RICE UNIVERSITY

COMPUTER SOLUTIONS OF MAGNETOPAUSE SHAPES

by

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ABSTRACT

Computer solutions are presented for the magnetopause shape of a configuration with axial symmetry.

Surface currents flowing on a magnetopause cancel the magnetic field outside the magnetopause. The general expression for the magnetic vector potential of the surface currents may be expanded via Legendre polynomials. Michel (1977) has shown that the summed variable (h) of the generating function for the Legendre polynomials may be introduced into the vector potential equation. When the correct expression for the shape of a magnetopause is inserted into the integrand, the integral is independent of h. A computer program has been developed to check this independence for various parameterized solutions for a magnetopause and thus determine an approximate solution.

The equations have been developed for the case of a solar wind interacting with a dipole magnetic field. The magnetic axis is aligned with the solar wind flow, so this problem has azimuthal symmetry. This configuration is called the Uranus problem. If Uranus had an intrinsic dipole magnetic field with axis aligned with its rotation

axis, one pole would point approximately into the solar wind flow every forty two years.

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INTRODUCTION

The selection of physical laws to approximate reality depends on the desired level of explanation. Cosmology models the universe as isotropic with evenly distributed matter. At the opposite end of the size scale, phenomena may be described by quantum mechanics and relativity. On this level the vacuum and its self-contortions form the relevant concepts. The elementary particles, comprising the matter which dominated the cosmological view, are first order perturbations on this vacuum. Between these two extremes, plasma physics encompasses both a discrete and a continuum description of nature.

When magnetic fields and particles interact, the properties of both are altered. The Maxwell equations indicate that a magnetic field changes the path of a moving charge, producing a different magnetic field, which again changes the particle's motion. This cause and effect analysis resembles the famous paradox of Zeno; it seems that each successive iteration only reaches half of the remaining distance to a solution. Fortunately, physics, like the lion which disproved the paradox by walking the full distance from the zoo to the classroom and devouring Zeno, has an infinite number of instances to adjust itself and consequently does so in a finite time. Some configurations reach an equilibrium between the magnetic field and the plasma. Once this equilibrium has been attained, a small perturbation could be described by restarting the iterations. This is a natural method to solve some problems, such as describing the electric and magnetic fields in a capacitor.[1] When the iterations converge easily, one obtains both an answer and reassurance that the universe may be described by nested cause and effect. Descriptions of the microstructure of the magnetopause do not always yield stable, selfconsistent solutions with this method.[2]

The problem of magnetopause stability may be analyzed in another framework. Consider a region of moving charged particles and a region containing a magnetic field. The plasma and fields may rearrange each other so that on the macroscopic scale two regions form and their interface is in equilibrium. The magnetopause is defined as the boundary between these regions. One region contains a net magnetic field which exactly cancels the unperturbed field in the region outside the magnetopause.

URANUS PROBLEM

Consider the solar wind flowing around an object with a dipole magnetic field. The magnetic axis is oriented parallel to the flow direction. See Fig. 1.

Equilibrium requires pressure balance along the magnetopause. The magnetic pressure just inside the magnetopause must balance the normal component of the ram pressure of the solar wind.

(1) P_{mag}=magnetic pressure

(2) P_g =gas pressure

(3) $P_{mag} = P_g$

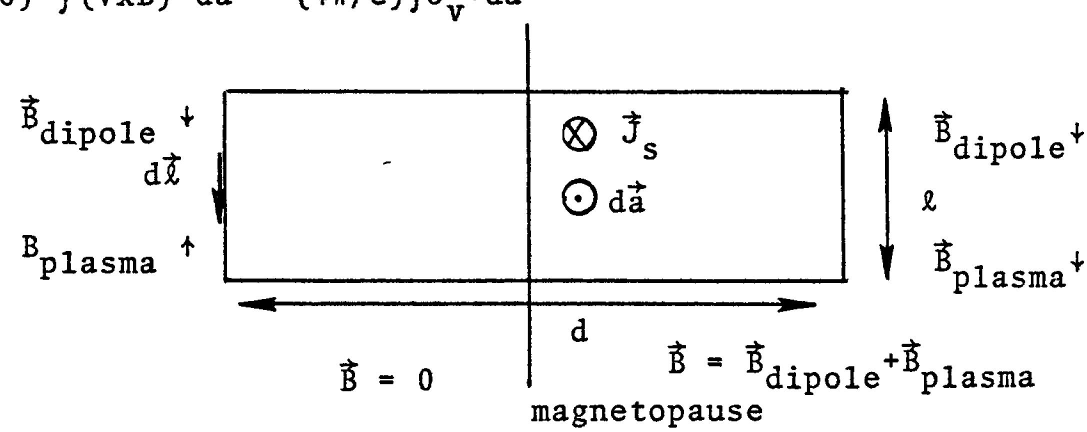
(4) $P_{mag} = B^2/(8\pi)$

One of the Maxwell equations expresses B in terms of J_v, the volume current density. The units of J_v are statamps/cm².

(5) $\nabla \times \vec{B} = (4\pi/c)\vec{J}_v$ in gaussian units.

See also appendix A.

(6) $\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = (4\pi/c)\int\vec{J}_v\cdot d\vec{a}$



The left hand side may be converted by Stoke's theorem.

(7) $\int(\nabla\times\vec{B})\cdot d\vec{a} = \int\vec{B}\cdot d\vec{\ell}=-B\ell$

Assume that all relevant plasma currents flow only on the magnetopause, which is sufficiently thin to consider a delta function an adequate representation for J_V. In the limit that the magnetopause goes to a delta function, the current thins such that $J_V d$ remains constant.

(8) $(4\pi/c)\int\vec{J}_V\cdot d\vec{a} = -(J_V d)\ell(4\pi/c)$

Define $J_V d = J_s$ under convention used in Midgley-Davis (1962).

(9) $B = (4\pi/c)J_s$

J_s has units of statamps/cm.

Consider that the particles have zero temperature. Pressure balance occurs when the ram pressure of the solar wind balances the pressure of the magnetic field of eq.(9) inside the magnetopause.

The normal flux of particles incident upon the magnetopause is $nv_0\cos\psi$. The change of momentum for each ion when reflected from the magnetopause is $2mv_0\cos\psi$. So for specular reflection of the particles, the ram pressure P_g equals[3]

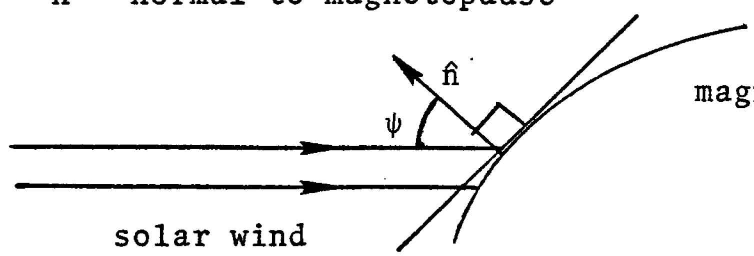
(10) $P_g = 2n(m_e + m_i)v_0^2\cos^2\psi$.

n = number density

v_0 = bulk velocity of the solar wind

m_i = mass of ion

m_e = mass of electrons



Let $P_g = P_0 \cos^2\psi$.

(11) $P_0 = 2n(m_i + m_e)v_0^2$

Using the pressure balance condition $P_g = P_{mag}$ gives

(12) $P_0 \cos^2\psi = B^2/(8\pi) = (2\pi/c^2)J_s^2$

(13) $J_s = (P_0\, c^2/(2\pi))^{1/2} \cos\psi = J_0 \cos\psi$

(14) $J_0 = (P_0\, c^2/(2\pi))^{1/2} = (n(m_i + m_e)v_0^2 c^2/\pi)^{1/2}$

As explained in Michel (1977) the correct solution for the shape of the magnetopause will give the magnetopause a magnetic moment equal but opposite to that of the intrinsic source. For a dipole magnetic field, only $I_1 = I_n$ (given by A-18) is nonzero.

