SUBMITTED TO *Modélisation Mathématique et Analyse Numérique*, 2001

THE STEEPEST DESCENT MINIMIZATION OF DOUBLE-WELL STORED ENERGIES DOES NOT YIELD VECTORIAL MICROSTRUCTURES

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ABSTRACT. We prove that the Steepest Descent algorithm applied to the minimization of total stored energies with rank-one related rotationally symmetric energy wells does not produce relaxing vectorial microstructures with non-trivial Young measures.

1. INTRODUCTION

Microstructures can be often obtained as weak limits of minimizing sequences of non-quasiconvex total stored energies with multiple, rotationally invariant local minima (wells). It is known that in many cases such minimizing sequences can be analytically constructed. Moreover, the scaling properties and geometrical structure of these sequences resemble properties of certain complex alloys. We ask the following question in the presented paper: *Is it possible to generate these minimizing sequences using well-known optimization algorithms such as e. g. the Steepest Descent applied to a reasonably rich class of non-(quasi/poly/rank-one)convex variational integrals?* The answer obtained in this paper is negative.

To be more precise, let us assume that we are given a stored energy E defined on some Sobolev space with values in $\mathbb{R}^+$. We will assume that the total stored energy is given in the form of variational integral, i.e.,

$$E(u) \stackrel{\text{def}}{=} \int_\Omega W(\nabla u(x))\, dx,$$

where W is a positive and *smooth* energy density with rotationally invariant wells, and a quadratic growth. The domain Ω is assumed to have a Lipschitz boundary. We will assume that the stored energy is minimized (relaxed) by *binomial microstructures*. Such patterns are generated by weakly differentiable maps $u_n : \mathbb{R}^N \to \mathbb{R}^N$ which can come arbitrarily close to having the following two properties:

$$\begin{aligned} &\nabla u_n \in \{F_1, F_2\}, \qquad \text{a.e. in } \Omega \subset \mathbb{R}^N, \quad N = 2, 3, \quad \text{and} \\ &u_n(x) = (\lambda_1 F_1 + \lambda_2 F_2)\, x, \quad x \in \partial\Omega, \quad \lambda_i > 0,\, i = 1, 2, \quad \text{and } \lambda_1 + \lambda_2 = 1. \end{aligned} \tag{1.1}$$

The gradients of these maps must have unbounded spatial frequency of oscillation between the given *variants* F_1 and F_2 as $n \to \infty$ to conform to the two contradictory conditions. The matrices $F_1 \in M^{N\times N}$ and

Date: April 19, 2001.

1991 *Mathematics Subject Classification.* 41A25, 65L60, 65L70, 65N12, 65N30, 73G05, 73S10.

Key words and phrases. Microstructures, Nonlinear Optimization, Relaxation.

This work was supported in part by the Los Alamos National Laboratory Computer Science Institute (LACSI) through LANL contract number 03891-99-23, as part of the prime contract W-7405-ENG-36 between the Department of Energy and the Regents of the University of California, by the grant NASA SECTP NAG5-8136, by the grant from Schlumberger Foundation, and by the grant from the Fonds National Belgique de la Recherche Scientifique through the Department of Mathematics, University Mons-Hainaut, Belgium.

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$F_2 \in M^{N\times N}$ are positive definite, linearly independent, and rank-one connected, i.e.,

$$F_1 = F_2 + a \otimes b, \tag{1.2}$$

where a and b are linearly independent vectors. The Hadamard jump condition (1.2) implies that there exists a continuous map $u : \mathbb{R}^N \mapsto \mathbb{R}^N$ such that $\nabla u \in \{F_1, F_2\}$.

We denote

$$\mathcal{F} \stackrel{\text{def}}{=} \{QF_iQ^T,\, Q \in SO(N),\, i = 1, 2\}$$

where $SO(N)$ is the space of proper rotations, i.e. a set of orthogonal matrices with positive determinant. We will use a projection $\Pi : M^{N\times N} \to \mathcal{F}$ defined by

$$\|A - \Pi A\| = \min_{M\in\mathcal{F}} \|A - M\| .$$

We require the energy density to have the following properties

$$\begin{aligned}
&W : M^{N\times N} \mapsto \mathbb{R}^+,\\
&W \in C^2(\mathbb{R}^{N^2}),\\
&W|_{\mathcal{F}} = 0, \qquad \text{and}\\
&D^2W(P)(v,v) \geq \lambda\,|v|^2, \qquad \text{for any } P \in \mathcal{F} \text{ and any } v \in \mathbb{R}^N, \lambda > 0,\\
&W(P) > 0, \qquad \text{for any } P \in M^{N\times N},\, P \notin \mathcal{F},\\
&W(P) \to +\infty \qquad \text{as } \|P\| \to \infty,
\end{aligned} \tag{1.3}$$

in order to relax E by binomial microstructures.

Let $\Lambda \in \mathbb{R}^+\setminus\{0\}$, $\mathbb{R}^+ = [0, +\infty)$, be the Lipschitz constant of the modulus of the energy density W, i.e.,

$$\left|\int_\Omega D^2W(\nabla u)(\varphi,\psi)\,dx\right| \leq \Lambda \int_\Omega \nabla\varphi(x) : \nabla\psi(x)\,dx, \tag{1.4}$$
$$\text{for any } \varphi,\psi \in W_0^{1,2}(\Omega,\mathbb{R}^N),\ \text{for any } u \in W^{1,2}(\Omega,\mathbb{R}^N), \qquad \Lambda \geq C > 0,$$

where

$$D^2W(P)(\varphi,\psi) \stackrel{\text{def}}{=} \frac{d}{dt}DW(u + t\psi)|_{t=0}\varphi.$$

We assume, in compliance with (1.4), that the stored energy E approaches the equilibrium states quadratically, i.e.,

$$E(u) \geq \Lambda \int_\Omega \|\nabla u(x) - \Pi\nabla u(x)\|^2\;dx. \tag{1.5}$$

Correspondingly, binomial microstructures, (1.1), are relaxing sequences of $\int_\Omega \|\nabla u(x) - \Pi\nabla u(x)\|^2\;dx$. Hence, in particular,

$$0 = \inf\left\{E(u)\,|\,u \in W^{1,\infty}(\Omega,\mathbb{R}^n),\, u(x) = Fx \stackrel{\text{def}}{=} (\lambda_1F_1 + \lambda_2F_2)\,x,\ x \in \partial\Omega\right\}. \tag{1.6}$$

The infimum in (1.6) is not attainable [3].

We have the following Lemma describing constitutive properties of the energy density W with respect to the macroscopic state F.

Lemma 1.1 (Constitutive restriction). *Let us assume that* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,2}(\Omega,\mathbb{R}^N)$ *is a minimizing sequence of the stored energy* E*, i.e.* $E(u_n)\downarrow 0$*. Then*

$$W(F) \le \Lambda \left\| R\,F_i - F \right\|^2, \qquad \textit{for any } R \in SO(N), \tag{1.7}$$

where

$$\Lambda = \left\| D^2 W(F) \right\|, \quad \textit{and } F = \lambda_1 F_1 + \lambda_2 F_2. \tag{1.8}$$

Proof. Since the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,2}(\Omega,\mathbb{R}^N)$ minimizes E we have [10]

$$u_n \rightharpoonup Fx \qquad \text{weakly in } W^{1,2}_0(\Omega,\mathbb{R}^N). \tag{1.9}$$

We obtain using the Taylor's expansion for any measurable subdomain e of Ω

$$\begin{aligned} &\int_e W(\nabla u_n(x))\,dx \\ &= \int_e W(F)\,dx + \int_e DW(F)(\nabla u_n(x) - F)\,dx \\ &\quad + \int_e \int_0^1 (1-\tau) D^2 W\left(F + \tau\left(\nabla u_n - F\right)\right)\,d\tau (\nabla u_n(x) - F, \nabla u_n(x) - F)\,dx. \end{aligned} \tag{1.10}$$

The convergence (1.9) yields

$$\int_e DW(F)(\nabla u_n(x) - F)\,dx \to 0 \qquad \text{as } n\to\infty.$$

We have from the assumption of the Lemma and (1.10)

$$\begin{aligned} 0 &\ge \int_e W(F) - \Lambda \lim_{n\to\infty} \int_e \left\|\Pi \nabla u_n(x) - F\right\|^2\,dx \\ &= \int_e W(F) - \Lambda \int_e \left\| R\,F_i - F\right\|^2\,dx, \qquad \text{for some } i = 1,2, \quad \text{and } R\in SO(N). \end{aligned} \tag{1.11}$$

The statement of the Lemma follows from (1.11).

□

Definition 1.2 ((Binomial) Laminate). *Let* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega,\mathbb{R}^N)$*,* $p > N$*, be a minimizing sequence of the stored energy* E*. Let* $\{w_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}(\Omega,\mathbb{R}^N)$ *be a sequence of functions with the following properties*

$$\begin{aligned} &\nabla w_n \in \{F_1, F_2\}, \qquad \textit{a.e. in } \Omega, \qquad \textit{and,} \\ &\|u_n - w_n\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \le C\,\|\nabla u_n - \Pi\nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}, \quad C \ne C(n). \end{aligned} \tag{1.12}$$

Then a maximal simply connected set e_n *of all points at which* $\Delta\, w_n$ *exists in the classical sense and where it is equal to zero is called a binomial laminate.*

Remark

We refer to the sequence $\{w_n\}_{n\in\mathbb{N}}$ as *sharp binomial microstructure*. The existence of the sharp binomial microstructures is established in Theorem 4.1. We set

$$\mathcal{E}_n \stackrel{\text{def}}{=} \{e_n \,|\, n\in\mathbb{N}\}. \tag{1.13}$$

We denote by $\operatorname{moh}(\mathcal{E}_n)$ the number of laminates in $\mathcal{E}_n$.

We note that laminates have Harnack-type "max-min-max" property proven in Lemma 4.2. Namely, there exists a positive constant C, independent of n, such that

$$\max_{e_n\in\mathcal{E}_n} \operatorname{meas}(e_n) \le C \min_{e_n\in\mathcal{E}_n} \operatorname{meas}(e_n). \tag{1.14}$$

The original version of the proof of this property can be found in [9]. We include here a simplified version of the proof. □

The purpose of this paper is to show that for any $u_0 \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > N$, *the sequence* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, *generated during the gradient navigated descent converges strongly to a function (representing a local minimum of the stored energy* E*), and that its derivatives do not converge to a measure-valued distribution even though the infimum of the stored energy is not attainable in any Sobolev space. Namely,*

$$\begin{aligned} &u_n \to u \qquad \text{strongly in } W^{1,2}(\Omega, \mathbb{R}^N), \quad \text{and, consequently,} \\ &\lim_{n\to\infty} E(u_n) = E(u) > 0. \end{aligned} \tag{1.15}$$

We anticipate that the following lower bounds holds

$$\lim_{n\to\infty} E(u_n) \geq C \ \mathrm{moh}(\mathcal{E}_0)^{-1}. \tag{1.16}$$

The set $\mathcal{E}_0$ contains the laminates of the initial function u_0.

Remark
The result (1.15) shows that minimization (relaxation) of non- quasiconvex stored energies that are relaxed by microstructures cannot be achieved without exploration of tools by which such patterns can be analytically constructed. The most prominent tools seem to be the self-similarity and Vitali's covering theorem [6]. This approach is studied in [9], [10], [5] by adding to the total stored energy E penalization term which has the form

$$\int_\Omega \|\nabla u_n(x) - \nabla \overline{u_n}(x)\|^2 \ dx.$$

Here, $\overline{u_n}(x)$ is self-similar scaling of u_n given by

$$\begin{aligned} &\overline{u_n}(x) = (\lambda_1 F_1 + \lambda_2 F_2)\, a_j + \epsilon_j u_n \left(\frac{x - a_j}{\epsilon_j}\right), \quad x \in Q_j \overset{\text{def}}{=} a_j + \epsilon_j \Omega, \\ &\Omega = \bigcup_j Q_j, \quad \mathrm{meas}(Q_j) \leq \epsilon_j, \quad \sum_j \epsilon_j^N = 1. \end{aligned}$$

It is possible to show [5] that the Steepest Descent algorithm yields the desired microstructure with properties (1.1) in this case. □

Remark
Questions similar to the one presented in this paper have been studied earlier in e.g. [2], [7], [12], [11] in the framework of visco-elasticity and gradient flow. These works indicate that dynamical mechanisms in combination with fast damping prevent formation of patterns with complicated Young measure as well. Moreover, question concerning dependence on the initial state (its norm) and questions concerning the footprint of the initial state in the asymptotic limit are studied in these papers. Though we assume that similar results can be established for the gradient based minimization, we do not address these issues here.

