The Acoustic Radiation Solution

William W. Symes and Tetyana Vdovina

ABSTRACT

The well-known radiation solution of the acoustic wave equation may also be viewed as the pressure field in the solution of the first-order system of linear acoustics, in two different ways. The first version casts in the source term as a defect in the acoustic constitutive law, the second presents it as an equivalent body source. The second form requires the addition of a parasitic stationary singular pressure field.

INTRODUCTION

Spherical waves expanding at speed $c > 0$ and centered at a source point $\mathbf{x}_s = (x_{1,s}, x_{2,s}, x_{3,s})$, of the form

$$\frac{g\left(t - \frac{r}{c}\right)}{r}, \; r = \sqrt{(x_1 - x_{1,s})^2 + (x_2 - x_{2,s})^2 + (x_3 - x_{3,s})^2}, \tag{1}$$

arise as causal solutions of the inhomogenous wave equation with a singular right-hand side. Courant and Hilbert (1962) call this inhomogenous equation the *radiation problem*, and we will refer to its solution (formula 1) as the *radiation solution*.

The dependent variable in the second order scalar wave equation may be interpreted as the pressure field (or alternately the potential field) of linear acoustics. A natural description of linear acoustics combines small-amplitude (linearized) versions of Newton's law and a constitutive law. The first states that the acceleration field is proportional to the pressure gradient, the second that the rate of change of pressure is proportional to the rate of change of volume (velocity divergence). These dynamical principles describe small motions of both gases and fluids, and constitute a first order symmetric hyperbolic system. Amongst many sources for the continuum physics of acoustics, we mention Friedlander (1958), who derives these relations from isentropic gas dynamics, and Gurtin (1981) who models waves in elastic fluids, that is, elastic materials which do not support shear stress. The SH wave system of plane strain elasticity also takes the same forms.

The purpose of this note is to explain precisely how the same radiation solution appears as a solution of both the second-order wave equation and the first order system of linear acoustics. In the latter case, the inhomogeneity giving rise to the radiation solution may appear either as a point defect in the constitutive law, or as an equivalent body force. While the arguments needed to relate these problem statements are entirely elementary, the relations don't seem to be easily available in explicit form in the literature. These relations are particularly important in calibrating computational radiation solutions. Numerical methods for acoustics rely on one formulation or the other - centered finite difference methods for both the second order wave equation (Alford et al., 1974; Cohen, 2001) and for the first order acoustic system (Virieux, 1984; Gustafsson and Wahlund, 2004) continue to prove useful in computational seismology, for example. If these various methods are to produce comparable results, the source mechanisms employed in them must be equivalent. In the continuum limit, these equivalent source models obey the relations presented here.

After deriving the various equivalent point radiator source representations, we end with a few remarks about the production of a radiation solution with a prescribed trace (spatial sample) at a given location.

Throughout the brief discussion, all physical parameters are independent of spatial location - that is, we model acoustic waves propagating in a homogeneous material. We consider explicitly only the case of three space dimensions. Analogous formulae for two space dimensions follow by the method of descent. Rather than carry along the source location in the notation, we presume that $\mathbf{x}_s = \mathbf{0}$. The general case follows from our formulas by the straightforward replacement $\mathbf{x} \rightarrow \mathbf{x} - \mathbf{x}_s$.

RADIATION SOLUTION OF THE WAVE EQUATION

According to Courant and Hilbert (1962), pp. 696-7, the 3D causal solution of the dimensionless radiation problem

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{s}} \right) u(\mathbf{s}, t) = 4\pi g(t) \delta(\mathbf{s}) \tag{2}$$

is

$$u(\mathbf{s}, t) = \frac{g(t-s)}{s}, \tag{3}$$

in which $s = \sqrt{s_1^2 + s_2^2 + s_3^2}$.

To begin the translation to dimensional quantities, we regard t as having units of time, and the "space" variable $\mathbf{s}$ has having units of time also. Note that the delta function on the RHS then has units of $1/(\text{time}^3)$, whereas the units of g and u are so-far unspecified. Introduce $c > 0$ having units of velocity; then $\mathbf{x} = c\mathbf{s}$ has units of length. Replace $\mathbf{s}$ with $\mathbf{x}$ in equation 2 to obtain

