Time-dependent coupling of Navier-Stokes and Darcy flows

Ayçıl Çeşmelioğlu, Vivette Girault, Béatrice Rivière

Abstract

A weak solution of the coupling of time-dependent Navier-Stokes equations with Darcy equations is defined. The interface conditions include the Beavers-Joseph-Saffman condition. Existence and uniqueness of the weak solution are obtained by a constructive approach. The analysis is valid for weak regularity interfaces.

1 Introduction

The coupling of Navier-Stokes with Darcy equations is encountered in many engineering problems such as groundwater contamination or blood flow in arteries. Their importance motivate the interest in understanding and solving this coupled system. The coupling is commonly modelled by the interface condition postulated by Beavers and Joseph in [1] or by its simplification introduced by Saffman in [2], and called the Beavers-Joseph-Saffman interface condition. We shall use this last condition in the present study. The reader can refer to the work of Jäger and Mikelić in [3] for the derivation by homogenization of the Beavers-Joseph-Saffman interface condition.

Many authors have studied the coupling of the Stokes and Darcy systems, and we can only list very few of them. For instance, the reader can refer to the works of Arbogast and Lehr in [4], of Arbogast and Brunson in [5], of Burman and Hansbo in [6], of Cao, Gunzburger, Hua and Wang in [7], of Kanschat and Rivière in [8], of Layton, Schieweck and Yotov in [9], of Vassilev and Yotov in [10], of Rivière and Yotov in [11], of Rivière in [12], of Mu and Xu in [13], of Mardal, Tai and Winther in [14], of Hanspal, Waghode, Nassehi and Wakeman in [15], of Discacciati and Quarteroni in [16], or of Discacciati, Quarteroni and Valli in [17].

In contrast, there are not many works on the coupling of Navier-Stokes and Darcy equations. The steady-state case has been mostly studied by Discacciati in [18], by Discacciati and Quarteroni in [16, 19], by Badea, Discacciati and Quarteroni in [20], by Girault and Rivière in [21], and by Chidyagwai and Rivière in [22, 23]. To our knowledge, the time-dependent coupled Navier-Stokes/Darcy problem, with Beavers-Joseph-Saffmann interface condition, has only been mathematically and numerically analyzed by

Çeşmelioğlu and Rivière in [24, 25]. In these references, the authors include inertial effects in the balance of forces at the interface. This simplifies the analysis because it brings a stronger control on the nonlinear convection term, but physical justification of this model is not clear, although it is meaningful from a mathematical point of view.

Therefore, following the work of Girault and Riviere in [21] who analyzed the steady state case without inertial effects on the interface, we propose here to study a time-dependent version of this problem without these inertial forces. The analysis of this model is not altogether straightforward because it does not satisfy an unconditional energy inequality, even if the data are smooth. As a consequence, we shall prove global existence in time of a solution for suitably small data, and uniqueness of a suitably small solution. Our proof is based on uniform a priori estimates for the solution of a Galerkin semi-discrete scheme in space, the full set of estimates being obtained by differentiating the scheme with respect to time. This approach is fairly robust and constructive in the sense that the theoretical analysis adapts easily to the numerical analysis of finite-element discretizations in space. Furthermore, it has the advantage of being independent of the Steklov-Poincaré operator used in [16, 19, 20] that does not seem to apply to rough interfaces. In contrast, we can handle the important case of interfaces with corners.

An outline of the paper follows. The rest of this section introduces the problem and states the main result. Section 2 gives the proof in several steps. Conclusions follow.

1.1 Statement of the problem

To simplify, we consider the case of one connected interface, but the extension to several interfaces is straightforward. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be an open, bounded domain with Lipschitz continuous boundary $\partial\Omega$. The surface region of Ω is denoted by Ω_1 and the subsurface region is denoted by Ω_2 with Lipschitz continuous boundaries $\partial\Omega_1$ and $\partial\Omega_2$. The interface separating the surface and the subsurface regions is denoted by $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. The boundary of Ω is split into $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$, $i = 1, 2$ corresponding to the outer boundaries of the surface and subsurface. Finally, the boundary Γ_2 is decomposed into two disjoint open sets: $\Gamma_2 = \Gamma_{2D} \cup \Gamma_{2N}$.

The partial differential equations governing the flow problem are given by

$$\begin{aligned} \boldsymbol{u}' - \nabla \cdot (2\mu \boldsymbol{D}(\boldsymbol{u}) - p_1 \boldsymbol{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \boldsymbol{f}_1, \quad \text{a.e. in} \quad \Omega_1 \times]0,T[, \quad (1.1)\\ \nabla \cdot \boldsymbol{u} &= 0, \quad \text{a.e. in} \quad \Omega_1 \times]0,T[, \quad (1.2)\\ -\nabla \cdot \big(\boldsymbol{K} \nabla p_2\big) &= f_2, \quad \text{a.e. in} \quad \Omega_2 \times]0,T[. \quad (1.3) \end{aligned}$$

The prime stands for the derivative with respect to time, $\boldsymbol{n}_{\Omega_i}$ is the exterior unit vector normal to $\partial\Omega_i$, $\boldsymbol{I}$ is the $d \times d$ identity tensor, $\boldsymbol{K}$ is the permeability tensor, and $\boldsymbol{D}(\boldsymbol{v})$ is the deformation tensor defined by

$$\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big).$$

System (1.1)–(1.3) is complemented by the boundary and the interface conditions below. As we are mostly interested in the coupling aspect of this problem, we prescribe standard academic conditions on Γ_i:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, \quad \text{a.e. on} \quad \Gamma_1 \times]0,T[, \qquad (1.4)\\ p_2 &= 0, \quad \text{a.e. on} \quad \Gamma_{2D} \times]0,T[, \qquad (1.5)\\ \boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_{\Omega_2} &= g, \quad \text{a.e. on} \quad \Gamma_{2N} \times]0,T[. \qquad (1.6) \end{aligned}$$

Here we assume that $|\Gamma_1| > 0$ and $|\Gamma_{2D}| > 0$. Now, let $\boldsymbol{n}_{12}$ be the unit normal vector to Γ_{12} pointing from Ω_1 to Ω_2 and let $\boldsymbol{\tau}_{12}^j$, $1 \le j \le d-1$, be an othonormal set of unit vectors on the tangent plane to Γ_{12}. On the interface Γ_{12}, we prescribe the following interface conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n}_{12} = -\boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_{12}, \ \text{a.e. on } \Gamma_{12} \times]0,T[, \quad (1.7)$$

$$\big((-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}\big) \cdot \boldsymbol{n}_{12} = p_2, \ \text{a.e. on } \Gamma_{12} \times]0,T[, \quad (1.8)$$

$$\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}^j = -2\mu G^j \big(\boldsymbol{D}(\boldsymbol{u}) \boldsymbol{n}_{12}\big) \cdot \boldsymbol{\tau}_{12}^j, 1 \le j \le d-1, \ \text{a.e. on } \Gamma_{12} \times]0,T[, \quad (1.9)$$

where

$$G^j = \frac{\mu\alpha}{(\boldsymbol{K}\boldsymbol{\tau}_{12}^j, \boldsymbol{\tau}_{12}^j)^{1/2}}.$$

Finally, to simplify the discussion, we prescribe a zero initial condition:

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{0}. \qquad (1.10)$$

The constant $\alpha > 0$ is given and is usually obtained from experimental data. We assume that the permeability tensor $\boldsymbol{K}$ is independent of time, uniformly bounded and positive definite: There exist $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ satisfying

$$\forall \boldsymbol{x} \in \overline{\Omega}_2\,,\ \lambda_{\min} \boldsymbol{x} \cdot \boldsymbol{x} \le \boldsymbol{K}\boldsymbol{x} \cdot \boldsymbol{x} \le \lambda_{\max} \boldsymbol{x} \cdot \boldsymbol{x}. \qquad (1.11)$$

Let

$$\boldsymbol{X} = \{\boldsymbol{v} \in H^1(\Omega_1)^d \,:\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},$$
$$M = \{q \in H^1(\Omega_2) \,:\, q = 0 \text{ on } \Gamma_{2D}\},$$

and

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X}\,;\, \nabla \cdot \boldsymbol{v} = 0 \text{ in } \ \Omega_1\};$$

see Section 1.3 for the definition of usual Sobolev spaces. Note that the symmetric deformation tensor satisfies

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}, (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} = (\boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1}. \tag{1.12}$$