□

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a bounded domain with Lipschitz boundary, and let $u = u(x) = \{u_i(x)\}_{i=1}^N$ be a weakly differentiable map. The gradient $\nabla u(x) \in M^{N\times N}$ is computed with respect to the coordinate system associated with the undeformed domain Ω. We denote by $M^{N\times N}$ the space of $N \times N$ real matrices. The matrix multiplication in this space is understood in the sense $A : B \stackrel{\text{def}}{=} \mathrm{Tr}(A^T B)$ where the matrix $A^T B$ is obtained by the standard matrix multiplication. Consequently, the matrix norm is given by $\|A\| = \sqrt{A : A}$. The corresponding norm in the space $L^p(\Omega, \mathbb{R}^{N\times N})$, $1 \le p < \infty$, is given by

$$\|\nabla u\|_{L^p(\Omega,\mathbb{R}^{N\times N})} \stackrel{\text{def}}{=} \left(\int_\Omega \|\nabla u(x)\|^p \, dx \right)^{1/p} = \left(\int_\Omega (\nabla u(x) : \nabla u(x))^{p/2} \, dx \right)^{1/p},$$

where $u : \mathbb{R}^N \mapsto \mathbb{R}^N$ belongs to some Sobolev space, i.e. $u \in W^{k,p}(\Omega, \mathbb{R}^N)$ for some $k \ge 1$. We will use $|.|$ to denote a norm on the space of vectors.

2.1. Steepest Descent Algorithms with respect to different topologies

We denote the weak gradient of the energy E by G. For any $u \in W^{1,2}(\Omega)$, the weak gradient $G \in W^{-1,2}(\Omega, \mathbb{R}^N)$ is given by the variational relation

$$\langle G(u), \varphi \rangle_{W^{-1,2}(\Omega,\mathbb{R}^N), W_0^{1,2}(\Omega,\mathbb{R}^N)} \stackrel{\text{def}}{=} \frac{d}{dt} E(u + t\varphi)\,|_{t=0} \tag{2.1}$$

$$= \int_\Omega DW(\nabla u(x)) : \nabla \varphi \, dx, \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{2.2}$$

Definition 2.1 (Strong Gradient)**.** *The strong gradient g is the $W_0^{1,2}(\Omega, \mathbb{R}^N)$-projection of G, i.e., for any $u \in W^{1,2}(\Omega)$ we compute*

$$g(u) = -\Delta^{-1} G(u), \tag{2.3}$$

where $-\Delta^{-1} : W^{-1,2}(\Omega, \mathbb{R}^N) \mapsto W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Remark

(i) The variational formulation for the strong gradient reads: Given $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, find the function $g(u) \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ by solving

$$\int_\Omega \nabla g(u(x)) : \nabla \varphi(x) \, dx = \langle G(u), \varphi \rangle_{W^{-1,2}(\Omega,\mathbb{R}^N), W_0^{1,2}(\Omega,\mathbb{R}^N)}, \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N). \tag{2.4}$$

(ii) We note that $-\Delta^{-1}$ mapping is isometric from $W^{-1,2}(\Omega, \mathbb{R}^N)$ to $W_0^{1,2}(\Omega, \mathbb{R}^N)$, [1]. Hence,

$$\|g(u)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} = \|G(u)\|_{W^{-1,2}(\Omega,\mathbb{R}^N)}. \tag{2.5}$$

□

Remark

The strong gradient g is unique for a given G. Namely, assuming that $\partial\Omega$ is piecewise C^∞, and taking $F \in W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*$, [1], represented by

$$\begin{aligned} \langle F, \varphi \rangle &\stackrel{\text{def}}{=} \int_\Omega \sum_{i=1}^N f_i(x) \frac{\partial \varphi(x)}{\partial x_i} + f_0(x)\varphi(x)\, dx, \qquad \varphi \in W_0^{1,p}(\Omega), \\ \|F\|_{W^{-1,p}(\Omega,\mathbb{R}^N)} &\stackrel{\text{def}}{=} \left(\sum_{i=0}^N \|f_i\|^p_{L^p(\Omega)} \right)^{1/p}, \end{aligned} \tag{2.6}$$

we have [[14], Theorem 3.8]: Let $p \in [2, \infty)$ then there exists unique solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, $u(x) = u_0(x)$ on $\partial\Omega$, $u_0 \in W^{1,p}(\Omega)$, of

$$\int_\Omega \nabla u(x) \nabla \varphi(x)\, dx = \langle F, \varphi \rangle, \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{2.7}$$

Moreover, there exists a constant $C(p, \Omega)$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(p, \Omega) \left(\|u_0\|_{W^{1,p}(\Omega)} + \|F\|_{W^{-1,p}(\Omega)} \right). \tag{2.8}$$

□

Definition 2.2 (Strong Steepest Descent Algorithm)**.** *Let* $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, *such that* $E(u_0) < 1$, *be given. Let us assume that the strong gradient given by* (2.3) *satisfies the regularity* $g(u_n) \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$. *Let* $\alpha_n \in (0, 1/\Lambda]$ *be a result of the following line search*

$$\min_{\alpha \in (0, 1/\Lambda]} E(u_n - \alpha g(u_n)). \tag{2.9}$$

We choose the new iteration to be

$$u_{n+1} = u_n - \alpha_n g(u_n), \qquad \text{in } \Omega. \tag{2.10}$$

We have $u_n \in C^0(\Omega, \mathbb{R}^N)$, $n \in \mathbb{N}$, in view of the continuity of the imbedding of $W^{1,p}(\Omega, \mathbb{R}^N)$ into continuous functions for $p > N$.

Remark

Frequently, we use a topologically weaker version of the Steepest Descent Algorithm, the *Weak Steepest Descent Algorithm*, in Finite Element calculations modeling the formation of microstructures [5], [9]. Finite element approximations disregard discontinuities in the first derivatives, regularizing thus the problem at hand. Microstructures ought to have spatial *white noise* characteristics in the deformation gradients. Consequently, the natural topology to be used for minimization in this framework is the topology of the space $W^{-1,2}(\Omega, \mathbb{R}^N)$. This would suggest the use of L^2-scalar product as opposed to the $W^{1,2}$-scalar product used in Definition 2.2. On the other hand, the L^2-scalar product is not coercive on the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^N)$ which means that we would not be able to guarantee the existence of new iterates if we were to change the scalar product to L^2- scalar product in Definition 2.2. We reconcile these difficulties by using L^2-topology but we restrict the algorithm to a finite dimensional subspace of $C^0(\overline{\Omega}, \mathbb{R}^N)$. This approach renders the minimization problem well-posed by restriction to a dense subset of $W^{1,\infty}(\Omega, \mathbb{R}^N)$. In practice, we use finite element spaces V_h which are imbedded into continuous functions by definition (e.g. $P1-$finite elements or $Q1-$finite elements on rectangular grid aligned with the coordinate system), i.e., we have $V_h \subset C^0(\overline{\Omega}, \mathbb{R}^N)$ for any positive h by construction.

□

Definition 2.3 (Weak Steepest Descent Algorithm)**.** *Let* $u_{0,h} \in V_h \subset W^{1,\infty}(\Omega, \mathbb{R}^N)$, *such that* $E(u_{0,h}) < 1$, *be given. Let* $\alpha_0 = 1/\Lambda$. *Let* $u_{n+1/2,h} \in V_h$, $u_{n+1/2,h}(x) = Fx$, $x \in \partial\Omega$, *be such that*

$$\int_\Omega \left(u_{n+1/2,h}(x) - u_{n,h}(x)\right) \varphi_h(x)\, dx = -\alpha_n \int_\Omega DW(\nabla u_{n,h}(x)) : \nabla \varphi_h(x)\, dx, \qquad \text{for all } \varphi_h \in V_h. \tag{2.11}$$

Let α_{n+1} *be a result of the following line search*

$$\overline{\min_{\alpha \in (\alpha, +\infty)}} E(u_{n,h} + \alpha(u_{n+1/2,h} - u_{n,h})), \overline{\quad} \alpha > 0. \tag{2.12}$$

We refer to [8] *and* [15] *for the line-search methods with a lower bound on the step size. We choose the new iteration to be*

$$u_{n+1,h} = u_{n,h} + \alpha_{n+1}\left(u_{n+1/2,h} - u_{n,h}\right), \qquad \textit{in } \Omega. \tag{2.13}$$

Remark

Both versions of the Steepest Descent Algorithm guarantee decrease of the energy when moving from u_n to u_{n+1}.

(i) We get using Taylor's expansion in the case of the Strong Steepest Descent Algorithm

$$\begin{aligned} E(u_{n+1}) &= E(u_n) + \alpha \int_\Omega DW(\nabla u_n(x))\nabla g(u_n)(x)\,dx \\ &\quad + \tfrac{1}{2}\alpha^2 \int_\Omega D^2W(\nabla u_n(x))\left(\nabla g(u_n)(x), \nabla g(u_n)(x)\right)\,dx \\ &\le E(u_n) - \alpha \int_\Omega \nabla g(u_n)(x)\nabla g(u_n)(x)\,dx + \tfrac{1}{2}\alpha^2\Lambda \int_\Omega \nabla g(u_n)(x)\nabla g(u_n)(x)\,dx. \end{aligned} \tag{2.14}$$

The right-hand side of the expression (2.14) has minimum at $\alpha = 1/\Lambda$. Thus, in particular,

$$\|g(u_n\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \le \sqrt{2\Lambda}E(u_n)^{1/2}. \tag{2.15}$$

(ii) The evaluation of $E(u_{n+1,h})$ in the case of the Weak Steepest Descent Algorithm is more involved. We drop the subscript h in what follows with understanding that u_n, $u_{n+1/2}$, $u_{n+1} \in V_h \subset W^{1,\infty}(\Omega,\mathbb{R}^N)$. We have, similar to (2.14), for some function $v \in V_h$

$$\begin{aligned} E(u_n + \alpha(u_{n+1/2} - u_n)) &= E(u_n) + \alpha \int_\Omega DW(\nabla u_n(x))\nabla\left(u_{n+1/2} - u_n\right)(x)\,dx \\ &\quad + \tfrac{1}{2}\alpha^2 \int_\Omega D^2W(\nabla v(x))\left(\nabla\left(u_{n+1/2} - u_n\right)(x), \nabla\left(u_{n+1/2} - u_n\right)(x)\right)\,dx \\ &\le E(u_n) - \frac{\alpha}{\alpha_n}\int_\Omega \left|u_{n+1/2} - u_n\right|^2(x)\,dx + \tfrac{1}{2}\alpha^2\Lambda \int_\Omega \left|\nabla\left(u_{n+1/2} - u_n\right)\right|^2(x)\,dx. \end{aligned} \tag{2.16}$$

Writing

$$\begin{aligned} &\Lambda \int_\Omega \left|\nabla\left(u_{n+1/2} - u_n\right)\right)\right|^2(x)\,dx \\ &= \Lambda \int_\Omega \left|\nabla u_{n+1/2} - \Pi\nabla u_{n+1/2} + \Pi\nabla u_{n+1/2} - \Pi\nabla u_n + \Pi\nabla u_n - u_n\right|^2(x)\,dx \\ &\le 2\Lambda \int_\Omega \left|\nabla u_{n+1/2} - \Pi\nabla u_{n+1/2}\right|^2\,dx + 2\Lambda \int_\Omega \left|\Pi\nabla u_{n+1/2} - \Pi\nabla u_n\right|^2\,dx + 2\Lambda \int_\Omega \left|\Pi\nabla u_n - u_n\right|^2(x)\,dx \end{aligned} \tag{2.17}$$

we can estimate the right-hand side of (2.17), using the constitutive assumption (1.5), and assuming that $\Pi\nabla u_{n+1/2} - \Pi\nabla u_n = 0$ a.e. in Ω, by

$$2\left(E(u_{n+1/2}) + E(u_n)\right).$$

Hence,

$$E(u_n+\alpha(u_{n+1/2}-u_n)) \le E(u_n) - \frac{\alpha}{\alpha_n}\int_\Omega \left|u_{n+1/2}-u_n\right|^2(x)\,dx + \alpha^2\left(E(u_{n+1/2})+E(u_n)\right). \tag{2.18}$$

It follows from (2.4) by taking $\varphi = g(u_n)$, that there exists a positive constant C_0, independent of n, such that

$$\left\|u_{n+1/2}-u_n\right\|_{L^2(\Omega,\mathbb{R}^N)} \ge C_0\,\alpha_n\,\|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}. \tag{2.19}$$

Thus it follows from (2.18) and (2.19)

$$E(u_n+\alpha(u_{n+1/2}-u_n)) \le E(u_n) - C_0\,\alpha\,\|g(u_n\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^N)} + \alpha^2\left(E(u_{n+1/2})+E(u_n)\right), \qquad \text{for any } \alpha\in\mathbb{R}^+. \tag{2.20}$$

Assuming that $E(u_n)>0$ we have for some $\alpha>0$

$$\alpha^2\left(E(u_{n+1/2})+E(u_n)\right) < \epsilon E(u_n+\alpha(u_{n+1/2}-u_n)), \qquad \epsilon>0.$$

Thus

$$(1-\epsilon)E(u_{n+1}) < E(u_n) - C_0\,\alpha\,\|g(u_n\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^N)}.$$

We find the parameter α by a line search,[8], [15]. In particular, we recover from the above inequality

$$\|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \le C\,E(u_n)^{1/2}, \quad C\ne C(n)>0. \tag{2.21}$$

□

3. Outline of the argument

The Steepest Descent Algorithms applied to relaxation of non-quasiconvex stored energies fail to produce microstructures with Young measure different from simple Dirac mass. The argument supporting this conclusion is based on the observation that the scaling properties of the *frequency* term $\mathrm{moh}\,(\mathcal{E}_n)$ are controlled by two different mechanisms. Namely, by the strong gradient $g(u_n)\in W^{1,2}(\Omega,\mathbb{R}^N)$, and by the *amplitude* term $\|u_n-Fx\|_{L^2(\Omega,\mathbb{R}^N)}$. The scaling properties of the strong gradient are determined by the Steepest Descent Algorithm; it follows from (2.15) as well as (2.21) and Theorem 5.1 that its $W^{1,2}$-norm scales as $E(u_n)^{1/2}$. The scaling of $\|u_n-Fx\|_{L^2(\Omega,\mathbb{R}^N)}$ is $E(u_n)^{1/4}$. The scaling of this term is determined by properties of binomial microstructures.

