RICE UNIVERSITY

ANALYTICAL SOLUTIONS TO SELECTED BOUNDARY
VALUE PROBLEMS AND THEIR APPLICATION TO
ROCKY MOUNTAIN FORELAND DEFORMATION

by

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ABSTRACT

ANALYTICAL SOLUTIONS TO SELECTED BOUNDARY VALUE PROBLEMS AND THEIR APPLICATION TO ROCKY MOUNTAIN FORELAND DEFORMATION

GARY DOUGLAS COUPLES

The Airy stress function is used, via the Principle of Superposition and the series summation concept, to obtain stress states in a static, self-gravitating elastic beam subjected to boundary stresses. The boundary conditions investigated are more complicated than those previously published and include cases with sawtooth-, step-, and sinusoidally-shaped lower boundary loads, with and without additional tectonic end loads. Potential shear fracture (fault) patterns derived from the calculated stress fields indicate co-existing (simultaneous) regions of lateral shortening and extension. Application of three of the cases to the study of the structural geometry of the Wind River, Owl Creek, and Beartooth Mountains of Wyoming yields a good "fit". For the case of the upthrust structures, these solutions provide a possible explanation for the observed rotations and zones of shortening and extension.
ACKNOWLEDGEMENTS

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I especially wish to thank my fellow graduate students for many stimulating discussions. Dave Weinberg deserves special credit for helping me develop my ideas about the "upthrust" problem.

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INTRODUCTION

Analytical solutions of the type presented by Hafner (1951) have proven useful as aids in understanding crustal deformations such as the uplifts of Precambrian basement blocks found in the Rocky Mountain foreland (Stearns, 1975). However, models of this type have seen only limited application, possibly because solutions for only a limited range of boundary conditions were available in the literature. New solutions were needed to provide a larger suite of loading conditions to choose from when selecting a model. The present study was undertaken to generate solutions (i.e., internal stress fields and associated features, such as fractures) for a suite of new, more complicated boundary conditions.

Why were new solutions desired? Final geometries or shapes of the uplifts in the foreland lead to the speculation that the "geometries" of the initial loading conditions may have had forms similar to those of the resulting structures. Asymmetric uplifts, like the Wind River Mountains, and plateau-like uplifts, such as the Beartooth Mountains, suggested that there was reason to investigate lower-boundary normal stresses with the shapes of sawtooth or step functions. The proximity of some of the uplifts to the Idaho-Wyoming thrust belt also suggested that horizontally-directed normal stress might be an important part of the imposed boundary conditions.
Therefore, plane strain solutions have been prepared for combinations of end loads and/or sawtooth-, step-, or sinusoidally-shaped normal stresses applied to the lower boundary of an elastic beam. Each stress system calculated in this study includes components related to the standard state stress field due to the effects of gravity. Potential shear fracture trajectories are drawn at ± 30° to the lines of most-compressive principal stress (σ₁ trajectories)\(^1\). The use of the new solutions is illustrated by applying the potential shear fracture (fault) trajectories resulting from three of the cases studied toward the explanation of the Laramide structural development of the Wind River, Owl Creek, and Beartooth Mountains of Wyoming.

The majority of this thesis has been published separately (Couples, 1977).

\(^1\) Compressive stresses are considered negative. Thus, σ₁ is the algebraically-least principal stress.
METHOD

Problems in structural geology have traditionally required reduction to some simplified concept. The general approach, taken by Hafner (1951) and followed here, is to investigate the mechanics of a rectangular beam of arbitrary thickness -- a two-dimensional problem. By appropriate arguments and assumptions, and by proper selection of material properties, etc., the theoretical elastic response of the two-dimensional beam can be extended to represent the equivalent distortions of an appropriate segment of the earth's crust. In the situations considered here, the appropriate segment of the crust is taken to be the upper 15 km of continental basement (see discussion in Stearns, 1971 for the definition of basement). The problems associated with that extension are discussed further below.

The rectangular beam under consideration in this study is depicted in Figure 1A, along with conventions for spatial coordinates. Figure 1B illustrates the sign convention for stresses used throughout this thesis.

Two-dimensional problems in the deformation of continua are either of the plane stress or plane strain type. Because plane stress rarely arises in natural geologic situations, plane strain is assumed, and the Airy stress function is used to solve for the stresses in the beam.