For the moment, take $\boldsymbol{f}_1 \in L^2(\Omega_1 \times]0, T[)^d$, $f_2 \in L^2(\Omega_2 \times]0, T[)$ and $g \in L^2(\Gamma_{2N} \times]0, T[)$. These assumptions will be progressively refined further on. We propose the following weak formulation for the problem (1.1)–(1.10):
Find $\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega_1)^d) \cap L^2(0, T; \boldsymbol{X})$ with $\boldsymbol{u}' \in L^1(0, T; L^{3/2}(\Omega_1)^d)$, $p_2 \in L^2(0, T; M)$ and $p_1 \in L^1(0, T; L^2(\Omega_1))$ such that

$$(P) \begin{cases} \forall \boldsymbol{v} \in \boldsymbol{X}, \forall q \in M, & (\boldsymbol{u}', \boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} \\ & +(\boldsymbol{K} \nabla p_2, \nabla q)_{\Omega_2} + (p_2, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \sum_{j=1}^{d-1} (\frac{1}{G^j} \boldsymbol{u} \cdot \boldsymbol{\tau}_{12}^j, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} \\ & -(\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2} + (g, q)_{\Gamma_{2N}} \text{ a.e. in }]0, T[, \\ \forall q \in L^2(\Omega_1), & (\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0 \text{ a.e. in }]0, T[, \\ & \boldsymbol{u}(0) = \boldsymbol{0} \text{ a.e. in } \Omega_1. \end{cases}$$

1.2 The interface condition

Before proceeding, it is necessary to make sure that the interface conditions (1.8) and (1.9) are meaningful for a solution of problem (P). In the steady-state case, they are interpreted as (cf. [21]):

$$\boldsymbol{\sigma}(\boldsymbol{u}, p_1)\boldsymbol{n}_{12} = \boldsymbol{b} \text{ with } \boldsymbol{b} = p_2 \boldsymbol{n}_{12} + \sum_{j=1}^{d-1} \frac{1}{G^j} (\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}^j) \boldsymbol{\tau}_{12}^j, \tag{1.13}$$

where the tensor $\boldsymbol{\sigma}(\boldsymbol{u}, p_1)$ is the Cauchy stress tensor:

$$\boldsymbol{\sigma}(\boldsymbol{u}, p_1) = -2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I}.$$

Since $\boldsymbol{b}$ belongs at least to $L^2(0, T; L^4(\Gamma_{12})^d)$, (1.13) makes sense provided the trace of $\boldsymbol{\sigma}(\boldsymbol{u}, p_1)\boldsymbol{n}_{12}$ on Γ_{12} can be defined, even if it is only in a weak sense.

Now, the assumption on $\boldsymbol{u}$ implies that $\boldsymbol{u} \in L^2(0,T;L^6(\Omega_1)^d)$; therefore the convection term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ belongs to $L^1(0,T;L^{3/2}(\Omega_1)^d)$. Then passing $\boldsymbol{u}'$ to the right-hand side of (1.1), our assumption on $\boldsymbol{f}_1$ yields

$$-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p_1) = \boldsymbol{f}_1 - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}' \in L^1(0,T;L^{3/2}(\Omega_1)^d).$$

The assumptions on $\boldsymbol{u}$ and p_1 also give

$$\boldsymbol{\sigma}(\boldsymbol{u},p_1) \in L^1(0,T;L^2(\Omega_1)^{d\times d}).$$

This means that each row of $\boldsymbol{\sigma}(\boldsymbol{u},p_1)$ belong to $L^1(0,T;H^{3/2}(\mathrm{div};\Omega_1))$ where

$$H^{3/2}(\mathrm{div};\Omega_1) = \{\boldsymbol{v}\in L^2(\Omega_1)^d\,;\,\nabla\cdot\boldsymbol{v}\in L^{3/2}(\Omega_1)\}.$$

A standard argument shows that $H^1(\Omega_1)^d$ is dense in $H^{3/2}(\mathrm{div};\Omega_1)$; therefore Green's formula:

$$\forall\boldsymbol{v}\in H^1(\Omega_1)^d, \forall w\in H^1(\Omega_1)\,,\,\langle\boldsymbol{v}\cdot\boldsymbol{n}_{\Omega_1},w\rangle_{\partial\Omega_1} = \int_{\Omega_1}(\nabla\cdot\boldsymbol{v})w + \int_{\Omega_1}\boldsymbol{v}\cdot\nabla w, \tag{1.14}$$

permits to define the normal trace $\boldsymbol{v}\cdot\boldsymbol{n}_{\Omega_1}$ in $H^{-1/2}(\partial\Omega_1)$ of functions in $H^1(\Omega_1)^d$ and extends by density to all $\boldsymbol{v}\in H^{3/2}(\mathrm{div};\Omega_1)$. As a consequence, each row of $\boldsymbol{\sigma}(\boldsymbol{u},p_1)\boldsymbol{n}_{12}$ belongs to $L^1\big(0,T;(H^{1/2}_{00}(\Gamma_{12}))'\big)$, where $(H^{1/2}_{00}(\Gamma_{12}))'$ is the dual space of $H^{1/2}_{00}(\Gamma_{12})$. Therefore we interpret (1.8) and (1.9) as:

$$\boldsymbol{\sigma}(\boldsymbol{u},p_1)\boldsymbol{n}_{12} = \boldsymbol{b},\ \boldsymbol{b} = p_2\boldsymbol{n}_{12} + \sum_{j=1}^{d-1}\frac{1}{G^j}(\boldsymbol{u}\cdot\boldsymbol{\tau}^j_{12})\boldsymbol{\tau}^j_{12}\ \text{a.e. on}\quad \Gamma_{12}\times(0,T). \tag{1.15}$$

Of course, (1.7) is meaningful because $\nabla\cdot(\boldsymbol{K}\nabla p_2)$ belongs to $L^2(\Omega_2\times]0,T[)$. Hence, if the solution is sought for in the spaces of problem (P), then problem (1.1)–(1.10) in set in $L^1(0,T;H^{-1}(\Omega_1)^d)$ and $L^2(0,T;H^{-1}(\Omega_2)^d)$. Then a standard argument shows that problems (1.1)–(1.10) and (P) are equivalent.

1.3 Notation, classical, and main results

The rest of this section is devoted to the definitions, inequalities, and results that will be used throughout the paper. To simplify the presentation, we set most definitions in dimension $d=3$.

Let (k_1, k_2, k_3) denote a triple of non-negative integers, set $|k| = k_1+k_2+k_3$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}.$$

Then, for any non-negative integer m, recall the classical Sobolev space (cf. Adams [26] or Nečas [27])

$$H^m(\Omega) = \{v \in L^2(\Omega)\,;\, \partial^k v \in L^2(\Omega) \,\forall\, |k| \leq m\},$$

equipped with the seminorm

$$|v|_{H^m(\Omega)} = \Big[\sum_{|k|=m} \int_\Omega |\partial^k v|^2 \, d\boldsymbol{x} \Big]^{1/2},$$

and norm (for which it is a Hilbert space)

$$\|v\|_{H^m(\Omega)} = \left[\sum_{0 \leq |k| \leq m} |v|^2_{H^k(\Omega)} \right]^{1/2}.$$

This definition is extended to any real number $s = m + s'$ for an integer $m \geq 0$ and $0 < s' < 1$ by defining the fractional semi-norm and norm:

$$|v|_{H^s(\Omega)} = \left(\sum_{|k|=m} \int_\Omega \int_\Omega \frac{|\partial^k v(\boldsymbol{x}) - \partial^k v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2\,s'}} \, d\boldsymbol{x} \, d\boldsymbol{y} \right)^{1/2},$$

$$\|v\|_{H^s(\Omega)} = \left(\|v\|^2_{H^m(\Omega)} + |v|^2_{H^s(\Omega)} \right)^{1/2}.$$

The reader can refer to Lions and Magenes [28] and Grisvard [29] for properties of these spaces. In the sequel we shall frequently use the fractional Sobolev spaces $H^{1/2}(\Gamma)$ and $H^{1/2}_{00}(\Gamma)$ for a Lipschitz surface Γ when $d = 3$ or curve when $d = 2$ with the following seminorms and norms:

$$|v|_{H^{1/2}(\Gamma)} = \Big(\int_\Gamma \int_\Gamma \frac{\big|v(\boldsymbol{x}) - v(\boldsymbol{y})\big|^2}{|\boldsymbol{x} - \boldsymbol{y}|^d} d\boldsymbol{x} d\boldsymbol{y} \Big)^{1/2}, \; \|v\|_{H^{1/2}(\Gamma)} = \Big(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{1/2}(\Gamma)} \Big)^{1/2}, \tag{1.16}$$

$$|v|_{H_{00}^{1/2}(\Gamma)} = \left(|v|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{|v(\boldsymbol{x})|^2}{\mathrm{d}_{\partial\Gamma}(\boldsymbol{x})} d\boldsymbol{x}\right)^{1/2}, \ \|v\|_{H_{00}^{1/2}(\Gamma)} = \left(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H_{00}^{1/2}(\Gamma)}\right)^{1/2}, \tag{1.17}$$

where $\mathrm{d}_{\partial\Gamma}(\boldsymbol{x})$ denotes the distance from $\boldsymbol{x}$ to $\partial\Gamma$. When Γ is a subset of $\partial\Omega$ with positive $d-1$ measure, $H_{00}^{1/2}(\Gamma)$ is the space of traces of all functions of $H^1(\Omega)$ that vanish on $\partial\Omega \setminus \Gamma$. The above norms (1.16) and (1.17) are not equivalent except when Γ is a closed surface or curve.