We construct the proof by *reductio ad absurdum*. Hence, let us assume that the sequence $\{u_n\}_{n=0}^\infty\subset W^{1,2}(\Omega,\mathbb{R}^N)$ is binomial microstructure generated by the Strong Steepest Descent Algorithms, i.e., let us assume that the sequence is given by Definition 2.2. Then Theorem 5.1 and the Definition 2.4 of the strong gradient yield

$$C_1E(u_n)^{\frac34} \le \|g(u_n)\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \le C_2E(u_n)^{\frac12}. \tag{3.1}$$

It follows from Theorem 5.3 that

$$\sum_{n=0}^{\infty}\alpha_n\,\|g(u_n)\|_{W^{1,2}(\Omega,\mathbb{R}^N)} < +\infty. \tag{3.2}$$

The summability of $\alpha_n \|g(u_n)\|_{W^{1,2}(\Omega,\mathbb{R}^N)}$ is enough to obtain the strong convergence of the iterates given by the Strong Steepest Descent Algorithm, Definition 2.2. Namely, denoting the weak limit of $\{u_n\}_{n\in\mathbb{N}}$ in $W^{1,2}(\Omega,\mathbb{R}^N)$ by u we have

$$\|u_n - u\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq \sum_{m=n}^{\infty} \alpha_n \|g(u_m)\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \to 0, \qquad \text{as } n\to\infty. \tag{3.3}$$

In order to show the strong convergence of the iterates generated by the Weak Steepest Descent Algorithm, Definition 2.3, we need additional scaling laws because we do not have a direct access to the gradients of this sequence, c.f. (2.13), (2.10). We note that the following arguments can be also applied to gradient flow associated with the energy E, i.e., to the equation $u_t = \operatorname{div} DW(\nabla u)$, [5].

We show in Theorem 6.1 that a simplified version of Lojasiewicz-Simon's inequality [16] providing the lower bound in (3.1) yields

$$0 < C \leq \|g(u_{n,h})\|^{2/3}_{W^{1,2}(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_{n,h}), \qquad N = 2,3, \tag{3.4}$$

for any sequence $\{u_{n,h}\}_{n,h} \subset V_h$ given by Definition 2.3. We recall that $\mathcal{E}_{n,h}$ is the set of laminates, c.f. (1.13), corresponding to $u_{n,h}$.

The binomial microstructures obey the following amplitude-frequency coupling proven in Theorem 5.5

$$\|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_n) < \infty. \tag{3.5}$$

Next, we derive in Theorem 7.2 and Lemma 7.1 the following upper and lower estimates valid for any binomial microstructure

$$C_1\, E(u_n)^{\frac{1}{4\gamma}} \leq \|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \leq C_2\, E(u_n)^{\frac{1}{4}}, \qquad \text{for any } \gamma \in (0,1). \tag{3.6}$$

Whence, assuming that the iterates generated by the Weak Steepest Descent Algorithm, Definition 2.3, yield minimizing binomial microstructure in V_h, i.e., $\lim_{n\to\infty} E(u_{n,h}) \leq C\, h^\alpha$, for some $\alpha > 0$, the contradiction is established upon comparison of the decay rates of the energy. Namely,

$$\begin{aligned} E(u_{n,h})^{\frac{1}{4\gamma}} &\overset{(3.6)}{\leq} C_1\, \|u_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\overset{(3.5)}{\leq} C_2\, \operatorname{moh}(\mathcal{E}_{n,h})^{-1} \\ &\overset{(3.4)}{\leq} C_3\, \|g(u_{n,h})\|^{\frac{2}{3}}_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\ &\overset{(2.21)}{\leq} C_4\, E(u_{n,h})^{\frac{1}{2}\frac{2}{3}}, \qquad \text{for any } \gamma\in(0,1), \quad h>0. \end{aligned} \tag{3.7}$$

The paper is organized according to the above steps. We prove existence of sharp microstructures in Section 4. We prove (3.1) in Section 5 and (3.4) in Section 6. We establish (3.5) in Section 4 and we verify (3.6) in Section 7. The last Section 8 contain the strong convergence result (1.15) for the Weak Steepest Descent Algorithm. The last Section 9 contains additional inequalities which might be of interest but which are not used in the paper.

4. SHARP BINOMIAL MICROSTRUCTURES

We established the existence of the sharp binomial microstructures $\{w_n\}_{n\in\mathbb{N}} \subset C^0(\Omega, R^N)$ in [10] within the framework of finite dimensional approximation. Here, we provide an extension of this result to infinite dimensional setting.

Theorem 4.1. *Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, be a minimizing sequence of*

$$\int_\Omega \|\Pi\nabla u_n(x) - \nabla u_n(x)\|^2 \, dx.$$

Then for every function u_n there exists a function $w_n \in C^0(\Omega, R^N)$ with $\nabla w_n \in \{F_1, F_2\}$ a.e. in Ω, such that for $n \geq n_0$, $n_0 \in \mathbb{N}$ sufficiently large, we have

$$\|u_n - w_n\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq C \, \|\nabla u_n - \Pi\nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} \,. \tag{4.1}$$

The constant C in (4.1) *is independent of n.*

Proof. The function u_n is weakly differentiable in Ω thus by Rademacher Theorem [[17], Theorem 2.2.1] u_n is classically differentiable for almost all points in Ω. Moreover, by Sobolev imbedding, $u_n \in C^0(\Omega,\mathbb{R}^N)$.

(i) Let x_0 be a point at which ∇u_n exists in the classical sense. Let us denote by $B_r(x_0)$ a ball at x_0 with radius $r > 0$. We define the function

$$w_{x_0,r}(x) \stackrel{\text{def}}{=} Fx_0 + \Pi_{1,2}\nabla u_n(x_0)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \,, \qquad x \in B_r(x_0). \tag{4.2}$$

The vector $r_{x_0} \in R^N$ will be determined later. We have for any $x \in B_r(x_0)$ and $i = 1$ or $i = 2$

$$|w_{x_0,r}(x) - Fx| = \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| . \tag{4.3}$$

Now for $x \in B_r(x_0)$ we get

$$\begin{aligned} &|w_{x_0,r}(x) - u_n(x)| \\ &\leq |Fx_0 - u_n(x_0) + u_n(x_0) - u_n(x) + F(x - x_0)| \\ &\quad + \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| \\ &\leq 3\,\|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} + \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| . \end{aligned} \tag{4.4}$$

If $r_{x_0} \neq 0$ we obtain

$$\begin{aligned} &\|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \\ &= \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} - (F_i - F)(x - x_0)\right| \\ &\leq \frac{1}{|r_{x_0}|} \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| + \frac{1}{|r_{x_0}|} \left|(F_i - F)(x - x_0)\right| . \end{aligned} \tag{4.5}$$

Taking $r > 0$ such that

$$\frac{\|F_i - F\|}{|r_{x_0}|} r \leq \tfrac{1}{2} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \tag{4.6}$$

yields

$$\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \leq \frac{2}{|r_{x_0}|} \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| . \tag{4.7}$$

Whence we get from (4.6) and (4.4)

$$\begin{aligned} &|w_{x_0,r}(x) - u_n(x)| \\ &\leq \left(1 + \frac{6}{|r_{x_0}|}\right) \left|(F_i - F)(x - x_0) + r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\right| \\ &= \left(1 + \frac{6}{|r_{x_0}|}\right) |w_{x_0,r}(x) - Fx| \,. \end{aligned} \tag{4.8}$$

Finally,

$$|w_{x_0,r}(x) - Fx| \le |(F_i - F)(x - x_0)| + \left| r_{x_0} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \right|. \tag{4.9}$$

Taking $r > 0$ such that

$$r \le |r_{x_0}| \, \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \tag{4.10}$$

we get from (4.9)

$$|w_{x_0,r}(x) - Fx| \le 2\, |r_{x_0}| \, \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} . \tag{4.11}$$

Thus we have for

$$r \le \min\left\{ 2\frac{|r_{x_0}|}{\|F - F_i\|}, |r_{x_0}| \right\} \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}$$

the following inequalities

$$\begin{aligned} |w_{x_0,r}(x) - u_n(x)| &\le \left(1 + \frac{6}{|r_{x_0}|}\right) |w_{x_0,r}(x) - Fx| \\ &\le 2\, |r_{x_0}| \left(1 + \frac{6}{|r_{x_0}|}\right) \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} , \qquad \text{for any } x \in B_r(x_0). \end{aligned} \tag{4.12}$$

Let us define

$$\begin{aligned} M &\overset{\text{def}}{=} \{x_0 \in \Omega \,|\, \nabla u_n(x_0) \text{ exists in the classical sense, } \|\nabla u_n(x_0) - \Pi_{1,2}\nabla u_n(x_0)\| < \delta < 1\}, \\ G(x_0, r) &\overset{\text{def}}{=} \{x \in B_r(x_0) \,|\, \operatorname{rank}(\nabla w_{x_0,r}(x) - \Pi_{1,2}\nabla u_n(x)) = 0\}, \quad x_0 \in M. \end{aligned} \tag{4.13}$$

The sets $G(x_0, r) \neq \emptyset$ have the following properties

$$\begin{aligned} &G(x_0, r) \subset B_r(x_0), \text{ closed}, \\ &\operatorname{meas}_N(G(x_0, r)) \ge C\, r^N, \\ &\operatorname{meas}_N(\partial G(x_0, r)) = 0. \end{aligned} \tag{4.14}$$

The constant C appearing in (4.14) is independent of r. The properties (4.14b,c) can be argued as follows. First, we verify the condition (4.14b). Let us assume that $\Pi_{1,2}\nabla u_n(x_0) = F_1$. Then

$$G_1(x_0, r) \overset{\text{def}}{=} G(x_0, r) = \{x \in B_r(x_0) \,|\, \Pi_{1,2}\nabla u_n(x) = F_1\}.$$

Without loss of generality we can assume that

$$\|\Pi_{1,2}\nabla u_n(x) - \nabla u_n(x)\| << 1 \qquad \text{a. e. in } B_r(x_0)$$

since $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence of $\int_\Omega \|\Pi\nabla u_n(x) - \nabla u_n(x)\|^2 \, dx$. The Young measure associated with this sequence converges to $\lambda_1\delta_{F_1} + \lambda_2\delta_{F_2}$ [13]. Thus, in particular,

$$B_r(x_0) = G_1(x_0, r) \cup G_2(x_0, r) \cup \mathcal{N}, \quad \operatorname{meas}(\mathcal{N}) < \epsilon,$$

where $G_2(x_0, r)$ is defined as $G_1(x_0, r)$ but with F_1 replaced by F_2. If the density of $G_1(x_0, r)$ would be vanishing, i.e., if

$$\lim_{r\to 0} \frac{\operatorname{meas}_N(G_1(x_0, r))}{r^N} = 0$$

then $\operatorname{meas}(G_2(x_0, r)) \to \operatorname{meas}(B_r(x_0)) + \epsilon$. But then

$$\frac{\lambda_1}{\lambda_2} \le \frac{\operatorname{meas}(G_1(x_0, r))}{\operatorname{meas}(G_2(x_0, r))} \to 0, \qquad \text{as } r \to 0,$$

which establishes the contradiction. Thus (4.14b) holds. [1]

As for the remaining conclusion, we proceed as follows. Since

$$\{x \in B_r(x_0) \mid [\Pi_{1,2}\nabla u_n(x)] = \pm a \otimes b\} \subset \partial G(x_0, r), \tag{4.15}$$

where $[.]$ denotes the jump of the projection of the derivatives of u_n, if

$$\text{meas}_N(\partial G(x_0, r)) > 0$$

there exists a measurable subset of Ω such that for any $x \in \Omega$ we have $[\Pi_{1,2}\nabla u_n(x)] \neq 0$. But then $u_n \notin W^{1,\infty}(\Omega, \mathbb{R}^N)$. Hence (4.14c) holds.

