A Posteriori Error Estimate for the 1D Wave Equation

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Abstract. The error of numerical schemes in heterogeneous media is difficult to analyse. In this paper, we derive an a posteriori estimate for the unidimensionnal wave equation in heterogeneous media. This estimates can be used as a tool to measure the precision of the numerical schemes in such media. The main result is based on a fondamental lemma given by Babuska and Rheinbolt for elliptic problems. We use this result to derive an error estimate for the semi-discrete problem. We also derive an estimate for the fully discretized problem.

Key Words. Wave Equation, A Posteriori Error Estimate, Heterogeneous Media

AMS(MOS) subject classifications. 65M15, 65N15, 65N30

1. Introduction.

The analysis of numerical schemes for wave propagation problems, is often reduced to the analysis of the dispersion relation. This phenomenon, takes place for linear equations, when we replace the partial derivatives by its finite difference or finite element approximations. Even though the control of the dispersion is of primary importance for the numerical simulation of wave propagation, the analysis of this effect is done in homogeneous media. Little is known about numerical phenomena in heterogeneous media for instance dispersion or the approximation of reflection coefficients.

In a two layers medium, we can carry out the analysis of dispersion and reflection coefficient (cf [6] and [7]), but for high order schemes this analysis becomes tedious, and for very heterogeneous media (like random media) impossible. That is why, we propose in this paper a tool to measure the quality of the numerical simulation, by deriving an a posteriori error estimate. Our work is based on the fondamental lemma obtained by Babuska and Rheinbolt, for elliptic problems (cf [1]). A lot of papers have followed in adaptative methods for differential equations (cf [4] or [5]), mainly for elliptic and parabolic problems.

The paper consists of the following parts. We start with the semi-discrete problem, for which the variational formulation gives an easy frame, to derive a semi-discrete estimate. Then, in the second part, we proceed to the fully discretized case where by interpolating the discrete sequence of approximations we derive the fully discrete estimate.

2. Estimation for the Semi-discrete Problem.

In this first paragraph we give an a posteriori error estimate for the semi-discrete problem. We use the fondamental lemma given by Babuska and Rheinbolt in [1]. We then deduce an a posteriori estimate in L^2 norm. Assuming the wave source has sufficient regularity, we deduce an estimate in H^1 norm.

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2.1. The Continuous Problem. We consider $\Omega =]0, L[$, and we seek the solution u of the radiation problem :

$$(1) \qquad \begin{cases} \frac{1}{K(x)}\frac{\partial^2 u}{\partial t^2} - \nabla(\frac{1}{\rho(x)}\nabla u) = f & x \in \Omega \\ u(x,0) = 0 & \frac{\partial u}{\partial t}(x,0) = 0 \\ u(0,t) = 0 & u(L,t) = 0 \end{cases}$$

We assume that :

$$(2) \qquad \begin{cases} K \in L^2(\Omega) & \rho \in L^2(\Omega) \\ 0 < K_m \leq K(x) \leq K_M < +\infty & 0 < \rho_m \leq \rho(x) \leq \rho_M < +\infty \end{cases}$$

To give the variational formulation of the problem, we will need the following spaces :

$$H = L^2(\Omega) \qquad V = H_0^1(\Omega)$$

On H we set the scalar product $(u,v) = \int 1/K uv dx$.
Then classically we consider the bilinear form $a(.,.)$ on $V \times V$ defined by :

$$(3) \qquad \begin{array}{rcl} a : V \times V & \longrightarrow & R \\ (u,v) & \longmapsto & a(u,v) = \int_\Omega \frac{1}{\rho}\nabla u \nabla v dx \end{array}$$

It is clear that with the assumptions (2) that $a(.,.)$ is bilinear, symetric, continuous and elliptic on $V \times V$.
Assuming $f \in C^2(0,T;H)$, we are led to look for the solution of the following variational problem :

$$(4) \qquad \begin{cases} f \in C^2(0,T;H) & u \in C^0(0,T;V) \cap C^2(0,T;H) \\ \frac{\partial^2}{\partial t^2}(u(t), \frac{v}{K}) + a(u(t),v) = (f(t),v) & \forall v \in V \\ u(x,0) = 0 & \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

