phi^2 + B_x^2 - B_y^2)] = 0.$$

Again, with considerable algebra, this simplifies to

$$\rho u^1 \frac{\partial}{\partial x}(u_3) + \rho u^3 \frac{\partial}{\partial y}(u_3) - \frac{E_y}{4\pi x}\frac{\partial}{\partial x}(xE_x)$$
$$- \frac{E_y}{4\pi}\frac{\partial}{\partial y}(E_y) - \frac{B_x}{4\pi}\frac{\partial}{\partial x}(B_y) + \frac{B_y}{4\pi}\frac{\partial}{\partial y}(B_x)$$
$$+ \frac{1}{8\pi}\frac{\partial}{\partial y}(B_\phi{}^2) = 0 \ . \qquad \text{(A9)}$$

APPENDIX III

In this section, equation (17) of the main text is derived. First, equation (14) is solved as follows:

$$\frac{x}{4\pi} B_m B_\phi = \frac{-\chi(\alpha)}{\Omega} - \frac{\rho}{\Omega} u_o u_m . \qquad \text{(A10)}$$

In Appendix II it was found that the conservation equation for angular momentum could be written as

$$\vec{\nabla}\cdot[\rho u_2 \vec{u}_m - \frac{x}{4\pi} \vec{B}_m B_\phi] = 0.$$

Substitution of equation (A10) into this gives

$$\vec{\nabla}\cdot[\rho u_2 \vec{u}_m + \frac{\rho}{\Omega} u_o \vec{u}_m + \frac{\chi(\alpha)}{\Omega} \vec{B}_m] = 0 ,$$

or

$$\nabla\cdot[\rho u_2 \vec{u}_m + \frac{\rho}{\Omega} u_o \vec{u}_m] = 0$$

which integrates to

$$u_2 + \frac{1}{\Omega} u_o = \text{constant} .$$

This is equivalent to

$$- \frac{c^2\gamma}{\Omega} + x\, u_\phi = \text{constant} ,$$

or

$$\gamma - \frac{\Omega}{c^2} x\, u_\phi = \text{constant} ,$$

and as $x \to 0$, $\gamma \longrightarrow 1$, so that the constant must be identically one:

$$\gamma - \frac{\Omega}{c^2} x\, u_\phi = 1 .$$

An alternative derivation of this equation can be accomplished by multiplying equation (A8) by u_x and equation (A9) by u_y and adding the resulting two equations. The result will be just equation (17), although much effort is required to reduce the equation to this simple form.

APPENDIX IV

In this section, equation (A9) is put into dimensionless coordinates. Consider each term individually:

$$\rho u^1 \frac{\partial}{\partial x}(u_3) + \rho u^3 \frac{\partial}{\partial y}(u_3) = (\rho \vec{u}_m \cdot \vec{\nabla})\, u_3$$

$$= F(f)\left[\frac{\Omega}{c}\vec{b}_m \cdot \frac{\Omega}{c}\vec{\nabla}\right] u_3$$

$$= \frac{\Omega^2}{c^2} F(f) b_\omega \frac{\partial}{\partial \omega}(u_z) + \frac{\Omega^2}{c^2} F(f) b_z \frac{\partial}{\partial z}(u_z).$$

$$-\frac{E_y}{4\pi x}\frac{\partial}{\partial x}(xE_x) = -\frac{1}{4\pi}\frac{\Omega}{c}\frac{1}{\omega}\frac{\Omega}{c} e_z\left[\frac{\Omega}{c}\frac{\partial}{\partial \omega}\left(\frac{\Omega}{c}\omega \frac{c}{\Omega} e_\omega\right)\right]$$

$$= -\frac{1}{4\pi}\frac{\Omega^3}{c^3}\frac{e_z}{\omega}\frac{\partial}{\partial \omega}(\omega e_\omega).$$

$$-\frac{E_y}{4\pi}\frac{\partial}{\partial y}(E_y) = \frac{-1}{4\pi}\frac{\Omega}{c} e_z \frac{\Omega}{c}\frac{\partial}{\partial z}\left(\frac{\Omega}{c} e_z\right)$$