Now we are in a position to apply Vitali Covering Theorem [[6], Theorem 10.5]. We obtain a (at most) countable subcollection of disjoint sets such that

$$\mathcal{M} = \bigcup_k G(x_k, r_k). \tag{4.16}$$

We define for any $x \in \mathcal{M}$

$$w_n(x) \stackrel{\text{def}}{=} w_{x_k, r_k}(x), \qquad x \in G(x_k, r_k). \tag{4.17}$$

(ii) Now we determine the vectors r_{x_k} in the definition (4.2) so that

$$\begin{aligned} &w_n \in C^0(\mathcal{M}, \mathbb{R}^N), \\ &|r_{x_n}| \geq C > 0. \end{aligned} \tag{4.18}$$

Assuming that $0 < \delta < 1$ implies that ∇u_n are a. e. $rank - one$ oriented in $\mathcal{M}$. We can assume that $\mathcal{M}$ is open and connected for otherwise we would apply the forthcoming argument to each of the components of $\mathcal{M}$. Hence application of [Proposition 1, [3]] yields a function $y_n \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that

$$\begin{aligned} &\nabla y_n \in \{F_1, F_2\}, \qquad \text{a. e. in } \Omega, \\ &y_n(x) = y_0 + F_2 x + g_n(x) a, \qquad x \in \mathcal{M}, \end{aligned} \tag{4.19}$$

where $y_0 \in \mathbb{R}^N$, $y_0 \perp a$, and $g_n \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ is such that $\nabla g_n(x) = \chi_{F_1}(x) b$ almost everywhere, χ_{F_1} denotes the characteristic function of the set of all points $x \in \mathcal{M}$ such that $\Pi_{1,2}\nabla u_n(x) = F_1$. Let $\mathcal{M}^c$ be a convex subset of $\mathcal{M}$. Then we can represent the function g_n in the form

$$g_n(x) = g(x \cdot b) + \beta \left\| u_n - Fx \right\|_{L^\infty(\Omega, \mathbb{R}^N)}, \qquad \beta > 0, \quad x \in \mathcal{M}^c, \tag{4.20}$$

where $g \in W^{1,\infty}(\mathbb{R})$ has derivatives either 0 or 1. Requiring

$$w_n(x_k) = y_n(x_k), \qquad x_k \in \mathcal{M},$$

give the following equation for r_{x_k}

$$y_0 + F_i x_k + \beta \left\| u_n - Fx \right\|_{L^\infty(\Omega, \mathbb{R}^N)} a = F x_k + r_{x_k} \left\| u_n - Fx \right\|_{L^\infty(\Omega, \mathbb{R}^N)}, \quad i = 1, 2. \tag{4.21}$$

Since $\nabla y_n \in \{F_1, F_2\}$ a. e. in $\mathcal{M}$, we obtain from (4.21)

$$w_n(x) = y_n(x), \qquad x \in G(x_k, r_k). \tag{4.22}$$

Consequently, (4.22) yields continuity of w_n in $\mathcal{M}$.

---

[1] If we would know that osc $u_n \geq \delta_n > 0$ then (4.14b) can be argued as follows. We have in view of the local Lipschitz property of u_n

$$\delta_n \leq |u_n(x_0) - u_n(y)| \leq \|F_i\| \, |x_0 - y|, \quad i = 1 \text{ or } 2.$$

Now we intersect some connected component of $B_r(x_0)$ with the set

$$\{y \mid |x_0 - y| \geq \delta_n / \|F_i\|\}.$$

Such set is contained in $G(x_0, r)$ and (4.14b) would be established. Unfortunately, it is not possible to rule out that osc u_n will vanish based solely on the information that $\|\Pi_{1,2}\nabla u_n - \nabla u_n\|_{L^2(\Omega)} \to 0$.

Now we provide the lower bound for $|r_{x_k}|$. Multiplying (4.21) by a, and recalling that $a \perp y_0$ and $a \perp b$, we get

$$\begin{aligned}&\beta \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} |a|^2 \\ &= r_{x_k} \cdot a \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} + (Fx_k - u_n(x_k)) \cdot a + (u_n(x_k) - F_i x_k) \cdot a \\ &= r_{x_k} \cdot a \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} .\end{aligned} \tag{4.23}$$

Hence $|r_{x_k}| \geq \beta |a|$. The assumption $\beta \geq$ const. > 0 yields the lower bound for r_{x_k}.

(iii) The last step consists in extending the function w_h from $\mathcal{M}^c$ onto Ω. The domain $\mathcal{M}^c$ is convex and closed. Without loss of generality we can assume that ∂M^c has uniform C^1-regularity, otherwise we would apply the argument to a proper subset of M^c. Moreover, we can chose M^c so that there exists a finite cover of ∂M^c. We extend w_h by method of images in the $a-$direction [[1], Theorem 4.26] and periodically in the b-direction. Since $a \perp b$ and $\{b, b^\perp\}$ forms a basis in $\mathbb{R}^N$, the reflection operator E brings w_h from M^c to $\mathbb{R}^N$. The reflection (extension) operator E mapping from M^c onto Ω is linear and it has the following properties

$$\begin{aligned}&Ew_n \in W^{1,2}(\Omega,\mathbb{R}^N), \\ &Ew_n = w_n, \quad \text{a.e. in } M^c, \\ &\nabla E w_n \in \{F_1, F_2\}, \qquad \text{a.e. in } \Omega, \text{ since } \nabla w_h \in \{F_1, F_2\}, \\ &\|Ew_n\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq C \|w_n\|_{W^{1,2}(M^c,\mathbb{R}^N)} .\end{aligned} \tag{4.24}$$

Combining (4.24) with (4.8) yields

$$\begin{aligned}\|u_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} &= \|u_n - Eu_n + Eu_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \|u_n - Eu_n\|_{L^2(\Omega,\mathbb{R}^N)} + \|E(u_n - w_h)\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} + C \|u_n - w_n\|_{L^2(M^c,\mathbb{R}^N)} \\ &\leq \|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} + C \|w_n - Fx\|_{L^2(M^c,\mathbb{R}^N)} .\end{aligned} \tag{4.25}$$

Since $\{u_n\}_{n\in N}$ is a binomial microstructure, we have

$$\text{meas}\,(\Omega\setminus M^c) \to 0. \tag{4.26}$$

Moreover, since $u_n \in W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, we can estimate using intepolation inequalities, $2 < p_1 < p$,

$$\begin{aligned}\|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} &\leq \|u_n - Eu_n\|_{L^{p_1}(\Omega\setminus M^c,\mathbb{R}^N)} \\ &\leq \|u_n - Eu_n\|^{\alpha}_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} \|u_n - Eu_n\|^{1-\alpha}_{L^{s_1}(\Omega\setminus M^c,\mathbb{R}^N)} ,\end{aligned} \tag{4.27}$$

where $1/p_1 = \alpha/2 + (1-\alpha)/s_1$, $2 \leq p_1 \leq s_1 \leq +\infty$. Thus

$$\begin{aligned}&\|u_n - Eu_n\|^{\alpha}_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} \|u_n - Eu_n\|^{1-\alpha}_{L^{s_1}(\Omega\setminus M^c,\mathbb{R}^N)} \\ &\leq \|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} \|u_n - Eu_n\|^{\alpha-1}_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} \text{meas}\,(\Omega\setminus M^c)^{\beta} \|u_n - Eu_n\|^{1-\alpha}_{L^{s_2}(\Omega\setminus M^c,\mathbb{R}^N)} ,\end{aligned} \tag{4.28}$$

where $\beta > 0$ and $p \geq s_2 > s_1 > 2$. Consequently, we have from (4.28) and (4.27)

$$\|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} \leq \text{meas}\,(\Omega\setminus M^c)^{\beta} \|u_n - Eu_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} , \quad \beta > 0. \tag{4.29}$$

We can estimate

$$\begin{aligned}\|u_n - Eu_n + Ew_n - Ew_n\|_{L^2(\Omega\setminus M^c,\mathbb{R}^N)} &\leq \|u_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} + \|E(u_n - w_n)\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \|u_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} + C \|u_n - w_n\|_{L^2(\Omega^c,\mathbb{R}^N)} \\ &\leq \|u_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} + C \|w_n - Fx\|_{L^2(\Omega^c,\mathbb{R}^N)} .\end{aligned} \tag{4.30}$$

Combining (4.30) with (4.25) and taking $\mathrm{meas}\,(\Omega\backslash M^{c})^{\beta} < 1$, we get the inequality

$$\left(1 - \mathrm{meas}\,(\Omega\backslash M^{c})^{\beta}\right) \|u_n - Ew_n\|_{L^2(\Omega,\mathbb{R}^N)} \leq C\, \|w_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)}\,. \tag{4.31}$$

The constant C is independent of n. We will denote Ew_n by w_n in what follows.

It follows from (4.11) that there exists a constant C, independent of n, such that

$$\begin{aligned} \|w_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} &\leq C \inf_{r_{x_k}\in\mathbb{R}^N, |r_{x_k}|\geq\beta>0} |r_{x_k}|\, \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \\ &= C\,\beta\, \|u_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)}\,, \quad \beta > 0. \end{aligned} \tag{4.32}$$

The constant β appears in view of the representation (4.20). Now we extend the upper bound in (4.32) to $\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)}$. Namely, we show that there exists a positive constant C, independent of n, such that

$$\|w_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \leq C\,\beta\, \|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)}\,, \qquad \beta > 0. \tag{4.33}$$

Let us denote $g(x) = w_n(x) - Fx$ and $f(x) = u_n(x) - Fx$ for brevity. We observe that (4.31) yields a constant C, independent of n, such that

$$\|f - g\|_{L^2(\Omega,\mathbb{R}^N)} \leq C\, \|g\|_{L^2(\Omega,\mathbb{R}^N)}\,. \tag{4.34}$$

Moreover, assuming that $0 < \beta < 1/2$, (4.11) guarantees

$$\|g\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq \|f\|_{L^\infty(\Omega,\mathbb{R}^N)}\,. \tag{4.35}$$

We note that $\beta > 0$ can be arbitrarily small since $\{u_n\}$ forms binomial microstructure. Hence, the Young's inequality and (4.34) yield for some $\epsilon > 0$

$$\begin{aligned} \|g\|^2_{L^2(\Omega,\mathbb{R}^N)} &= \|f\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2\int_\Omega (g(x) - f(x))\, f(x)\, dx + \|f - g\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \|f\|^2_{L^2(\Omega,\mathbb{R}^N)} + 4\epsilon\, \|g\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac{1}{\epsilon}\, \|f\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|f - g\|^2_{L^2(\Omega,\mathbb{R}^N)}\,. \end{aligned} \tag{4.36}$$

We have in view of (4.35) $\int_\Omega (g(x) - f(x))\, g(x)\, dx < 0$. Thus, using (4.34), we get

$$\begin{aligned} \|f - g\|^2_{L^2(\Omega,\mathbb{R}^N)} &= \int_\Omega (g(x) - f(x))\, f(x)\, dx + \int_\Omega (g(x) - f(x))\, g(x)\, dx \\ &\leq \int_\Omega (g(x) - f(x))\, f(x)\, dx \\ &\leq 4\epsilon\, \|g\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac{1}{\epsilon}\, \|f\|^2_{L^2(\Omega,\mathbb{R}^N)}\,. \end{aligned} \tag{4.37}$$

The upper estimate (4.33) follows from (4.37) and (4.36).

