Transfer of approximation and numerical homogenization of hyperbolic boundary value problems with a continuum of scales

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ABSTRACT
Galerkin approximate solutions of two self-adjoint systems with the same right-hand side have errors that mutually dominate each other, provided that the approximating subspaces contain exact solutions to problems with the same right-hand sides for their respective systems. This “transfer of approximation” property was first formulated by Berlyand and Owhadi in a more specialized setting. It provides a very general framework for numerical homogenization, that is, construction of optimal-order finite element approximations for linear elliptic boundary value problems with bounded and measurable coefficients, relying neither for formulation nor proof of convergence on assumed scale separation, periodicity, or ergodicity of coefficients. The construction extends to hyperbolic problems provided that the data is smooth in time. In its basic form, the transfer construction of optimal order approximation has severe practical drawbacks: it produces bases with global support, hence dense stiffness matrices, and at the cost of many solutions of problems as difficult in principle as the original. Numerical experiments suggest that a localization procedure suggested by Owhadi and Zhang still requires basis functions of large support (producing stiffness matrices of high bandwidth) to be effective. The harmonic coordinate construction of Owhadi and Zhang, on the other hand, can be viewed as a transfer-of-approximation instance, and produces stiffness matrices with the same sparsity pattern as standard conforming finite element methods.

INTRODUCTION
Introduction goes here.

The 2nd order divergence form operator has played a central role in much work on homogenization, both analytical and numerical, and figures prominently in the work of Berlyand and Owhadi (2010).

An obvious method to reduce the computational cost of transfer basis construction is localization of the basis, that is, solution of the transfer problem on smaller domains. Owhadi and Zhang (2010) propose to construct an approximation subspace by solving the transfer problem for each basis function on a sudomain somewhat larger than
the support of the corresponding piecewise polynomial basis function, with artificial Dirichlet conditions at the boundary of the subdomain. The error so introduced is controlled via clever decay estimates for Green’s functions.

I explain a different approach to localization in this paper, applicable to specifically to hyperbolic initial-boundary value problems. The foundation for this concept was established in (Blazek et al., 2008): even for coefficients which are merely bounded and measurable, weak solutions of symmetric hyperbolic systems propagate at finite speed, determined by the principal part, just as is the case for systems with smooth coefficients. I divide the domain into overlapping subdomains, and localize the problem data further to subsets separated from the boundaries of the subdomains. For a limited time, the solutions of these subdomain problems do not interact with the boundaries of the subdomains, therefore are global solutions on the entire domain. Furthermore, because of the construction of the subproblem data, the subdomain solutions add up to the global solution over the same limited time interval. After short-time extrapolation of the solution based on this principle, the global solution can be synthesized, the subdomain data reconstructed, and another short-time extrapolation performed. In this way, finitely many local solves lead to a global solution. The subdomain solutions are approximated by the transfer construction in the subdomains.

THE TRANSFER LEMMA

The developments to follow hinge on a trivial fact about isomorphisms of Banach spaces, that is, continuous linear maps with continuous inverses. If $V, W$ are Banach spaces with norms $\| \cdot \|_V, \| \cdot \|_W$, and $A : V \to W$ is an isomorphism, then the pull-back $\| \cdot \|_A : V \to \mathbb{R}$ by $A$ of the norm on $W$, $\| u \|_A = \| Au \|_W$, is a norm equivalent to $\| \cdot \|_V$.

**Theorem 1. (basic transfer lemma):** Suppose that $V, W$ are Banach spaces, $A_0, A_1 : V \to W$ are isomorphisms, and $S_0, S_1 \subset V$ are subspaces satisfying the condition $A_0 S_0 = A_1 S_1$. Then for any $f \in W$,

$$\inf_{v \in S_0} \| A_0^{-1} f - v \|_{A_0} = \inf_{v \in S_1} \| A_1^{-1} f - v \|_{A_1}.$$ 