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2_{\mathbf{x}} \right) u(\mathbf{x}, t) = 4c^3 \pi g(t) \delta(\mathbf{x}), \tag{4}$$

in which the delta has now acquired units of 1/volume. Setting $r = cs = \sqrt{x_1^2 + x_2^2 + x_3^2}$, obtain for the solution as a function of $\mathbf{x}, t$ the relation

$$u(\mathbf{x}, t) = c \frac{g\left(t - \frac{r}{c}\right)}{r}. \tag{5}$$

Set

$$p(\mathbf{x},t) = \frac{1}{c}u(\mathbf{x},t) = \frac{g\left(t - \frac{r}{c}\right)}{r}.$$

Then p is the causal solution of

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) p(\mathbf{x},t) = 4\pi g(t)\delta(\mathbf{x}) \tag{6}$$

We have not yet assigned units to the dependent variable. Suppose that p is a pressure, with units of force per unit area or mass/(length $\cdot$ time^2). Then g has units of energy/area. The RHS of equation 6 has units of energy/area/volume.

Introduce the mass density ρ and the bulk modulus κ, so that $c = \sqrt{\frac{\kappa}{\rho}}$. Then equation 6 can be rewritten

$$\left(\frac{1}{\kappa}\frac{\partial^2}{\partial t^2} - \frac{1}{\rho}\nabla^2_{\mathbf{x}}\right) p(\mathbf{x},t) = \frac{4\pi}{\rho}g(t)\delta(\mathbf{x}). \tag{7}$$

The RHS of equation 7 has units of specific energy (energy per unit mass) per unit area.

CONSTITUTIVE POINT DEFECT REPRESENTATION

Define

$$\mathbf{v}(\mathbf{x},t) = -\frac{1}{\rho}\int_{-\infty}^{t} dt_1 \, \nabla p(\mathbf{x},t_1)$$

$$= \frac{\mathbf{x}}{\rho}\left(\frac{1}{cr^2}g\left(t - \frac{r}{c}\right) + \frac{1}{r^3}\int_{-\infty}^{t-\frac{r}{c}} dt_1 \, g(t_1)\right). \tag{8}$$

Then the vector quantity $\mathbf{v}$ has units of velocity, and can indeed be identified with the particle velocity field of an elastic fluid.

Differentiate both sides of equation 8 with respect to t, and integrate both sides of equation 7 from $-\infty$ to t, to obtain the system

$$\frac{1}{\kappa}\frac{\partial p(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{v}(\mathbf{x},t) = \frac{4\pi}{\rho}\int_{-\infty}^{t} dt_1 \, g(t_1)\delta(\mathbf{x}), \tag{9}$$

$$\rho\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \nabla p(\mathbf{x},t) = 0. \tag{10}$$

The causal solution of the linear acoustics system 9 and 10 is equivalent to that of the wave equation 7, by construction, in the sense that the two share the same pressure field. The RHS of equation 9 has a natural interpretation as a defect location for the standard acoustic constitutive law.

EQUIVALENT DILATATIONAL BODY FORCE

Define the stationary singular (infinite energy) pressure field

$$p_s(\mathbf{x},t) \equiv -4\pi c^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, g(t_2)\delta(\mathbf{x})$$

and the modified pressure field $\bar{p} = p + p_s$. Then from equation 7 one sees immediately that $\bar{p}, \mathbf{v}$ form the causal solution of the system

$$\frac{1}{\kappa}\frac{\partial \bar{p}(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{v}(\mathbf{x},t) = 0, \quad (11)$$

$$\rho \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \nabla \bar{p}(\mathbf{x},t) = -4\pi c^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, g(t_2)\nabla\delta(\mathbf{x}). \quad (12)$$

The RHS of equation 12 has units of force density. System 11, 12 may be interpreted as a body force equivalent to the defect system 9, 10, in that the velocity fields are the same and the pressure fields differ only at the source point $\mathbf{x} = \mathbf{0}$.

TARGET PULSE

We end by deducing the source mechanisms requred to produce a radiation pressure field with a prescribed trace $f(t)$ at a prescribed location $\mathbf{x}_r$. This method for calibrating point radiator sources is used for example in the SEG's SEAM project.

The required relation is $p(\mathbf{x}_r, t) = f(t)$, whence

$$g(t) = x_r f\left(t + \frac{x_r}{c}\right),\ x_r = \sqrt{x_{1,r}^2 + x_{2,r}^2 + x_{3,r}^2}. \quad (13)$$

Note that since f is a pressure, g has units of energy/area, as required.

Insertion of g defined by equation 13 in the right-hand sides of 7, 9 or 12 will produce a pressure field with the desired property.

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