The gradient of w_n is a projection of ∇u_n onto $\{F_1, F_2\}$. Consequently, we have in view of [[13], proof of Theorem 4]

$$\|\nabla u_n - \nabla w_n\|^2_{L^2(\Omega,\mathbb{R}^{N\times N})} \leq C\, \|\nabla u_n - \Pi\nabla u_n\|_{L^2(\Omega,\mathbb{R}^{N\times N})}\,. \tag{4.38}$$

We have from (4.33) and (4.38)

$$\begin{aligned} C\, \|w_n - u_n\|^2_{W^{1,2}(\Omega,\mathbb{R}^N)} &\leq \|w_n - Fx\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|\nabla u_n - \nabla w_n\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq C\, \left(\|u_n - Fx\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|\nabla u_n - \Pi\nabla u_n\|_{L^2(\Omega,\mathbb{R}^{N\times N})}\right). \end{aligned} \tag{4.39}$$

The Poincaré-Friedrichs inequality and [[13], Theorem 2] yield

$$\begin{aligned} \|u_n - Fx\|^2_{L^2(\Omega,\mathbb{R}^N)} &\le C\ \|(\nabla u_n - F)\,m\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ &\le C\ \|(\nabla u_n - F)\,m\|^2_{L^2(\Omega,\mathbb{R}^N)}, \quad m \in b^\perp, \\ &\le C\ \|\nabla u_n - \Pi\nabla u_n\|_{L^2(\Omega,\mathbb{R}^N)}. \end{aligned} \tag{4.40}$$

The proof of (4.1) follows from (4.39) and (4.40). □

Lemma 4.2. *Let the sequence* $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, *be a binomial microstructure. Let* $\mathcal{E}_n$ *be its set of laminates. Then there exists a positive constant* C, *independent of* n, *such that*

$$\max_{e\in\mathcal{E}_n} \operatorname{meas}(e_n) \le C\ \min_{e\in\mathcal{E}_h} \operatorname{meas}(e_n). \tag{4.41}$$

Proof. Let $\{w_n\}_{n=0}^{\infty}$ be the sharp approximate binomial microstructure. Let e be an arbitrary open subset of Ω, and let

$$e_i \stackrel{\text{def}}{=} \{x \in e \,|\, \Pi_{1,2}\nabla w_n(x) = F_i\}, \quad i = 1,2. \tag{4.42}$$

We have for $i = 1,2$

$$\int_e \lambda_i F_i\,dx + (\operatorname{meas}(e_i) - \lambda_i \operatorname{meas}(e))\,F_i = \int_{e_i} \Pi_{1,2}\nabla w_n(x)\,dx. \tag{4.43}$$

Thus

$$\left\|\sum_{i=1}^{2} (\operatorname{meas}(e_i) - \lambda_i \operatorname{meas}(e))\,F_i\right\| = \left\|\int_e \nabla w_n(x) - F\,dx\right\|. \tag{4.44}$$

Moreover, taking $e_{1,2} = e_1 \cup e_2$, for any $e_1, e_2 \in \mathcal{E}_n$, we have

$$\begin{aligned} &\sum_{i=1}^{2} (\operatorname{meas}(e_i) - \lambda_i \operatorname{meas}(e_{1,2}))\,F_i \\ &= (\lambda_2 \operatorname{meas}(e_1) - \lambda_1 \operatorname{meas}(e_2))\,F_1 + (\lambda_1 \operatorname{meas}(e_2) - \lambda_2 \operatorname{meas}(e_1))\,F_2 \\ &= (\lambda_2 \operatorname{meas}(e_1) - \lambda_1 \operatorname{meas}(e_2))\,(F_1 - F_2)\,. \end{aligned} \tag{4.45}$$

Hence

$$\begin{aligned} |\lambda_1 \operatorname{meas}(e_2) - \lambda_2 \operatorname{meas}(e_1)|\,\|F_1 - F_2\| &= \left\|\sum_{i=1}^{2} (\operatorname{meas}(e_i) - \lambda_i \operatorname{meas}(e_{1,2}))\,F_i\right\| \\ &= \left\|\int_{e_{1,2}} \nabla w_n(x) - F\,dx\right\|. \end{aligned} \tag{4.46}$$

We obtain using integration by parts the inequality

$$\left\|\int_{e_{1,2}} \nabla w_n(x) - F\,dx\right\| \le \|w_n - Fx\|_{L^\infty(e_{1,2},\mathbb{R}^N)} \operatorname{meas}_{N-1}(\partial e_{1,2}). \tag{4.47}$$

The inequality (4.47) and (4.46) yield

$$\begin{aligned} \left|\frac{\operatorname{meas}(e_2)}{\operatorname{meas}(e_1)} - \frac{\lambda_2}{\lambda_1}\right| &= \frac{1}{\|F_1 - F_2\|\,\lambda_1 \operatorname{meas}(e_1)} \left\|\int_{e_{1,2}} \nabla w_n(x) - F\,dx\right\| \\ &\le C\ \|w_n - Fx\|_{L^\infty(e_{1,2},\mathbb{R}^N)} \operatorname{meas}(e_1)^{-1}. \end{aligned} \tag{4.48}$$

The sharp binomial microstructure is a continuous affine function hence, by comparing the gradients,

$$\|w_n - Fx\|_{L^\infty(e_{1,2},\mathbb{R}^N)} \operatorname{meas}(e_1)^{-1} \le C. \tag{4.49}$$

Consequently, there exists a constant C, independent of $\mathcal{E}_n$ such that

$$\left| \frac{\operatorname{meas}(e_2)}{\operatorname{meas}(e_1)} - \frac{\lambda_2}{\lambda_1} \right| \le C. \tag{4.50}$$

The proof follows from the arbitrariness of the choice of $e_i \in \mathcal{E}_n$, $i = 1, 2$.

□

5. Estimates of the strong gradient

The following Theorem is a simplified version of [[16], Theorem 3].

Theorem 5.1. *Let the sequence $\{u_n\}_{n=0}^\infty \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, be a binomial microstructure relaxing the variational integral $\int_\Omega W(\nabla u(x))\, dx$. Then there exists a positive constant C, independent of $n \in \mathbb{N}$, such that we have*

$$C\,\Lambda^{\frac{1}{4}}\, E(u_n)^{\frac{3}{4}} \le \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \le \sqrt{2\Lambda}\, E(u_n)^{\frac{1}{2}}. \tag{5.1}$$

Proof. (i) The upper estimate of the strong gradient is derived in (2.15).

(ii) The lower estimate in (5.1) follows by using the sharp binomial microstructures delivered by Theorem 4.1, and by using the constitutive assumption (1.5). We obtain

$$\begin{aligned}
E(u_n) &= |E(u_n) - E(w_n)| \\
&= \left| \int_0^1 \int_\Omega DW(\nabla u_n + t(\nabla w_n - \nabla u_n))(x) : \nabla(w_n(x) - u_n(x))\, dx\, dt \right| \\
&= \left| \int_0^1 \int_\Omega \nabla g(u_n + t(w_n - u_n))(x) : \nabla(w_n(x) - u_n(x))\, dx\, dt \right| \\
&\le \int_0^1 \|g(u_n + t(w_n - u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\, dt\, \|\nabla(w_n - u_n)\|_{L^2(\Omega,\mathbb{R}^{N\times N})} \\
&\le C \int_0^1 \|g(u_n + t(w_n - u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\, dt\, \|\nabla u_n - \Pi \nabla u_n\|^{\frac{1}{2}}_{L^2(\Omega,\mathbb{R}^{N\times N})} \\
&\le C \left(\frac{1}{\Lambda}\right)^{\frac{1}{4}} \int_0^1 \|g(u_n + t(w_n - u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\, dt E(u_n)^{\frac{1}{4}}.
\end{aligned} \tag{5.2}$$

We show below that there exists a constant C, independent of n, such that

$$\int_0^1 \|g(u_n + t(w_n - u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\, dt \le C\, \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}. \tag{5.3}$$

The proof then follows from (5.2) and (5.3).

We have, in view of the assumption (1.4),

$$\begin{aligned} &\frac{d}{dt}\|g(u_n+t(w_n-u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\ &= -\|g(u_n+t(w_n-u_n))\|^{-1}_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\ &\quad 2\int_\Omega D^2W(\nabla u_n+t(\nabla w_n-\nabla u_n))(x)\left(\nabla(w_n(x)-u_n(x)),\nabla g(u_n+t(w_n-u_n))\right)\,dx \\ &= -\|g(u_n+t(w_n-u_n))\|^{-1}_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\, 2\Lambda_1\int_\Omega \nabla\left(w_n(x)-u_n(x)\right):\nabla g(u_n+t(w_n-u_n))\,dx, \end{aligned} \tag{5.4}$$

for some $\Lambda_1>0$. Since, recalling again the assumption (1.4), for some $\tau\in(0,1)$ and any $\varphi\in W_0^{1,2}(\Omega,\mathbb{R}^N)$,

$$\begin{aligned} &\int_\Omega \nabla g(u_n)(x):\nabla\varphi(x)\,dx = \int_\Omega DW(\nabla u_n(x)):\nabla\varphi(x)\,dx \\ &= \int_\Omega \left(DW(\nabla u_n(x))-DW(\nabla w_n(x))\right):\nabla\varphi(x)\,dx \\ &= \int_\Omega D^2W\left(\nabla u_n(x)+\tau\nabla\left(w_n-u_n\right)(x)\right)\right)\left(\nabla\left(w_n-u_n\right)(x),\nabla\varphi(x)\right)\,dx \\ &= \Lambda_2\int_\Omega \nabla\left(w_n-u_n\right)(x):\nabla\varphi(x)\,dx, \qquad \Lambda_2>0, \end{aligned} \tag{5.5}$$

we obtain from (5.5) and (5.4) with $\varphi=g(u_n+t(w_n-u_n))$ a positive constant C, independent of n, such that

$$\left|\frac{d}{dt}\|g(u_n+t(w_n-u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\Big|_{t=\tau}\right| \le C\,\|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,. \tag{5.6}$$

The inequality (5.3) follows from (5.6) and the proof is completed.

□

Remark
The lower estimate in (5.1) can be improved to $E(u_n)^{\frac{1}{2}}$ using differential geometry arguments if the energy density W were analytic [16].

□

Now we prove the following two results, Lemma 5.2 and Theorem 5.3, yielding the summability of the $W^{1,2}(\Omega,\mathbb{R}^N)$-norm of $g(u_n)$ which is essential to obtain the strong convergence of the minimizing sequences generated by the Strong Steepest Descent Algorithm.

Lemma 5.2. *Let $\{g(u_n)\}_{n=0}^\infty\subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p>N$, be the sequence of strong gradients, defined by* (2.1)*, corresponding to the sequence $\{u_n\}_{n=0}^\infty\subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p>N$, generated by the Strong Steepest Descent Algorithm. Then there exists a positive finite constant β, independent of n, such that for any $n\ge n_0$ we have for $t\in(0,\alpha_n)$*

$$\begin{aligned} &\int_\Omega \nabla g\left(u_{n+1}+tg(u_n)\right)(x)\nabla g(u_n)(x)\,dx \\ &\ge \beta\,\|g(u_{n+1}+tg(u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,\|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,, \qquad t\in[0,\alpha_n]. \end{aligned} \tag{5.7}$$

Proof. We denote

$$g_n(t)(x) \stackrel{\text{def}}{=} g\left(u_{n+1} + tg(u_n)\right)(x).$$

We will not indicate the dependence on x in the course of the proof to make the formulas shorter and easier to read. We have

$$\begin{aligned}(5.8)\quad & \int_\Omega \|\nabla g_n(t)\|^2 \, dx + \int_\Omega \|\nabla g(u_n)\|^2 \, dx \\ &= \int_\Omega \nabla g(u_n) \nabla g_n(t)\, dx - \int_\Omega \nabla\left(g(u_n) - g_n(t)\right) \nabla g_n(t)\, dx + \int_\Omega \|\nabla g(u_n)\|^2 \, dx.\end{aligned}$$

Writing

$$\begin{aligned}(5.9)\quad & \int_\Omega \nabla\left(g(u_n) - g_n(t)\right) \nabla g_n(t)\, dx \\ &= (t-\alpha_n) \int_\Omega \int_0^1 D^2 W\left(\nabla u_n + \tau \nabla\left(u_{n+1} - u_n + tg(u_n)\right)\right) d\tau \left(\nabla g(u_n), \nabla g_n(t)\right) dx,\end{aligned}$$

and recalling (1.4) we obtain from (5.8) and (5.9)

(5.10)

$$\begin{aligned}& 2\|g_n(t)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\ &\le \int_\Omega \nabla g_n(t) \nabla g(u_n)\, dx + \Lambda |t-\alpha_n| \|g_n(t)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} + \int_\Omega \|\nabla g(u_n)\|^2 \, dx.\end{aligned}$$

Since $\alpha_n \le 1/\Lambda$ we have

$$(5.11)\quad \|g_n(t)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \le \int_\Omega \nabla g_n(t) \nabla g(u_n)\, dx + \int_\Omega \|\nabla g(u_n)\|^2 \, dx.$$

Since

$$\int_\Omega \|\nabla g(u_n)\|^2 \, dx = \int_\Omega \nabla\left(g(u_n) - g_n(t)\right) \nabla g(u_n)\, dx + \int_\Omega \nabla g_n(t) \nabla g(u_n)\, dx$$

we obtain from (5.9) and Cauchy's inequality

$$(5.12)\quad \int_\Omega \|\nabla g(u_n)\|^2 \, dx \le \Lambda |t-\alpha_n| \|g_n(t)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} + \int_\Omega \nabla g_n(t) \nabla g(u_n)\, dx.$$

Since $\Lambda |t-\alpha_n| < 1$ for $t \in (0, \alpha_n)$ the proof follows from (5.12) and (5.11).