**Proof.**

$$\{ \| A_0^{-1} f - v_0 \|_{A_0} : v_0 \in S_0 \} = \{ \| f - w \|_W : w \in A_0 S_0 = A_1 S_1 \}$$

$$= \{ \| A_1^{-1} f - v_1 \|_{A_1} : v_1 \in S_1 \},$$

so the infima are the same. $\square$
Berlyand and Owhadi (2010) introduced the transfer-of-approximation concept via a special cases of the Hilbert triple setting popularized by Lions and Magenes (1972). Suppose that $V, H$ are Hilbert spaces with norms $\| \cdot \|_V, \| \cdot \|_H$ Denote by $D : V \to H$ a bounded linear map, defining a coercive bilinear form $u, v \mapsto \langle Du, Dv \rangle_H$. Without loss of generality, we may assume that $\|v\|_V = \|Dv\|_H$. According to the Lax-Milgram theorem (Yosida (1996), p. 92 ff), $D^*D$ is an isomorphism $V \to V^*$. The dual norm on $V^*$ is also Hilbert, and for $f \in V^*$, 

$$\|f\|_{V^*} = \|D(D^*D)^{-1}f\|_H.$$ 

Since $D$ is coercive, $\text{Rng}(D)$ is closed. Denote by $\Pi = D(D^*D)^{-1}D^*$ the orthogonal projection in $H$ onto $\text{Rng}(D)$.

For self-adjoint bounded positive definite $C : H \to H$, define $\| \cdot \|_{C^{-\text{flux}}} : V \to \mathbb{R}$ by 

$$\|v\|_{C^{-\text{flux}}} = \|\Pi CDv\|_H, \ v \in V.$$ 

(This is the abstract definition of the Berlyand-Owhadi flux norm (Berlyand and Owhadi, 2010).)

Note that $D^*CD$ is an isomorphism $: V \to V^*$, thanks to the Lax-Milgram Theorem. Since $\Pi CD = D(D^*D)^{-1}D^*CD$, the flux norm for $C$ is in fact identical to the $A$-norm for $A = D^*CD$. Thus Theorem 1 implies

**Corollary 1.** (Berlyand-Owhadi transfer lemma), Suppose that $V, H$ are Hilbert spaces, $D \in \mathcal{B}(V,W)$ a coercive linear map. Assume that $C_0, C_1 \in \mathcal{B}(H,H)$ are self-adjoint with bounded inverses, so that $A_0 = D^*C_0D$ and $A_1 = D^*C_1D$ are bounded and coercive. Suppose finally that $S_0, S_1$ are subspaces of $V$ for which $A_0S_0 = A_1S_1$. Then for any $f \in V^*$,

$$\inf_{v \in S_0} \|A_0^{-1}f - v\|_{C_0^{-\text{flux}}} = \inf_{v \in S_1} \|A_1^{-1}f - v\|_{C_1^{-\text{flux}}}.$$ 

**Corollary 2.** Suppose in addition that $X \subset V^*$ is a subspace with norm $\| \cdot \|_X$, and $\Gamma > 0$, so that 

$$\inf_{v \in S_0} \|A_0^{-1}f - v\|_V \leq \Gamma \|f\|_X \text{ for all } f \in X.$$ 

Then there exists $L > 0$ so that 

$$\inf_{v \in S_1} \|A_1^{-1}f - v\|_V \leq L \Gamma \|f\|_X \text{ for all } f \in X.$$ 

**Proof.** If one replaced the $V$ norms with the respective flux norms in the preceding two inequalities, the conclusion would follow from Corollary 1, with $L = 1$. However the $C_0, C_1$ flux norms are actually the $A_0, A_1$ norms as defined at the beginning of this section, and these are equivalent to the $V$ norm as already remarked. 