Since $a(.,.)$ is bilinear, symetric and V-elliptic, there exists an orthonormal basis of eigenvectors $(v_j)_{j=1..+\infty} \in H$, associated to the eigenvalues $(\mu_j)_{j=1..+\infty}$ such that ([8]):

$$\forall v \in V \qquad a(v, v_j) = \mu_j (v, v_j)$$

We can therefore decompose $u(t)$ on the basis $(v_j)_{j=1..+\infty}$ into $u_j(t) = (u(t), v_j)$ which verifies the following ordinary differential equation :

$$(5) \qquad \begin{cases} \frac{d^2 u_j}{dt^2}(t) + \mu_j u_j(t) = f_j(t) \\ u_j(0) = 0 & \frac{\partial u_j}{\partial t}(0) = 0 \end{cases}$$

The solution of (5) is well known (cf [8]) and given by :

$$(6) \qquad u_j(t) = \frac{1}{\sqrt{\mu_j}} \int_0^t \sin(\sqrt{\mu_j}(t-\tau)) f_j(\tau) d\tau$$

whence $u(t)$ by summation on all the u_j.
Under this form it is clear that

$$f \in C^2(0,T;H) \Longrightarrow u \in C^2(0,T;H)$$

This result will be used in the following for the error estimate in section 2.3.

2.2. The Semi-discrete Problem . We discretize $\Omega =]0, L[$ in a regular grid with space step h in $I+2$ points

$$0 = x_0 < x_1 < ... < x_I < x_{I+1} = L$$

We define the functions $(\varphi_i)_{i=0..I+1} \in P_1(\Omega)$ by :

$$\varphi_i(x_j) = \delta_{i,j}$$

and the subspace V_h of V spanned by the $(\varphi_i)_{i=1..I}$ which vanish at $x=0$ and $x=L$. Therefore

$$V_h = [\varphi_i]_{i=1..I}$$

The interpolation operator is classically defined (cf [8]) by :

$$\begin{array}{rcl} \Pi_h : C^0(\Omega) & \longrightarrow & V_h \\ v & \longmapsto & \Pi_h v(x) = \sum_{i=1}^{I} v(x_i)\varphi_i(x) \end{array}$$

We now seek u_h solution of the following variational problem

$$(7) \qquad \begin{cases} f \in C^2(0,T;H) & u_h \in C^0(0,T;V_h) \cap C^2(0,T;H) \\ \dfrac{\partial^2}{\partial t^2}(u_h(t), \dfrac{v_h}{K}) + a(u_h(t), v_h) = (f(t), v_h) & \forall v_h \in V_h \\ u_h(x,0) = 0 & \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

Exactly for the same reasons than for the continous problem, there exists an orthonormal basis of eigenvectors $(v_{j,h})_{j=1..I} \in H$ associated to the eigenvalues $(\mu_{j,h})_{j=1..I}$ such that :

$$\forall v_h \in V_h \qquad a(v_h, v_{j,h}) = \mu_{j,h}(v_h, v_{j,h})$$

Again we can decompose $u_h(t)$ on the basis $(v_{j,h})$ into $u_{j,h}(t) = (u(t), v_{j,h})$ which is the solution of the following ordinary differential equation :

$$(8) \qquad \begin{cases} \dfrac{d^2 u_{j,h}}{dt^2}(t) + \mu_{j,h} u_{j,h}(t) = f_{j,h}(t) \\ u_j(0) = 0 \qquad \dfrac{\partial u_j}{\partial t}(0) = 0 \end{cases}$$

Whence

$$u_{j,h}(t) = \frac{1}{\sqrt{\mu_{j,h}}}\int_0^t \sin(\sqrt{\mu_{j,h}}(t-\tau))f_{j,h}(\tau)d\tau \tag{9}$$

whence $u_h(t)$ by summation on all the $u_{j,h}$. It is important to note that

$$f \in C^2(0,T;H) \Longrightarrow u_h \in C^2(0,T;H)$$

2.3. Semi-discrete A-posteriori Error Estimate. We are now going to use the preceeding variational formulations to write down the equation verified by the error $e(t) = u(t) - u_h(t)$. The essential tool is the Lax-Milgram lemma, which allows us to get an estimate of the norm of $e(t)$ as a function of a new function $z(t)$. Then using the fondamental lemma of [1], we can compute this new unknown $z(t)$ given $u_h(t)$ on each interval $]x_i, x_{i+1}[$. We derive an estimate in $L^2(\Omega)$ norm in proposition 1, which is the main result of this section. Assuming some regularity on the source function f, by the same procedure, we also derive an estimate of the error in $H^1(\Omega)$ norm in proposition 2.