Hafner (1951) outlined the pertinent theoretical aspects and the basic approach of using the Airy stress function in
Figure 1. A. Geometry and dimensions of rectangular beam studied in this thesis. Coordinate system with origin at upper-left corner of beam. Arrow at right labeled "g" indicates gravitational attraction parallel to Z. B. Components of two-dimensional stress field acting in the X - Z plane. Shear and normal stresses considered positive as shown.
geologic deformations. Further details can be found in many
texts on the theory of elasticity (e.g., Jaeger, 1969; Jaeger
and Cook, 1970; Timoshenko and Goodier, 1970). The math-
ematical analysis requires the selection of the Airy stress
function, $\phi(x,z)$, such that

(1) \[ \nabla^2 \left\{ \nabla^2 \phi(x,z) \right\} = \nabla^4 \phi(x,z) = 0. \]

There are infinitely many functions, $\phi$, which satisfy equa-
tion (1), but the "trick" of the analysis is to choose $\phi$
which yields the desired boundary stresses, determined by
evaluating

(2)

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 \phi(x,z)}{\partial z^2} \\
\sigma_{zz} &= \frac{\partial^2 \phi(x,z)}{\partial x^2} \\
\tau_{xz} &= \frac{\partial^2 \phi(x,z)}{\partial x \partial z}
\end{align*}
\]

at the boundary values of $X$ and $Z$. This semi-inverse tech-
nique amounts to starting in the "middle" of the problem
and working toward two ends: the determination of the bound-
ary conditions, and the resulting stress state (Serra, 1973).
Experience with the method indicates which forms of the
Airy stress function are appropriate beginning points for a
particular problem.

Many different forms of the Airy stress function may
satisfy equation (1), but of primary interest to this study
are the polynomial and hyperbolic-trigonometric (sinusoidal)
forms. Specifically, let us consider the choices

\[ \phi(x,z) = Dxz^2 \]  
and  
\[ \phi(x,z) = \sin \alpha x \left( C_1 \cosh az + C_2 \sinh az + C_3 z \cosh az + C_4 z \sinh az \right) \]  

Equation (3) describes a supplementary stress system consisting of a constant load applied to the end of the beam, with static equilibrium being maintained by shear stresses acting on the base and end surfaces. Figure 6 of Hafner (1951) depicts the result of superposing a "standard state" stress system (see below) and the supplementary stress system described by equation (3). Similarly, the stress field due to equation (4) superposed with the "standard state" results in stress and shear fracture trajectories as illustrated in Plates 1A (also shown as Fig. 2 of this thesis), 1B-1D of Hafner (1951).

The terminology "standard state" was introduced by Anderson (1942) who used it as meaning the hydrostatic state, or an isotropic stress state where

\[ \sigma_{\text{vert}} = \sigma_{\text{horiz}} = \rho gz. \]

\[ \rho = \text{density} \]

\[ g = \text{acceleration due to gravity} \]

---

1 A supplementary stress system is one which is the result of a set of imposed boundary loads. It is distinct from a stress system related to the effects of gravity; gravity does not influence the supplementary stress system.
Figure 2. Stress trajectory (a) and potential fault (b) diagrams for variable vertical and shearing stress along the bottom of a block 10 miles thick. Solid heavy lines in (a) are maximum principal stress trajectories. Dashed lines are minimum principal stress trajectories. Light weight solid lines are lines of equal shearing stress. Arrows in (b) indicate sense of shear along potential faults (after Hafner, 1951).
By "standard state", I mean that state of stress extant in a body (in this case, a portion of the crust) in the absence of imposed loads. The "elastic" stresses in a laterally-constrained, self-gravitating body are given by

\[
\begin{align*}
\sigma_{\text{vert}} &= \rho g z \\
\sigma_{\text{horiz}} &= \frac{u}{1 - u} \sigma_{\text{vert}}.
\end{align*}
\]

In such a stress system, \(\sigma_{\text{horiz}} < \sigma_{\text{vert}}\) unless \(u = 0.5\), where \(u\) is Poisson's ratio. Hafner (1951) was aware of this theoretical state of stress (equations (6)) and pointed out that the hydrostatic state (equations (5)) is achieved in the body only by the addition of additional horizontal stress (unless \(u = 0.5\)). Nevertheless, he chose to use the hydrostatic state as his standard state, opting for a definition consistent with that used by Anderson (1942).

Unfortunately, there is no generally accepted state of stress for the "natural" standard state. We must choose to use either a theoretical state (equations (6)) or an assumed state (equations (5)). Heim's rule -- that the natural state of stress tends to become hydrostatic over long periods of time (Jaeger, 1969, p. 172) -- does not resolve the problem. This is because Heim's conclusion was based on the assumption that the crustal rocks would flow, thereby removing the lateral motion constraints and allowing a hydrostatic state of stress to be achieved.

Reference to data published by Handin (1966) and by
Borg and Handin (1966) indicates that "granitic" rocks are able to sustain only a limited amount of permanent deformation before failing when subjected to the maximum temperatures (up to 500° C) and confining pressures (about 5 kb) that would be experienced by rocks at the bottom of a 15 km thick portion of continental crust. This being so, these rocks are considered to be "brittle". It should be noted that the differential stresses necessary to deform the specimens in those experiments was on the order of 5 - 10+ kb. Equations (6) predict about 3 kb maximum differential stress in the standard state stress field. I interpret this information to mean that "granitic" rocks do not flow as assumed in Heim's rule, and that the natural standard state is not necessarily the hydrostatic state.