Rewriting (A-18) with $x = \cos\theta$ gives

(15) $I_n = \int_{-1}^{1} J_s(\theta)\; r^{n+1}\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{1/2} P_n^{1}(x)\,dx$

(eq.1 Michel(1977))

I_1 may be expressed in terms of R_g, obtained from the generating function for the Legendre polynomials.

(16) $I_1 = I(h) = -\int_{-1}^{1} J_s(\theta)\, r^2\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{1/2} \sin\theta\, R_g^{-3/2}\,dx$

(17) $R_g = 1 - 2xrh + r^2h^2$

Since the present problem will involve symmetry in $\theta \to -\theta$, but not about $\theta = \pi/2$, it is more convenient to change the coordinate system to (ρ, Z) rectangular cordinates.

(18) $d\ell = (R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2)^{1/2} d\theta$

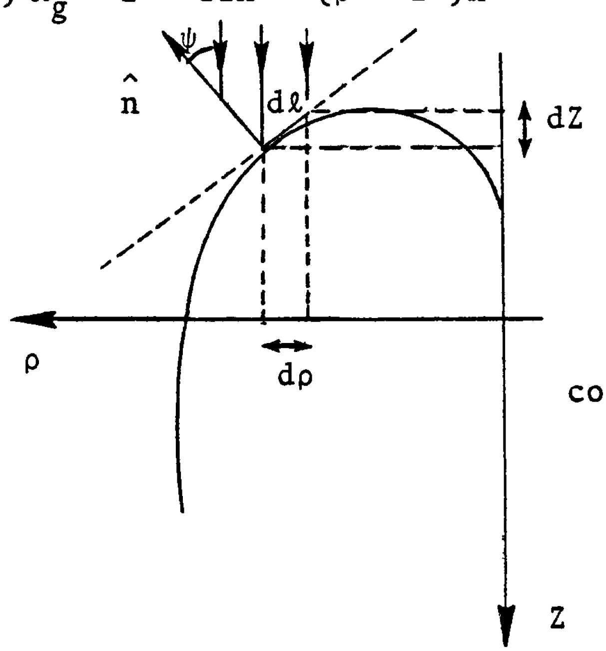
(19) $I(h) = I_1 = -\int_{\pi}^{0} J_s(\theta) \frac{\rho^2}{\sin^2\theta} \frac{d\ell}{d\theta} \sin\theta R_g^{-3/2} (-\sin\theta d\theta)$

(20) $I(h) = \int_{\pi}^{0} J_s(\theta) \rho^2 \, d\ell R_g^{-3/2}$

$J_s(\theta) = J_o \cos\psi$, which simplifies the integral.

(21) $I(h) = J_o \int_{\rho max}^{0} \rho^2 d\rho R_g^{-3/2}$

(22) $R_g = 1 + 2Zh + (\rho^2 + Z^2)h^2$



$R = \sqrt{\rho^2 + Z^2}$

$\rho = R\sin\theta$

$-Z = R\cos\theta$

$\cos\psi = d\rho/d\ell$

As shown in Michel (1977), although h appears explicitly in this integral, when the correct expression for $Z(\rho)$ is inserted, the integral is independent of h. To determine the value of I_1 take $h = 0$.

$$(23)\quad I(h=0) = J_0 \int_{\rho max}^{0} \rho^2 d\rho (1)^{3/2} = -(1/3)J_0$$

$\rho_{max} = 1$ has been used to scale the system in the computer program. Table I shows the computed values for one of the solutions for the magnetopause shape. From this it is obvious that the computer solution gives a value for the moment that is close to the correct value of 1/3 and nearly independent of h.

To use in calculation on the computer, determine $I'(h)$.

$$(24)\quad I'(h) = \frac{dI}{dh} = 3J_0 \int_0^{\rho max} d\rho\ \rho^2 (Z + h(\rho^2 + Z^2))(1+2Zh+(\rho^2+Z^2)h^2)^{-5/2}$$

When the correct expression for $Z(\rho)$ is inserted in the integrand, this integral should equal zero, independently of h.

By analogy with the gross features of the earth's magnetosphere, the general shape of the axisymmetric case is shown in Fig.1. Consider the function $Z = 1/(a^2+\rho^2)$ which has the shape shown below.