2.3.1. L^2 Norm Error Estimate. We want to majorate $e(t) = u(t) - u_h(t)$. To write down the equation verified by $e(t)$, we introduce for t fixed, the continuous linear form $F(t)$ on V defined by :

$$\begin{aligned} F(t) : V &\longrightarrow R \\ v &\longmapsto F(t)(v) = (f(t),v) - (\frac{\partial^2 u_h(t)}{\partial t^2}, \frac{v}{K}) - a(u_h(t),v) \end{aligned}$$

F is well defined. Since $f \in C^2(0,T;H)$ we have $u_h \in C^2(0,T;H)$. So according to Lax-Milgram lemma, we know that there exists $z(t) \in V$ such that

$$a(z(t),v) = F(t)(v) \tag{10}$$

that is to say, using (4):

$$a(z(t),v) = (\frac{1}{K}\frac{\partial^2 e(t)}{\partial t^2}, v) - a(e(t),v) \qquad \forall v \in V \tag{11}$$

Therefore $e(t)$ is solution of the the following variational problem :

$$\left\{ \begin{array}{ll} \frac{\partial^2}{\partial t^2}(e(t), \frac{v}{K}) + a(e(t),v) = a(z(t),v) & \forall v \in V \\ e(x,0) = 0 & \frac{\partial e}{\partial t}(x,0) = 0 \end{array} \right. \tag{12}$$

Using the orthonormal basis $(v_j)_{j=1..+\infty}$, we can write that $e_j(t) = (e(t), v_j)$ verifies the following ordinary differential equation :

$$\left\{ \begin{array}{ll} \frac{d^2 e_j}{dt^2}(t) + \mu_j e_j(t) = \mu_j z_j(t) & \\ e_j(0) = 0 & \frac{\partial e_j}{\partial t}(0) = 0 \end{array} \right. \tag{13}$$

whence

$$e_j(t) = \alpha_j \int_0^t \sin(\alpha_j(t-\tau)) z_j(\tau) d\tau \tag{14}$$

with $\alpha_j = \sqrt{\mu_j}$. Using Parseval formula

$$||e(t)||_{0,\Omega}^2 = \sum_{j=1}^{+\infty} |e_j(t)|^2$$

we derive the following majoration

$$||e(t)||_{0,\Omega}^2 \leq t \int_0^t a(z(\tau), z(\tau)) d\tau \tag{15}$$

From (15) we are going to get an a posteriori estimate. We will write a series of local elliptic equation verified by the function z on each interval $]x_i, x_{i+1}[$. Then we will get computable quantities in terms of u_h which will give the a posteriori estimate.
By definition of z we have

$$\forall v \in V \qquad a(z,v) = (f,v) - \frac{\partial^2}{\partial t^2}(u_h, \frac{v}{K}) - a(u_h, v) \tag{16}$$

therefore $\forall v_h \in V_h \quad a(z, v_h) = 0$. So we can write

$$\begin{aligned} a(z,z) &= a(z, z - v_h) \\ &= a(Pz, z - v_h) \qquad v_h \ / \ \Pi_h v_h = \Pi_h z \end{aligned}$$

where P is the orthogonal projection for the scalar product $a(.,.)$ on the subspace

$$W = \{v \in V \cap C^0(\Omega) \ / \ \Pi_h v = 0\}$$

In fact

$$\begin{aligned} a(z,z) &= a(z, z - v_h) \\ &= a(Pz, z - v_h) + a(Pz^\perp, z - v_h) \end{aligned}$$

but $w = z - v_h \in W$ since $\Pi_h v_h = \Pi_h z$, therefore $a(Pz^\perp, z - v_h) = 0$

<u>Remark</u>
To define $\Pi_h z$ we must have $z \in C^0(\Omega)$. This possible since in one dimension: $H^1(\Omega) \subset C^0(\Omega)$ (cf [9]).

Therefore with $||z||_E^2 = a(z,z)$ we have

$$||z||_E^2 \leq ||Pz||_E \cdot ||z - v_h||_E \tag{17}$$

We are now ready to apply the fondamental lemma of Babuska and Rheinbolt (cf [1]).