Inasmuch as the in situ state of stress in the crust prior to Laramide deformation is unknown, I have chosen the stress system given by equations (6) as more likely than the hydrostatic state. Determination of a value to be used for Poisson's ratio was made by reference to Jaeger (1969, p. 58), from which I selected $\nu = 0.25$ as being typical for "granitic" rocks.

One additional consideration was "built into" the standard state as used in this study. I planned to apply the results to the Rocky Mountain foreland, where, prior to Laramide deformation, the basement rocks were overlain by a sedimentary veneer some 2.5 to 4.5 km thick. This depth of burial was simulated by adding a uniform 0.5 kb compressive
stress acting in the Z-direction. With \( u = 0.25 \), and assuming a constant \( \rho = 2.75 \) for the basement rocks, the standard state used in this study is calculated by

\[
\sigma_{xx} = 2.75gz - 0.5 \text{ kb} \\
\sigma_{zz} = \frac{\nu}{1 - \nu} \sigma_{xx} = \frac{\sigma_{xx}}{3} \\
\tau_{xz} = 0.
\]

(7)

In the examples referred to above (Fig. 6 and Plates 1A-1D of Hafner, 1951), the Principle of Superposition was employed. For linearly-elastic materials, this principle is: if \( \psi_1 \) and \( \psi_2 \) are valid static stress fields defined over the same physical region, a linear addition of the two fields (i.e., summing corresponding components of the stress tensor) results in another valid, statically-admissible stress field, \( \psi_3 \), whose boundary conditions are the sum of the boundary conditions that would have caused \( \psi_1 \) and \( \psi_2 \) individually.

All of the solution reported in this thesis contain the standard state stress system described above (equations (7)). The stress fields due to one or more of a variety of boundary loads are superposed onto this standard state, resulting in stress fields related to boundary conditions more complicated than those for which solutions have been published previously. The basic development of the equations necessary to calculate the interior stresses caused by the end-load boundary condition is found in Hafner (1951).

The components of the supplementary stress system due
to an end load are determined by applying equations (2) to equation (3)

\[ \sigma_{xx} = 2Dx \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xz} = -2Dz. \]

Substitution of \((L - x)\) for \(x\) in equation (3) allows the end load to be applied to the opposite end of the beam. Notice that the supplementary stress system described by equations (8) contributes nothing to the vertical normal stress component, \(\sigma_{zz}\).

The equation for the lower-boundary (i.e., \(z = -c\)) supplementary normal stress in the case of a simple, sinusoidally-varying load has the form

\[ \sigma_{zz}(x,-c) = A \sin \alpha x. \]

However, this study employs the Fourier series approach discussed in Timoshenko and Goodier (1970, p. 53-61) to obtain lower-boundary supplementary normal stresses that vary laterally as sawtooth- and step-shaped functions. Briefly summarized, this approach proceeds as follows.

Consider \(p\) different cases of sinusoidal supplementary normal stress, each with its own amplitude, \(A_n\), and wave number, \(\alpha_n\). For each of them we have

\[ \sigma_{zzn}(x,-c) = A_n \sin \alpha_n x. \]

Using the Principle of Superposition, let us sum all the
different stress fields (i.e., $\psi_1 + \psi_2 + \ldots + \psi_p$), each with boundary stresses as in equation (10). Simultaneously, we sum their boundary stresses

\begin{equation}
\sigma_{zz}(x,-c) = \sum_{n=1}^{p} A_n \sin \alpha_n x,
\end{equation}

where $A_n$ is a function of $Z$, being constant at a given depth. Thus, the $\sigma_{zz}$ component of the stress tensor has the form

\begin{equation}
\sigma_{zz}(x,z) = \sum_{n=1}^{p} A_n (z) \sin \alpha_n x.
\end{equation}

Equation (11) forms a Fourier sine series approximation for $\sigma_{zz}(x,-c)$ of some shape. The process described above (equations (9) – (12)) can be repeated for the case where $\cos \alpha x$ is substituted for $\sin \alpha x$ in equation (4) so that there are series in $\cos \alpha_n x$ and $\sin \alpha_n x$ available for approximation. This allows $\sigma_{zz}(x,-c)$ to take any shape, or, any $\sigma_{zz}(x,-c)$ may be approximated, as long as the shape function is "balanced" so that it requires no constant term in its approximation.

The other stress components ($\sigma_{xx}$, $\tau_{xz}$) can be analysed in similar fashion, generating expressions for their boundary and interior values. Other Airy stress functions with different forms could be used to generate a suite of "base elements" for all the stress components. Thus, any statically-admissible set of supplementary boundary loads could be approximated by a sufficiently complicated scheme of super-

position using the "base" boundary conditions, with a consequent ability to determine the interior stress state of the body. Therefore, any statically-loaded beam which meets the formal mathematical-mechanical assumptions of the analysis can be modeled for interior stress states by using a suite of Airy stress functions and the Principle of Superposition, although this approach may not always prove to be practical.

