An Interior-Point Algorithm for the Maximum-Volume Ellipsoid Problem*

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Abstract

In this report, we consider the problem of finding the maximum-volume ellipsoid inscribing a given full-dimensional polytope in $\Re^n$ defined by a finite set of affine inequalities. We present several formulations for the problem that may serve as algorithmic frameworks for applying interior-point methods. We propose a practical interior-point algorithm based on one of the formulations and present preliminary numerical results.

1 Introduction

Since Karmarkar's 1984 ground-breaking work [8], the area of interior-point methods has matured considerably, as evidenced by a string of recently appeared books in this area (see, for example [19, 20, 24, 29, 30, 31]) which

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contain comprehensive lists of references. Even so, there are still many research topics that need to be further studied, especially in extending and applying interior-point methodology to solving practically important optimization problems.

In this paper, we are concerned with the problem of finding the maximum-volume ellipsoid (MVE) inscribed in a polytope in $\Re^n$ defined by a finite set of affine inequalities. We will call this problem the MVE problem in short.

The MVE problem has its root in rounding of convex bodies in $\Re^n$. One of the earliest studies was due to F. John [6]. In particular, John's results implies that once the maximum-volume inscribing ellipsoid $\mathcal{E}$ is found in $\mathcal{P}$, then

$$\mathcal{E} \subset \mathcal{P} \subset n\mathcal{E}.$$

That is, $\mathcal{E}$ provides an n-bounding for $\mathcal{P}$. Moreover, if $\mathcal{P}$ is centrally symmetric around the origin, then the rounding factor can be reduced to $\sqrt{n}$.

In general, ellipsoids are much easier to handle, both theoretically and computationally, than polytopes are. For example, the global minimum of any quadratic in an ellipsoid can be easily located in polynomial time, while finding such a global minimum in a polytope is generally a NP-hard problem. Not surprisingly, for many problems a fruitful and effective approach is to use ellipsoids to approximate polytopes in various theoretic and algorithmic settings. A celebrated example is Khachiyan's ellipsoid method for linear programming [11] — the first polynomial-time algorithm for linear programming. Other applications include optimal design [22, 25], computational geometry (for example, [27]) and algorithm construction (for example, [23]).

Recently, several randomized polynomial-time algorithms ([3, 16, 7], for example) have been proposed for estimating the volume of convex bodies (computing the volume itself is NP-hard). In the case of polytopes, these algorithms require approximating polytopes by ellipsoids.

It is well known that the rounding of a polytope can be accomplished by the (shallow-cut) ellipsoid method in polynomial time (see, for example, [21, 5]). It is also well known, however, that ellipsoid method is not a practically efficient algorithm. A number of polynomial-time interior-point algorithms have been proposed in the recent years for the MVE problem, for example, by Nesterov and Nemirovskii [18], Khachiyan and Todd [12] (also see [10] for a related problem), and Nemirovskii [17]. All these works are primarily concerned with the computational complexity issues and the proposed algorithms are in theoretic nature. Vandenberghe, Boyd and Wu [26] proposed an algorithm for the class of problems called MAXDET problems to which the MVE problem belong. However, their algorithm does not take into account of the special structure of the MVE problem.

The MVE problem plays a major role in the well-known Lenstra's algorithm [14] for integer programming, which is the first polynomial algorithm for integer programming when the number of integer variables is fixed . In the Lenstra algorithm, a MVE problem need to be solved at every node of the branch-and-bound search tree. The efficiency of the algorithm used to solve this problem, therefore, will dominate the calculation. Though theoretically efficient, Lenstra's algorithm has never been implemented, nor studied from a computational point of view. One of the primary reasons for the lack of computational results for Lenstra's algorithm is the lack of practically efficient algorithms and software for the MVE problem involved.

It is curiously noticeable that computational results on the MVE problem are scarce. In fact, at the writing we are not aware of any published computational results for this problem. This work constitutes a first effort from the author to construct practically algorithms for solving the MVE problem.