LEMMA 1. *If* $1/\rho \in C^1(\Omega)$ *then there exists a constante* $C = C(h)$ *such that*

$$\inf_{\{v_h \in V_h \ / \ \Pi_h v_h = \Pi_h z\}} ||z - v_h||_E \leq C(h) ||z||_E \tag{18}$$

PROOF : see Appendix A

With the preceeding result and the lemma we get

$$(19)\qquad ||Pz||_E \leq ||z||_E \leq C(h)\,||Pz||_E$$

therefore an estimation of $||Pz||_E$ will give an estimation of $a(z,z)$, and an error estimate. By definition of Pz, we can write

$$a(Pz,w) = a(z,w) \qquad \forall w \in W$$

therefore

$$a(Pz,w) = F(w) \qquad \forall\, w \in W$$

in particular $\forall\, i = 1..I-1\ \ \forall\, \varphi \in \mathcal{D}(]\S_), \S_{)+\infty}[)$, we have $\varphi \in W$ and we can write :

$$(20)\int_{x_i}^{x_{i+1}} \frac{1}{\rho}\frac{\partial Pz}{\partial x}\frac{\partial \varphi}{\partial x}dx = \int_{x_i}^{x_{i+1}} f(x,t)\varphi(x)dx - \int_{x_i}^{x_{i+1}} \frac{1}{\rho}\frac{\partial u_h}{\partial x}\frac{\partial \varphi}{\partial x}dx - \frac{1}{K}\frac{\partial^2 u_h}{\partial t^2}\varphi(x)dx$$

Therefore we have in the sense of distributions on $]x_i, x_{i+1}[$ (cf [10])

$$(21)\begin{cases} -\dfrac{d}{dx}(\dfrac{1}{\rho}\dfrac{dPz}{dx}) = f(x,t) + \dfrac{\partial}{\partial x}(\dfrac{1}{\rho}\dfrac{\partial u_h}{\partial x})(x,t) - \dfrac{1}{K}\dfrac{\partial^2 u_h}{\partial t^2} & x \in]x_i, x_{i+1}[\ \ t \geq 0 \\ Pz(x_i) = 0 \qquad Pz(x_{i+1}) = 0 & \end{cases}$$

Assuming somme regularity on the coefficients, one can write (21) as an ordinary differential equation. We assume that $\forall i = 1..I-1$

$$(22)\qquad f(t) \in C^0(]x_i, x_{i+1}[) \qquad 1/K \in C^0(]x_i, x_{i+1}[) \qquad 1/\rho \in C^1(]x_i, x_{i+1}[)$$

and write (21) as

$$(23)\qquad \begin{cases} A(Pz) = r_i & x \in]x_i, x_{i+1}[\\ Pz(x_i) = 0 & Pz(x_{i+1}) = 0 \end{cases}$$

with

$$Av = -\nabla(\tfrac{1}{\rho}\nabla v)$$

$$r_i(x,t) = f(x,t) - Au_h(x,t) - \frac{1}{K}\frac{\partial^2 u_h}{\partial t^2}(x,t) \qquad x \in]x_i, x_{i+1}[$$

We easily deduce that

$$(24)\qquad ||Pz||_{0,]x_i,x_{i+1}[} \leq \lambda_{i,min}^{-1}||r_i||_{0,]x_i,x_{i+1}[}$$

where $\lambda_{i,min}$ is the smallest eigenvalue of A on $]x_i, x_{i+1}[$. Thus

$$(25)\qquad \begin{aligned} a(Pz,Pz) &= \sum_{i=1}^{I-1}\int_{x_i}^{x_{i+1}} Pz\ r_i dx \\ &\leq \sum_{i=1}^{I-1}\lambda_{i,min}^{-1}||r_i||^2_{0,]x_i,x_{i+1}[} \end{aligned}$$

Finaly we obtain the error estimate given by the following proposition

PROPOSITION 1. *Let $u(t)$ be the solution of the continuous problem (1) and $u_h(t)$ the solution of the semi-discrete problem (7) at time t. Assuming*

$$\forall i = 1..I-1 \qquad f(t) \in C^0(]x_i, x_{i+1}[) \quad 1/K \in C^0(]x_i, x_{i+1}[) \quad 1/\rho \in C^1(]x_i, x_{i+1}[)$$

then we have the following error estimate :

$$||u(t) - u_h(t)||^2_{0,\Omega} \leq C^2(h)t \sum_{i=1}^{I-1} \lambda^{-1}_{i,min} \int_0^t (\int_{x_i}^{x_{i+1}} |r_i(x,\tau)|^2 dx) d\tau \tag{26}$$

with

$$r_i(x,t) = f(x,t) - Au_h(x,t) - \frac{1}{K}\frac{\partial^2 u_h}{\partial t^2}(x,t) \qquad x \in]x_i, x_{i+1}[$$