These formal assumptions can be stated in an hierarchical sequence. The body is (1) a continuum (2) in static equilibrium. The body (3) behaves in a linear elastic fashion and is homogeneous and isotropic. The geometry of the body is that of (4) a rectangular beam of arbitrary thickness. Elastic distortions of the beam occur under a condition of (5) plane strain. Assumptions made in extending the results to explain natural deformations do not affect the theoretical development and are treated in the Discussion below.

The discussion above has been general, aimed only at understanding the concept of series approximation. The equations below (see also equations (16) through (23) of Hafner (1951)) are sufficient for calculating the total stress field due to those lower-boundary supplementary normal stresses that can be approximated by sine series. The stress components of each term in the series are given by
\[
\sigma_{xx_n}(x,z) = \sin \alpha_n x (-K_1 F_1 + K_2 F_2)
\]
(13)

\[
\sigma_{zz_n}(x,z) = \sin \alpha_n x (-K_1 F_3 - K_2 F_4)
\]

\[
\tau_{xz_n}(x,z) = \cos \alpha_n x (K_1 F_4 - K_2 F_1),
\]

with

\[
F_1(z) = \sinh \alpha_n z + \alpha_n z \cosh \alpha_n z
\]

\[
F_2(z) = 2 \cosh \alpha_n z + \alpha_n z \sinh \alpha_n z
\]

\[
F_3(z) = \sinh \alpha_n z - \alpha_n z \cosh \alpha_n z
\]

\[
F_4(z) = \alpha_n z \sinh \alpha_n z,
\]

and

\[
K_1 = \frac{A_n \alpha c \cosh \alpha_n c + A_n \sinh \alpha_n c}{\sinh^2 \alpha_n c - \alpha_n^2 c^2}
\]

(15)

\[
K_2 = \frac{A_n \alpha c \sinh \alpha_n c}{\sinh^2 \alpha_n c - \alpha_n^2 c^2}.
\]

The constant \(c\) is the depth of the beam, and \(\ell\) is its length (refer to Fig. 1A). \(\alpha_n\) is determined by

\[
\alpha_n = \frac{n\pi}{\ell}
\]

(16)

The \(A_n\) terms of the series (in equations (15)) allow for the necessary variation to calculate stress systems for various shapes of lower-boundary normal stress. For a simple sawtooth wave (see Fig. 3A) we find

\[
f(x) = \frac{2W}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{\ell},
\]

(17)
Figure 3. A. Sawtooth wave of period $\ell$ and amplitude $\pm W$.
B. Step wave of period $\ell$ and amplitude $\pm W$. 
and for a step function (Fig. 3B)

\[ f(x) = \frac{4W}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{\lambda} \]

(18)

Using these relationships, the following expressions are derived for the sawtooth load

\[ A_n = 2A \frac{(-1)^{n+1}}{n\pi} \]

(19)

and for the step load

\[ A_n = \frac{4A}{n\pi} \] \hspace{1cm} (n=1,3,5,...).

(20)

For the sinusoid, there is only one term

\[ A_1 = A. \]

(21)

In these expressions, \( A \) is the amplitude of the wave being approximated. In the case of the step and the sawtooth, it is measured at the discontinuity -- when \( A = 1.0 \) kb, the \( \sigma_{zz} \) value jumps from 1.0 kb below standard state to 1.0 kb above standard state at the discontinuity point.

In order to obtain smooth shapes for the lower-boundary normal stresses, I made preliminary calculations which showed that a 15 term series gave the smoothest sawtooth before the appearance of a Gibb's spike.\(^1\) A 10 term series gave an acceptably smooth step function with minimal spike.

\(^1\) A Gibb's spike is a numerical phenomenon that is observed when approximating a function that has a discontinuity point. It becomes manifest as more terms are added to the series approximation.
In this study, the state of stress is calculated at grid points distributed across an \( X - Z \) plane of the beam. The grid is rectangular, with grid points spaced at 2.0 km horizontally and 1.5 km vertically. Thus, there are 51 grid points in the \( X \)-direction and 11 in the \( Z \)-direction for a total of 561. The components of the standard state stress field (equations (7)) are calculated at each grid point, and equations (8) (with possible variations) and/or equations (13) - (15) are then added to the respective stress components at each grid point.

From the stress components at each point,
\[
\sigma_{xx} \quad \tau_{xz} \\
\tau_{xz} \quad \sigma_{zz}
\]
the principal stresses and their orientations are calculated by the Mohr circle method. Potential shear fracture trajectories (geologically representative of fault trajectories) are drawn at a characteristic angle, \( \theta \), relative to the local direction of the most-compressive principal stress, \( \sigma_1 \). For the purposes of this thesis, I choose as representative a value of \( \phi \) (angle of internal friction) of 30° (Handin, 1966), resulting in \( \theta = 45° - \frac{\phi}{2} = 30° \).