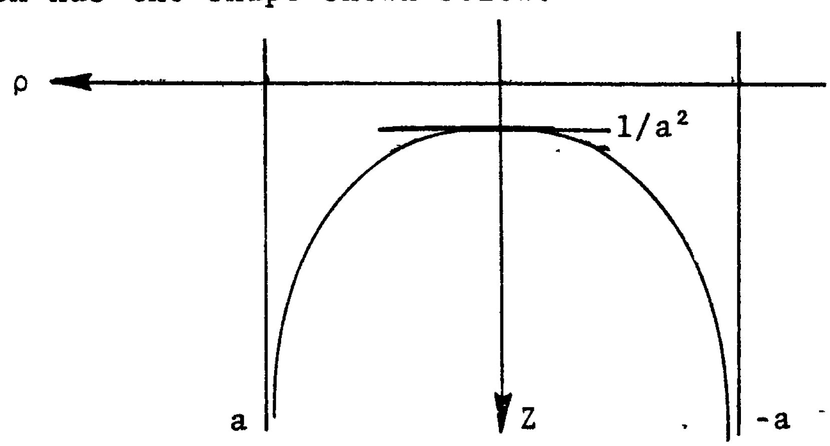


TABLE I

I_1 as a Function of h for Minimized ε

$B = 1.0$

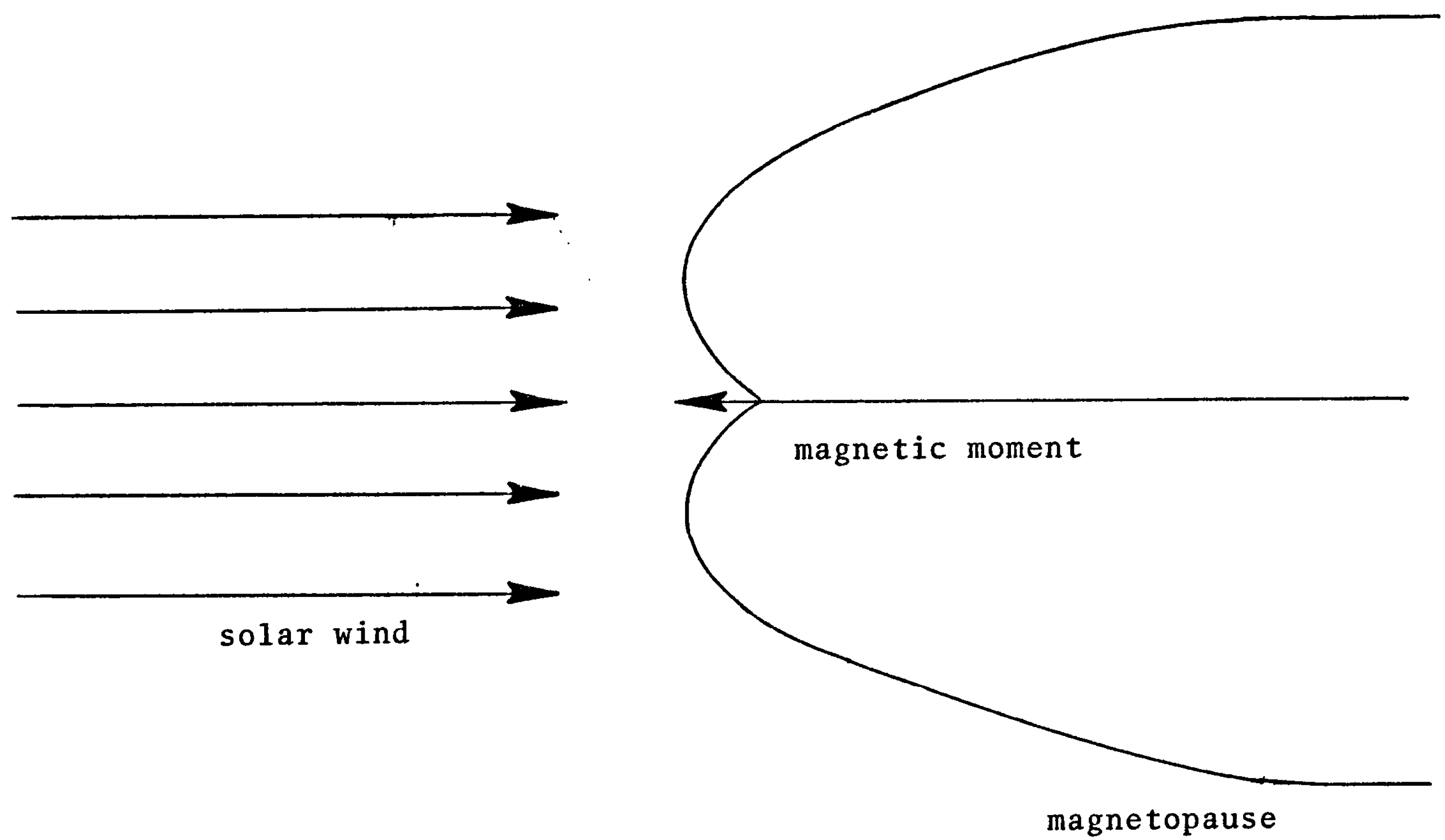
$P = 0.5$

$Z_o = 0.4$

$\rho_o = 0.35$

h	$-I_1(h) \cdot J_o^{-1}$
0.1	0.331
0.2	0.335
0.3	0.339
0.4	0.343
0.5	0.344
0.6	0.344
0.7	0.341
0.8	0.336
0.9	0.328
1.0	0.319

note: I_1 is nearly equal to 1/3, independently of h.



GENERAL SHAPE OF MAGNETOPAUSE EXPECTED FOR URANUS PROBLEM

FIGURE 1

Transform the coordinate system: $Z \rightarrow Z+Z_0$:

$\rho \rightarrow \rho-\rho_0$

so the function becomes $Z+Z_0 = (a^2-(\rho-\rho_0)^2)^{-1/2}$

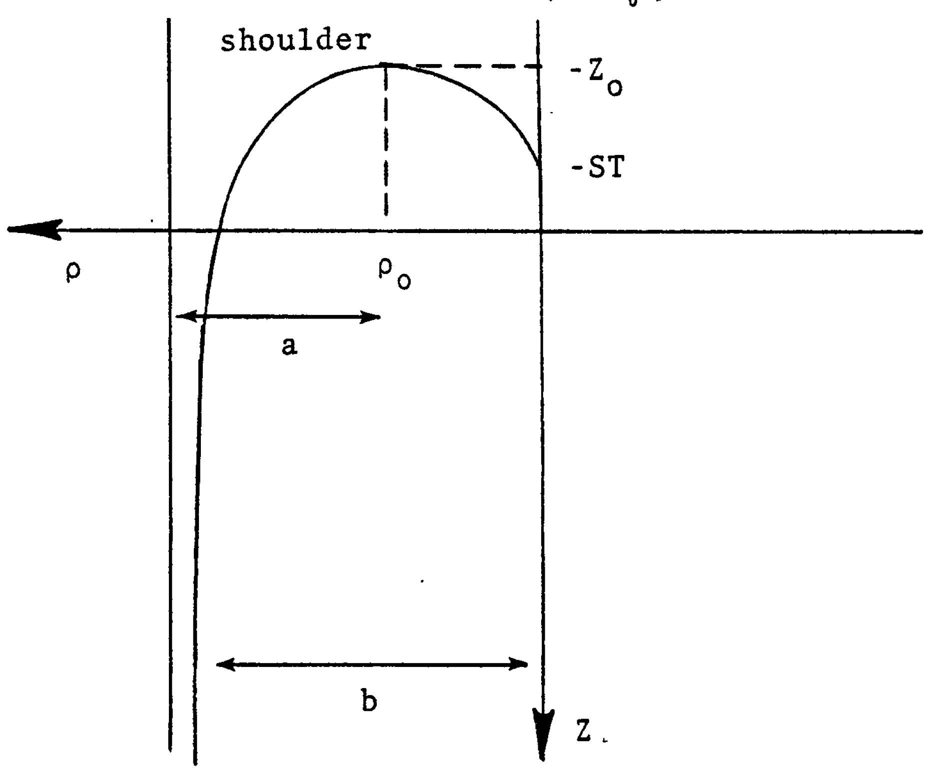
Now generalize this function so that the denominator has an arbitrary exponent P, which was expected to be 1/2 or 1. Then multiply by powers of a so that both ρ and Z will have units of length when scaled by r_0.

$$(25)\quad Z = \frac{a^{2P+1}}{(a^2-(\rho-\rho_0)^2)^P} - a - Z_o.$$

Define $b = a+\rho_0$ = asymptotic width of the tail.

The standoff distance at the symmetry axis = ST.

$$(26)\quad ST = -Z(\rho=0) = a + Z_o - \frac{a^{2P+1}}{(a^2-\rho_0^2)^P}$$



When the expression for $Z(\rho)$ is inserted into $I'(h)$, the denominator might tend to zero and cause difficulties in programming. Consider $1 + 2Zh + (\rho^2+Z^2)$. In the tail region the h^2 term will dominate, making the denominator positive there. If the expression remains positive, the denominator will not become zero. The only negative quantity is Z, which is less than zero around the shoulder of the magnetopause. Require $|2Zh| < 1+h^2(\rho^2+Z^2)$ for the denominator to be nonzero everywhere. The most negative value of Z is $-Z_o$. Thus if $Z_o < 1/(2h)$, it is sufficient to insure that the denominator is nonzero everywhere. Since $h<1$, if $Z_o \leq 1/2$ no problems will be encountered. In all cases, useful values for Z_o were less than or equal to 1/2 anyway. Also, the computer program contained a check to insure that the denominator did not go to zero when the shape was further refined by adding more functions to $Z(\rho)$.

The shape contains four adjustable parameters: b, ρ_o, Z_o, and P. It is convenient to set the scale by defining an asymptotic tail width b to be unity, leaving three parameters.

For the correct solution to the magnetopause shape, $I'(h)$ should equal zero. Following Michel (1977) define an error $\varepsilon = \sum_h \left(I'(h)\right)^2$. In the accretion problem described in Michel (1977) numerical difficulties required a summation on h from 0 to 0.8 instead of to

1.0. In this case, however, the integral was easy to calculate around 1.0 but difficult to calculate numerically about 0. Ten equal steps in h were used from 0.1 to 1.0.

The ρ integral was evaluated by Simpson's rule. To determine the number of grid spaces necessary, two tests were made. Table II shows $J_o^{-1} I'(h)$ for two typical values of h as a function of grid size. Table III shows $J_o^{-2} \varepsilon$ as a function of grid size in ρ. From this it is clear that 121 points will give at least three significant figures in ε. The programs were run with 121 points in the ρ grid and 10 points in the h grid.