$$= -\frac{1}{4\pi}\frac{\Omega^3}{c^3} e_z \frac{\partial}{\partial z}(e_z).$$

$$-\frac{B_x}{4\pi}\frac{\partial}{\partial x}(B_y) = -\frac{1}{4\pi}\frac{\Omega}{c} b_\omega \frac{\Omega}{c}\frac{\partial}{\partial \omega}\left(\frac{\Omega}{c} b_z\right)$$

$$= -\frac{1}{4\pi}\frac{\Omega^3}{c^3} b_\omega \frac{\partial}{\partial \omega}(b_z).$$

$$+\frac{B_x}{4\pi}\frac{\partial}{\partial y}(B_x) = \frac{1}{4\pi}\frac{\Omega^3}{c^3} b_\omega \frac{\partial}{\partial z}(b_\omega).$$

$$\frac{1}{8\pi}\frac{\partial}{\partial y}(B_\phi 2) = \frac{1}{8\pi}\frac{\Omega^3}{c^3}\frac{\partial}{\partial z}(b_\phi^{\,2}).$$

Then equation (A9) becomes

$$\frac{\Omega^2}{c^2} F(f) b_\omega \frac{\partial}{\partial \omega} (u_z) + \frac{\Omega^2}{c^2} F(f) b_z \frac{\partial}{\partial z} (u_z)$$

$$- \frac{1}{4\pi} \frac{\Omega^3}{c^3} e_z \left[\frac{1}{\omega} \frac{\partial}{\partial \omega} (\omega e_\omega) + \frac{\partial}{\partial z} (e_z)\right]$$

$$- \frac{1}{4\pi} \frac{\Omega^3}{c^3} b_\omega \left[\frac{\partial}{\partial \omega} (b_z) - \frac{\partial}{\partial z} (b_\omega)\right]$$

$$+ \frac{1}{8\pi} \frac{\Omega^3}{c^3} \frac{\partial}{\partial z} (b_\phi{}^2) = 0.$$

Dividing through by $\frac{\Omega^3}{4 \pi c^3}$ and using the definition of $\sigma(f)$, this becomes

$$\frac{1}{c\sigma} b_\omega \frac{\partial}{\partial \omega} (u_z) + \frac{1}{c\sigma} b_z \frac{\partial}{\partial z} (u_z) - \frac{e_z}{\omega} \frac{\partial}{\partial \omega}(\omega e_\omega)$$

$$- e_z \frac{\partial}{\partial z}(e_z) - b_\omega \frac{\partial}{\partial \omega} (b_z) + b_\omega \frac{\partial}{\partial z} (b_\omega)$$

$$+ \frac{1}{2} \frac{\partial}{\partial z} (b_\phi{}^2) = 0 .$$

This is identically equation (61).