Diagrams are drawn to depict principal stress trajectories in the beam; this is a standard means of presentation. In this study, I use a Calcomp 1136 plotter to draw tangent lines to each of the principal stresses and each of the shear fracture trajectories at each grid point. Families
of curves are drawn by hand on the machine output using the
tangent-line information. The curves are drawn with a
spacing designed to aid visual clarity, but the spacing
shown here has no exact physical significance. This
discussion of technique is necessary only to point out
the possibility for subjective bias and human error in
the drafting process.

Appendix A briefly discusses the organization of the
computations involved in the study. Also included there
is a description of the steps involved in plotting the
results.
RESULTS

Table 1 summarizes the suite of boundary conditions investigated in this thesis. For each case calculated, principal stress trajectories and potential shear fracture trajectories have been drawn. The location of each set of stress trajectory/shear fracture trajectory diagrams is indicated in Table 1.

In Figures 4 through 12, the boundary stresses are illustrated on the stress trajectory diagram in each case. All boundary normal stresses are compressive with their magnitudes depicted by scaled arrows directed toward the beam. The normal stresses on the base of each beam consist of a standard state component and a supplementary component. The sawtooth or other shape can be visualized as having been added to a constant. Thus the positive and negative variation of the supplementary stress becomes a variation of magnitude, but not of sign, when superposed with the standard state.

Shear stresses acting on the boundaries of the beam are portrayed by half-arrows to indicate sense of shear. The half-arrows are drawn to the same scale as the normal-stress arrows. No shear stresses are present on the top of any of the beams. Shear stresses are always present on the ends of the beams -- hence these planes are not principal planes. Shear stresses are present along the bottom of the beam only when there is an applied end load. The absence of
Table 1

Summary of Imposed Boundary Conditions

<table>
<thead>
<tr>
<th>shape of lower-boundary normal stress</th>
<th>end loads</th>
<th>location of stress and fracture trajectory diagrams</th>
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<tbody>
<tr>
<td>sawtooth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4 kb</td>
<td>left</td>
<td>0.0 kb 0.0 kb</td>
</tr>
<tr>
<td>1.0</td>
<td>right</td>
<td>0.0 0.0</td>
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<td>right</td>
<td>2.0 1.0</td>
</tr>
<tr>
<td>step</td>
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<td></td>
</tr>
<tr>
<td>0.4 kb</td>
<td>left</td>
<td>0.0 kb 0.0 kb</td>
</tr>
<tr>
<td>1.0</td>
<td>right</td>
<td>0.0 0.0</td>
</tr>
<tr>
<td>sinusoid</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0 kb</td>
<td>left</td>
<td>0.0 kb 1.0 kb</td>
</tr>
</tbody>
</table>

1 All imposed boundary conditions in addition to the standard state as described in text.

2 End loads necessitate basal shear; 0.45 kb for 3.0 kb end load, and 0.15 kb for 1.0 kb end load.

3 Net result of 1.0 kb uniform horizontal compression and 1.0 kb left end load with associated 0.15 kb basal shear.
shear stresses on the base of the beam is a function of the loading system selected — i.e., the absence is a designed absence. Other loading systems could, and possibly should, be calculated to investigate the effects of variable shear stresses distributed on the base of the beam.

The shear fracture trajectory diagram in each Figure is, in one sense, easily explained. The two families of lines (for right-lateral and left-lateral senses of shear) are nothing more than orientations drawn at ± 30° to the \( \sigma_1 \) trajectory. On a phenomenological basis, these lines do not represent the prediction of specific shear fractures, either in location or orientation. They represent a field of trajectories of orientations for those fractures which may possibly form. As described here, no knowledge is implied about the specific location of an individual fracture that forms as a result of the imposed loads.

This disclaimer is not to be taken in reverse: that the shear fractures which do form will not have orientations like the drawn trajectories. Arguments can be advanced (see Discussion) which "elevate" the trajectories into a status of near-prediction. For the purpose of describing the diagrams containing the results of this thesis, let us assume that the shear fracture trajectories as drawn represent possible orientations for shear fractures related to the imposed loads.

Figures 4 and 5 form a pair. Both of these cases have a sawtooth-shaped lower-boundary supplementary normal stress.
LOAD SUMMARY
1. LOWER BOUNDARY - SAWTOOTH WITH ±0.4 kb DEVIATION
2. NO TECTONIC END LOAD IN ADDITION TO STANDARD STATE

BLOCK DIMENSIONS
100 km x 15 km.
LOAD SUMMARY
1. LOWER BOUNDARY - SAWTOOTH WITH ± 1.0 kb DEVIATION
2. NO TECTONIC END LOAD IN ADDITION TO STANDARD STATE