TABLE II

Behavior of I´(h) as a Function of Grid Size in ρ for Typical Values of h

h	Number of Points in Grid	$I'(h)/J_o$
0.8	21	-2.59445 E-02
	31	-2.59320 E-02
	51	-2.59287 E-02
	121	-2.59281 E-02
0.1	21	9.047389 E-02
	31	8.889210 E-02
	51	8.697861 E-02
	71	8.652818 E-02
	91	8.642495 E-02
	121	8.639902 E-02

$B = 1.0$

$P = 1.0$

$Z_o=0.5$

$\rho_o=0.25$

TABLE III

Behavior of ε as a Function of Grid size in ρ

$B = 1.0$

$P = 1.0$

$Z_o = 0.5$

$\rho_o = 0.25$

$h = 0.1$ to 1.0 in ten steps

Number of points in GRID	ε/J_o^2
31	1.215 E-02
51	1.180 E-02
71	1.172 E-02
91	1.170 E-02
121	1.170 E-02

RESULTS FOR THE URANUS PROBLEM

The parameter P determines the asymptotic shape of the tail. Write $\rho = a+\rho_0-\varepsilon$. As $\varepsilon\to 0$, $Z\to\infty$, describing the tail.

$$(27) \quad \lim_{\varepsilon\to 0} Z = \lim_{\varepsilon\to 0} \frac{a^{2P+1}}{(a^2-(a+\rho_0-\varepsilon-\rho_0)^2)^P} - a - Z_o$$

$$(28) \quad = \lim_{\varepsilon\to 0} \frac{a^{2P+1}}{(+2a\varepsilon+\text{order}(\varepsilon^2))^P} - a - Z_o \to \varepsilon^{-P}$$

For each of five values of P, ρ_0 and Z_o were varied to give the minimum error. Table IV shows the values for the best fit for each P. P = 1/2 gave the smallest error. Fig. 2 shows the dependence of the solution shape upon P.

The error has now been minimized with a three parameter fit. Since ρ_0 and Z_o are relatively uncoupled and thus easy to minimize, more terms may be added to get a better fit. ρ_0 was stable,so the expansion should not alter $Z(\rho_0)$. $\sum_n C_n(\rho-\rho_0)^n$ keeps $Z(\rho_0)$ as before. Table 5 summarizes the results for $Z(\rho)$ with the new function added on to try to reduce the error and improve the solution shape.

First, Z_o was held constant and the error was minimized with C_2 and ρ_0. Then C_1 and ρ_0 were varied. Both minima gave the same value for ρ_0; however, with C_1 and C_2 both varied, the C's appeared to be coupled.

TABLE IV

Determination of the Asymptotic Tail Shape

$B = 1.0$

$h = 0.1$ to 1.0 in 10 steps

$\rho = 0.0$ to 1.0 in 121 steps

P	ε/J_0^2		ρ_0	Z_0	ST	
0.25	0.364	E-02	0.05	0.4	0.399	
0.50	0.182	E-02	0.35	0.4	0.279	***
0.75	0.240	E-02	0.40	0.4	0.068	
1.00	0.326	E-02	0.47	0.4	- 1.55	
2.00	0.572	E-02	0.495↑	0.4	-328	