Throughout the paper, we shall use the following Sobolev, Korn and trace inequalities: For any $\boldsymbol{v} \in \boldsymbol{X}$, there exist constants $S_2, S_4, T_2, T_4, C_D > 0$ depending only on Ω_1 such that

$$\|\boldsymbol{v}\|_{L^2(\Omega_1)} \le S_2 |\boldsymbol{v}|_{H^1(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Omega_1)} \le S_4 |\boldsymbol{v}|_{H^1(\Omega_1)},$$

$$\|\boldsymbol{v}\|_{L^2(\Gamma_{12})} \le T_2 |\boldsymbol{v}|_{H^1(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Gamma_{12})} \le T_4 |\boldsymbol{v}|_{H^1(\Omega_1)},$$
$$|\boldsymbol{v}|_{H^1(\Omega_1)} \le C_D \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}.$$

Also, for any $q \in M$, there exist $\tilde{S}_2, T_{12}, T_N > 0$ depending only on Ω_2 satisfying

$$\|q\|_{L^2(\Omega_2)} \le \tilde{S}_2 |q|_{H^1(\Omega_2)},$$
$$\|q\|_{H^{\frac{1}{2}}(\Gamma_{12})} \le T_{12} |q|_{H^1(\Omega_2)}, \quad \|q\|_{L^2(\Gamma_{2N})} \le T_N |q|_{H^1(\Omega_2)}.$$

Let $E \in \mathcal{L}(M, H^1(\Omega))$ be an extension operator and let C_E denote the continuity constant of the extension.

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a,b[$ with values in a functional space, say X. More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number r, $1 \le r \le \infty$, we define

$$L^r(a,b;X) = \{f \text{ measurable in }]a,b[; \int_a^b \|f(t)\|_X^r dt < \infty\},$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r dt\right)^{1/r},$$

with the usual modification if $r = \infty$. It is a Banach space if X is a Banach space. Here X is usually a Sobolev space. In particular, $L^2(a,b;H^m(\Omega))$ is a Hilbert space and $L^2(a,b;L^2(\Omega))$ coincides with $L^2(\Omega\times]a,b[)$. In addition, we shall also use spaces with derivatives in time, such as

$$H^1(a,b;X) = \{f \in L^2(]a,b[;X)\,;\ f' \in L^2(]a,b[;X)\}\,,$$

equipped with the graph norm

$$\|f\|_{H^1(a,b;X)} = \Big(\|f\|^2_{L^2(a,b;X)} + \|f'\|^2_{L^2(a,b;X)}\Big)^{1/2},$$

for which is is a Hilbert space. The following result establishes compact imbeddings in space and time and generalizes the Aubin-Lions Lemma, see Aubin [30], or Lions [31]. Its proof, due to Simon, is written in [32].

Theorem 1.1 *Let X, W, Y be three Banach spaces with continuous imbeddings: $X \subset W \subset Y$, the imbedding of X into W being compact. Then for any number $q \in [1,\infty]$, the space*

$$\{v \in L^q(0,T;X)\,;\ \frac{\partial v}{\partial t} \in L^1(0,T;Y)\} \tag{1.18}$$

is compactly imbedded into $L^q(0,T;W)$.

The following theorem is the main result of this work. Recall that

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X}\,;\ \nabla\cdot\boldsymbol{v} = 0 \text{ in } \ \Omega_1\}.$$

Theorem 1.2 *In addition to the basic assumptions on the data $\boldsymbol{f}_1, f_2, g$ and $\boldsymbol{K}$ stated above, suppose that $\boldsymbol{f}_1 \in H^1(0,T;L^2(\Omega_1)^d)$, $f_2 \in H^1(0,T;L^2(\Omega_2))$, $g \in H^1(0,T;L^2(\Gamma_{2N}))$. Let*

$$\mathcal{A} = \frac{C_D^2 S_2^2}{2\mu}\big(\|\boldsymbol{f}_1\|^2_{H^1(0,T;L^2(\Omega_1)^d)} + \|\boldsymbol{f}_1\|^2_{L^\infty(0,T;L^2(\Omega_1)^d)}\big) + \frac{\tilde{S}_2^2}{\lambda_{\min}}\big(\|f_2\|^2_{H^1(0,T;L^2(\Omega_2))} + \|f_2\|^2_{L^\infty(0,T;L^2(\Omega_2))}\big)$$
$$+ \frac{T_N^2}{\lambda_{\min}}\big(\|g\|^2_{H^1(0,T;L^2(\Gamma_{2N}))} + \|g\|^2_{L^\infty(0,T;L^2(\Gamma_{2N}))}\Big), \tag{1.19}$$

$$\mathcal{C} = \|\boldsymbol{f}_1(0)\|^2_{L^2(\Omega_1)} + \frac{C_E^2}{\lambda_{\min}}(\tilde{S}_2^2+1)\big(\tilde{S}_2\|f_2(0)\|_{L^2(\Omega_2)} + T_N\|g(0)\|_{L^2(\Gamma_{2N})}\big)^2. \tag{1.20}$$

If the data satisfy

$$\mathcal{A} + 2\mathcal{C} < \big(\frac{\mu}{C_D^2}\big)^3 \frac{1}{4S_4^4}, \tag{1.21}$$

then problem (P) has one and only one solution $(\boldsymbol{u}, p_1, p_2) \in H^1(0,T;\boldsymbol{V}) \times L^\infty(0,T;L^2(\Omega_1)) \times H^1(0,T;H^1(\Omega_2))$ *satisfying:*

$$\|\boldsymbol{D}(\boldsymbol{u})\|_{L^\infty(0,T;L^2(\Omega_1)^{d\times d})} < \frac{\mu}{2C_D^3 S_4^2}, \tag{1.22}$$

$$\|\boldsymbol{u}'\|^2_{L^\infty(0,T;L^2(\Omega_1)^d)} \leq \frac{C_D^2 S_2^2}{2\mu}\|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1\times]0,T[)} + 2\mathcal{C} + \frac{1}{\lambda_{\min}}\big(\tilde{S}_2^2\|f_2\|^2_{L^2(\Omega_2\times]0,T[)} + T_N^2\|g\|^2_{L^2(\Gamma_{2N}\times]0,T[)}\big), \tag{1.23}$$

$$\|\boldsymbol{D}(\boldsymbol{u}')\|^2_{L^2(\Omega_1\times]0,T[)} \leq \big(\frac{C_D S_2}{\mu}\big)^2\|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1\times]0,T[)} + \frac{2}{\mu}\mathcal{C} + \frac{1}{\mu\lambda_{\min}}\big(\tilde{S}_2^2\|f_2\|^2_{L^2(\Omega_2\times]0,T[)} + T_N^2\|g\|^2_{L^2(\Gamma_{2N}\times]0,T[)}\big), \tag{1.24}$$

$$\|\boldsymbol{K}^{\frac{1}{2}}\nabla p_2\|_{L^\infty(0,T;L^2(\Omega_2))} \leq \frac{1}{\lambda_{\min}^{\frac{1}{2}}}\big(\tilde{S}_2\|f_2\|_{L^\infty(0,T;L^2(\Omega_2))} + T_N\|g\|_{L^\infty(0,T;L^2(\Gamma_{2N}))} + C_E(1+\tilde{S}_2^2)^{\frac{1}{2}}(1+S_2^2)^{\frac{1}{2}}\frac{\mu}{2C_D^2 S_4^2}\big), \tag{1.25}$$

$$\|\boldsymbol{K}^{\frac{1}{2}}\nabla p'_2\|^2_{L^2(\Omega_2\times]0,T[)} \leq \frac{C_D^2 S_2^2}{2\mu}\|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1\times]0,T[)} + 2\mathcal{C} + \frac{2}{\lambda_{\min}}\big(\tilde{S}_2^2\|f'_2\|_{L^2(\Omega_2\times]0,T[)} + T_N^2\|g'\|_{L^2(\Gamma_{2N}\times]0,T[)}\big), \tag{1.26}$$

$$\|p_1\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C\big(\|\boldsymbol{u}'\|_{L^\infty(0,T;L^2(\Omega_1)^d)} + \|\boldsymbol{D}(\boldsymbol{u})\|_{L^\infty(0,T;L^2(\Omega_1)^{d\times d})} + \|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1(\Omega_1)^d)} + \|p_2\|_{L^\infty(0,T;H^1(\Omega_2))} + \|\boldsymbol{f}_1\|_{L^\infty(0,T;L^2(\Omega_1)^d)}\big), \tag{1.27}$$

where C *is a constant that only depends on* $\mu, S_2, S_4, \lambda_{\max}, \lambda_{\min}, T_{12}, T_2$ *and* α.

The next section gives the proof of this result by considering first a reduced problem.

2 Existence and uniqueness of weak solution

Consider the reduced formulation of problem (P) in $\boldsymbol{V}$:
Find $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega_1)^d) \cap L^2(0,T;\boldsymbol{V})$ with $\boldsymbol{u}' \in L^1(0,T;L^{3/2}(\Omega_1)^d)$ and $p_2 \in L^2(0,T;M)$ solution of

$$(P_V)\begin{cases} \forall \boldsymbol{v} \in \boldsymbol{V}, \forall q \in M, & (\boldsymbol{u}',\boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2,\nabla q)_{\Omega_2} \\ & +(p_2,\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \sum_{j=1}^{d-1}(\frac{1}{G^j}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}^j,\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} \\ & = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1} + (f_2,q)_{\Omega_2} + (g,q)_{\Gamma_{2N}} \text{ a.e. in }]0,T[, \\ & \boldsymbol{u}(0) = \boldsymbol{0} \text{ a.e. in } \Omega_1. \end{cases}$$

Clearly Problem (P) implies this problem. The converse is established in Section 2.4.