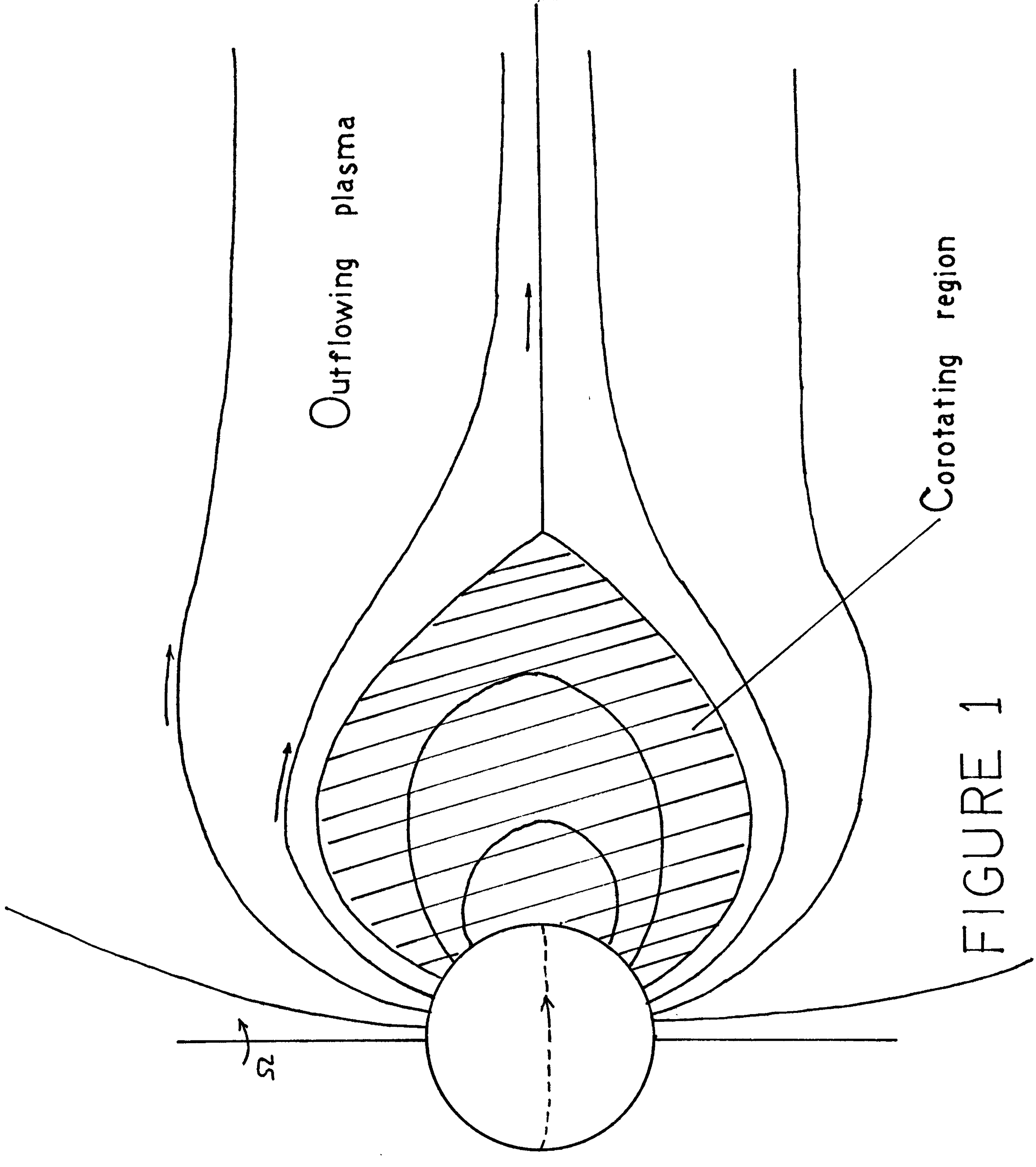

FIGURE 1

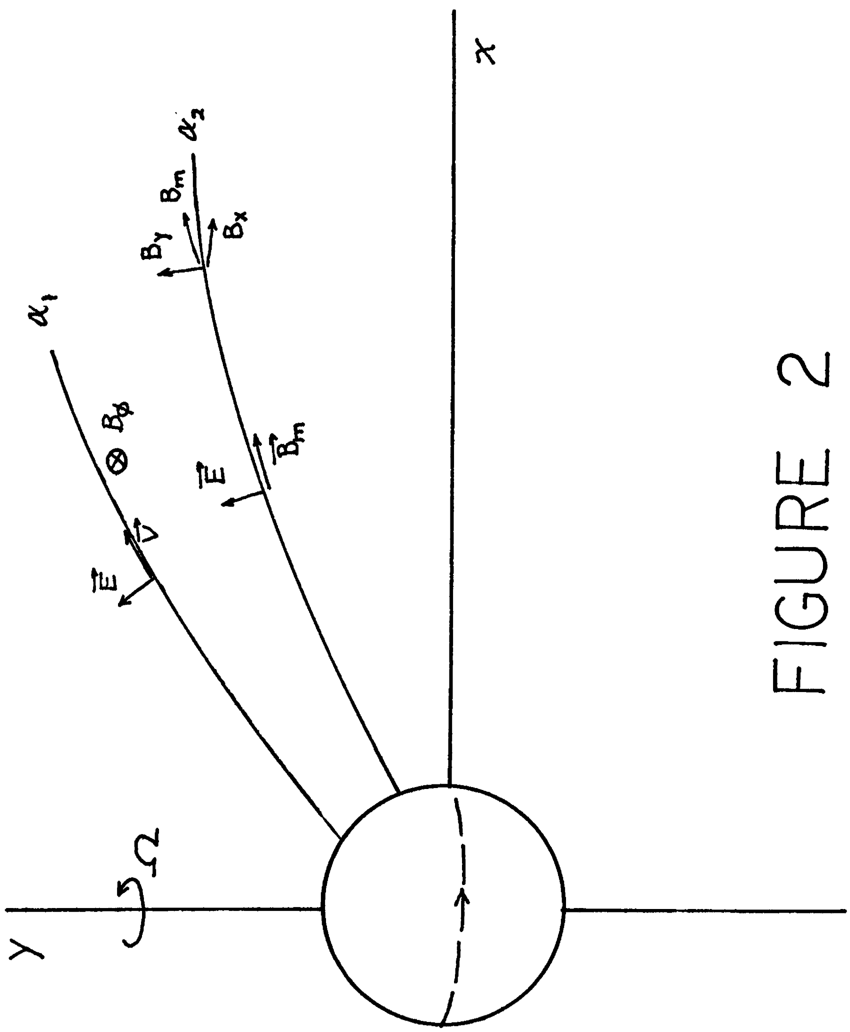

FIGURE 2