***best fit

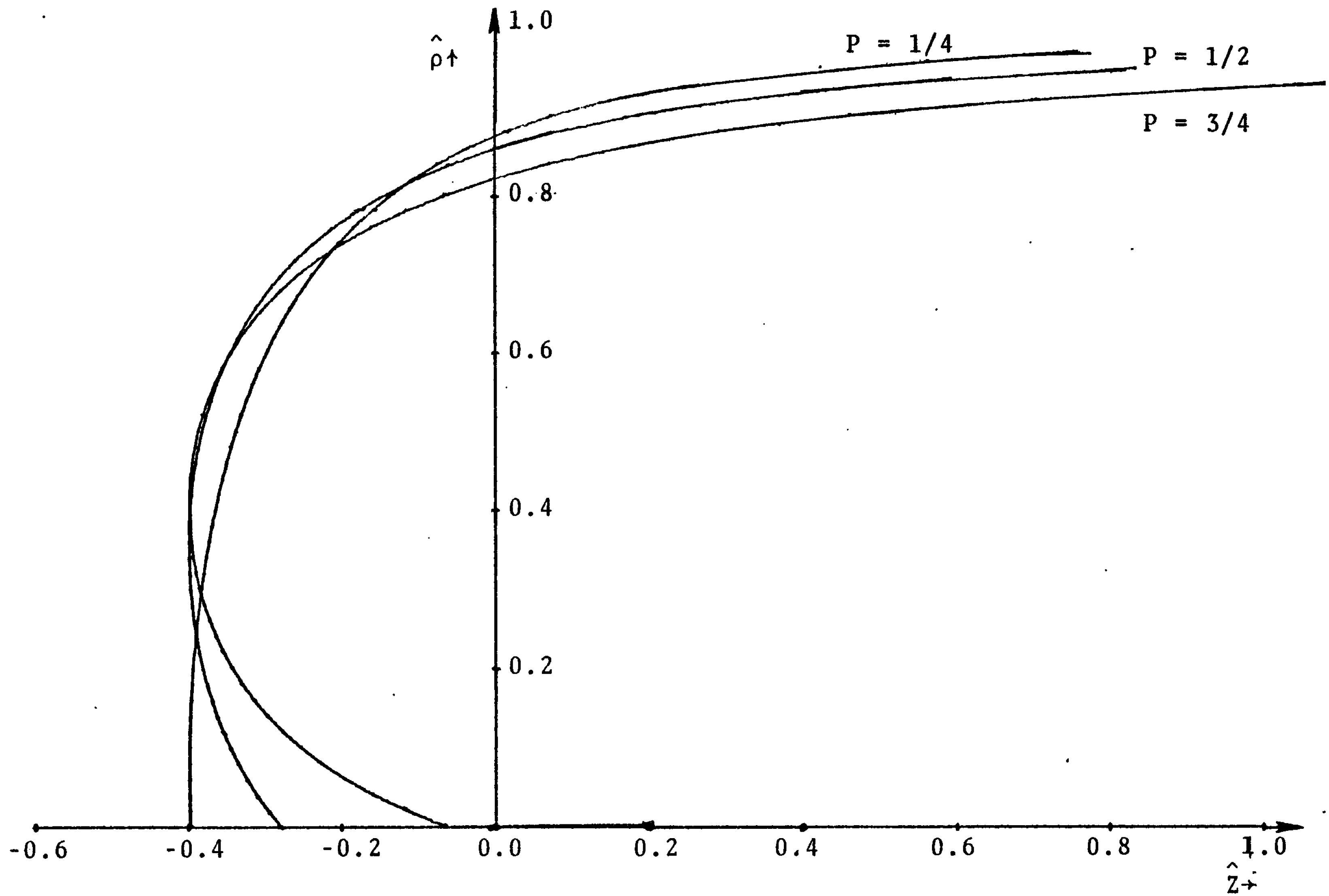


FIGURE 2

TABLE V

B = 1.0

P = 0.5

h = 0.1 to 1.0 in 10 steps

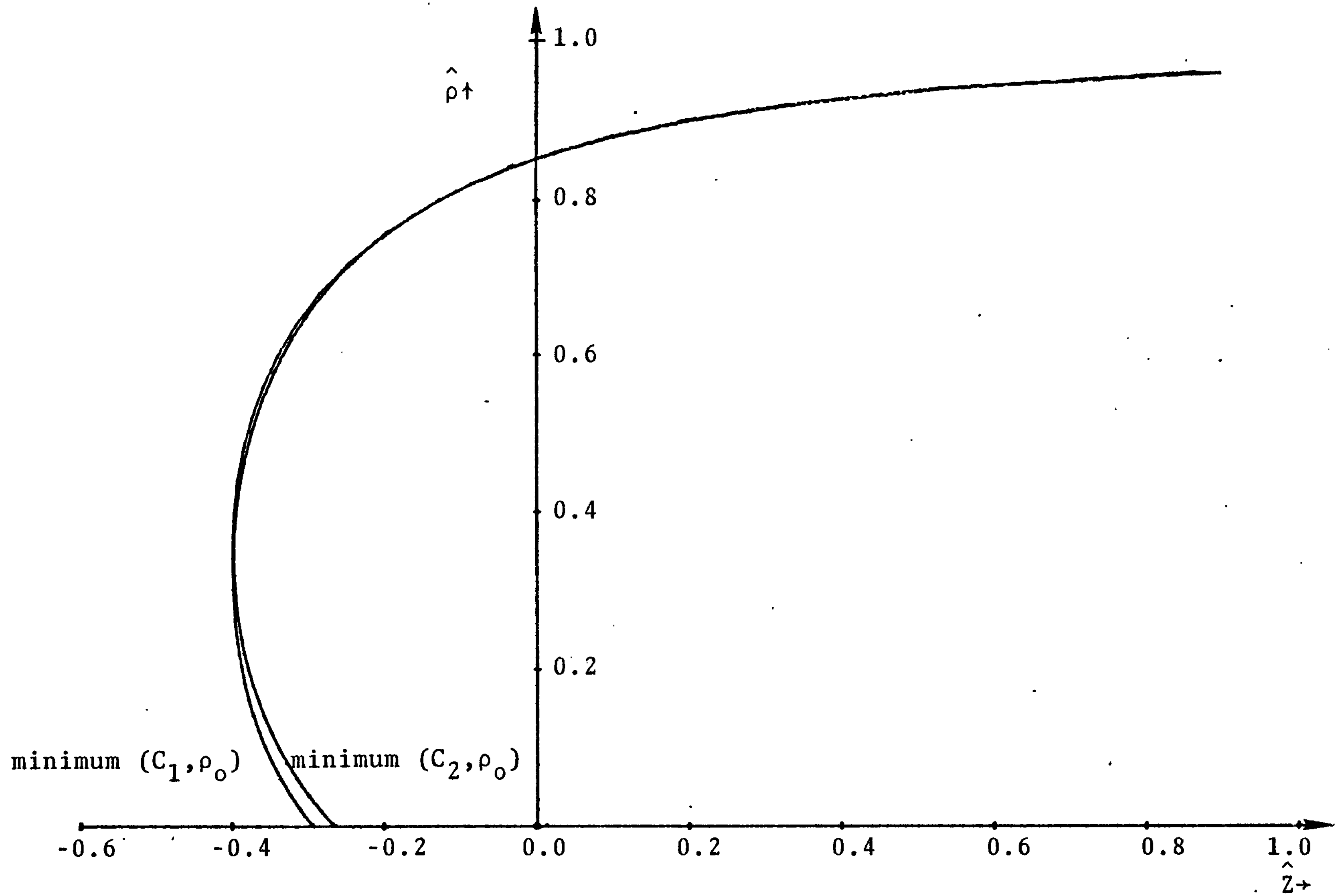
ρ = 0.0 to 1.0 in 121 steps

ρ_o	Z_0	C_1	C_2	ε/J_0^2	
0.35	0.4			1.82	E-03
0.35	0.4*		0.1	1.75	E-03
0.35	0.4*	0.05		1.69	E-03
0.35*	0.4*	0.6	-1.4	4.24	E-04
0.35	0.4	0.6*	-1.4*	4.24	E-04

* Parameters held constant during that minimization.

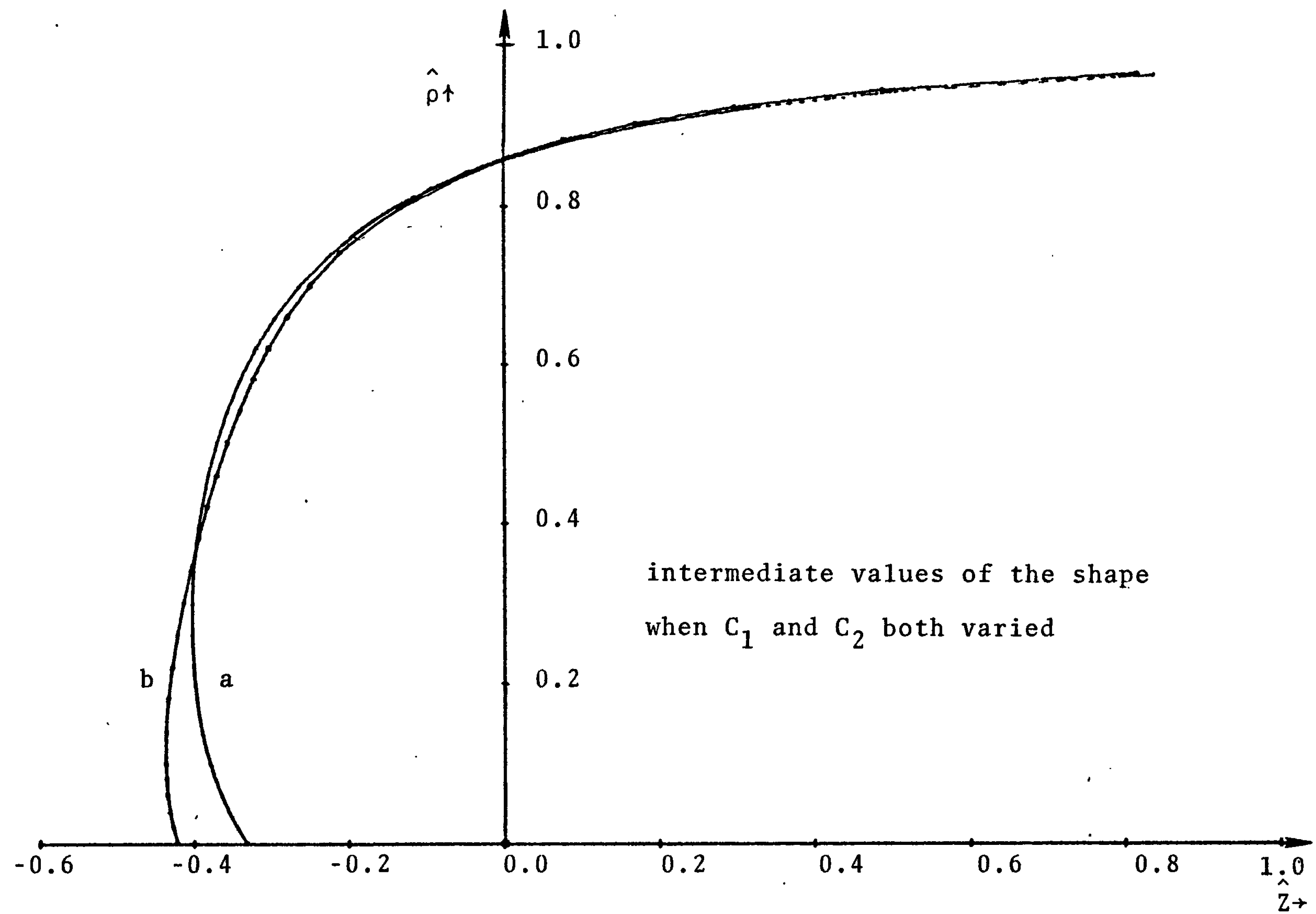
The values for this minimum changed the shape of the magnetopause drastically from the other minima. See Fig. 3,4,5. This last solution is obviously nonphysical, so the choice of an expansion in $(\rho-\rho_0)$ does not give a better shape, at least to two parameters.