2.1 A Galerkin solution

Let us refine the assumptions on the data: let $\boldsymbol{f}_1 \in \mathcal{C}^0(0,T;L^2(\Omega_1)^d)$, $f_2 \in \mathcal{C}^0(0,T;L^2(\Omega_2))$ and $g \in \mathcal{C}^0(0,T;L^2(\Gamma_N))$. We construct a solution by Galerkin's method. As $\boldsymbol{V}\times M$ is separable, it has a basis of smooth functions $\{(\boldsymbol{\Phi}_m,\varphi_m)\}_{m\geq 0}$. Denote by $\boldsymbol{V}_m = \text{span}\{\boldsymbol{\Phi}_i,\, i = 1,\dots,m\}$ and by $M_m = \text{span}\{\varphi_i,\, i = 1,\dots,m\}$, the spaces spanned by the first m basis functions. The following problem is a semi-discretization of (P_V) in this basis: Find $\boldsymbol{u}_m \in \mathcal{C}^1(0,T;\boldsymbol{V}_m)$ and $p_m \in \mathcal{C}^0(0,T;M_m)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}_m$ and for all $q \in M_m$,

$$(\boldsymbol{u}_m',\boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}_m),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m,\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_m,\nabla q)_{\Omega_2} + (p_m,\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$+\sum_{j=1}^{d-1}(\frac{1}{G^j}\boldsymbol{u}_m\cdot\boldsymbol{\tau}_{12}^j,\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{u}_m\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1} + (f_2,q)_{\Omega_2} + (g,q)_{\Gamma_{2N}}, \quad \text{for all } t \in]0,T[\tag{2.1}$$

$$\boldsymbol{u}_m(0) = \boldsymbol{0}. \tag{2.2}$$

Problem (2.1) can be reformulated by observing that p_m is determined by $\boldsymbol{u}_m$: Indeed, for a given $\boldsymbol{u} \in H^1(\Omega_1)^d$, the problem : Find $p_m \in M_m$ such that

$$\forall q \in M_m\,,\ (\boldsymbol{K}\nabla p_m,\nabla q)_{\Omega_2} = (f_2,q)_{\Omega_2} + (g,q)_{\Gamma_{2N}} + (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}, \tag{2.3}$$

has a unique solution, say $p_m(\boldsymbol{u})$. The next lemma gives a bound for $p_m(\boldsymbol{u})$.

Lemma 2.1 *The mapping $\boldsymbol{u} \mapsto p_m(\boldsymbol{u})$ is linear and continuous, uniformly in m: There exists a constant C independent of m such that for all $\boldsymbol{u}$ in $\boldsymbol{X}$:*

$$\|\boldsymbol{K}^{1/2}\nabla\, p_m(\boldsymbol{u})\|_{L^2(\Omega_2)} \le C\big(\|f_2\|_{L^2(\Omega_2)} + \|g\|_{L^2(\Gamma_{2N})} + \|\boldsymbol{u}\|_{H(\mathrm{div};\Omega_1)}\big). \quad (2.4)$$

Proof. Linearity follows immediately from (2.3) and from uniqueness. The bound (2.4) relies on a good estimate for $(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}$. To this end, we use the extension operator $E \in \mathcal{L}(M, H^1(\Omega))$ introduced in Section 1.3. Then for all $\boldsymbol{u}$ in $\boldsymbol{X}$

$$(\nabla\cdot\boldsymbol{u}, E(q))_{\Omega_1} + (\boldsymbol{u}, \nabla\, E(q))_{\Omega_1} = (\boldsymbol{u}\cdot\boldsymbol{n}_{\Omega_1}, E(q))_{\partial\Omega_1} = (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

Hence

$$\big|(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}\big| \le \|\boldsymbol{u}\|_{H(\mathrm{div};\Omega_1)}\|E(q)\|_{H^1(\Omega_1)} \le C_E\|\boldsymbol{u}\|_{H(\mathrm{div};\Omega_1)}\|q\|_{H^1(\Omega_2)}. \quad (2.5)$$

Therefore

$$\|\boldsymbol{K}^{1/2}\nabla\, p_m\|_{L^2(\Omega_2)} \le \frac{1}{\lambda_{\min}^{1/2}}\big(\tilde{S}_2\|f_2\|_{L^2(\Omega_2)} + T_N\|g\|_{L^2(\Gamma_{2N})} + C_E(1+\tilde{S}_2^2)^{1/2}\|\boldsymbol{u}\|_{H(\mathrm{div};\Omega_1)}\big), \quad (2.6)$$

whence (2.4). ∎

When $\boldsymbol{u}$ depends on t, the statement of Lemma 2.1 is valid for any t for which $\boldsymbol{u}(t)$ exists. In particular, since $\boldsymbol{u}(0) = \boldsymbol{0}$, we have for $t = 0$,

$$\|\boldsymbol{K}^{1/2}\nabla\, p_m(\boldsymbol{0})\|_{L^2(\Omega_2)} \le \frac{1}{\lambda_{\min}^{1/2}}\big(\tilde{S}_2\|f_2(0)\|_{L^2(\Omega_2)} + T_N\|g(0)\|_{L^2(\Gamma_{2N})}\big). \quad (2.7)$$

Lemma 2.2 *For each m, there exists a time T_m with $0 < T_m \le T$ such that Problem (2.1),(2.2) has a unique maximal solution $\boldsymbol{u}_m \in \mathcal{C}^1(0, T_m; \boldsymbol{V}_m)$ and $p_m \in \mathcal{C}^0(0, T_m; M_m)$.*

Proof. Let us write

$$\boldsymbol{u}_m(\boldsymbol{x}, t) = \sum_{j=1}^{m} \alpha_j(t)\boldsymbol{\Phi}_j(\boldsymbol{x}), \quad p_m(\boldsymbol{x}, t) = \sum_{j=1}^{m} \beta_j(t)\varphi_j(\boldsymbol{x}).$$

The functions α_j and β_j are the unknowns of problem (2.1),(2.2) and it can be expressed in matrix form as

$$\begin{cases} \boldsymbol{A}\boldsymbol{\alpha}' + \boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{F}(\boldsymbol{\alpha}) + \boldsymbol{D}\boldsymbol{\beta} = \boldsymbol{b}, \\ \boldsymbol{M}\boldsymbol{\beta} - \boldsymbol{C}\boldsymbol{\alpha} = \boldsymbol{c}, \\ \text{with } \boldsymbol{\alpha}(0) \text{ given,} \end{cases}$$

and with the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ containing the components α_i and β_i respectively. The matrices are defined by

$$\boldsymbol{A}_{ij} = (\boldsymbol{\Phi}_j, \boldsymbol{\Phi}_i)_{\Omega_1}, \quad \boldsymbol{B}_{ij} = 2\mu(\boldsymbol{D}(\boldsymbol{\Phi}_j), \boldsymbol{D}(\boldsymbol{\Phi}_i))_{\Omega_1} + \sum_{k=1}^{d-1} (\frac{1}{G^k} \boldsymbol{\Phi}_j \cdot \boldsymbol{\tau}_{12}^k, \boldsymbol{\Phi}_i \cdot \boldsymbol{\tau}_{12}^k)_{\Gamma_{12}},$$

$$\boldsymbol{D}_{ij} = (\boldsymbol{\Phi}_i \cdot \boldsymbol{n}_{12}, \varphi_j)_{\Gamma_{12}}, \quad \boldsymbol{M}_{ij} = (\boldsymbol{K} \nabla \varphi_j, \nabla \varphi_i)_{\Omega_2}, \quad \boldsymbol{C}_{ij} = (\boldsymbol{\Phi}_j \cdot \boldsymbol{n}_{12}, \varphi_i)_{\Gamma_{12}} = \boldsymbol{D}_{ji},$$

and the vectors are given by

$$(\boldsymbol{F}(\boldsymbol{\alpha}))_i = \boldsymbol{N}_i \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}, \quad \boldsymbol{b}_i = (\boldsymbol{f}_1, \boldsymbol{\Phi}_i)_{\Omega_1}, \quad \boldsymbol{c}_i = (f_2, \varphi_i)_{\Omega_2} + (g, \varphi_i)_{\Gamma_{2N}},$$

where $\boldsymbol{N}_i = \big((\Phi_j \cdot \nabla \Phi_k, \Phi_i)_{\Omega_1}\big)_{1 \le j,k \le m}$ is a matrix for each $i = 1, \dots m$. As suggested by Lemma 2.1, $\boldsymbol{\alpha}$ is the only unknown here. Indeed, since $\boldsymbol{M}$ is symmetric positive definite, we can solve for $\boldsymbol{\beta} = \boldsymbol{M}^{-1}(\boldsymbol{c} + \boldsymbol{C}\boldsymbol{\alpha})$ and substitute this into the first equation, i.e. we eliminate $\boldsymbol{\beta}$. This gives

$$\boldsymbol{A}\boldsymbol{\alpha}' + \boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{F}(\boldsymbol{\alpha}) + \boldsymbol{C}^T \boldsymbol{M}^{-1}(\boldsymbol{c} + \boldsymbol{C}\boldsymbol{\alpha}) = \boldsymbol{b}.$$