<u>Remark</u>

We know that the smallest eigenvalue of A with vanishing boundary condition on $]x_i, x_{i+1}[$ is greater than the smallest eigenvalue of the operator $B = -1/\rho_M \Delta$ on the same interval. Thus

$$\lambda^{-1}_{i,min} \leq \frac{h^2}{\rho_M \pi^2}$$

For a homogeneous medium with $K(x) = C_0$ et $\rho = 1$ we have

$$A = -\Delta \qquad C(h) = 1 \qquad \lambda^{-1}_{i,min} = \frac{h^2}{\pi^2}$$

Furthermore we have $r_i = o(1)$, since $u_h \longrightarrow u$ when $h \rightarrow 0$, we have

$$\sum_{i=1}^{I-1} \lambda^{-1}_{i,min} \int_{x_i}^{x_{i+1}} |r_i(x,\tau)|^2 dx = o(h^2)$$

We have therefore the following order of convergence

$$||u(t) - u_h(t)||_{0,\Omega} = o(h) \tag{27}$$

2.3.2. H^1 Norm Error Estimate. It is possible to get a better estimate by assuming somme regularity of the source function f. Since

$$f \in C^3(0,T;H) \Longrightarrow u_h \in C^3(0,T;H)$$

we can define

$$\begin{array}{rcl} F'(t): V & \longrightarrow & R \\ v & \longmapsto & F'(t)(v) = (\frac{\partial f(t)}{\partial t}, v) - (\frac{1}{K}\frac{\partial^3 u_h(t)}{\partial t^3}, v) - a(\frac{\partial u_h(t)}{\partial t}, v) \end{array} \tag{28}$$

and since $z(t)$ is the solution of

$$a(z(t), v) = F(t)(v) \tag{29}$$

$z'(t) = \frac{\partial z(t)}{\partial t}$ is the solution of

$$a(z'(t), v) \quad = \quad F'(t)(v) \tag{30}$$

According to (14) we can write e_j as

$$e_j(t) \quad = \quad \alpha_j \int_0^t \sin(\alpha_j(t-\tau)) z_j(\tau) d\tau$$

with $\alpha_j = \sqrt{\mu_j}$. Integrating by parts gives :

$$e_j(t) \quad = \quad [\, -z_j(\tau)\cos(\alpha_j(t-\tau))\,]_0^t + \int_0^t \cos(\alpha_j(t-\tau)) z_j'(\tau) d\tau \tag{31}$$

whence

$$e_j^2(t) \quad \leq \quad 3\left(z_j^2(0) + z_j^2(t) + t\int_0^t (z_j')^2(\tau) d\tau\right) \tag{32}$$

we deduce after multiplication by μ_j

$$a(e(t), e(t)) \quad \leq \quad 3\left(a(z(0), z(0)) + a(z(t), z(t)) + t\int_0^t a(z_j'(\tau), (z_j'(\tau)) d\tau\right) \tag{33}$$

Then since

$$a(z'(t), v_h) = 0 \qquad \forall v_h \in V_h$$

we can apply the fondamental lemma

$$a(e(t), e(t)) \quad \leq \quad 3\left(a(Pz(0), Pz(0)) + a(Pz(t), Pz(t)) + t\int_0^t a(Pz_j'(\tau), (Pz_j'(\tau)) d\tau\right) \tag{34}$$

which like before gives

$$\left\{ \begin{array}{rcl} a(Pz(t), Pz(t)) & \leq & \sum_{i=1}^{I-1} \lambda_{i,min}^{-1} ||r_i(t)||^2_{0,]x_i, x_{i+1}[} \\ \\ a(Pz'(t), Pz'(t)) & \leq & \sum_{i=1}^{I-1} \lambda_{i,min}^{-1} ||r_i'(t)||^2_{0,]x_i, x_{i+1}[} \end{array} \right. \tag{35}$$

with

$$\left\{ \begin{array}{rcl} r_i(x,t) & = & f(x,t) - Au_h(x,t) - \frac{1}{K}\frac{\partial^2 u_h}{\partial t^2}(x,t) \qquad x \in]x_i, x_{i+1}[\\ \\ r_i'(x,t) & = & \frac{\partial r_j}{\partial t}(x,t) \end{array} \right. \tag{36}$$