This last result explains the term “transfer of approximation”: in effect, the XXXXXXXXXXX
SCALAR ELLIPTIC PROBLEM

Suppose that $C \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ is uniformly symmetric positive definite. According to the standard $L^2$ theory of elliptic boundary value problems (Gilbarg and Trudinger, 1983), the bilinear form

$$a_C(u, v) = \int_\Omega (\nabla u)^T C \nabla v$$

is continuous and coercive on $H^1_0(\Omega)$, hence defines an isomorphism $A_C : H^1_0(\Omega) \to H^{-1}(\Omega)$. Multiplication by $C$ defines a self-adjoint positive definite map on $(L^2(\Omega))^n$, so this problem fits into the framework of Berlyand-Owhadi transfer lemma (Corollary 2), with $V = H^1_0(\Omega)$, $H = (L^2(\Omega))^n$, and $D = \nabla$. Of course, for $C = I$, $a_I$ is the standard Dirichlet form and $A_I$ is the Laplacian.

**Theorem 2.** (compare XXXX) Suppose that $\Omega \subset \mathbb{R}^n$ is smoothly bounded, and $C \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ satisfies

$$0 < c_* |\xi|^2 \leq \xi^T C(x) \xi \leq c^* |\xi|^2$$

for almost every $x \in \Omega, \xi \in \mathbb{R}^n$.

Choose a sequence $\{T^h\}$ of tetrahedral meshes of diameter $h \to 0$, constructed by successive refinement in such a way that the aspect ratios are uniformly bounded in $h$. Denote by $S^h_I \subset H^1_0(\Omega)$ piecewise linear functions on $T^h$, and by $B^h_I$ the standard nodal basis of $S^h_I$. With $A_C, A_I$ as above, define

$$B^h_C = \{A^{-1}_C A_I \phi : \phi \in B^h_I\},$$

and $S^h_C = \text{span} B^h_C$. Then there exists $\Gamma \geq 0$ so that

$$\inf_{v \in S^h_C} \|A^{-1}_C f - v\|_{H^1(\Omega)} \leq \Gamma h \|f\|_{L^2(\Omega)}$$

**Proof.** The fundamental regularity theorem for the Dirichlet problem in smoothly bounded domains states that $A^{-1}_I$ defines a bounded map $: L^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega)$, whence standard estimate in the theory of finite element approximation establishes the existence of $\Gamma > 0$ depending only on $\Omega$ so that for $f \in L^2(\Omega)$,

$$\inf_{v \in S^h_I} \|A^{-1}_I f - v\|_{H^1(\Omega)} \leq \Gamma h \|f\|_{L^2(\Omega)}$$

Note that the subspaces $S^h_I$ are nested, that is, $S^h_{I_1} \subset S^h_{I_2}$ if $h_1 \leq h_2$; thus the transfer subspaces $S^h_C$ are nested as well. The conclusion follows by appeal to Corollary 1.

**Corollary 3.** Suppose that $f \in L^2(\Omega)$. The Galerkin approximation $u^h \in S^h_C$ to $A^{-1}_C f$, defined by

$$a_C(u^h, v) = \langle f, v \rangle \text{ for all } v \in S^h_C,$$

satisfies

$$\|A^{-1}_C f - u^h\|_{H^1(\Omega)} \leq \Gamma h \|f\|_{L^2(\Omega)},$$

with the same constant $\Gamma$ as in the transfer approximation estimate 2.
Proof. This conclusion follows from Céa’s Lemma (Ciarlet (2002), Chapter 3).

The importance of this result lies in the total lack of any requirement on the coefficient matrix $C$ other than that it be measurable, symmetric, essentially bounded and with an essentially bounded inverse. Thus Galerkin approximation via the transfer basis construction (1) yields numerical homogenization, without any assumption of scale separation or ergodicity.

The practical drawbacks of the transfer-of-approximation approach are also visible: construction of the transfer basis entails solution of $O(h^{-n})$ elliptic problems, each as difficult as the original - essentially, the construction of a discrete Green’s function. Moreover, the stiffness matrix is in general dense.

SCALAR WAVE EQUATION

For the sake of concreteness, this section treats the solution of an initial-boundary value problem for a scalar wave equation in a domain $\Omega \subset \mathbb{R}^n$: given symmetric positive definite $C \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ and $f \in L^2(\mathbb{R}, L^2(\Omega))$, $f(t) = 0$ for $t < 0$, seek a distribution $u$ on $\Omega \times \mathbb{R}$ of appropriate regularity so that $u = 0$ for $t < 0$, and

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot C \nabla u = f$$

in an appropriate sense. This problem is a stand-in for a more general class of problems, including linear elastodynamics, which can be treated similarly.