Figure 5

BLOCK DIMENSIONS
100 km x 15 km.
Both have the same standard state components. The difference in loading conditions is in the magnitude of the sawtooth. Figures 4 and 5 are examples of ±0.4 kb and ±1.0 kb differential loads, respectively. Notice that the stress fields induced by the loads are quite similar in terms of the orientation of the principal stresses. The main difference is that the increased differential load in Figure 5 causes a singular point (i.e., where principal stresses undergo a 90° rotation, or a point of zero differential stress) to "move up" into the beam on the right side. The fracture trajectories likewise necessarily exhibit great similarity. "Shallow" fractures (near the top of the beam) imply lateral shortening (in the X-direction) on the left side and lateral extension on the right side of both beams. "Deep" fractures indicate lateral expansion on the left side in both cases, but opposite senses of gross displacement on the right side. A very important feature to note is that a single mechanical system (i.e., the loaded beam) indicates or requires simultaneous lateral shortening and lateral expansion.

Figures 6 and 7 form another pair. This time the supplementary normal stress on the lower boundary has the form of a step. Differential loads are ±0.4 kb and ±1.0 kb, respectively. Similar comments are in order with respect to singular points and regions of shortening and extension. Notice that the main difference between the pairs (Figs. 4 and 5 and Figs. 6 and 7) is a smoother curvature of trajec-
LOAD SUMMARY
1. LOWER BOUNDARY - STEP WITH ±0.4 kb DEVIATION
2. NO TECTONIC END LOAD IN ADDITION TO STANDARD STATE

Figure 6

BLOCK DIMENSIONS
100 km x 15 km.
tories in the case of a step-shaped loading condition. This difference is most apparent when comparing the stress trajectories of Figures 5 and 7.

The four cases discussed thus far have had no imposed end loads. Because of this, they are periodic in the X-direction. One end of each diagram matches the other. However, the addition of end loads removes that symmetry. Figure 8 has a sinusoidal lower-boundary supplementary normal stress with a difference of ±1.0 kb; it also has an end load of 1.0 kb imposed on the right. Notice that the normal stress magnitudes drawn on the end of the beam are separated into: (1) a part due to the imposed end load, and (2) a part related to the standard state. All of the diagrams which follow are consistent in this way. In Figure 8, the relative "smoothness" of the loading stress is reflected in the smoother curvature of the trajectories in the beam; there are certainly fewer "shoulders" (or regions of high curvature) present along the stress trajectory lines of Figure 8 than in either Figure 5 or Figure 7.

Figures 9 and 10 form another pair. Here, the difference is the magnitude of the end load, imposed on the right. In Figure 9, the end load is 1.0 kb with 0.15 kb shear stress on the base of the beam. The end load is 3.0 kb in Figure 10, and the shear stress on the base is also higher at 0.45 kb. The appearance of Figure 9 is very similar to Figure 4; the differences in trajectories are close to the right end of the beam where the end load has its greatest
LOAD SUMMARY
1. LOWER BOUNDARY - SINUSOID WITH ±1.0 kb DEVIATION
2. 1.0 kb TECTONIC END LOAD IN ADDITION TO STANDARD STATE,
   WITH 0.15 kb SHEAR ALONG BASE

BLOCK DIMENSIONS
100 km x 15 km

Figure 8
effect (Recall from equations (8) that $\sigma_{xx}$ decreases linearly away from the end load). However, Figure 10 is not similar to Figure 4 at all. The end load has caused the region of shortening on the left side to increase in area (the singular point has "moved" down). The right side of the beam is totally different; in Figure 10 it is a broad region containing fractures that would lead to lateral shortening. The top of the beam indicates all shortening, except for a small extension region about the singular point in right-center. Note an increase in the number of singular points as the complexity of the loading system increases.

Figure 11 is included to demonstrate the effects of a left end load on a $\pm 1.0$ kb sawtooth. Notice that the upper-left region of lateral shortening is much increased in size relative to Figure 5. Also note the singular point developed in the lower-left corner. Figure 12 is another case illustrating the effects of end loads on the trajectories. The loading combination of 1.0 kb uniform horizontal compression, 1.0 kb left end load, and $\pm 1.0$ kb sawtooth leads to trajectories that are quite similar to those in Figure 11. The lower-left singular point of Figure 11 is not observed in Figure 12, but a singular point is fully developed on the right side of the beam.

It is significant to note that, by-and-large, the imposition of horizontal loads does not greatly alter the potential shear fracture trajectories, particularly in the
LOAD SUMMARY
1. LOWER BOUNDARY—SAWTOOTH WITH ±1.0 kb DEVIATION
2. 1.0 kb TECTONIC END LOAD IN ADDITION TO STANDARD STATE, WITH 0.15 kb SHEAR ALONG BASE
3. 1.0 kb UNIFORM HORIZONTAL COMPRESSION

BLOCK DIMENSIONS
100 km x 15 km.