This paper is organized as follows. In Section 2, we will introduce the formulation of the MVE problem as a convex program. In Section 3, we present the optimality conditions (or system) for the problem, as well as the perturbations of the optimality conditions. Section 4 contains several equivalent optimality systems resulting from eliminating different variables from the original optimality system. Based on one of the optimality systems, an interior-point algorithm is proposed in Section 5 and preliminary numerical results presented in Section 6. In the last section, some concluding remarks are offered.

2 Problem Description

Consider a given polytope $\mathcal{P}$ in $\Re^n$,

$$\mathcal{P} = \{y \in \Re^n : Ay \le b\}, \tag{1}$$

where $A \in \Re^{m\times n}$ and $b \in \Re^m$. For convenience of discussion, we will assume that

1. A has full rank containing no zero-rows;

2. There is a point $y \in \Re^n$ satisfying $Ay < b$.

Given a center $x \in \Re^n$ and a nonsingular scaling matrix $E \in \Re^{n\times n}$, an ellipsoid in $\Re^n$ can be defined as

$$\mathcal{E}(x, E) = \{y \in \Re^n : (y - x)^T (EE^T)^{-1}(y - x) \leq 1\};$$

or equivalently,

$$\mathcal{E}(x, E) = \{y \in \Re^n : y = x + Es \text{ and } \|s\| \leq 1\}, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm in $\Re^n$. Clearly, the shape of the ellipsoid is uniquely determined by the symmetric positive definite matrix EE^T, but not uniquely by E since the same ellipsoid can also be generated by EQ for any orthogonal matrix $Q \in \Re^{n\times n}$. Without loss of generality, we can assume that E itself is symmetric positive definite.

The ellipsoid $\mathcal{E}(x, E)$ is contained in $\mathcal{P}$ if and only if

$$\sup_{\|s\|=1} a_i^T (x + Es) \leq b_i, \quad i = 1 : m$$

where a_i^T is the i-th row of A, or equivalently

$$a_i^T x + \|Ea_i\| \leq b_i, \quad i = 1 : m.$$

Introducing the notation

$$h(E) = (\|Ea_1\|, \cdots, \|Ea_m\|)^T \in \Re^m, \tag{3}$$

we have

$$\mathcal{E}(x, E) \subset \mathcal{P} \iff b - Ax - h(E) \geq 0. \tag{4}$$

Since the volume of $\mathcal{E}(x, E)$ is a multiple of $\det E$, the maximum-volume ellipsoid contained in $\mathcal{P}$ is the solution $(x^*, E^*) \in \Re^n \times \Re^{n\times n}$ to the following optimization problem

$$\begin{array}{cc} \max & \log\det E \\ \text{s.t.} & b - Ax - h(E) \geq 0 \\ & E \succ 0 \end{array} \tag{5}$$

where $E \succ 0$ means that E is symmetric positive definite. It is well known that (5) is a convex program with a unique pair of solution $(x^*, E^*) \in \Re^n \times$

$\Re^{n\times n}$ which is uniquely determined by the first-order optimality, or Karush-Kuhn-Tucker (KKT), conditions for the problem. The maximum volume ellipsoid (MVE) problem is classified as in the class of MAXDET problems by [26].

We introduce some notation. For any given vector $v = (v_1, v_2, \cdots v_p)^T \in \Re^p$, we denote the $p \times p$ diagonal matrix with v on its diagonal either by $\mathbf{diag}(v)$, or by its upper-case letter V whenever no confusion can occur. On the other hand, for a square matrix M, $\mathbf{diag}(M)$ is the vector consisting of the diagonal elements of M.

3 Optimality Conditions and Perturbations

The Lagrangian function of (5) is

$$L(x, E, u) = \log\det E + u^T(b - Ax - h(E))$$

where $u \in \Re^m$ is the vector of Lagrange multipliers.