A causal weak solution of (4) is a function $u \in C^1(\mathbb{R}, L^2(\Omega)) \cap C^0(\mathbb{R}, H^1_0(\Omega))$ for which $u(t) = 0$ for $t < 0$, and

$$0 = \int dt \left( \left\langle u(t), \frac{\partial^2 v}{\partial t^2}(t) \right\rangle_{L^2(\Omega)} + a_C(u(t), v(t)) - \left\langle f(t), v(t) \right\rangle_{L^2(\Omega)} \right)$$

for all $v \in C^2(\mathbb{R}, H^1_0(\Omega))$.


For $u \in C^1(\mathbb{R}, L^2(\Omega)) \cap C^0(\mathbb{R}, H^1_0(\Omega))$, define the energy $e_C[u]$ by

$$e_C[u](t) = \frac{1}{2} \left( \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)}^2 + a_C(u(t), u(t)) \right),$$

Stolk (2000) also showed that if $u$ is a weak solution of (4), then for any $T > 0$,

$$e_C[u](t) \leq K_T \int_0^T \| f(s) \|_{L^2(\Omega)} ds, \quad 0 \leq t \leq T.$$  (5)
In (5), the $T$-dependent constant $K_T$ grows exponentially with $T$ at a rate which depends on the various other bounds in the problem. I will use the same notation $K_T$ for any other such constant.

It follows from (5) that if $f$ has one $L^2$ derivative in time, then

$$\left\| \frac{\partial^2 u}{\partial t^2}(t) \right\|_{L^2(\Omega)}^2 \leq K_T \int_0^T \left\| \frac{\partial f}{\partial t}(s) \right\|_{L^2(\Omega)}^2 \, ds, \ 0 \leq t \leq T.$$  

and in particular $u \in C^2(\mathbb{R}, H^1_0(\Omega))$, so that (4) holds pointwise in time. Thus

$$A_C u = \frac{\partial^2 u}{\partial t^2} - f$$

holds pointwise in time.

In the notation of the last section, an error estimate for approximation in the transfer subspaces $W^k_C$ follows from the Poisson estimate (2):

$$\inf_{v \in W^k_C} \| u(t) - v \|_{H^1(\Omega)} \leq K_T h^k \left( \int_0^T \left\| \frac{\partial f}{\partial t}(s) \right\|_{L^2(\Omega)}^2 \, ds \right)^{1/2}, \ 0 \leq t \leq T.$$  

Similarly, assuming that $f$ has three $L^2$ time derivatives,

$$\inf_{v \in W^k_C} \left\| \frac{\partial u}{\partial t}(t) - v \right\|_{H^1(\Omega)} \leq K_T h^k \left( \int_0^T \left\| \frac{\partial f}{\partial t^2}(s) \right\|_{L^2(\Omega)}^2 \, ds \right)^{1/2}, \ 0 \leq t \leq T,$$

and

$$\inf_{v \in W^k_C} \left\| \frac{\partial^2 u}{\partial t^2}(t) - v \right\|_{H^1(\Omega)} \leq K_T h^k \left( \int_0^T \left\| \frac{\partial f}{\partial t^3}(s) \right\|_{L^2(\Omega)}^2 \, ds \right)^{1/2}, \ 0 \leq t \leq T.$$  

The Galerkin approximate solution $u^k \in C^2(\mathbb{R}, W^k_C)$ is the solution of the system of ODEs

$$\frac{d^2}{dt^2} \langle u^k, v^k \rangle + a_C(u^k, v^k) = \langle f, v^k \rangle, \ v^k \in W^k_C$$