Finally, ρ_0 and Z_o were varied, holding C_1 and C_2 constant. This showed that ρ_0 and Z_o were the best values for those parameters. This demonstrated that the main part of the expansion was very stable about the determined values of Z_o and ρ_0.



1.0
ρ̂↑
0.8
0.6
0.4
0.2
minimum (C1,ρo)
minimum (C2,ρo)
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Ẑ→

FIGURE 3



1.0
ρ̂↑
0.8
0.6
0.4
0.2
intermediate values of the shape
when C1 and C2 both varied
b
a
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Ẑ→

FIGURE 4

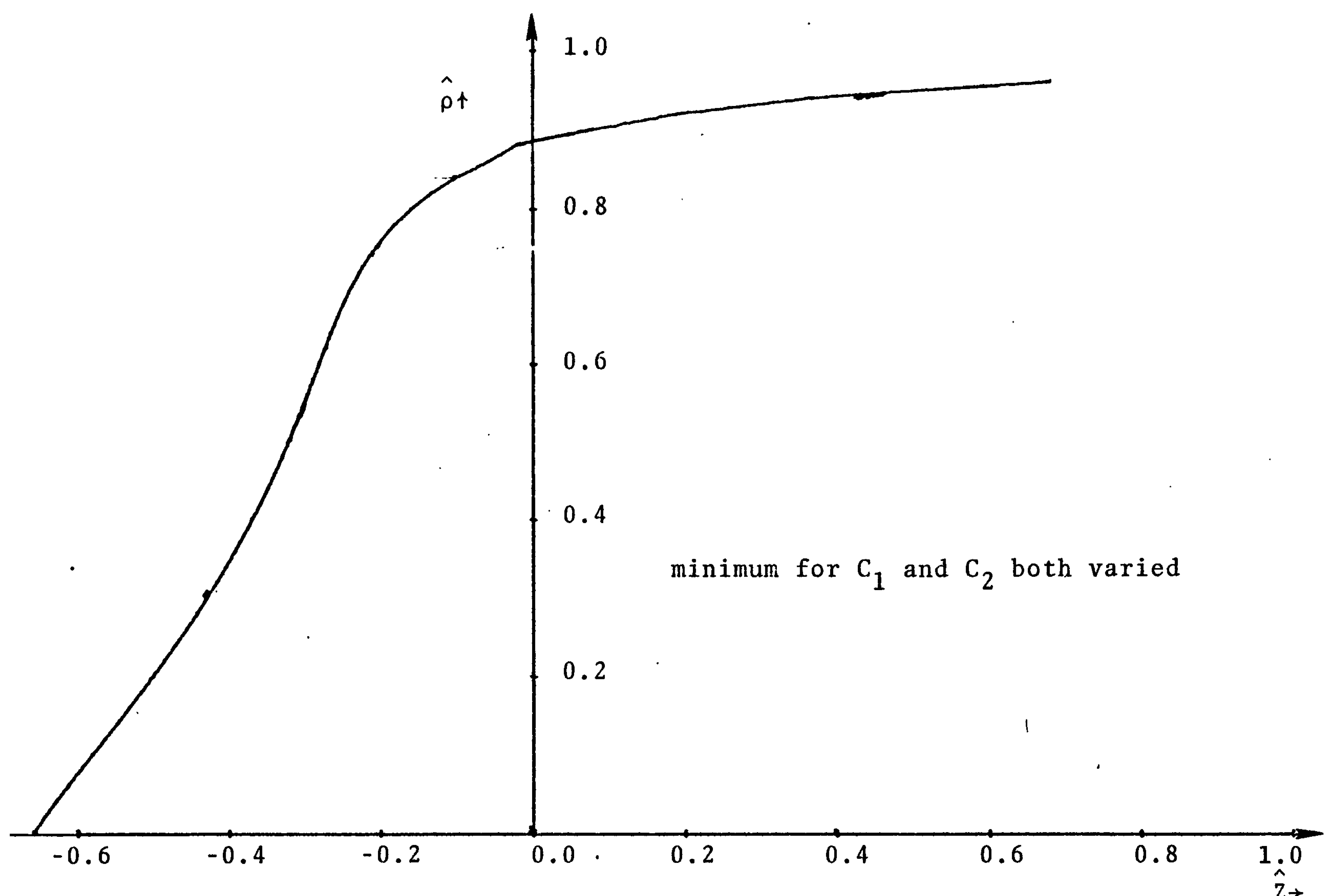

$\hat{\rho}\uparrow$
1.0
0.8
0.6
0.4
0.2
minimum for C_1 and C_2 both varied
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
$\hat{Z}\rightarrow$

FIGURE 5

APPENDIX A

DERIVATION OF MAGNETOPAUSE MOMENT FOR THE VECTOR POTENTIAL

Gaussian units will be used. In this section the equations leading to the expression for the moment integral will be derived from the Maxwell equations and basic definitions. This derivation is not original to this thesis, but summarizes and expands upon work reported in Midgley and Davis (1962,1963) and Michel (1977). The scaling factors have been kept in a selfconsistent system of units and will be used in Appendix D to scale the magnetopause shape to reasonable conditions for Uranus.

Begin with one of the Maxwell equations.

(A- 1) $\nabla \times \vec{H} = (4\pi/c)\vec{J}_v + 1/c \frac{\partial \vec{D}}{\partial t}$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{M}(\vec{x}) = \sum_i N_i \langle\vec{m}_i\rangle$$

where N_i = average number / volume of molecules of species i

$\langle\vec{m}_i\rangle$ = average molecular moment due to species i in a small volume at point $\vec{x}$

Since the solution will be developed for the region outside the magnetopause, where there are no magnetic materials, take $\langle\vec{m}_i\rangle = 0$.

A time independent solution is sought, so all time derivatives are zero.

(A- 2) $\nabla\times\vec{B} = (4\pi/c)\vec{J}_v$

where $\vec{J}_v$ is the volume density of free currents in statamps / cm²

Since $\nabla\cdot\vec{B} = 0$, define the magnetic vector potential $\vec{A}$.

(A- 3) $\vec{B} = \vec{\nabla}\times\vec{A}$

Written in terms of $\vec{A}$, Ampere's law becomes

(A- 4) $\nabla\times(\nabla\times\vec{A}) = (4\pi/c)\vec{J}_v$

Use the vector identity

(A- 5) $\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times\nabla\times\vec{A}$

Choose the Coulomb Gauge so that $\nabla\cdot\vec{A} = 0$.

(A- 6) $\nabla^2\vec{A} = -(4\pi/c)\vec{J}_v$

equation (5.31) of Jackson

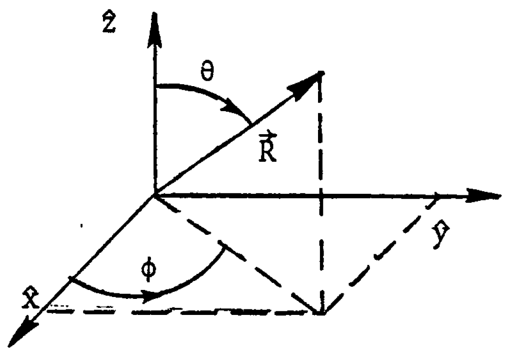
This means that each rectangular component of the vector potential satisfies the Poisson equation.