Using the formulas

$$\nabla[\log\det E] = E^{-1} \tag{6}$$

and

$$\frac{d}{dE}h_i(E) = \frac{Ea_ia_i^T + a_ia_i^TE}{2h_i(E)}$$

we can derive the optimality (KKT) conditions as

$$\begin{aligned} A^Tu &= 0, && \text{(7a)}\\ E^{-1} - [E(A^TYA) + (A^TYA)E]/2 &= 0, && \text{(7b)}\\ z - (b - Ax - h(E)) &= 0, && \text{(7c)}\\ Uz &= 0, && \text{(7d)}\\ u, z &\geq 0, && \text{(7e)} \end{aligned}$$

where $Y = \mathbf{diag}(h(E))^{-1}U$, $U = \mathbf{diag}(u)$ and z is a slack variable.

A large class of primal-dual interior point algorithms can be viewed as a damped Newton's method applied to so-called perturbed KKT conditions. Following this approach, we replace the zero right-hand-side of (7d) by a positive vector $c \in \Re^m$, i.e.,

$$Uz = c, \tag{8}$$

and resulting system is the so-called perturbed KKT (PKKT) conditions. The PKKT conditions are equivalent to the gradient of a strictly concave function being zero, and have a unique solution for each $c > 0$. This fact is stated in the following proposition.

Proposition 1 *For any positive vector $c \in \Re^m$, the perturbed KKT conditions have a unique solution (E, x, u, z) where $E \succ 0$ and $u, z > 0$.*

Proof: Consider the strictly concave function (both in x and in E) parameterized by $0 < c \in \Re^m$

$$B_c(x, E) = \log\det E + \sum_{i=1}^{m} c_i \log(b - Ax - h(E))_i.$$

For any give $c > 0$, the function has a unique minimizer pair $x_c, E_c)$ where

$$\nabla B_c(x, E) = 0, \quad b - Ax - h(E) > 0, \quad E \succ 0.$$

It is straightforward to verify that $\nabla B_c(x, E) = 0$ is equivalent to the perturbed KKT conditions if we introduce the variables

$$z = b - Ax - h(E), \quad u = \mathbf{diag}(z)^{-1} c,$$

which completes the proof.

4 Elimination of Variables

In this section, we present several equivalent forms of the KKT or the perturbed KKT (PKKT) system through the elimination of different variables. In principle, interior-point Newton's method may be applied to anyone of these systems.

4.1 Eliminating Variable E

To eliminate the matrix E from the PKKT system, we let

$$y = \mathbf{diag}(h(E))^{-1} u, \tag{9}$$

and solve E from (7b) as a function of y to obtain

$$E(y) = (A^T Y A)^{-1/2}. \tag{10}$$

As a result, $h_i(E) = \|Ea_i\|$ becomes a function of y which we denote by $h_i(y)$; namely,

$$h(y) = (\|E(y)a_1\|, \cdots, \|E(y)a_m\|)^T = \sqrt{\mathbf{diag}(H(y))}, \tag{11}$$

where

$$H(y) = A(A^T Y A)^{-1} A^T. \tag{12}$$

After eliminating E from the PKKT conditions, we arrive at a set of equivalent PKKT conditions

$$\begin{aligned} A^T u &= 0, & \text{(13a)} \\ y - \mathbf{diag}(h(y))^{-1} u &= 0, & \text{(13b)} \\ z - (b - Ax - h(y)) &= 0, & \text{(13c)} \\ Uz &= c, & \text{(13d)} \\ u, z &\geq 0. & \text{(13e)} \end{aligned}$$

4.2 Eliminating Variable u

Solving (13b) for u,

$$u = Yh(y),$$

and substituting the above result into (13c), and noting that y has the same zero-nonzero pattern as u, we obtain a set of equivalent perturbed KKT conditions involving only variables x, y and z:

$$\begin{aligned} A^T Y h(y) &= 0, & \text{(14a)} \\ z - (b - Ax - h(y)) &= 0, & \text{(14b)} \\ Yz &= c, & \text{(14c)} \\ y, z &\geq 0, & \text{(14d)} \end{aligned}$$