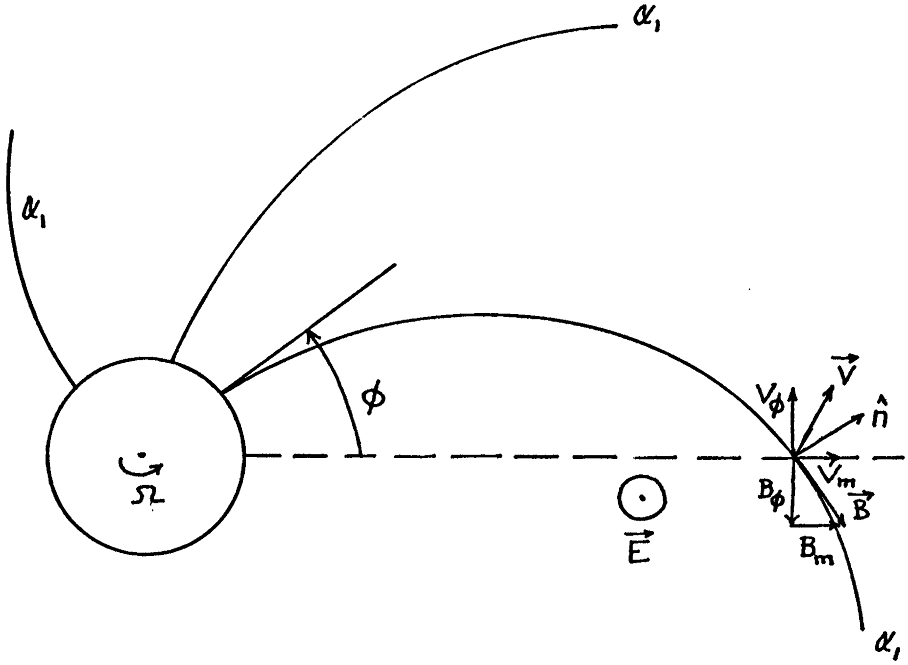

FIGURE 3A

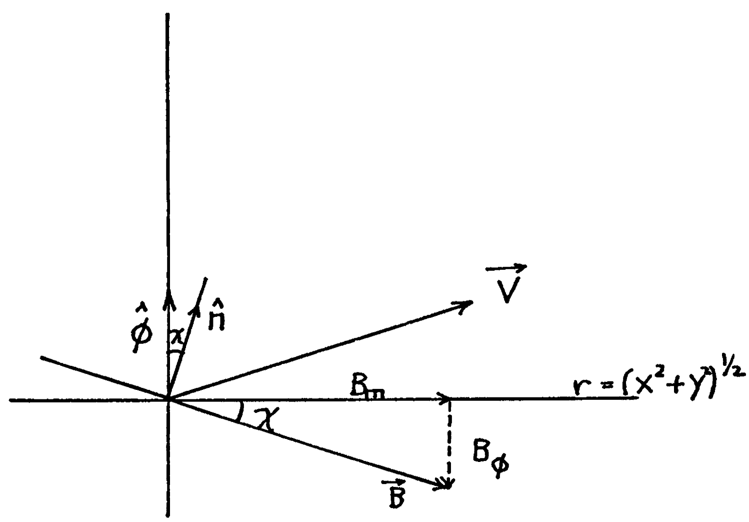

FIGURE 3B