Following Jackson, if there are no sources of the magnetic field at infinity, the solution may be written

$$\text{(A-7)}\quad A(x) = 1/c\int d^3x' \frac{\vec{J}_V(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

equation (5.32) of Jackson

Define the following coordinate system:



current source at primed coordinates

field observed at unprimed coordinates

The dipole moment is alligned with the $\hat{Z}$ axis, so all quantities are independent of ϕ.

$$\text{(A-8)}\quad \vec{A}(\vec{x}) = 1/c\int R'^2 \sin\theta' dR' d\theta' d\phi' \frac{\vec{J}_V(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

As argued in the text, it is reasonable to consider that the currents flow only on the magnetopause, which is of negligible thickness. The volume integral becomes a surface integral over the magnetopause.

$$\text{(A-9)}\quad \vec{A}(\vec{x}) = 1/c\int dS' \frac{\vec{J}_S(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

dS' is a surface element of the magnetopause and its expression in spherical coordinates is derived in Appendix B.

$$\text{(A-10)}\quad dS' = R'\sin\theta'\left(R'^2 + \left(\frac{\partial R'}{\partial \theta'}\right)^2\right)^{1/2} d\theta' d\phi'$$

$R' = R'(\theta')$ describes the magnetopause.

Following Midgley and Davis (1963), $|\vec{x}-\vec{x}'|$ may be expanded via associated Legendre polynomials for the region outside the magnetopause.

$$\text{(A-11)}\quad \frac{1}{|\vec{x}-\vec{x}'|} = 1/c\sum_{n=0}^{\infty}\sum_{m=0}^{n}(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\frac{R'^n}{R^{n+1}}$$

$$\cdot P_n^m(\cos\theta)P_n^m(\cos\theta')\ \cos m(\phi-\phi')$$

Reverse the order of summation and integration.

$$\text{(A-12)}\quad \vec{A}(\vec{x}) = 1/c\sum_n\sum_m(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\frac{1}{R^{n+1}}P_n^m(\cos\theta)$$

$$\cdot\int_0^{\pi}d\theta' P_n^m(\cos\theta')R'\sin\theta'\left(R'^2+\left(\frac{\partial R'}{\partial\theta'}\right)^2\right)\int_0^{2\pi}d\phi'\vec{J}_s(\vec{x}')\cos m(\phi-\phi')$$

Since $\vec{B} = \nabla \times \vec{A}$ and $\vec{B}$ is a dipole field, $\vec{A} = -A\hat{\phi}$.

$\vec{J}$ is parallel to $\vec{A}$ locally, so $\vec{J}_s = -J_s(\theta)\hat{\phi}$. (magnetopause)

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\text{(A-13)}\quad \cos m(\phi-\phi) = \cos m\phi\cos m\phi + \sin m\phi\sin m\phi$$

Combining these equations, perform the ϕ integration of

(A-14) $\int_0^{2\pi} \vec{J}_s(\vec{x}') \cos m(\phi-\phi')d\phi'$

Only m=1 terms survive.

(A-15) $= J_s(\theta')(-\hat{x}\sin\phi + \hat{y}\cos\phi) \quad = \quad J_s(\theta')\hat{\phi}$

It will be convenient to convert the length R in the integrand into a dimensionless variable. Define r_o.

(A-16) $R = r_o r$

r_o has been calculated in Appendix C. It will be used to scale the problem to describe a physical situation.

The vector potential may now be written in terms of the moment integral I_{n1}. Since $P_o^1 = 0$, the lower limit of the summation may be changed to 1.

(A-17) $\vec{A}(\vec{x}) = -\hat{\phi}1/c\sum_{n=1}^{\infty} \frac{1}{n(n+1)} I_{n1}P_n^{\ 1}(\cos\theta) \frac{r_o^{\ n+2}}{R^{n+1}}$

(magnetopause)

(A-18) $I_{n1} = \int_o^{\pi} d\theta' P_n^1(\cos\theta')r'^{n+1}\sin\theta'(r'^2 + \left(\frac{\partial r'}{\partial \theta}\right)^2)^{1/2} J_s(\theta')$

APPENDIX B

DERIVATION OF AREA DIFFERENTIAL

This derivation follows the notation of Olmsted for parametrically represented functions and for the partial derivatives of these function.

S is a surface defined parametrically:

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

u and v are coordinates whose domains form a bounded rectangle R.

The area of this surface A(S) is

$$\text{(B-1)} \quad A(S) = \iint_R \{ (j_1)^2 + (j_2)^2 + (j_3)^2 \}^{1/2} du dv$$

A typical Jacobian is

$$j_3 = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

where $M_n = \dfrac{\partial M}{\partial n}$

This problem involves the transformation:

$$x = x(R,\theta,\phi) = R(\theta)\sin\theta\cos\phi = \tilde{x}(\theta,\phi)$$
$$y = y(R,\theta,\phi) = R(\theta)\sin\theta\sin\phi = \tilde{y}(\theta,\phi)$$
$$z = z(R,\theta,\phi) = R(\theta)\cos\phi = \tilde{z}(\theta,\phi)$$

(B-2) $dS = \{j_1^2 + j_2^2 + j_3^2\}^{1/2}\, d\theta d\phi$

(B-3) $j_1 = \begin{vmatrix} \tilde{y}_\theta & \tilde{y}_\phi \\ z_\theta & z_\phi \end{vmatrix} = R^2\sin^2\theta\cos\phi - RR_\theta\sin\theta\cos\theta\cos\phi$

(B-4) $j_2 = \begin{vmatrix} \tilde{z}_\theta & \tilde{z}_\phi \\ \tilde{x}_\theta & \tilde{x}_\phi \end{vmatrix} = R^2\sin^2\theta\sin\phi - RR_\theta\sin\theta\cos\theta\sin\phi$

(B-5) $j_3 = \begin{vmatrix} \tilde{x}_\theta & \tilde{x}_\phi \\ \tilde{y}_\theta & \tilde{y}_\phi \end{vmatrix} = R^2\sin\theta\cos\theta + RR_\theta\sin^2\theta$

(B-6) $j_1^2 + j_2^2 + j_3^2 = R^2\sin^2\theta(R^2 + R_\theta^2)$

For symmetry about the z axis, the surface element is

(B-7) $dS = R\sin\theta\{R^2 + R_\theta^2\}^{1/2} d\theta d\phi$

APPENDIX C

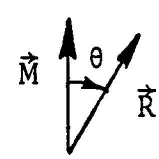
DERIVATION OF THE SCALING DISTANCE r_o

Following Midgley and Davis (1962), define the scaling length r_o. The magnetic pressure of just the dipole field in its equatorial plane at radius r_o equals the pressure that the solar wind would exert on an object perpendicular to its flow.

P_{g_o} = gas pressure on an object oriented perpendicularly to the solar wind

P_{dip_o} = magnetic pressure due to dipole field at r_o

(C-1) $P_{g_o} = 2n(m_p+m_e)v_o^2$



(C-2) $\vec{B}_{dip} = (M/R^3)(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$

M = intrinsic magnetic moment of the object

other symbols defined in text

(C-3) $P_{dip_o} = B_{dip}^2(r_o)/(8\pi)$

Equating (C-1) and (C-3) determines r_o.

(C-4) $r_o = M^{1/3}v_o^{-1/3}(16\pi nm_p)^{-1/6}$

APPENDIX D

r_o SCALED TO URANUS

In this appendix r_o will be evaluated for the condition of the solar wind predicted around Uranus and with a magnetic moment for Uranus guessed empirically.