Multiplying (14b) by $A^T Y$ and using (14a) and (14c), we obtain another set of equivalent PKKT conditions

$$\begin{aligned} A^T Y(b - Ax) &= A^T c, & \text{(15a)} \\ z - (b - Ax - h(y)) &= 0, & \text{(15b)} \\ Yz &= c, & \text{(15c)} \\ y, z &\geq 0. & \text{(15d)} \end{aligned}$$

We note that (15a) is less nonlinear than (14a) is.

4.3 Eliminating Variable x

We can further eliminate the variable x. To do so, we first solve (15a) for x and obtain

$$x(y,c) = (A^T Y A)^{-1} A^T (Yb - c). \tag{16}$$

Substituting $x(y,c)$ into $b - Ax$, we obtain a function of y and c:

$$g(y,c) = b - x(y,c) = b - A(A^T Y A)^{-1} A^T (Yb - c).$$

Recall from (12) that $H(y) = A(A^T Y A)^{-1} A^T$. Hence,

$$g(y,c) = [I - H(y)Y]b + H(y)c. \tag{17}$$

Replacing $b - Ax$ by $g(y,c)$ in (15c) and deleting (15a), we arrive at another set of equivalent PKKT conditions:

$$\begin{aligned} z - (g(y,c) - h(y)) &= 0, && (18a)\\ Yz &= c, && (18b)\\ y, z &\geq 0, && (18c) \end{aligned}$$

In the case of $c = 0$, the above system represents a nonlinear complementarity problem:

$$0 \leq y \perp z = f(y) \geq 0, \tag{19}$$

where $f(y) = g(y,0) - h(y)$. This nonlinear complementarity problem is equivalent to the optimality conditions of the maximum-volume ellipsoid problem (5).

4.4 Alternative equations

The components of the function $h(y)$ are defined as Euclidean norm of vectors and hence involve square roots. However, it is possible to use alternative equations involving $h(y)^2$ only. Observe that for $Ax \leq b$, the inequality $(b - Ax) - h(y) \geq 0$ is equivalent to, after adding a nonnegative slack variable z,

$$z - \left[(b - Ax)^2 - h(y)^2\right] = 0, \; z \geq 0. \tag{20}$$

Similarly, whenever $g(y,c) \geq 0$, $g(y,c) - h(y) \geq 0$ is equivalent to

$$z - \left[g(y,c)^2 - h(y)^2\right] = 0, \; z \geq 0. \tag{21}$$

These alternative equations are free of square roots, but require additional constraints in order to maintain equivalence to the PKKT or KKT conditions.

4.5 Derivative formulas

In applying interior-point Newton's method to different formulations presented above, we will need derivative information of various functions. Here we collect derivative formulas for possible future use.

Lemma 1 *Let $H(y), h(y)$ and $g(y)$ be defined as in (12), (11) and (17), respectively; and let v be any vector in $\Re^m$. Then*

$$\begin{aligned}
\partial H_{ij}(y)/\partial y_k &= -H_{ik}(y)H_{jk}(y), && (22)\\
\nabla_y[H(y)v] &= -H(y)\mathbf{diag}(H(y)v), && (23)\\
\nabla h(y) &= -(1/2)\mathbf{diag}(h(y))^{-1}[H(y)\circ H(y)], && (24)\\
\nabla_y g(y,c) &= -H(y)\mathbf{diag}(g(y,c)), && (25)\\
\nabla[h(y)^2] &= -H(y)\circ H(y), && (26)\\
\nabla_y[g(y,c)^2] &= -2\mathbf{diag}(g(y,c))H(y)\mathbf{diag}(g(y,c)), && (27)
\end{aligned}$$

where "$\circ$" denotes the element-wise or Hadamard product of matrices.