Denote by $P^k_C$ the projection of $H^1_0(\Omega)$ onto $W^k_C$ with respect to the inner product $a_C(\cdot, \cdot)$. A standard calculation (see Strang and Fix (1973), or Terentyev and Symes (2009), equation A-4) shows that

$$\frac{d}{dt} e_C[u^k - P^k_C u] \leq \frac{2}{\alpha} e_C[u^k - P^k_C u] + 2\alpha \left\| \frac{d^2}{dt^2} (u - P^k_C u) \right\|_{L^2(\Omega)}^2$$

for any $\alpha > 0$. From a standard differential inequality,

$$e_C[u^k - P^k_C u](t) \leq K_T \int_0^T \left\| \frac{d^2}{dt^2} (u - P^k_C u) \right\|_{L^2(\Omega)}^2$$  

(8)
Since the $H^1$ norm dominates the $L^2$ norm on $H^1_0(\Omega)$, and since the inner product defined by $a_C$ is equivalent to the $H^1$ inner product, it follows that for each $t \in [0, T]$,

\[
\left\| \frac{d^2}{dt^2} (u - P^k_C u) \right\|_{L^2(\Omega)}^2 \leq C_1 \left\| \frac{d^2}{dt^2} (u - P^k_C u) \right\|_{H^1(\Omega)}^2
\]

\[
\leq C_2 a_C \left( \frac{d^2}{dt^2} (u - P^k_C u), \frac{d^2}{dt^2} (u - P^k_C u) \right)
\]

\[
= C_2 \inf_{v \in W^k_C} a_C \left( \frac{d^2 u}{dt^2} - v, \frac{d^2 u}{dt^2} - v \right)
\]

\[
\leq C_2 C_3 \inf_{v \in W^k_C} \left\| \frac{d^2 u}{dt^2} - v \right\|_{H^1(\Omega)}^2.
\]

Similarly,

\[
a_C(u - P^k_C u, u - P^k_C u) = \inf_{v \in W^k_C} a_C(u - v, u - v) \leq \inf_{v \in W^k_C} \| u - v \|_{H^1(\Omega)}^2
\]

In these inequalities, $C_1, C_2, C_3$ are $t$-independent constants depending on $\Omega$ and $C$.

Combine the last two inequalities with (6), (7), and (8) to obtain

\[
ee_C[u^k - P^k_C u](t) \leq K_T(h^k)^2 \int_0^T \left\| \frac{\partial^3 f}{\partial t^3} (s) \right\|_{L^2(\Omega)}^2 ds, \quad 0 \leq t \leq T,
\]

and

\[
ee_C[u - P^k_C u](t) \leq K_T(h^k)^2 \int_0^T \left\| \frac{\partial^2 f}{\partial t^2} (s) \right\|_{L^2(\Omega)}^2 ds, \quad 0 \leq t \leq T;
\]

whence

\[
ee[u - u^k]^{1/2}(t) \leq K_T h^k \left\| \frac{\partial^3 f}{\partial t^3} \right\|_{L^2(\Omega \times [0,T])}, \quad 0 \leq t \leq T,
\]

which demonstrates the optimal order of approximation in energy.

Optimal order approximation in $L^2$ follows by an adaptation of the Nitsche-Aubin lemma (Ciarlet (2002), pp. 136 ff) to time-dependent problems - see Terentyev and Symes (2009), Appendix B for details. The estimate depends on yet more smoothness in time of the source term $f$:

\[
\| u - u^k \|_{L^2(\Omega)}(t) \leq K_T(h^k)^2 \left\| \frac{\partial^3 f}{\partial t^3} \right\|_{L^2(\Omega \times [0,T])}, \quad 0 \leq t \leq T.
\]
LOCALIZATION FOR WAVES

As mentioned in the introduction, formally hyperbolic systems such as the simple second-order scalar equation (4) propagate the support of solutions at finite speed (Blazek et al., 2008). For (4), the maximum speed of propagation is \( v[C] = \text{ess sup} \sqrt{\rho[C]} \), in which \( \rho \) denotes spectral radius. That is, for any weak solution \( u \), \( \tau > 0 \),

\[
\text{supp} \ u(t + \tau) \subset \left( \left( \text{supp} \ u(t) \cup \text{supp} \frac{\partial u}{\partial t}(t) \right) + B_{v[C] \tau}(0) \right) \cap \Omega
\]

In the remainder of this section, I will use this property to define a localization of the transfer basis construction which produces an approximate solution of (4) with optimal order error in energy.