Figure 12
region immediately above the "jump" in the lower-boundary normal stress where deformation might be expected to be most likely to occur. Figure 10 is the only case investigated in which the potential fault trajectories in a substantial portion of the beam show primarily horizontal motions. However, the magnitudes of the loads depicted in Figure 10 seem quite "disproportionate" when compared to the load. "ratios" in the other cases investigated.
DISCUSSION

The impetus for generating this larger suite of stress and shear fracture trajectory diagrams arose from a longer-term study of Rocky Mountain foreland deformation. However, the solutions are certainly "independent" of the Rockies and can be used in studies of similar deformations elsewhere.

One of the more important aspects involved in solving boundary-value problems is the selection of a solution technique. For problems in statics, analytic methods are easiest to implement. When specifying boundary conditions in terms of stresses (tractions), the Airy stress function is used. Mixed boundary conditions (tractions and displacements) can be handled with equivalent mathematics (Sanford, 1959; Howard, 1966). Numerical methods can also be used to generate the desired solutions, but their use for static problems amounts to "overkill", as well as being somewhat more expensive\(^1\) (J.-Cl. DeBremaecker, personal communication, 1976).

A feature of the stress and fracture trajectory diagrams worth noting can be seen by comparing Figures 4 and 5. Although the loading conditions in these two cases differ only in the magnitude of the sawtooth, the fracture trajectories are not coincident. Hafner (1951) attempted to

\(^{1}\) At the current rate for computer time ($360./hr) on an IBM 370/155, each case costs only $5. to compute using the Airy stress function. Plotting costs average about $12. (based on $36./wall hour) when plotted on a Calcomp 1136.
present the results of such a sequence of loading conditions by drawing only one diagram and indicating regions of "stability" for each load increment. By inference, each loading system in the sequence resulted in identical fracture trajectories. The differences in trajectories evidenced by Figures 4 and 5 rule out that data-presentation shortcut used by Hafner.

In general terms, the shear fracture trajectories assume rather unusual patterns in the regions around singular points. The trajectories tend to splay around or into these areas developing a vague resemblance to feathers. That these patterns seem unrealistic is not surprising. The singular points are locations of zero differential stress, and their immediate surroundings have low stress differences. Thus, it is quite unlikely that fractures would initiate in these regions. Hafner (1951) did not attempt to depict fracture trajectories in these regions of low stress difference, but they are shown here for: (1) completeness, and (2) because the problem of fracture propagation, especially into or through such regions, is very poorly understood. Drawing these trajectories may provide information that may aid in advancing that understanding.

The formal assumptions listed above (p. 13) are quite restrictive. *Sensu stricto* they limit the analysis to loading conditions below a "failure load". In other words,
the formation of the first crack strictly violates the assumption of continuum, at least in the immediate vicinity of the crack. It is possible to circumvent this problem by defining continuum to include cracks -- or, that in bulk, the material behaves as a continuum. Then, the question affecting this study becomes: does the presence or formation of a fracture alter the stress field such that the calculated fracture trajectories are invalid? Or, restated, do the actually-formed fractures propagate along the trajectories as initially calculated, or along other paths?

The means of answering that question calls upon empiricism -- the question cannot be answered on a theoretical basis at this time. The approach appeals to the degree of ordering found among natural fractures. Stearns has presented field evidence pertinent to this point. For the Bonita Fault near Tucumcari, New Mexico, Stearns (1972) has shown that there is a high degree of ordering for macroscopic shear fractures both parallel and conjugate to the normal fault. He has also shown that fractures associated with folded rocks exhibit remarkable degrees of ordering (Stearns, 1968).

It is very difficult to conceive that these fractures (those in ordered sets) all formed simultaneously; they must have developed sequentially. Thus for real rocks, the ordered orientations of fractures indicate that the formation of one crack, and even the formation of many fractures, does not appreciably alter the orientation of the causative stresses, otherwise, the ordering would not be obtained.
As a corollary, it seems reasonable to accept that fracture orientations predicted from theoretical solutions such as those presented here are not invalidated by the limiting assumptions encountered in the formal analysis.

The ultimate test of results such as those presented here will not come from theoretical considerations. To be sure, advances in the understanding of theoretical mechanics and modeling techniques will be incorporated by future workers and will lead to more sophisticated solutions. But the test of the results must again come from empiricism -- from the usefulness of the solutions. If theoretical solutions aid in the understanding of geologic problems, if they help explain a possible origin for a previously enigmatic structure, then the solutions attain value and a sense of validity. For systems as complex as those of the Rocky Mountain foreland, the solutions will never be "correct"; they can only be improved.

Stearns (1975) used one of Hafner's (1951) solutions to explain a possible mode of origin for the structures observed in a cross-section drawn across the northern Bighorn Basin of Wyoming (Fig. 13). In this region, a "rigid" basement was overlain by a "passive" sedimentary veneer prior to Laramide deformation. The observed (and interpreted) final motions of the rotated and differentially uplifted and downdropped basement blocks can be explained as resulting from the rotation of a sequence of rigid blocks along curved faults. The locations, curvatures, and senses
of shear of the faults is remarkably well matched by Hafner's solution.