Using these formulas, we have

$$\begin{aligned}
\nabla_y[g(y,c)-h(y)] &= (1/2)\mathbf{diag}(h(y))^{-1}H(y)\circ H(y)\\
&\quad -H(y)\mathbf{diag}(g(y,c)),\\
\nabla_y[g(y,c)^2-h(y)^2] &= H(y)\circ H(y)\\
&\quad -2\mathbf{diag}(g(y,c))H(y)\mathbf{diag}(g(y,c)).
\end{aligned}$$

We note that the second matrix is symmetric while the first is not.

5 An Interior Point Algorithm

So far our limited computational experiments seem to favor the conditions (15). Let $c = \mu e$ for scalar $\mu > 0$, where e is the vector of all ones. We can write (15) as

$$F(x,y,z) = \mu \begin{pmatrix} A^T e \\ 0 \\ e \end{pmatrix} \qquad (28)$$

and $y, z \geq 0$, where

$$F(x,y,z) \equiv \begin{pmatrix} A^T Y(b-Ax) \\ z-(b-Ax-h(y)) \\ Yz \end{pmatrix}$$

Besides the nonnegativity $y, z \geq 0$, (28) is a square, nonlinear system of $n + 2m$ variables. It is known that as $\mu \to 0$, the solution of (28) converges to a solution from which we can reconstruct the solution to the original problem.

By direct calculation, we obtain

$$F'(x, y, z) = \begin{bmatrix} -A^T Y A & A^T \mathbf{diag}(b - Ax) & 0 \\ A & \nabla h(y) & I \\ 0 & Z & Y \end{bmatrix}$$

To solve the linear system

$$F'(x, y, z) \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix},$$

we can use the following block Gaussian elimination procedure:

$$\begin{aligned} \delta y &= M^{-1}(r_2 - Y^{-1} r_3 + A(A^T Y A)^{-1} r_1), \\ \delta z &= Y^{-1}(r_3 - Z \delta y), \\ \delta x &= (A^T Y A)^{-1}(r_1 - A^T \mathbf{diag}(b - Ax) \delta y), \end{aligned}$$

where

$$M = H(y) \mathbf{diag}(b - Ax) - \mathbf{diag}(2h(y))^{-1} H(y) \circ H(y) - Y^{-1} z.$$

Choose $(x^0 \in \mathcal{P}, y^0 > 0, z^0 > 0)$, set $k = 0$.

1. Choose $\sigma^k \in (0, 1)$ and set $\mu^k = \sigma^k \frac{(y^k)^T z^k}{m}$.

2. Solve for $(\delta x, \delta y, \delta z)$ from

$$F'(x, y, z) \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \mu \begin{pmatrix} A^T e \\ 0 \\ e \end{pmatrix} - F(x, y, z)$$

for $(x, y, z) = (x^k, y^k, z^k)$ and $\mu = \mu^k$.

3. Choose step-length $\alpha^k \in (0, 1]$ and update

$$(x^{k+1}, y^{k+1}, z^{k+1}) = (x^k, y^k, z^k) + \alpha^k (\delta x, \delta y, \delta z),$$

such that

$$x^{k+1} \in \mathcal{P},\ y^{k+1} > 0,\ z^{k+1} > 0.$$

4. Increment k and go to Step 1.

6 Numerical Results

We have implemented this prototype algorithm in Matlab and performed limited numerical experiments on randomly generated problems (description of parameter choices is skipped here).

Figure 1 below illustrates how a 2-dimensional problem is solved. The small circles in the picture represents the positions of the x-iterates, starting from close to the boundary and converging to the center of the ellipse in a few steps.