As $\boldsymbol{A}$ is also invertible, solving the problem defined by (2.1) and (2.2) is equivalent to solving

$$\begin{cases} \boldsymbol{\alpha}' + \boldsymbol{A}^{-1}(\boldsymbol{B} + \boldsymbol{C}^T \boldsymbol{M}^{-1} \boldsymbol{C})\boldsymbol{\alpha} = \boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{F}(\boldsymbol{\alpha}) - \boldsymbol{C}^T \boldsymbol{M}^{-1} \boldsymbol{c}), \\ \text{with } \boldsymbol{\alpha}(0) \text{ given.} \end{cases}$$

Note that the matrix multiplying $\boldsymbol{\alpha}$ in the above left-hand side is the product of two symmetric positive definite matrices with constant coefficients. By assumption, the coefficients in the right-hand side are continuous in time and locally Lipschitz with respect to $\boldsymbol{\alpha}$. Therefore it stems from the theory of ordinary differential equations [33] that this system has a unique maximal solution $\boldsymbol{\alpha}$ in the interval $[0, T_m]$ for some T_m such that $0 < T_m \le T$ and each component of $\boldsymbol{\alpha}$ belongs to $\mathcal{C}^1(0, T_m)$. Then the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and the regularity in time of the data imply that each component of $\boldsymbol{\beta}$ is in $\mathcal{C}^0(0, T_m)$. ∎

We need a uniform in m a priori bound on $(\boldsymbol{u}_m, p_m)$ to conclude that $T_m = T$.

2.2 A priori estimates for Galerkin solution

A first a priori estimate is obtained by choosing $\boldsymbol{v} = \boldsymbol{u}_m$ and $q = p_m$ in (2.1). Cauchy-Schwarz and Hölder's inequalities yield

$$
\begin{aligned}
(\boldsymbol{u}_m', \boldsymbol{u}_m)_{\Omega_1} + 2\mu \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla p_m\|^2_{L^2(\Omega_2)} + \sum_{j=1}^{d-1} \|\frac{1}{\sqrt{G^j}} \boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}^j\|^2_{L^2(\Gamma_{12})} \\
\le C_D^3 S_4^2 \|\boldsymbol{D}(\boldsymbol{u}_m)\|^3_{L^2(\Omega_1)} + C_D S_2 \|\boldsymbol{f}_1\|_{L^2(\Omega_1)} \|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} \\
+ \frac{1}{\lambda_{\min}^{1/2}} \big(\tilde{S}_2 \|f_2\|_{L^2(\Omega_2)} + T_N \|g\|_{L^2(\Gamma_{2N})}\big) \|\boldsymbol{K}^{1/2} \nabla p_m\|_{L^2(\Omega_2)}. \quad (2.8)
\end{aligned}
$$

The cubic term in the right-hand side of (2.8) is problematic because it cannot be absorbed by the second term in the left-hand side unless it is small enough. Observe that under the assumption $\boldsymbol{u}_m(0) = \boldsymbol{0}$, the continuity of the solution guarantees that $\boldsymbol{D}(\boldsymbol{u}_m)$ will stay as small as we wish in an interval $[0, \overline{T}_m]$, where $0 < \overline{T}_m \le T_m$ depends upon the smallness condition we prescribe. We propose the following smallness condition on $\boldsymbol{D}(\boldsymbol{u}_m)$:

$$
\forall t \in [0, \overline{T}_m], \quad \|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} < \frac{\mu}{2 C_D^3 S_4^2}, \quad (2.9)
$$

that is in fact a bound for $\boldsymbol{u}_m$ in $L^\infty(0, \overline{T}_m; H^1(\Omega_1)^d)$. Note that it implies that

$$
C_D^3 S_4^2 \|\boldsymbol{D}(\boldsymbol{u}_m)\|^3_{L^2(\Omega_1)} < \frac{\mu}{2} \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)}.
$$

Our aim is to show that (2.9) holds for all $t \in [0, T_m]$. This will give a uniform in m a priori bound for the Galerkin solution $(\boldsymbol{u}_m, p_m)$ thus enabling us to conclude that $T_m = T$. We proceed by contradiction: Assume that there is a time $T^* \in]0, T_m]$ such that

$$
\forall t \in [0, T^*[\,, \ \|\boldsymbol{D}(\boldsymbol{u}_m)(t)\|_{L^2(\Omega_1)} < \frac{\mu}{2 C_D^3 S_4^2}, \quad \|\boldsymbol{D}(\boldsymbol{u}_m)(T^*)\|_{L^2(\Omega_1)} = \frac{\mu}{2 C_D^3 S_4^2}. \quad (2.10)
$$

By using (2.10) and applying Young's inequality to (2.8), we obtain:

$$
\begin{aligned}
(\boldsymbol{u}_m', \boldsymbol{u}_m)_{\Omega_1} + \frac{\mu}{2} \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \sum_{j=1}^{d-1} \|\frac{1}{\sqrt{G^j}} \boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}^j\|^2_{L^2(\Gamma_{12})} \\
\le \frac{C_D^2 S_2^2}{4\mu} \|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{1}{2\lambda_{\min}} \big(\tilde{S}_2^2 \|f_2\|^2_{L^2(\Omega_2)} + T_N^2 \|g\|^2_{L^2(\Gamma_{2N})}\big). \quad (2.11)
\end{aligned}
$$

This yields for all $t \in [0, T^*]$:

$$\frac{\mu}{2}\|\boldsymbol{D}(\boldsymbol{u}_m)(t)\|^2_{L^2(\Omega_1)} \leq \|\boldsymbol{u}'_m(t)\|_{L^2(\Omega_1)}\|\boldsymbol{u}_m(t)\|_{L^2(\Omega_1)} + \frac{C_D^2 S_2^2}{4\mu}\|\boldsymbol{f}_1\|^2_{L^\infty(0,T;L^2(\Omega_1)^d)} + \frac{1}{2\lambda_{\min}}\big(\tilde{S}_2^2\|f_2\|^2_{L^\infty(0,T;L^2(\Omega_2))} + T_N^2\|g\|^2_{L^\infty(0,T;L^2(\Gamma_{2N}))}\big). \quad (2.12)$$

Now we need a bound for $\|\boldsymbol{u}'_m\|_{L^2(\Omega_1)}$. Note that the most straightforward approach that consists in choosing $\boldsymbol{v} = \boldsymbol{u}'_m$ in (2.1), is here inconclusive because of the nonlinear term. The approach of [34] that consists in choosing $\boldsymbol{v} = \Delta\,\boldsymbol{u}_m$ in (2.1) is not appropriate because it requires either a smooth boundary or no reentrant corners and this restricts artificially the interface. However, if the data are sufficiently smooth in time, a bound for $\|\boldsymbol{u}'_m\|_{L^2(\Omega_1)}$ can be derived by differentiating equation (2.1) with respect to t; see [31] for the procedure. To this end, assume that $\boldsymbol{f}_1 \in H^1(0,T;L^2(\Omega_1)^d)$, $f_2 \in H^1(0,T;L^2(\Omega_2))$ and $g \in H^1(0,T;L^2(\Gamma_{2N}))$. Then the conclusions of Lemma 2.2 hold and this extra regularity implies that each component of $\boldsymbol{u}'_m$ belongs to $H^1(0,T_m)$; in turn, Lemma 2.1 implies that p_m belongs to $H^1(0,T_m)$. Therefore, we can differentiate each term of (2.1) with respect to t. Let p'_m denote the time derivative of p_m and choose $\boldsymbol{v} = \boldsymbol{u}'_m$ and $q = p'_m$:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} + 2\mu\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla p'_m\|^2_{L^2(\Omega_2)} + \sum_{j=1}^{d-1}\|\frac{1}{\sqrt{G^j}}\boldsymbol{u}'_m\cdot\boldsymbol{\tau}^j_{12}\|^2_{L^2(\Gamma_{12})} = -(\boldsymbol{u}'_m\cdot\nabla\boldsymbol{u}_m + \boldsymbol{u}_m\cdot\nabla\boldsymbol{u}'_m, \boldsymbol{u}'_m)_{\Omega_1} + (\boldsymbol{f}'_1, \boldsymbol{u}'_m)_{\Omega_1} + (f'_2, p'_m)_{\Omega_2} + (g', p'_m)_{\Gamma_{2N}}.$$