Suppose that \( \mathcal{U} = \{ \Omega_i, i = 1, ..., N_d \} \) is an open cover of \( \Omega \), and \( \mathcal{P} = \{ \chi_i : i = 1, ..., N_d \} \) a subordinate partition of unity. Choose \( \tau > 0 \), define \( r = v[C] \tau \), and choose a second open cover \( \mathcal{U}^+ = \Omega_i^+ i = 1, ..., N_d \), so that

\[
\Omega_i + B_r(0) \subset \Omega_i^+, \ i = 1, ..., N_d
\]

(11)

Select \( \gamma \in C^\infty(\mathbb{R}) \) for which \( \gamma(t) = 0 \) for \( t < -\tau/2 \), \( \gamma(t) = 1 \) for \( t > 0 \). For \( T \in \mathbb{R} \), denote by \( \gamma_T(t) = \gamma(t - T) \).

Suppose that \( u \in C^1(\mathbb{R}, L^2(\Omega)) \cap C^0(\mathbb{R}, H^1_0(\Omega)) \) is a weak solution of (4). Then \( \gamma_T u \) is a weak solution of

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla \cdot C \nabla \right) \gamma_T u = \gamma_T f - 2 \frac{\partial \gamma_T}{\partial t} \frac{\partial u}{\partial t} - \frac{\partial^2 \gamma_T}{\partial t^2} u,
\]

(12)

and of course \( \gamma_T u(t) = 0 \) for \( t < T - \tau/2 \), \( \gamma_T u(t) = u(t) \) for \( t \geq T \).

Denote by \( f_T[u] \) the right-hand side of (12). Note that \( f_T[u] \) depends on \( u \) only in the interval \( T - \tau/2 < t < T \), and for \( t > T \), \( f_T[u] = f \). Also note that the basic energy estimate (5) implies that \( f_T[u] \) has as many time derivatives in \( L^2_{loc}(\mathbb{R}, L^2(\Omega)) \) as does \( f \).

Denote by \( w_i \in C^1(\mathbb{R}, L^2(\Omega_i^+)) \cap C^0(\mathbb{R}, H^1_0(\Omega_i^+)) \) the weak solution in \( \Omega_i^+ \) of

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla \cdot C \nabla \right) w_i = \chi_i f_T[u]; w_i(t) = 0, t < T - \tau/2.
\]

(13)

Extension by zero embeds \( L^2(\Omega_i^+) \) in \( L^2(\Omega) \), and \( H^1_0(\Omega_i^+) \) in \( H^1_0(\Omega) \). With this identification, \( w_i \in C^1(\mathbb{R}, L^2(\Omega)) \cap C^0(\mathbb{R}, H^1_0(\Omega)) \). Also, because the support of the right-hand side of (13) is contained in \( \Omega_i^+ \) and \( w_i = 0 \) for \( t < T - \tau/2 \), (11) implies that the support of \( w_i(t) \) is contained in \( \Omega_i^+ \) for \( t < T + \tau/2 \), hence \( w_i \) is actually a weak solution of (4) for \( t < T + \tau/2 \). The uniqueness of weak solutions implies that

\[
\sum_{i=1}^{N_d} w_i(t) = \gamma_T(t) u(t), \ t \leq T + \tau/2
\]
hence
\[ \sum_{i=1}^{N_d} w_i(t) = u(t), \quad T \leq t \leq T + \tau/2. \] (14)

Equation (14) suggests an algorithm for time extrapolation of weak solutions of (4): given the data \( f, C \) and a weak solution \( u(t), t \leq T \),

1. compute \( f_T[u] \) (localize in time);

2. multiply by the cutoff functions \( \chi_i, i = 1, \ldots, N_d \) (localize in space);

3. construct the weak solutions of (13) in the domains \( \Omega_i^{++}, i = 1, \ldots, N_d \) over the time interval \([T - \tau/2, T + \tau/2]\);

4. synthesize \( u(t), t < T + \tau/2 \) via (14).