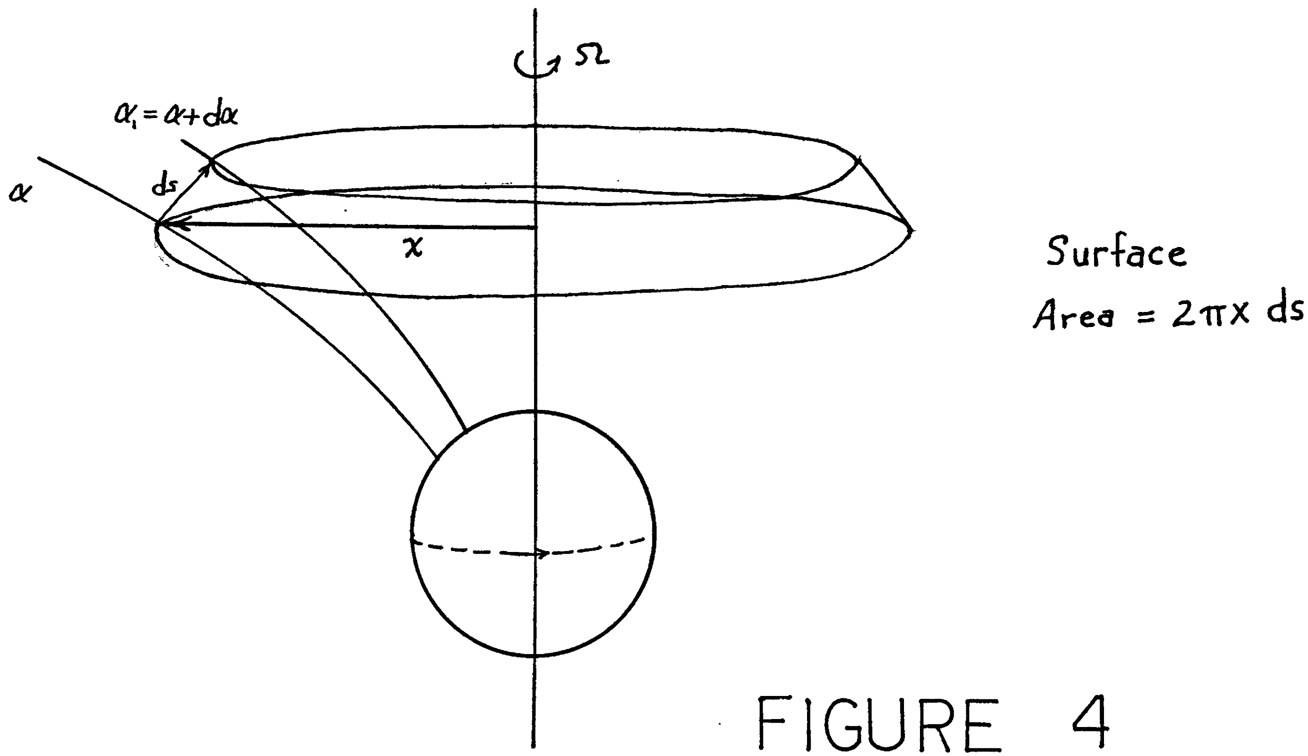

FIGURE 4

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