However, the structural style in the more southwesterly portions of Wyoming is not as well characterized by high-angle faults as is the northern region studied by Stearns (1971; 1975). In this southern part of the state, there are many more low-angle faults and basement "overhangs". This geometry has variously been termed "upthrust". Attempts to explain this geometry via the solutions of Hafner (1951), Sanford (1959), and Howard (1966) have seen only limited success. The curving reverse fault, and, in some cases, even the normal-fault zone, have been either predicted in solutions or modeled in sandbox analogs. But the combination of proper rotations of the blocks, and proper locations and geometries of faults has not previously been explained.

Figure 14A depicts a typical upthrust structure -- the Wind River Mountains of Wyoming. The cross-section as shown is very schematic. Surface and near-surface attitudes and geometries are honored as far as they are known. The configuration shown at depth for the faults is purely speculative and merely represents my concept of one possible geometry. Note that the correct rotations are produced with no "room" problems, as are the regions of shortening and extension.

Figure 14B is a simplified version of Figure 11 with only two faults drawn. Suppose that these two faults form
Figure 14. A. Cross-section of Wind River Mountains. B. Best-fit theoretical solution.
in the beam and that movement occurs along them with the indicated senses of shear. The resulting geometry would be very similar to the geometry of the "real" world shown directly above.

Figure 15A is a simplified cross-section of the Owl Creek Mountains of Wyoming. Again, the section is very schematic, although greater portions of this geometry can be seen in the field. Note particularly the rotations of the basement blocks, the upthrust fault, and the zone of extension.

In a similar fashion, Figure 15B is a simplified version of Figure 12. Only two of the potential faults are illustrated. If movement occurs on the near-circular faults, a geometry nearly identical to that of Figure 15A results.

A final model is illustrated in Figure 16A. This is a cross-section drawn across the eastern flank of the Beartooth Mountains at the Montana-Wyoming border. The lower portions of the section are speculative, but the upper reaches are better documented from outcrops along the flank of the uplift. The flat surface of the uplift is particularly noticeable, as are the splinter blocks and the sloping basin block.

Figure 16B is the best-fit solution for this structure. It is simplified from Figure 7. Only one potential fault is shown in this case; it is slightly concave to the left. If this fault undergoes motion, a "room problem" ensues.
Figure 15. A. Cross-section of Owl Creek Mountains. B. Best-fit theoretical solution.
Figure 16. A. Cross-section of Beartooth Mountains. B. Best-fit theoretical solution.
Gravitational forces would tend to eliminate that problem near the top of the beam by breaking off the small piece left hanging at the corner. The splinter blocks are expected to result from this alteration of the fault geometry.
CONCLUSIONS

An inexpensive and relatively simple technique has been described which allows for the calculation of stress states in a self-gravitating elastic beam subjected to a complicated set of boundary stresses. Procedures for presenting the stress field information and for deriving the associated shear fracture trajectory pattern have also been outlined.

The conclusions of this thesis can be summarized as follows:

1. Loads with similar geometry but of different magnitude result in non-identical stress trajectories, and, consequently, in different shear fracture trajectory patterns.

2. Potential fault patterns for "vertical uplift" are not greatly altered even by imposing large lateral loads.

3. Arguments derived from the ordering observed for natural fractures indicate that the unsolved problem of fracture initiation and propagation may not be critical in limiting the usefulness of static solutions such as those presented here.

4. Previously enigmatic structures like the Wind River, Owl Creek, and Beartooth Mountains of the Rocky Mountain foreland can be more nearly explained by loading systems with geometries similar to those of the resulting
uplifts. The important features, such as the rotations, splinter blocks, and fault zones, are "predicted" by the solutions. Regardless of whether or not the solutions used here "exactly" explain the precise details of the structures, the hypothetical geometries suggested by the solutions "explains" a greater portion of the observed natural relationships of the foreland than did previous attempts.
APPENDIX A

Serra (1973) presented a computer program for the calculation and plotting of stress trajectories and their associated fracture trajectories based on the previous work of Hafner (1951). His program provided a very helpful guide in organizing my calculations, but certain changes were necessary. Serra's program calculated the total stress components at each grid point and immediately plotted a trajectory symbol. In the method described in this thesis, the stress components are calculated as series sums. This necessitates storing the values in arrays in the memory of the computer. After the total stress field is calculated, the arrays are carried into a separate Mohr circle and plotting routine.

Changes were also necessary in the plotting steps. Serra plotted both principal stress trajectory tangents at the same time. I found it much easier to plot each one separately, draw in the trajectory lines, and then superimpose them into one diagram. The benefits of following this procedure become even more apparent when drawing the shear fracture trajectories which are non-orthogonal. One other modification was made in Serra's program. I plotted symbols from left to right on one pass and from right to left on the return. This was done to minimize costly free-travel of the plotter.
REFERENCES