□

Theorem 5.3. *Let the sequence $\{u_n\}_{n=0}^\infty \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, be binomial microstructure generated by the Strong Steepest Descent Algorithms, i.e., let us assume that the sequence is given by Definition 2.2. Then there exists a finite constant C such that*

$$(5.13)\quad \sum_{n=0}^\infty \alpha_n \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \le C < +\infty.$$

Proof. Theorem 5.1 and the definition of the Steepest Descent Algorithm (2.9) yield for any $\theta \in [0, 1/4]$

$$\begin{aligned}
&\frac{d}{dt}E(u_{n+1} + tg(u_n))^\theta \\
&= \theta E(u_{n+1} + tg(u_n))^{\theta-1} \int_\Omega DW(\nabla u_{n+1}(x) + t\nabla g(u_n)(x))\nabla g(u_n)(x)\,dx \\
&= \theta E(u_{n+1} + tg(u_n))^{\theta-1} \int_\Omega \nabla g\left(u_{n+1} + tg(u_n)\right)(x)\nabla g(u_n)(x)\,dx \qquad (5.14)\\
&\overset{Lemma\ 5.2}{\geq} \beta\,\theta E(u_{n+1} + tg(u_n))^{\theta-1} \|g(u_{n+1} + tg(u_n))\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\
&\overset{Theorem\ 5.1}{\geq} \beta\,\theta E(u_{n+1} + tg(u_n))^{\theta-1} E(u_{n+1} + tg(u_n))^{1-\theta} \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,.
\end{aligned}$$

Integrating $\frac{d}{dt}E(u_{n+1} + tg(u_n))^\theta$ over $(0, \alpha_n)$ we obtain

$$E(u_{n+1} + \alpha_n g(u_n))^\theta - E(u_{n+1}))^\theta \geq C\,\beta\,\theta\alpha_n \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,. \tag{5.15}$$

Since $E(u_{n+1} + \alpha_n g(u_n)) = E(u_n)$ summing up (5.15) over $n \in \mathbb{N}$ we obtain

$$E(u_0)^\theta - \lim_{n\to\infty} E(u_n)^\theta \geq \beta\,\theta \sum_{n=0}^{\infty} \alpha_n \|g(u_n)\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \tag{5.16}$$

which proves (5.13). □

Theorem 5.4 (Strong Convergence of the Strong Steepest Descent Algorithm)**.** *Let the sequence* $\{u_n\}_{n=0}^\infty \subset W^{1,p}(\Omega, \mathbb{R}^N)$*,* $p > N$*, be generated by the Strong Steepest Descent Algorithm, i.e., let us assume that the sequence is given by the Definition 2.2. Then there exists a constant* C*, independent of* n*, and there exists a function* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *such that*

$$\lim_{n\to\infty} E(u_n) = E(u) \geq C > 0. \tag{5.17}$$

Moreover,

$$u_n \to u, \qquad \textit{strongly in } W^{1,2}(\Omega, \mathbb{R}^N). \tag{5.18}$$

Proof. Let us assume that the sequence $\{u_n\}_{n=0}^\infty \subset W^{1,2}(\Omega, \mathbb{R}^N)$ generated by the Strong Steepest Descent Algorithm, Definition 2.2, minimizes the energy E. Then there exists its weak limit $u \in W^{1,2}(\Omega, \mathbb{R}^N)$. It follows from Theorem 5.3 that

$$\|u_n - u\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq \sum_{m=n}^{\infty} \alpha_n \|g(u_m)\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \to 0, \qquad \text{as } n \to \infty. \tag{5.19}$$

Since

$$0 < inf_{v\in W^{1,2}(\Omega,\mathbb{R}^N)} E(v) \leq E(u) \tag{5.20}$$

the proof follows.

□

We extend the result of Theorem 5.19 for the Weak Steepest Descent Algorithm, Definition 2.3. We need additional scaling laws to show that even the Weak Steepest Descent Algorithm returns deterministic assymptotics for the deterministic initial guesses.

5.1. FREQUENCY AND AMPLITUDE

Theorem 5.5. *Let the sequence* $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, *be a binomial microstructure. Let* $\mathcal{E}_n$ *be the set of laminates of* u_n. *Then there exists a positive constant* C, *independent of* n, *such that*

$$\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_n) \le C. \tag{5.21}$$

Proof. Let $\{w_n\}_{n=0}^{\infty} \subset W^{1,\infty}(\Omega, \mathbb{R}^N)$ be the sharp binomial microstructure corresponding to $\{u_n\}_{n=0}^{\infty}$. Let us denote

$$\epsilon_n \stackrel{\text{def}}{=} \|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} . \tag{5.22}$$

Then there exists a point $\overline{x} \in \Omega$ and $e_n \in \mathcal{E}_n$ such that

$$\|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} = |w_n(\overline{x}) - F\overline{x}| = \epsilon_n, \qquad \overline{x} \in \overline{e_n}. \tag{5.23}$$

We obtain using the Taylor expansion at $x \in e_n$ with $x_0 \notin e_n$

$$\begin{aligned} w_n(x) &= w_n(x_0) + F_i(x - x_0) \\ &= w_n(x_0) - Fx + Fx - Fx_0 + Fx_0 + F_i(x - x_0), \qquad i = 1 \text{ or } 2. \end{aligned} \tag{5.24}$$

Without loss of generality we can assume that

$$|w_n(x_0) - Fx_0| \le \tfrac{1}{2} |w_n(\overline{x}) - F\overline{x}| . \tag{5.25}$$

Hence, there exists a positive constant C, independent of n, such that

$$\begin{aligned} \epsilon_n &= |w_n(\overline{x}) - F\overline{x}| = |w_n(x_0) + F(x_0 - \overline{x}) - F_i(\overline{x} - x_0)| \\ &\le |w_n(x_0) - Fx_0| + \|F\| \, |\overline{x} - x_0| + \|F_i\| \, |\overline{x} - x_0| \\ &\le \tfrac{1}{2}\epsilon_n + \max\{\|F\|, \|F_i\|\} \, |\overline{x} - x_0| \\ &\le \tfrac{1}{2}\epsilon_n + \max\{\|F\|, \|F_i\|\} \operatorname{diam}(e_n) \\ &\le C \left(\tfrac{1}{2}\epsilon_n + \max\{\|F\|, \|F_i\|\} \operatorname{meas}(e_n)\right), \qquad 0 < C < 1. \end{aligned} \tag{5.26}$$

The last inequality follows in view of the fact that e_n is laminate, i. e. $\operatorname{diam}(e_n) < 1$ and $\operatorname{meas}(e_n) = \operatorname{diam}(e_n) \times \operatorname{meas}_{N-1}(e_n)$. Thus,

$$\|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \operatorname{meas}(e_n)^{-1} = \epsilon_n \operatorname{meas}(e_n)^{-1} \le C, \quad C \ne C(n). \tag{5.27}$$

Since

$$\operatorname{moh}(\mathcal{E}_n) \le \left(\min_{e_n \in \mathcal{E}_n} \operatorname{meas}(e_n) \right)^{-1} \tag{5.28}$$

we have

$$\begin{aligned} &\|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_n) \le \|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \left(\min_{e_n \in \mathcal{E}_n} \operatorname{meas}(e_n) \right)^{-1} \\ &\le \|w_n - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \operatorname{meas}(e_n)^{-1} \le C, \quad C \ne C(n). \end{aligned} \tag{5.29}$$

It follows from (4.33) of Theorem 4.1 that

$$\begin{aligned} \|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} &\le \|u_n - w_n\|_{L^2(\Omega,\mathbb{R}^N)} + \|w_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\le C \, \|w_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} . \end{aligned} \tag{5.30}$$

The proof follows from (5.30) and (5.29).

□

6. Scaling of the frequency with respect to the Strong Gradient

The goal of this section is to show that

$$0 < C \ \leq \|g(u_{n,h})\|^{2/3}_{W^{1,2}(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_{n,h}) \,, \tag{6.1}$$

where $g(u_{n,h})$ is the strong gradient of the total energy at the point $u_{n,h}$, and $\mathcal{E}_{n,h}$ is the set of laminates corresponding to $u_{n,h}$.

Theorem 6.1. *Let the sequence* $\{u_{n,h}\}_{n=0}^{\infty} \subset V_h \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, *be binomial microstructure generated by the Weak Steepest Descent Algorithms, i.e., let us assume that the sequence is given by Definition 2.3, and let us assume that*

$$\lim_{h\to 0_+} \lim_{n\to+\infty} E(u_{n,h}) = 0. \tag{6.2}$$

Then there exists a positive constant C*, independent of* n *and* h*, such that*

$$C \ \leq \|g(u_{n,h})\|^{2/3}_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_{n,h}) \,, \tag{6.3}$$

where $g(u_{n,h})$ *is the strong gradient defined by* (2.4)*.*

Proof. Let $\{w_{n,h}\}_{n=0}^{\infty}$ be the sharp binomial microstructure corresponding to $\{u_{n,h}\}_{n=0}^{\infty} \subset V_h$, $h > 0$. The sharp binomial microstructure is a sequence of continuous functions with gradients in $\{F_1, F_2\}$. Thus there exists a constant C, independent of n, such that

$$C \ \operatorname{meas}(\Omega) \leq \int_\Omega \|\nabla w_{n,h}(x) - F\|^2 \ dx. \tag{6.4}$$

Integration by parts in (6.4) leads to the following estimate

$$\begin{aligned} C \ \operatorname{meas}(\Omega) &\leq \int_\Omega \|\nabla w_{n,h}(x) - F\|^2 \ dx = \sum_{E_{n,h}\in\mathcal{E}_{n,h}} \int_{e_n} \|\nabla w_{n,h}(x) - F\|^2 \ dx \\ &= \sum_{e_n\in\mathcal{E}_{n,h}} \int_{\partial e_n} (w_{n,h}(x) - Fx) \frac{\partial}{\partial n}(w_{n,h}(x) - F)\, ds \\ &= (F_i - F)n \cdot \sum_{e_n\in\mathcal{E}_{n,h}} \int_{\partial e_n} (w_{n,h}(x) - Fx) \ ds \\ &\leq |(F_i - F)n| \operatorname{moh}(\mathcal{E}_{n,h}) \|w_{n,h} - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \operatorname{meas}_{N-1}(\partial e_n). \end{aligned} \tag{6.5}$$

The function $w_{n,h} - Fx$ is a piece-wise affine function imbedded into the finite dimensional space V_h with $\dim(V_h) = \operatorname{moh}(\mathcal{E}_{n,h})$. Since this function is $w_{n,h}$ constant in the direction(s) $\{b^\perp\}$, its oscillations occur only in one spatial dimension. Consequently, it follows from the discrete inverse inequalities [[4], Theorem 17.2] that

$$\|w_{n,h} - Fx\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq \|w_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \operatorname{moh}(\mathcal{E}_{n,h})^{1/2} \,. \tag{6.6}$$

Since $\operatorname{meas}_{N-1}(\partial e_n) \geq C > 0$ in view of the Hadamard jump condition, we get from (6.6), (6.5) and Lemma 4.2 a positive constant C, independent of n and h, such that

$$C \ \operatorname{meas}(\Omega) \leq \operatorname{moh}(\mathcal{E}_{n,h})^{3/2} \|w_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \,. \tag{6.7}$$

We have in view of (4.33) of Theorem 4.1

$$\|w_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \leq C \ \|u_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \,. \tag{6.8}$$

Now we show that the Definition 2.3 yields

$$\|u_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \le C\ \|g(u_{n,h})\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,. \tag{6.9}$$

The proof of the Theorem thus will follow from (6.8), (6.9) and (6.7).