Using assumption (2.10), Hölder's and Cauchy-Schwarz inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} + 2\mu\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla p'_m\|^2_{L^2(\Omega_2)} + \sum_{j=1}^{d-1}\frac{1}{G^j}\|\boldsymbol{u}'_m\cdot\boldsymbol{\tau}^j_{12}\|^2_{L^2(\Gamma_{12})} \leq \mu\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)} + C_D S_2\|\boldsymbol{f}'_1\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{u}'_m)\|_{L^2(\Omega_1)} + \frac{1}{\lambda_{\min}^{1/2}}\big(\tilde{S}_2\|f'_2\|_{L^2(\Omega_2)} + T_N\|g'\|_{L^2(\Gamma_{2N})}\big)\|\boldsymbol{K}^{1/2}\nabla p'_m\|_{L^2(\Omega_2)}. \quad (2.13)$$

Then suitable applications of Young's inequality yield

$$\frac{d}{dt}\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} \leq \frac{C_D^2 S_2^2}{2\mu}\|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1)} + \frac{\tilde{S}_2^2}{\lambda_{\min}}\|f'_2\|^2_{L^2(\Omega_2)} + \frac{T_N^2}{\lambda_{\min}}\|g'\|^2_{L^2(\Gamma_{2N})}.$$

By integrating from 0 to t for any $t \in [0, T^*]$ this becomes

$$\|\boldsymbol{u}'_m(t)\|^2_{L^2(\Omega_1)} \leq \frac{C_D^2 S_2^2}{2\mu} \|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1 \times]0,T[)} + \|\boldsymbol{u}'_m(0)\|^2_{L^2(\Omega_1)} + \frac{1}{\lambda_{\min}} \big(\tilde{S}_2^2 \|f'_2\|^2_{L^2(\Omega_2 \times]0,T[)} + T_N^2 \|g'\|^2_{L^2(\Gamma_{2N} \times]0,T[)}\big). \quad (2.14)$$

To bound the term $\|\boldsymbol{u}'_m(0)\|^2_{L^2(\Omega_1)}$, we use $\boldsymbol{v} = \boldsymbol{u}'_m(0)$ and $q = 0$ in (2.1) at time $t = 0$. Since $\boldsymbol{u}_m(0) = \boldsymbol{0}$, this yields:

$$\|\boldsymbol{u}'_m(0)\|^2_{L^2(\Omega_1)} + (p_m(0), \boldsymbol{u}'_m(0) \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} = (\boldsymbol{f}_1(0), \boldsymbol{u}'_m(0)).$$

By applying (2.5) and using the fact that $\nabla \cdot \boldsymbol{u}' = 0$, this gives

$$\|\boldsymbol{u}'_m(0)\|_{L^2(\Omega_1)} \leq \|\boldsymbol{f}_1(0)\|_{L^2(\Omega_1)} + C_E \|p_m(0)\|_{H^1(\Omega_2)}.$$

Then (2.7) implies

$$\|\boldsymbol{u}'_m(0)\|_{L^2(\Omega_1)} \leq \|\boldsymbol{f}_1(0)\|_{L^2(\Omega_1)} + \frac{(\tilde{S}_2^2 + 1)^{1/2}}{\lambda_{\min}^{1/2}} C_E \big(\tilde{S}_2 \|f_2(0)\|_{L^2(\Omega_2)} + T_N \|g(0)\|_{L^2(\Gamma_{2N})}\big). \quad (2.15)$$

By substituting into (2.14), this yields for all $t \in [0, T^*]$

$$\|\boldsymbol{u}'_m(t)\|^2_{L^2(\Omega_1)} \leq \frac{C_D^2 S_2^2}{2\mu} \|\boldsymbol{f}'_1\|^2_{L^2(\Omega_1 \times]0,T[)} + 2\mathcal{C} + \frac{1}{\lambda_{\min}} \big(\tilde{S}_2^2 \|f'_2\|^2_{L^2(\Omega_2 \times]0,T[)} + T_N^2 \|g'\|^2_{L^2(\Gamma_{2N} \times]0,T[)}\big), \quad (2.16)$$

where $\mathcal{C}$ is defined by (1.20). This gives a bound for $\boldsymbol{u}'_m$ in $L^\infty(0, T^*; L^2(\Omega_1)^d)$. To get a bound for the other factor $\|\boldsymbol{u}_m\|_{L^2(\Omega_1)}$ in (2.12), we revert to (2.8) and use assumption (2.10); then we integrate both sides from 0 to t for all $0 \leq t \leq T^*$ and use (2.2). We obtain for all $0 \leq t \leq T^*$:

$$\|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega_1)} \leq \frac{C_D^2 S_2^2}{3\mu} \|\boldsymbol{f}_1\|^2_{L^2(\Omega_1 \times]0,T[)} + \frac{1}{\lambda_{\min}} \big(\tilde{S}_2^2 \|f_2\|^2_{L^2(\Omega_2 \times]0,T[)} + T_N^2 \|g\|^2_{L^2(\Gamma_{2N} \times]0,T[)}\big). \quad (2.17)$$

Combining (2.16) and (2.17), writing

$$\|\boldsymbol{u}'_m\|_{L^2(\Omega_1)} \|\boldsymbol{u}_m\|_{L^2(\Omega_1)} \leq \frac{1}{2} \|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} + \frac{1}{2} \|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)},$$

and substituting into (2.12), we derive that

$$\forall t \in [0,T^*]\,,\ \mu \|\boldsymbol{D}(\boldsymbol{u}_m(t))\|^2_{L^2(\Omega_1)} \le \mathcal{A} + 2\mathcal{C},$$

where $\mathcal{A}$ is defined in (1.19). Since this inequality is valid for $t = T^*$ and because we have made the assumption (1.21) on the data, we conclude that

$$\|\boldsymbol{D}(\boldsymbol{u}_m(T^*))\|_{L^2(\Omega_1)} < \frac{\mu}{2C_D^3 S_4^2},$$

which is a contradiction. Thus we have the following result.

Lemma 2.3 *Under the assumptions of Theorem 1.2, the Galerkin system* (2.1)*,* (2.2) *has a unique solution* $(\boldsymbol{u}_m, p_m)$ *in the interval* $[0,T]$*. Moreover* $\boldsymbol{u}_m$ *is bounded uniformly in* $L^\infty(0,T;H^1(\Omega_1)^d)$ *by* (2.17) *and* (2.9)*,* $\boldsymbol{u}'_m$ *is bounded uniformly in* $L^\infty(0,T;L^2(\Omega_1)^d)$ *by* (2.16)*. In addition*

$$\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1\times]0,T[)} \le \frac{C_D^2 S_2^2}{\mu^2}\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1\times]0,T[)} + \frac{2\mathcal{C}}{\mu} + \frac{1}{\mu\lambda_{\min}}\big(\tilde{S}_2^2\|f'_2\|^2_{L^2(\Omega_2\times]0,T[)} + T_N^2\|g'\|^2_{L^2(\Gamma_{2N}\times]0,T[)}\big), \quad (2.18)$$

$$\|\boldsymbol{K}^{1/2}\nabla p_m\|_{L^\infty(0,T;L^2(\Omega_2))} \le \frac{1}{\lambda_{\min}^{1/2}}\big(\tilde{S}_2\|f_2\|_{L^\infty(0,T;L^2(\Omega_2))} + T_N\|g\|_{L^\infty(0,T;L^2(\Gamma_{2N}))} + C_E(1+\tilde{S}_2^2)^{1/2}\|\boldsymbol{u}_m\|_{L^\infty(0,T;L^2(\Omega_1)^d)}\big), \quad (2.19)$$

$$\|\boldsymbol{K}^{1/2}\nabla p'_m\|^2_{L^2(\Omega_2\times]0,T[)} \le \frac{C_D^2 S_2^2}{2\mu}\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1\times]0,T[)} + 2\mathcal{C} + \frac{2}{\lambda_{\min}}\big(\tilde{S}_2^2\|f'_2\|^2_{L^2(\Omega_2\times]0,T[)} + T_N^2\|g'\|^2_{L^2(\Gamma_{2N}\times]0,T[)}\big). \quad (2.20)$$

Proof. The argument developed above gives global existence of the maximal solution $(\boldsymbol{u}_m, p_m)$ in the whole interval $[0,T]$. Both bounds (2.18) and (2.20) follow easily from (2.13), and (2.19) is an immediate consequence of Lemma 2.1 (see (2.6)). ∎

2.3 Passing to the limit

As both $\boldsymbol{V}$ and M are reflexive, it follows from (2.18), (2.9), (2.19) and (2.20) that there exists a pair of functions $(\boldsymbol{u}, p_2)$ in $H^1(0,T;\boldsymbol{V}) \times H^1(0,T;M)$ and a subsequence, still denoted $(\boldsymbol{u}_m, p_m)$, such that

$$\boldsymbol{u}_m \to \boldsymbol{u}, \ \text{ weakly in } H^1(0,T;H^1(\Omega_1)^d), \tag{2.21}$$

$$p_m \to p_2, \ \text{ weakly in } H^1(0,T;H^1(\Omega_2)). \tag{2.22}$$

We now apply Theorem 1.1 with the choices $q = 4$, $X = H^1(\Omega_1)^d$, $E = L^4(\Omega_1)$ and $Y = L^2(\Omega_1)$. The convergence (2.21) implies in particular that

$$\boldsymbol{u}_m \to \boldsymbol{u}, \ \text{ strongly in } L^4(\Omega_1 \times]0,T[). \tag{2.23}$$