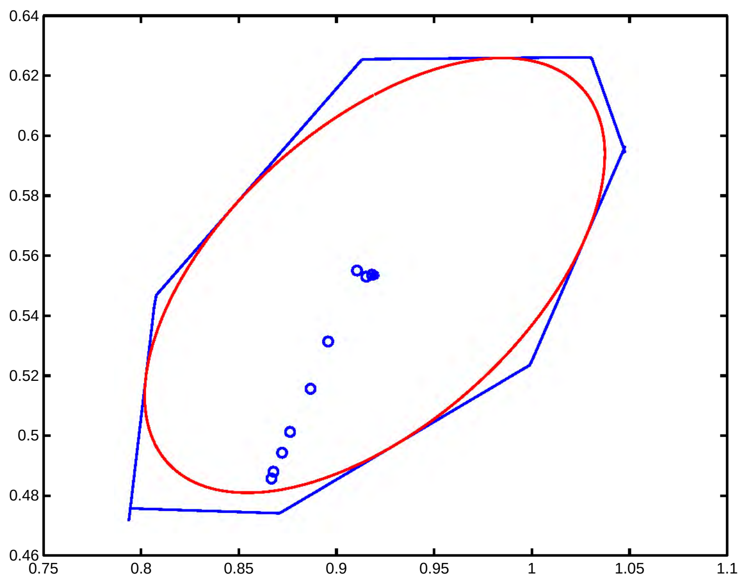

Figure 1: A 2-D example

Figure 2 contains an edited session of a Matlab run of the implemented algorithm on a random problem with 100 variables and a polytope defined by 500 inequalities. The problem is solved on a Sun UltraSparc-1 workstation in about 100 seconds.

We emphasize that since the matrix variable E has been eliminated, system (28) contains only $2m+n$ variables. The linear algebra cost per iteration is no more expensive than algorithms based on the volumetric barrier as described in [28, 1], for example. In general, the prototype algorithm seems to be quite robust and reasonably efficient for medium-size problems.

Figure 2: Result on a Random Problem

```
[m n] = [500 100]
iter   0   residual = 2.91e+03
iter   1   residual = 1.28e+03
iter   2   residual = 7.58e+02
iter   3   residual = 6.86e+02
       :             :
iter  17   residual = 5.74e-03
iter  18   residual = 7.25e-04
iter  19   residual = 1.13e-04
iter  20   residual = 1.80e-05
Converged!
CPU time: 97.74 seconds
```

We also conducted experiments on polytopes derived from traveling salesman problem (TSP). The largest polytope in our tests is a convex hull of tours through 7 nodes that is defined by 3437 inequality constraints and 7 equality constraints in 21 variables. This polytope is then projected onto a full-dimensional polytope in 14 variables defined by 3437 inequality constraints. For this problem, the origin is an extremely good starting point for the center of the maximum volume ellipsoid. Starting from the origin, the algorithm converges rather quickly; see a edited session in Figure 3. The result was obtained on an AlphaServer 4100 5/400 with 1GB of memory, and on a 400MHz Alpha 21164 processor (the machine has 4 processors, but only one was used).

We note that the algorithm is not particularly suitable for problems where m is far greater than n since it solves an $m \times m$, and usually dense, linear system at each iteration. This is clearly demonstrated by the result in Figure 3 where about three hours of CPU time was consumed.

7 Concluding Remarks

Our preliminary numerical results indicate that the infeasible, primal-dual, interior-point approach seems promising for problems of small to medium sizes in terms of both the numbers of variables and constraints. This research

Figure 3: Result on a TSP polytope

```
[m n] = [3437 14]
iter   0   residual = 5.99e-01
iter   1   residual = 6.82e+00
iter   2   residual = 5.60e+00
iter   3   residual = 1.58e+00
iter   4   residual = 5.22e-01
iter   5   residual = 1.09e-01
iter   6   residual = 1.22e-02
iter   7   residual = 1.53e-04
iter   8   residual = 2.39e-08
Converged!
CPU time: 11012 seconds
```

is still at a preliminary stage and many issues remain to be investigated for the development of efficient and robust algorithms for the MVE problem, especially large-scale problems in terms of either the number of variables or the number of constraints.

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