In order to verify (6.9) we proceed as follows. Taking

$$\varphi = \left(u_{n+1/2,h} - u_n\right)\left\|u_{n+1/2,h} - u_n\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}$$

in the Definition 2.3 we get

$$\begin{aligned}
&\left\|u_{n+1/2,h} - u_{n,h}\right\|^2_{L^2(\Omega,\mathbb{R}^N)}\\
&\le \left\|u_{n+1/2,h} - u_{n,h}\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \alpha_n \sup_{\|\varphi\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\le 1} \left|\langle G(u_{n,h}),\varphi\rangle_{W^{-1,2}(\Omega,\mathbb{R}^N),W_0^{1,2}(\Omega,\mathbb{R}^N)}\right|\\
&= \left\|u_{n+1/2,h} - u_{n,h}\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \alpha_n \|G(u_{n,h})\|_{W^{-1,2}(\Omega,\mathbb{R}^N)}\,.
\end{aligned} \tag{6.10}$$

We have from (2.16) for some $\lambda > 0$ and $n \in \mathbb{N}$ large, and some $v_{n,h} \in V_h$,

$$\begin{aligned}
&\frac{\alpha}{\alpha_n}\int_\Omega \left|u_{n+1/2,h} - u_{n,h}\right|^2(x)\,dx\\
&= E(u_{n,h}) - E(u_{n,h} + \alpha(u_{n+1/2,h} - u_{n,h}))\\
&\quad + \tfrac{1}{2}\alpha^2\int_\Omega D^2W(\nabla v_{n,h}(x))\left(\nabla\left(u_{n+1/2,h} - u_{n,h}\right)(x), \nabla\left(u_{n+1/2,h} - u_{n,h}\right)(x)\right)\,dx\\
&\ge \tfrac{1}{2}\alpha^2\lambda\int_\Omega \left\|\nabla\left(u_{n+1/2,h} - u_{n,h}\right)(x)\right\|^2\,dx.
\end{aligned} \tag{6.11}$$

It follows from (6.10) and (6.11) with $\alpha = \alpha_{n+1}$ that

$$\left\|u_{n+1/2,h} - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} \le \sqrt{\frac{2}{\lambda}}\frac{\alpha_n}{\sqrt{\alpha_n\alpha_{n+1}}}\|g(u_{n,h})\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,. \tag{6.12}$$

We can assume that there exists constant C, independent of n, such that

$$\alpha_n \le C\alpha_{n+1}. \tag{6.13}$$

Consequently,

$$\left\|u_{n+1/2,h} - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} \le \sqrt{\frac{2C}{\lambda}}\|g(u_{n,h})\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}\,. \tag{6.14}$$

Let $w_{n+1/2,h}$ be the sharp binomial microstructure, given by Theorem 4.1, corresponding to $u_{n+1/2,h}$. Let us define the function

$$\overline{w_{n+1/2,h}}(x) = w_{n+1/2,h}(x) - \tfrac{1}{\mathrm{meas}(\Omega)}\int_\Omega w_{n+1/2,h}(x) - u_{n+1/2,h}(x)\,dx. \tag{6.15}$$

Then $\nabla\overline{w_{n+1/2,h}} \in \{F_1, F_2\}$ and $\int_\Omega \overline{w_{n+1/2,h}}(x) - u_{n+1/2,h}(x)\,dx = 0$. Moreover, it follows from (4.33) that

$$\left\|w_{n+1/2,h} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)} \le \tfrac{1}{2}\left\|u_{n+1/2,h} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)}\,. \tag{6.16}$$

Since

$$\tfrac{1}{\mathrm{meas}(\Omega)}\int_\Omega w_{n+1/2,h}(x) - u_{n+1/2,h}(x)\,dx \le \mathrm{meas}(\Omega)^{-1/2}\left\|w_{n+1/2,h} - u_{n+1/2,h}\right\|_{L^2(\Omega,\mathbb{R}^N)}$$

we get from (6.16)

$$\left\|\overline{w_{n+1/2,h}} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)} \leq (1-\epsilon)\left\|u_{n+1/2,h} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)}, \quad \epsilon > 0, \tag{6.17}$$

provided $\mathrm{meas}(\Omega) > 1/4$. Since we assume that $\{u_{n,h}\}_{n\in\mathbb{N}}$ is a minimizing sequence of the energy E, we have

$$\left\|u_{n+1/2,h} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)} \leq \left\|u_{n,h} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{6.18}$$

Writing

$$\begin{aligned} &\left\|Fx - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\leq \left\|u_{n+1/2,h} - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} + \left\|u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right\|_{L^2(\Omega,\mathbb{R}^N)} + \left\|\overline{w_{n+1/2,h}} - Fx\right\|_{L^2(\Omega,\mathbb{R}^N)} \end{aligned} \tag{6.19}$$

it suffices to show that

$$\left\|u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right\|_{L^2(\Omega,\mathbb{R}^N)} \leq C\left\|u_{n+1/2,h} - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} \tag{6.20}$$

for some constant C, independent of n and h, in order to conclude (6.9). The proof of the inequality (6.9) follows from (6.19)-(6.20) and (6.14). In order to prove (6.20) we recall the Definition 2.3 and we set $\varphi = u_{n+1/2,h} - \overline{w_{n+1/2,h}}$. This yields for some $v_{n,h} \in V_h$ such that $\|\nabla v_{n,h} - \Pi\nabla v_{n,h}\|_{L^2(\Omega,\mathbb{R}^{N\times N})}$ is small

(6.21)

$$\begin{aligned} &\int_\Omega \left(u_{n+1/2,h} - u_{n,h}\right)(x)\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)(x)\,dx \\ &= -\alpha_n \int_\Omega DW(\nabla u_n) : \nabla\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)(x)\,dx \\ &\quad + \alpha_n \int_\Omega DW(\nabla \overline{w_{n+1/2,h}}) : \nabla\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)(x)\,dx \\ &= \alpha_n \int_\Omega D^2W(\nabla v_{n,h})\left(\nabla\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)(x), \nabla\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)(x)\right)\,dx \\ &\geq \alpha_n\lambda\left\|\nabla\left(u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right)\right\|^2_{L^2(\Omega,\mathbb{R}^{N\times N})} \geq C\alpha_n\left\|u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right\|^2_{L^2(\Omega,\mathbb{R}^N)},\ C>0. \end{aligned}$$

We obtain from (6.21) and (2.12)

$$\left\|u_{n+1/2,h} - u_{n,h}\right\|_{L^2(\Omega,\mathbb{R}^N)} \geq C\sqrt{\alpha}\left\|u_{n+1/2,h} - \overline{w_{n+1/2,h}}\right\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{6.22}$$

The proof of (6.9) follows. □

7. Scaling of the amplitude with respect to the Energy

The goal of this section is show that for any binomial microstructure $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, there exists a positive constant C, independent of n, such that for some $n_0 \in \mathbb{N}$ large we have for any $n \geq n_0$ and $\gamma \in (0,1)$

$$E(u_n)^{\frac{1}{4\gamma}} \leq C\,\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{7.1}$$

We note that this result complements the inequality proven in [13]

$$\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \leq C\,E(u_n)^{\frac{1}{4}}$$

which we re-establish for the sake of completeness first.

Lemma 7.1. *Let the sequence* $\{u_n\}_{n=0}^{\infty} \subset W^{1,2}(\Omega, \mathbb{R}^N)$ *be a binomial microstructure. Then there exists a positive constant* C*, independent of* n*, such that*

$$\|u_n - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \le C\, E(u_n)^{\frac{1}{4}} \tag{7.2}$$

Proof. We have from [13] and (1.5)

$$\begin{aligned} E(u_n)^{\frac{1}{2}} &\ge \sqrt{\Lambda} \left(\int_\Omega \|\nabla u(x) - \Pi \nabla u(x)\|^2 \, dx \right)^{\frac{1}{2}} \\ &\ge C \int_\Omega \|(\nabla u(x) - F)\, m\|^2 \, dx \\ &\ge C\, \|u_n - Fx\|^2_{L^2(\Omega,\mathbb{R}^N)}, \qquad m \in b^\perp. \end{aligned} \tag{7.3}$$

The second inequality follows from [[13], Theorem 2]. The last inequality is the Poincaré-Friedrichs inequality with respect to directional derivatives. □

Theorem 7.2. *Let the sequence* $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, *be a binomial microstructure. Then there exists a positive constant* C*, independent of* n*, such that for any* $n \in \mathbb{N}$ *sufficiently large we have*

$$E(u_n)^{\frac{1}{4}} \le C\, \|u_n - Fx\|^\gamma_{L^2(\Omega,\mathbb{R}^N)}, \qquad \gamma \in (0,1), \tag{7.4}$$

provided that $\Lambda > 0$ *in the constitutive assumption* (1.5) *is sufficiently large.*

Proof. First, we show that there exists a constant C, independent of n, such that

$$\begin{aligned} &\left\| \int_D \nabla u_n(x) - F \, dx \right\| \\ &\le C \operatorname{meas}(D)^{1/2} \|\nabla u_n - \Pi \nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} + C \operatorname{meas}(D)^{1/2} \|u_n - Fx\|^\gamma_{L^2(\Omega,\mathbb{R}^N)}, \\ &\quad \gamma \in (0,1), \qquad D \subset \Omega, \text{ open}. \end{aligned} \tag{7.5}$$

Then the proof is as follows. Let w_n be the sharp microstructure corresponding to u_n constructed in Theorem 4.1. Then, again in view of Theorem 4.1, we obtain a positive constant C, independent of n, such that

$$\begin{aligned} &\left\| \int_D \nabla u_n(x) - F \, dx \right\| \\ &\ge \left\| \int_D \nabla w_n(x) - F \, dx \right\| - \left\| \int_D \nabla u_n(x) - \nabla w_n(x) \, dx \right\| \\ &\ge \|F_i - F\| \operatorname{meas}(D) - C \operatorname{meas}(D)^{1/2} \|\nabla u_n - \Pi \nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}, \quad i = 1, 2. \end{aligned} \tag{7.6}$$

Combining (7.5) and (7.6) we obtain

$$\|\nabla u_n - \Pi \nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} + \|u_n - Fx\|^\gamma_{L^2(D,\mathbb{R}^N)} \ge C \operatorname{meas}(D)^{1/2}, \qquad D \in \mathcal{E}_n. \tag{7.7}$$

Taking $D \in \mathcal{E}_n$, where $\mathcal{E}_n$ is the set of laminates of u_n, the Harnack-type “max-min-max” properties of the laminates, Lemma 4.2, yield

$$\operatorname{meas}(D)^{1/2} \ge \operatorname{moh}(\mathcal{E}_n)^{-1/2}, \qquad D \in \mathcal{E}_n. \tag{7.8}$$

Since Theorem 6.1 and Theorem 5.1 give [2]

$$\operatorname{moh}(\mathcal{E}_n)^{-1} \geq C\, \|g(u_n)\|^{2/3}_{W^{1,2}(\Omega,\mathbb{R}^N)} \geq C\, E(u_n)^{1/2} \tag{7.9}$$

we get from (7.9), (7.8) and (7.7)

$$\|\nabla u_n - \Pi\nabla u_n\|^{1/2}_{L^2(D,\mathbb{R}^{N\times N})} + \|u_n - Fx\|^{\gamma}_{L^2(D,\mathbb{R}^N)} \geq C\, E(u_n)^{1/4}. \tag{7.10}$$

Thus, recalling the constitutive assumption (1.5), we obtain the claim of the Theorem in view of the inequality

$$\|\nabla u_n - \Pi\nabla u_n\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} \leq \frac{1}{\Lambda} E(u_n)^{1/4} \tag{7.11}$$

provided that $\Lambda > 0$ is sufficiently large.

Proof of the estimate (7.5). Let the mapping R_ε be the Friedrichs' mollifier with a kernel ϕ_0, and let ε be arbitrary and positive. We have

$$\left\| \int_D (\nabla u_n(x) - F)\, dx \right\| \leq \left\| \int_D \nabla u_n(x) - R_\varepsilon \nabla u_n(x)\, dx \right\| + \left\| \int_D R_\varepsilon\,(\nabla u_n(x) - F)\, dx \right\|. \tag{7.12}$$

We estimate each of the integrals on the right-hand side of (7.12) separately.

We have

$$\begin{aligned} &\left\| \int_D \nabla u_n(x) - R_\varepsilon \nabla u_n(x)\, dx \right\| \\ &\leq \left\| \int_D \nabla u_n(x) - \Pi_{1,2}\nabla u_n(x) - R_\varepsilon\left(\nabla u_n(x) - \Pi_{1,2}\nabla u_n(x)\right)\, dx \right\| \\ &\quad + \left\| \int_D \Pi_{1,2}\nabla u_n(x) - R_\varepsilon \Pi_{1,2}\nabla u_n(x)\, dx \right\|. \end{aligned} \tag{7.13}$$

Since in view of [[13], Theorem 4] we have

$$\|\Pi_{1,2}\nabla u_n(x) - \nabla u_n(x)\|_{L^2(\Omega,\mathbb{R}^{N\times N})} \leq C\, \|\Pi\nabla u_n(x) - \nabla u_n(x)\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}, \tag{7.14}$$

we obtain a constant C, independent of D and n, such that

$$\begin{aligned} &\|\Pi_{1,2}\nabla u_n(x) - \Pi\nabla u_n(x)\|_{L^1(D,\mathbb{R}^{N\times N})} \\ &\leq \operatorname{meas}(D)^{1/2}\, \|\Pi_{1,2}\nabla u_n(x) - \Pi\nabla u_n(x)\|_{L^2(\Omega,\mathbb{R}^{N\times N})} \\ &\leq \operatorname{meas}(D)^{1/2} \left(\|\Pi_{1,2}\nabla u_n(x) - \nabla u_n(x)\|_{L^2(\Omega,\mathbb{R}^{N\times N})} + \|\nabla u_n(x) - \Pi\nabla u_n(x)\|_{L^2(\Omega,\mathbb{R}^{N\times N})} \right) \\ &\leq C\, \operatorname{meas}(D)^{1/2}\, \|\nabla u_n(x) - \Pi\nabla u_n(x)\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}. \end{aligned} \tag{7.15}$$

[2]We can apply Theorem 6.1 here. We assume that $\{u_n\}_{n\in\mathbb{N}}$ is a binomial microstructure thus (6.2) of Theorem 6.1 is satisfied.