Now, let us multiply both sides of (2.1) by any function Φ in $L^2(0,T)$ and integrate over $]0,T[$; we obtain

$$\begin{aligned}
&\int_0^T (\boldsymbol{u}_m'(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + 2\mu \int_0^T (\boldsymbol{D}(\boldsymbol{u}_m(t)), \Phi(t)\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} dt \\
&\quad + \int_0^T (\boldsymbol{u}_m(t) \cdot \nabla \boldsymbol{u}_m(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (\boldsymbol{K}\nabla p_m(t), \Phi(t)\nabla q)_{\Omega_2} dt \\
&+ \int_0^T (p_m(t), \Phi(t)\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} dt + \sum_{j=1}^{d-1} \int_0^T (\frac{1}{G^j}\boldsymbol{u}_m(t)\cdot\boldsymbol{\tau}_{12}^j, \Phi(t)\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} dt \\
&\qquad - \int_0^T (\boldsymbol{u}_m(t)\cdot\boldsymbol{n}_{12}, \Phi(t)q)_{\Gamma_{12}} dt \\
&= \int_0^T (\boldsymbol{f}_1(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (f_2(t), \Phi(t)q)_{\Omega_2} dt + \int_0^T (g(t), \Phi(t)q)_{\Gamma_{2N}} dt,
\end{aligned}$$

for any $\boldsymbol{v} \in \boldsymbol{V}_k$ and $q \in M_k$, with $m \geq k$. Passing to the limit with respect to m in each linear term of the above equation is easy owing to (2.21) and (2.22). The strong convergence in (2.23) allows to pass to the limit with respect to m in the nonlinear term. Therefore, we readily derive that

$$\int_0^T (\boldsymbol{u}'(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + 2\mu \int_0^T (\boldsymbol{D}(\boldsymbol{u}(t)), \Phi(t)\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} dt$$
$$+ \int_0^T (\boldsymbol{u}(t) \cdot \nabla \boldsymbol{u}(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (\boldsymbol{K}\nabla p_2(t), \Phi(t)\nabla q)_{\Omega_2} dt$$
$$+ \int_0^T (p_2(t), \Phi(t)\boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} dt + \sum_{j=1}^{d-1} \int_0^T (\frac{1}{G^j}\boldsymbol{u}(t) \cdot \boldsymbol{\tau}_{12}^j, \Phi(t)\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} dt$$
$$- \int_0^T (\boldsymbol{u}(t) \cdot \boldsymbol{n}_{12}, \Phi(t)q)_{\Gamma_{12}} dt$$
$$= \int_0^T (\boldsymbol{f}_1(t), \Phi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (f_2(t), \Phi(t)q)_{\Omega_2} dt + \int_0^T (g(t), \Phi(t)q)_{\Gamma_{2N}} dt,$$

for any $\boldsymbol{v} \in \boldsymbol{V}_k$ and $q \in M_k$. As we can approximate the elements of $\boldsymbol{V}$ and M by the elements of $\boldsymbol{V}_k$ and M_k, this equation holds for any $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in M$. Furthermore, since this is true for any $\Phi \in L^2(0,T)$, it implies that a.e. in $]0,T[$ and for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times M$:

$$(\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}(t)), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2(t), \nabla q)_{\Omega_2} + (p_2(t), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$+ \sum_{j=1}^{d-1} (\frac{1}{G^j}\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12}^j, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} = (\boldsymbol{f}_1(t), \boldsymbol{v})_{\Omega_1} + (f_2(t), q)_{\Omega_2} + (g(t), q)_{\Gamma_{2N}}. \quad (2.24)$$

Finally, to recover the initial value, take $\Phi \in H^1(0,T)$, with $\Phi(T) = 0$. As $\boldsymbol{u}_m(0) = \boldsymbol{0}$, we have

$$0 = \int_0^T (\boldsymbol{u}_m'(t), \boldsymbol{v})_{\Omega_1} \Phi(t) dt + \int_0^T (\boldsymbol{u}_m(t), \boldsymbol{v})_{\Omega_1} \Phi'(t) dt.$$

When passing to the limit, this reads

$$0 = \int_0^T (\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1} \Phi(t) dt + \int_0^T (\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1} \Phi'(t) dt = (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} \Phi(0).$$

Therefore

$$\forall \boldsymbol{v} \in \boldsymbol{V}\ ,\ (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = 0.$$

Since $\boldsymbol{u}(0) \in \boldsymbol{V}$, this yields the initial condition. Thus we have proved the following intermediate theorem.

Theorem 2.1 *Under the assumptions of Theorem 1.2, the reduced problem* (P_V) *has at least one solution* $(\boldsymbol{u}, p_2)$ *in* $H^1(0,T;\boldsymbol{V}) \times H^1(0,T;H^1(\Omega_2))$.

2.4 Recovering the Stokes pressure

In contrast to the familiar situation of the Navier-Stokes equation, recovering the pressure is easy owing to the stronger regularity in time of the solution. Indeed, consider the following bilinear form on $(\boldsymbol{X} \times M) \times L^2(\Omega_1)$:

$$\forall (\boldsymbol{v}, q) \in \boldsymbol{X} \times M, \forall \lambda \in L^2(\Omega_1)\,,\ b\big((\boldsymbol{v}, q), \lambda\big) = \int_{\Omega_1} (\nabla \cdot \boldsymbol{v})\lambda. \tag{2.25}$$

This bilinear form is continuous on $(\boldsymbol{X} \times M) \times L^2(\Omega_1)$ and satisfies the following inf-sup condition (cf. for example [21]): There exists a constant $\beta > 0$ such that

$$\forall \lambda \in L^2(\Omega_1)\,,\ \sup_{\boldsymbol{v} \in \boldsymbol{X}} \frac{b\big((\boldsymbol{v}, q), \lambda\big)}{\|\boldsymbol{v}\|_{H^1(\Omega_1)}} \geq \beta \|\lambda\|_{L^2(\Omega_1)}. \tag{2.26}$$

The inequality is unchanged if we replace the supremum over $\boldsymbol{v}$ by the supremum over the pair $(\boldsymbol{v}, q)$ with any q in M; in fact the supremum is attained for $q = 0$. Note also that

$$\boldsymbol{V} \times M = \{(\boldsymbol{v}, q) \in \boldsymbol{X} \times M\,;\, \forall \lambda \in L^2(\Omega_1), b\big((\boldsymbol{v}, q), \lambda\big) = 0\}.$$

Now, let $(\boldsymbol{u}, p_2)$ be one solution to problem (P_V) and ℓ denote the mapping defined for a.e. t in $]0, T[$:

$$(\boldsymbol{v}, q) \mapsto \ell(\boldsymbol{v}, q) = (\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}(t)), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}(t) \cdot \nabla \boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}$$
$$+ (\boldsymbol{K}\nabla p_2(t), \nabla q)_{\Omega_2} + (p_2(t), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \sum_{j=1}^{d-1} (\frac{1}{G^j}\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12}^j, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}$$
$$- (\boldsymbol{f}_1(t), \boldsymbol{v})_{\Omega_1} - (f_2(t), q)_{\Omega_2} - (g(t), q)_{\Gamma_{2N}}.$$

For a.e. t in $]0, T[$, this mapping is linear, continuous on $\boldsymbol{X} \times M$ and vanishes on $\boldsymbol{V} \times M$. Therefore the inf-sup condition (2.26) and the Babuska-Brezzi's theory imply that for a.e. $t \in]0, T[$, there exists a unique function $p_1 \in L^2(\Omega_1)$ satisfying

$$\forall (\boldsymbol{v}, q) \in \boldsymbol{X} \times M\,,\ b\big((\boldsymbol{v}, q), p_1\big) = \ell(\boldsymbol{v}, q).$$

In other words

$$(\nabla \cdot \boldsymbol{v}, p_1(t))_{\Omega_1} = (\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{u}(t)), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}(t) \cdot \nabla \boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}$$

$$+(\boldsymbol{K}\nabla p_2(t), \nabla q)_{\Omega_2} + (p_2(t), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \sum_{j=1}^{d-1}(\frac{1}{G^j}\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12}^j, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}$$

$$- (\boldsymbol{f}_1(t), \boldsymbol{v})_{\Omega_1} - (f_2(t), q)_{\Omega_2} - (g(t), q)_{\Gamma_{2N}},$$

i.e. we recover the first equation of Problem (P). The estimate (1.27) for p_1 in $L^\infty(0,T; L^2(\Omega_1))$ is an immediate consequence of this equation (with the choice $q = 0$) and (2.26). This finishes the existence part of the proof of Theorem 1.2.