(D-1) $r_o = (16\pi m_p)^{-1/6} M^{1/3} (n v_o^2)^{-1/6}$

One theory predicts that the intrinsic magnetic moment of a planet is proportional to its rotational angular momentum.

(D-2) $M_U = (m_u/m_E)(R_U/R_E)^2 (T_U/T_E)^{-1} M_E$

m_U = mass of Uranus = 14.6 m_E

R_U = radius of Uranus = 3.69 R_E

T_U = rotational period of Uranus = 0.448 T_E

M_E = magnetic moment of Earth = 8.1 E 25 gauss·cm³

(D-3) M_U = 3.6 E 28 gauss·cm³

Observation with satellites shows that the radial variation of the solar wind velocity at planetary distances is less than its variation in time. A typical value for the velocity is 400 km/sec = 4 E 7 cm/sec.

Since the flux of the solar wind through a shell surrounding the sun is constant, for a constant wind velocity the number density scales as X_p^{-2} , where X_p is the distance of the planet from the sun.

(D-4) $n_U = (X_E/X_U)^2 n_E$

X_E = distance of Uranus from sun = 19.2 AU

n_E = number density of solar wind at 1 AU

= $5/cm^3$

(D-5) n_U = 1.4 E-2 $/cm^3$

(D-6) r_o = 40 R_U = 9.4 E10 cm

The most uncertain parameter in this estimate of r_o is M_U, the intrinsic magnetic moment of Uranus. Freeman (1976) has shown that the sun and the planets of known magnetic moment lie close to the graph of

$\log_{10} M/M_E = 1.25 \log_{10}(\omega V/(\omega_E V_E)$ where ω = angular frequency of rotation of the planet and V = the volume of the planet

This relationship may be used to scale the magnetic moment of Uranus from that of the earth.

(D-7) $M_U = M_E (T_U/T_e)^{-1.25} (R_U/R_E)^{3.75}$

This gives M_U = 3.0 E 28 gauss·cm³ and slightly reduces r_o from the previous value to 8.8 E 10 cm.

(D-8) $r_o = 38\ (M_U/(3.0\text{ E }28\text{ gauss}\cdot\text{cm}^3)^{1/3}\ R_U$

APPENDIX E

COMPUTER PARAMETERS SCALED TO URANUS

Equation (23) defined $\rho_{max} = 1$ for the computer solutions. To get a physical value for any distance, ρ_{max} must have a numerical value which will make $\vec{A}(\vec{x})$ equal to the vector potential for a dipole field.

(E-1) $\vec{A}(\vec{x}) = \hat{\phi}\ M/R^2\ P_1^1(\cos\theta)$ for the magnetopause, which is opposite that of the planet.

Comparing equation (E-1) with (A-17) and (23) for n = 1 gives

$$-(1/2c) I_1 P_1^1(\cos\theta) r_o^3/R^2 = M/R^2\ P_1^1(\cos\theta)$$

By definition $r_o^3 = M/(8\pi P_o)^{1/2}$

$$-(1/2c) I_1/(8\pi P_o)^{1/2} = 1$$

(E-2) $I_1 = -8\pi J_o$ by eq. (14)

(23) gives $I_1 = -1/3\ J_o(\rho_{max})^3$

(E-3) $\rho_{max} = (24\pi)^{1/3} = 4.22$

To scale a solution to physical distances, multiply the computer solution parameters by $r_o \rho_{max} = 38R_U \cdot 4.2 = 160\ R_U$.

	computer	scaled (in R_U)
B	1.0	160
ρ_o	0.35	56
Z_o	0.40	64
ST	0.28	45

THE MAGNETOSPHERE OF URANUS

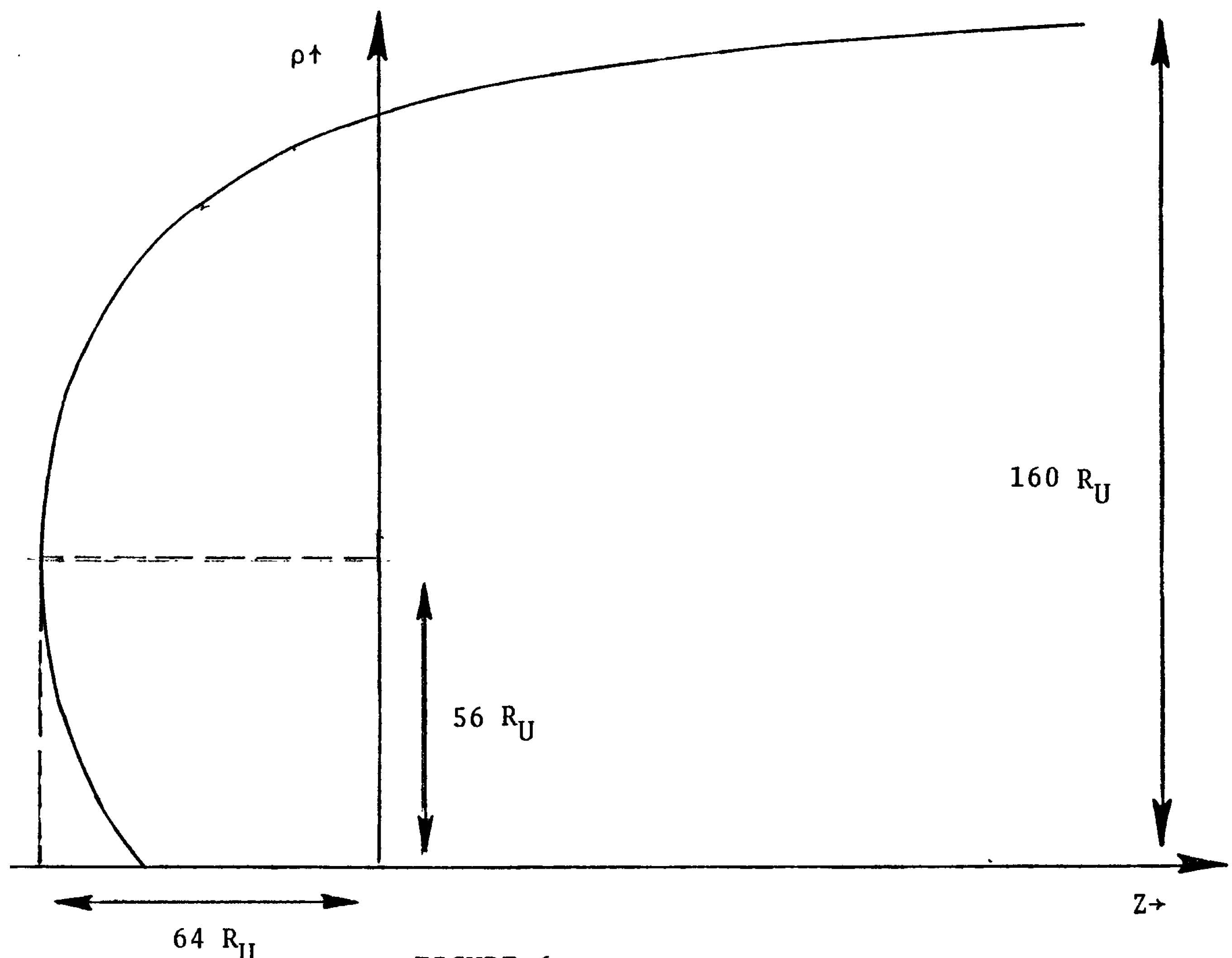


FIGURE 6

FOOTNOTES

[1]R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics, 2, 4th ed., New York, 1966, section 23-2.

[2]D.M. Willis, "Structure of the Magnetopause", Rev. Geophys. and Spac. Phys., 9, 953, 1971.

[3]Ibid.

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