Thus using the fact that $\Pi^{\varepsilon}_{1,2} \stackrel{\text{def}}{=} \Pi_{1,2}\nabla u_n(x+\varepsilon z)$ is again a projection onto $\{F_1, F_2\}$ for given $\varepsilon > 0$, we obtain a constant C, independent of D and n, such that

$$\begin{aligned} &\left\| \int_D \Pi_{1,2}\nabla u_n(x) - R_\varepsilon \Pi_{1,2}\nabla u_n(x)\,dx \right\| \\ &= \left\| \int_D \Pi_{1,2}\nabla u_n(x) - \Pi\nabla u_n(x) + \Pi\nabla u_n(x) - R_\varepsilon \Pi_{1,2}\nabla u_n(x)\,dx \right\| \\ &\leq C \text{ meas}(D)^{1/2} \left\| \nabla u_n(x) - \Pi\nabla u_n(x) \right\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}. \end{aligned} \tag{7.16}$$

Hence we have constants $C_1 - C_3$, independent of D, n and ε, such that

$$\begin{aligned} &\left\| \int_D \left(\nabla u_n(x) - R_\varepsilon \nabla u_n(x)\right)\,dx \right\| \leq \int_D \left\| \nabla u_n(x) - \Pi_{1,2}\nabla u_n(x)\right\|\,dx \\ &\quad + \int_D \left\| R_\varepsilon\left(\nabla u_n(x) - \Pi_{1,2}\nabla u_n(x)\right)\right\|\,dx + C_1 \text{ meas}(D)^{1/2} \left\| \nabla u_n(x) - \Pi\nabla u_n(x)\right\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} \\ &\leq \int_D \left\| \nabla u_n(x) - \Pi_{1,2}\nabla u_n(x)\right\|\,dx \\ &\quad + C_2 \int_D \left\| \nabla u_n(x) - \Pi_{1,2}\nabla u_n(x)\right\|\,dx + C_1 \text{ meas}(D)^{1/2} \left\| \nabla u_n(x) - \Pi\nabla u_n(x)\right\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})} \\ &\leq C_3 \text{ meas}(D)^{1/2} \left\| \nabla u_n(x) - \Pi\nabla u_n(x)\right\|^{1/2}_{L^2(\Omega,\mathbb{R}^{N\times N})}. \end{aligned} \tag{7.17}$$

The second integral on the right-hand side of (7.12) can be estimated using integration by parts and the Hölder inequality. Namely,

$$\begin{aligned} &\left\| \int_D R_\varepsilon\left(\nabla u_n(x) - F\right)\,dx \right\| \leq \frac{1}{\varepsilon} \left\| \nabla\phi_0 \right\|_{L^\infty(\Omega,\mathbb{R}^N)} \int_D \left| u_n(x-\varepsilon y_0(x)) - F(x-\varepsilon y_0(x))\right|\,dx \\ &\leq C\,\frac{1}{\varepsilon} \left\| \nabla\phi_0 \right\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \text{meas}(D)^{1/2} \left\| u_h(x) - Fx \right\|_{L^2(\Omega,\mathbb{R}^N)}. \end{aligned} \tag{7.18}$$

The estimate (7.5) follows from (7.12), (7.17) and (7.18) by taking □

$$\varepsilon = \|u_n - Fx\|^{1-\gamma}_{L^2(\Omega,\mathbb{R}^N)} \qquad \text{for any } 0 < \gamma < 1.$$

8. Strong convergence of the Weak Steepest Descent Algorithm

Theorem 8.1. *Let the sequence $\{u_{n,h}\}_{n=0}^{\infty} \subset V_h \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, $h > 0$, be generated by the Weak Steepest Descent Algorithm, i.e., let us assume that the sequence is given by Definition 2.3. Let us assume that the constant $\Lambda > 0$ in* (1.4) *is sufficiently large. Then there exists a constant C, independent of n and h, and there exists a function $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ such that*

$$\lim_{h\to 0}\lim_{n\to\infty} E(u_{n,h}) = E(u) \geq C > 0. \tag{8.1}$$

Moreover,

$$u_{n,h} \to u, \qquad \textit{strongly in } W^{1,2}(\Omega, \mathbb{R}^N), \qquad \textit{as } h \to 0_+, \textit{ and } n \to +\infty. \tag{8.2}$$

Proof. Let us assume first that

$$\lim_{h\to 0_+} \lim_{n\to\infty} E(u_{n,h}) = 0. \tag{8.3}$$

Thus

$$\lim_{h\to 0_+} \lim_{n\to\infty} \int_\Omega \|\nabla u_{n,h}(x) - \Pi\nabla u_{n,h}(x)\| \, dx = 0. \tag{8.4}$$

It follows from (8.4) and [13], [9] that

$$\nabla u_{n,h} \rightharpoonup Fx, \qquad \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^N) \text{ as } n \to +\infty, h \to 0_+. \tag{8.5}$$

We obtain from Theorem 7.2, Theorem 5.5, Corollary 6.1, and Theorem 5.1 that

$$\begin{aligned} E(u_{n,h})^{\frac{1}{4\gamma}} &\overset{Theorem\ 7.2}{\leq} C_1 \, \|u_{n,h} - Fx\|_{L^2(\Omega,\mathbb{R}^N)} \\ &\overset{Theorem\ 5.5}{\leq} C_2 \, \text{moh}\,(\mathcal{E}_n)^{-1} \\ &\overset{Theorem\ 6.1}{\leq} C_3 \, \|g(u_{n,h})\|^{\frac{2}{3}}_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\ &\overset{(2.21)}{\leq} C_4 \, E(u_{n,h})^{\frac{1}{2}\frac{2}{3}}, \qquad \text{for any } \gamma \in (0,1). \end{aligned} \tag{8.6}$$

Thus there exists a positive constant C, independent of $n \in \mathbb{N}$ such that

$$C \leq E(u_{n,h})^{\frac{1}{12} - \frac{1}{4}\left(\frac{1-\gamma}{\gamma}\right)}$$

which contradicts (8.3) for

$$\frac{1}{12} - \frac{1}{4}\left(\frac{1-\gamma}{\gamma}\right) > 0$$

if γ is sufficiently close to 1. Thus we have

$$\lim_{h\to 0_+} \lim_{n\to+\infty} E(u_{n,h}) > 0. \tag{8.7}$$

Consequently, there exists a function $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\lim_{h\to 0_+} \lim_{n\to+\infty} E(u_{n,h}) = E(u). \tag{8.8}$$

Hence, for some $t \in (0,1)$ we have

$$E(u) - E(u_{n,h}) = \int_\Omega DW\left(\nabla u_{n,h}(x) + t\left(\nabla u(x) - \nabla u_{n,h}(x)\right)\right) : \left(\nabla u(x) - \nabla u_{n,h}(x)\right) \, dx. \tag{8.9}$$

Since,

$$\int_\Omega DW\left(\nabla u(x)\right) : \left(\nabla u(x) - \nabla u_{n,h}(x)\right) \, dx = 0 \tag{8.10}$$

we obtain from (8.9) and (8.10) for some $\tau \in (0,1)$

$$\begin{aligned} &0 \leftarrow E(u) - E(u_{n,h}) \\ &= \int_\Omega D^2W\left(\nabla w_n(\tau)(x)\right)\left(\nabla u(x) - \nabla u_{n,h}(x), \nabla u(x) - \nabla u_{n,h}(x)\right)\,dx \\ &\geq \Lambda \left\|\nabla u - \nabla u_{n,h}\right\|^2_{W^{1,2}(\Omega,\mathbb{R}^N)}, \qquad \text{for some } \Lambda > 0, \end{aligned} \tag{8.11}$$

where

$$\nabla w_n(\tau) = \nabla u_{n,h} + t\left(\nabla u - \nabla u_{n,h}\right) + \tau\left(\nabla u_{n,h} + t\left(\nabla u - \nabla u_{n,h}\right) - \nabla u\right).$$

The convexity of D^2W in the direction $\nabla u(x) - \nabla u_{n,h}(x)$ follows from the fact the Steepest Descent Algorithm must stagnate at the function u. The proof is finished.

□

9. AUXILIARY INEQUALITIES

Lemma 9.1. *Let the sequence $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega,\mathbb{R}^N)$, $p > N$, be binomial microstructure generated by one of the Steepest Descent Algorithms, i.e., let us assume that the sequence is given either by Definition 2.2 or Definition 2.3. Let $\mathcal{E}_n$ be a set of laminates corresponding to u_n. Let us assume that $\alpha_n \geq \lambda > 0$ for any $n \in \mathbb{N}$. Then there exist $n_0 \in \mathbb{N}$, and a positive constant C, independent of n, such that for any $n \geq n_0$ we have*

$$\operatorname{moh}(\mathcal{E}_n)^{-\frac{1}{2}} \geq C \sum_{m=n}^{\infty} E(u_m). \tag{9.1}$$

Thus, in particular, there exists a constant C, independent of n, such that

$$C \geq \operatorname{moh}(\mathcal{E}_n)^{\frac{1}{2}} \sum_{m=n}^{\infty} \left\|u_n - Fx\right\|^4_{L^2(\Omega,\mathbb{R}^N)}. \tag{9.2}$$

Proof. We have

$$C \geq \sum_{n=0}^{\infty} \alpha_n \left\|g(u_n)\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \geq \lambda \sum_{n=0}^{\infty} \left\|g(u_n)\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \geq C \sum_{n=0}^{\infty} E(u_n)^{\frac{3}{4}}. \tag{9.3}$$

Hence, taking $n \in \mathbb{N}$ such that

$$1 \geq \sum_{m=n}^{\infty} E(u_m)^{\frac{3}{4}} \tag{9.4}$$

we obtain from (9.4)

$$E(u_n)^{\frac{1}{4}} \geq C \sum_{m=n}^{\infty} E(u_m). \tag{9.5}$$

Next, we have from Theorem 6.1, Theorem 5.1, (9.4) and Lemma 7.2

$$\operatorname{moh}(\mathcal{E}_n)^{-1} \geq C \left\|g(u_n)\right\|^{\frac{2}{3}}_{W^{1,2}(\Omega,\mathbb{R}^N)} \geq C\,E(u_n)^{\frac{1}{2}} \geq C\left(\sum_{m=n}^{\infty} E(u_m)\right)^2. \tag{9.6}$$

The proof follows.

□

Lemma 9.2. *Let the sequence* $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega, \mathbb{R}^N)$, $p > N$, *be binomial microstructure generated by one of the Steepest Descent Algorithms, i.e., let us assume that the sequence is given either by Definition 2.2 or Definition 2.3. Then*

$$E(u_0)^\theta \geq \lim_{n\to\infty} E(u_n)^\theta + \frac{1}{4}\beta\,\theta \sum_{n=0}^{\infty} E(u_n)^{1/2} \int_0^{\alpha_n} E(u_{n+1} + tg(u_n))^{\theta-1/4}\,dt, \qquad \theta \in \mathbb{R}. \tag{9.7}$$

The positive constant β *in* (9.7) *comes from the estimate* (5.7) *of Lemma 5.2.*

Proof. The proof is similar to the proof of Theorem 5.3. We have for any $\theta \in \mathbb{R}^+$

$$\begin{aligned}
&\frac{d}{dt}E(u_{n+1} + tg(u_n))^\theta \\
&= \theta E(u_{n+1} + tg(u_n))^{\theta-1} \int_\Omega DW(\nabla u_{n+1}(x) + t\nabla g(u_n)(x))\nabla g(u_n)(x)\,dx \\
&= \theta E(u_{n+1} + tg(u_n))^{\theta-1} \int_\Omega \nabla g\left(u_{n+1} + tg(u_n)\right)(x)\nabla g(u_n)(x)\,dx \qquad (9.8)\\
&\overset{Lemma\ 5.2}{\geq} \beta\,\theta E(u_{n+1} + tg(u_n))^{\theta-1} \left\|g(u_{n+1} + tg(u_n))\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \left\|g(u_n)\right\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \\
&\overset{Theorem\ 5.1}{\geq} \frac{1}{4}\beta\,\theta E(u_{n+1} + tg(u_n))^{\theta-1} E(u_{n+1} + tg(u_n))^{3/4} E(u_n))^{1/2}.
\end{aligned}$$

Integrating (9.8) over $(0, \alpha_n)$ and summing over $n \in \mathbb{N}$ we get

$$E(u_0)^\theta - \lim_{n\to\infty} E(u_n)^\theta \geq \frac{1}{4}\beta\,\theta \sum_{n=0}^{\infty} E(u_n)^{1/2} \int_0^{\alpha_n} E(u_{n+1} + tg(u_n))^{\theta-1/4}\,dt. \tag{9.9}$$

which concludes the proof.

□

Acknowledgement: The author would like to thank the Department of Mathematics, University Mons-Hainaut, Belgium, and the Department of Mathematics, Federal Institute of Technology, Lausanne, Switzerland, for the hospitality during his work on the material presented in this paper. The author would like to thank Alfred Wagner for numerous helpful discussions of the subject.

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