2.5 Uniqueness

We cannot establish unconditional uniqueness, even when $d = 2$, because we cannot prove that all solutions of Problem (P_V) are bounded. However, we can prove that this problem has no more than one solution $(\boldsymbol{u}, p_2)$ satisfying:

$$\|\boldsymbol{D}(\boldsymbol{u})\|_{L^\infty(0,T;L^2(\Omega_1)^{d\times d})} \leq \frac{\mu}{C_D^3 S_4^2}, \tag{2.27}$$

a condition that is slightly sharper than (1.22).

Theorem 2.2 *Problem (P_V) has no more than one solution $(\boldsymbol{u}, p_2)$ in $L^\infty(0,T;\boldsymbol{V})\times L^2(0,T;M)$ satisfying* (2.27). *In particular, under the assumptions of Theorem 1.2, Problem* (1.1)–(1.10) *has one and only one solution.*

Proof. Let $(\boldsymbol{u}, p_2)$ and $(\tilde{\boldsymbol{u}}, \tilde{p}_2)$ be two solutions to (P_V). Set $\boldsymbol{w} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$ and $\varphi = p_2 - \tilde{p}_2$; then we have for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times M$

$$(\boldsymbol{w}', \boldsymbol{v})_{\Omega_1} + 2\mu(\boldsymbol{D}(\boldsymbol{w}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2}$$

$$+ (\varphi, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \sum_{j=1}^{d-1}(\frac{1}{G^j}\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}^j, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} - (\boldsymbol{w}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} = 0,$$

$$\boldsymbol{w}(0) = \boldsymbol{0}.$$

The choice $\boldsymbol{v} = \boldsymbol{w}$ and $q = \varphi$ gives

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + 2\mu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2\Omega_1} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u} + \tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(\Omega_2)} + \sum_{j=1}^{d-1}\|\frac{1}{\sqrt{G^j}}\boldsymbol{w}\cdot\boldsymbol{\tau}^j_{12}\|^2_{L^2(\Gamma_{12})} = 0.$$

Now suppose that $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ are bounded by (2.27). Then for a.e. t in $]0,T[$

$$\left|(\boldsymbol{w}\cdot\nabla\boldsymbol{u} + \tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1}\right| \leq 2\mu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}.$$

Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(\Omega_2)} + \sum_{j=1}^{d-1}\|\frac{1}{\sqrt{G^j}}\boldsymbol{w}\cdot\boldsymbol{\tau}^j_{12}\|^2_{L^2(\Gamma_{12})} \leq 0.$$

Integrating this from 0 to t and using $\boldsymbol{w}(0) = \boldsymbol{0}$, we get

$$\frac{1}{2}\|\boldsymbol{w}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(\Omega_2)}dt + \sum_{j=1}^{d-1}\int_0^t\|\frac{1}{\sqrt{G^j}}\boldsymbol{w}\cdot\boldsymbol{\tau}^j_{12}\|^2_{L^2(\Gamma_{12})}dt \leq 0. \tag{2.28}$$

This yields $\boldsymbol{w} = \boldsymbol{0}$, $\varphi = 0$. The uniqueness of the pressure p_1 comes from Section 2.4. ∎

3 Conclusions and perspectives

In this work, we have proved that a time-dependent Navier-Stokes system coupled with a Darcy model with suitably small data has one and only one solution, even in the presence of a rough interface. The proof, based on a Galerkin discretization in space, lends itself readily to a variety of finite-element discretizations which will be the object of future work. The study of other boundary and initial conditions, as well as other Darcy models, are in progress. It would also be very interesting to extend the work in [7] on Beavers-Joseph conditions to a rough interface.

References

[1] G. Beavers and D. Joseph, “Boundary conditions at a naturally impermeable wall,” *J. Fluid. Mech*, vol. 30, pp. 197–207, 1967.

[2] P. Saffman, “On the boundary condition at the surface of a porous media,” *Stud. Appl. Math.*, vol. 50, pp. 292–315, 1971.

[3] W. Jäger and A. Mikelić, “On the interface boundary condition of Beavers, Joseph, and Saffman,” *SIAM J. Appl. Math.*, vol. 60, pp. 1111–1127, 2000.

[4] T. Arbogast and H. Lehr, “Homogenization of a Darcy-Stokes system modeling vuggy porous media,” *Computational Geosciences*, vol. 10, pp. 291–302, 2006.

[5] T. Arbogast and D. Brunson, “A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium,” *Computational Geosciences*, vol. 11, no. 3, pp. 207–218, 2007.

[6] E. Burman and P. Hansbo, “A unified stabilized method for Stokes and Darcy’s equations,” *Journal of Computational and Applied Mathematics*, vol. 198, no. 1, pp. 35–51, 2007.

[7] Y. Cao, M. Gunzburger, F. Hua, and X. Wang, “Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition,” *Commun. Math. Sci.*, vol. 8, pp. 1–25, 2010.

[8] G. Kanschat and B. Rivière, “A strongly conservative finite element method for the coupling of Stokes and Darcy flow,” *Journal of Computational Physics*, vol. 229, pp. 5933–5943, 2010.

[9] W. Layton, F. Schieweck, and I. Yotov, “Coupling fluid flow with porous media flow,” *SIAM J. Numer. Anal.*, vol. 40, no. 6, pp. 2195–2218, 2003.

[10] D. Vassilev and I. Yotov, “Coupling Stokes-Darcy flow with transport,” *SIAM J. Sci. Comp.*, vol. 31, no. 5, pp. 3661–3684, 2009.

[11] B. Rivière and I. Yotov, “Locally conservative coupling of Stokes and Darcy flow,” *SIAM J. Numer. Anal.*, vol. 42, pp. 1959–1977, 2005.

[12] B. Rivière, "Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems," *Journal of Scientific Computing*, vol. 22, pp. 479–500, 2005.

[13] M. Mu and J. Xu, "A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow," *SIAM Journal on Numerical Analysis*, vol. 45, pp. 1801–1813, 2007.

[14] K. A. Mardal, X.-C. Tai, and R. Winther, "A robust finite element method for Darcy-Stokes flow," *SIAM J. Numer. Anal.*, vol. 40, no. 5, pp. 1605–1631 (electronic), 2002.

[15] N. Hanspal, A. Waghode, V. Nassehi, and R. Wakeman, "Numerical analysis of coupled Stokes/Darcy flows in industrial filtrations," *Transport in Porous Media*, vol. 64, no. 1, pp. 1573–1634, 2006.

[16] M. Discacciati and A. Quarteroni, "Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations," in *Numerical Analysis and Advanced Applications - ENUMATH 2001* (B. et al, ed.), pp. 3–20, Springer, Milan, 2003.

[17] M. Discacciati, A. Quarteroni, and A. Valli, "Robin-Robin domain decomposition methods for the Stokes-Darcy coupling," *SIAM J. Numer. Anal.*, vol. 45, no. 3, pp. 1246–1268, 2007.

[18] M. Discacciati, *Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows*. PhD thesis, Ecole Polytechnique Fédérale de Lausanne, Switzerland, 2004.

[19] M. Discacciati and A. Quarteroni, "Navier-Stokes/Darcy coupling: Modeling, analysis, and numerical approximation," *Rev. Mat. Comput.*, vol. 22, pp. 315–426, 2009.

[20] L. Badea, M. Discacciati, and A. Quarteroni, "Mathematical analysis of the Navier-Stokes/Darcy coupling," *Numerische Mathematik*, vol. 115, no. 2, pp. 195–227, 2010.

[21] V. Girault and B. Rivière, "DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition," *SIAM Journal on Numerical Analysis*, vol. 47, pp. 2052–2089, 2009.

[22] P. Chidyagwai and B. Rivière, "Numerical modelling of coupled surface and subsurface flow systems," *Advances in Water Resources*, vol. 33, pp. 92–105, 2010.

[23] P. Chidyagwai and B. Rivière, "On the solution of the coupled Navier-Stokes and Darcy equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 198, pp. 3806–3820, 2009.

[24] A. Çeşmelioğlu and B. Rivière, "Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow," *Journal of Numerical Mathematics*, vol. 16, pp. 249–280, 2008.

[25] A. Çeşmelioğlu and B. Rivière, "Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow," *Journal of Scientific Computing*, vol. 40, pp. 115–140, 2009.

[26] R. Adams, *Sobolev Spaces.* Academic Press, New-York, 1975.

[27] J. Nečas, *Les méthodes directes en théorie des équations elliptiques.* Masson, Paris, 1967.

[28] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I.* New York: Springer-Verlag, 1972.

[29] P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics.* Boston, MA: Pitman, 1985.

[30] J. Aubin, "Un théorème de compacité," *CRAS Paris Série I*, vol. 256, pp. 5042–5044, 1963.

[31] J.-L. Lions, *Equations differentielles operationnelles.* Springer-Verlag, Berlin, Heidelberg, New York, 1961.

[32] J. Simon, "Compact sets in the space $L^p(0,T;B)$," *Ann. Math. Pures Appl.*, vol. 146, pp. 1093–1117, 1990.

[33] E. A. Coddington and N. Levinson, *Theory of differential equations.* McGraw-Hill, New-York, 1955.

[34] J. Heywood and R. Rannacher, "Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order estimates for spatial discretization," *SIAM J. of Numer. Anal.*, vol. 19, pp. 275–311, 1982.