**ECONOMICS**

- \( N_d \) = number of subdomains
- \( N_c \) = number of coarse cells per subdomain
- \( N_f \) = number of fine cells per coarse cell
- \( M_f = N_d \times N_c \times N_f \) = number of fine cells in domain
- \( N_r \) = number of right-hand sides in multi-simulation
- \( M_c = N_d \times N_c \) = number of coarse cells in domain
- \( S[N] \) = number of flops for elliptic solve on domain with \( N \) elements - depends on many factors other than \( N \), ranges from \( O(N^3) \) for dense matrix LU to \( O(N^2) \) for dense matrix Krylov methods with exceptionally good preconditioning, or \( O(N \log N) \) for Krylov methods with Fourier preconditioning applied to low-contrast problems with local stencils.

Number of coarse cells = size of transfer basis = \( M_c \). Number of fine grid cells per domain = \( N_c \times N_f \). Cost of computing transfer basis = \( M_c \times S[N_c \times N_f] \).

Number of time steps for fine grid computation = \( O((M_f)^{1/d}) \). Cost per time step (assuming usual local stencil) = \( O(M_f) \). Total cost = \( O((M_f)^{1+1/d}) \).

Number of time steps for coarse grid computation = \( O((M_c)^{1/d}) \). Cost per time step, assuming that subdomain stiffness matrices are dense = \( O(N_d \times N_c^2) = O(M_c \times N_c) \). Total cost = \( O((M_c)^{1+1/d} \times N_c) \).

Thus the ratio of costs (sparse fine grid time stepping)/(dense coarse grid time stepping) = \( O(N_f^{1+1/d} / N_c) \).
Successful amortization of transfer basis computation over time steps means that the cost of basis computation should not dominate the cost of time stepping. The basis construction cost amortizes over time steps, also over right-hand sides. That is,

\[ M_c * S[N_c * N_f] \leq O(N_r M_c^{1+1/d} * N_c), \]

hence

\[ S[N_c * N_f] \leq O(N_r M_c^{1/d} * N_c). \tag{15} \]

This condition couples \( N_f, N_c, \) and \( N_d, \) in a somewhat nontransparent way.

Assuming optimal preconditioned Krylov and local stencils for the transfer basis construction, (15) boils down to

\[ N_f (\log N_c + \log N_f) \leq O(N_r M_c^{1/d}) \]

For a single simulation \( (N_r = 1), \) this means that doubling the number of fine grid cells per coarse grid cell implies increasing the coarse grid size \( M_c \) by \( 2^d \) in order to keep the transfer basis construction cost equilibrated with the cost of time-stepping. For less efficient elliptic solver methodology, the constraint on \( N_f \) is even more strict.

On the other hand, equilibration is more than recovered if \( N_r \simeq N_f. \)

To understand the implications of this observation, take \( d = 3, N_d = N_c = N_f = 1000, N_r = 1000. \) Assuming a local FE stencil with 10 flops per node and error reduction of 0.5 per Krylov subspace iteration, unit initial error, and required relative error of \( 10^{-3}, \) the cost of basis construction is \( \simeq 10^{14} \) flops. The cost of fine-grid time stepping is roughly \( 10^{17} \) flops, accounting for \( N_r = 1000 \) and assuming an effective Courant number of 0.1. The cost of coarse-grid time stepping is \( \simeq 10^{12} \) flops per right-hand side. or \( 10^{15} \) for \( N_r = 1000 \) right-hand sides.

If on the other hand \( N_f \) is increased to \( 10^6, \) then the cost of computing the transfer basis goes up by \( 10^3, \) to \( 10^{17} \) flops, whereas the coarse grid time-stepping cost is independent of \( N_f \) hence remains \( \simeq 10^{15} \) flops, so that basis construction dominates the cost of solving even \( N_r = 1000 \) time stepping problems. The cost of fine-grid time stepping, on the other hand, goes up to \( \simeq 10^{21} \) flops.

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