Therefore we can state the following proposition

PROPOSITION 2. *Let $u(t)$ be the solution of the continuous problem (1) and $u_h(t)$ the solution of the semi-discrete problem (7) at time t. With the assumptions of the* *proposition 1, and if furthermore $f \in C^3(0,T;H)$ then we have the following error estimate :*

$$
\begin{aligned}
|u(t) - u_h(t)|^2_{1,\Omega} \quad \leq \quad & 3\rho_M C^2(h) \left\{ \sum_{i=1}^{I-1} \lambda^{-1}_{i,min} \left(||r_i(0)||^2_{0,]x_i,x_{i+1}[} + ||r_i(t)||^2_{0,]x_i,x_{i+1}[} \right) \right. \\
& \left. + \int_0^t ||r_i'(\tau)||^2_{0,]x_i,x_{i+1}[} d\tau \right\}
\end{aligned} \tag{37}
$$

with

$$
\left\{ \begin{array}{lcll} r_i(x,t) & = & f(x,t) - Au_h(x,t) - \dfrac{1}{K}\dfrac{\partial^2 u_h}{\partial t^2}(x,t) & x \in]x_i, x_{i+1}[\\ r_i'(x,t) & = & \dfrac{\partial r_i}{\partial t}(x,t) & \end{array} \right. \tag{38}
$$

Remark

For a homogeneous medium with $K(x) = K_0$ and $\rho = 1$ we have

$$
A = -\Delta \qquad C(h) = 1 \qquad \lambda^{-1}_{i,min} = \frac{h^2}{\pi^2}
$$

Furthermore we have $r_i = o(1)$ whence

$$
\sum_{i=1}^{I-1} \lambda^{-1}_{i,min} \int_{x_i}^{x_{i+1}} |r_i(x,\tau)|^2 dx = O(h^2)
$$

Therefore

$$
||u(t) - u_h(t)||_{1,\Omega} \quad = \quad o(h) \tag{39}
$$

We can compare this result to the classical a-priori error estimate for the finite element method (cf [8]) :

$$
||u(t) - u_h(t)||_{1,\Omega} \quad \leq \quad C\ h \left\{ ||u(t)||_{2,\Omega} + ||\frac{du(t)}{dt}||_{2,\Omega} + \int_0^t ||\frac{d^2u(s)}{dt^2}||_{2,\Omega} ds \right\} \tag{40}
$$

3. Fully Discretised Problem. When the problem is fully discretised we look for an estimation of the quantity $||u(t^n) - u_h^n||$. By decomposition we can write this error as a sum of the spatial error and temporal error, so that we only have to estimate the time error $||u_h(t^n) - u_h^n||$. In order to do so we will define an interpolation $\tilde{u}$ of $(u_h^n)_{n=1..N}$. Then we will get an a posteriori error estimate of $||u_h(t^n) - \tilde{u}(t^n)||$, and then an estimation of $||u(t^n) - \tilde{u}(t^n)||$. We start with a special case which will be used latter.

3.1. Single Eigenvalue Case. In the following we will have to treat the following system :

$$
\left\{ \begin{array}{l} \dfrac{d^2u}{dt^2}(t) \ + \ \omega^2 u(t) \ = \ f(t) \\ u(0) \ = \ \dfrac{du}{dt}(0) \ = \ 0 \end{array} \right. \tag{41}
$$

For discrete integration we use Newmark's scheme

$$\text{(42)} \qquad \begin{cases} u^{n+1} &= u^n + \Delta t \dot{u}^n + \Delta t^2 \left((\frac{1}{2} - \theta)\ddot{u}^n + \theta \ddot{u}^{n+1} \right) \\ \dot{u}^{n+1} &= \dot{u}^n + \Delta t^2 \left((1-\delta)\ddot{u}^n + \delta \ddot{u}^{n+1} \right) \\ \ddot{u}^n &+ \ \omega^2 u^n = f^n = f(t^n) \end{cases}$$

This corresponds to the scheme

$$\text{(43)} \begin{cases} \dfrac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 \left(\theta u^{n+1} + (\dfrac{1}{2} + \delta - 2\theta)u^n + (\dfrac{1}{2} - \delta + \theta)u^{n-1} \right) \\ = \theta f^{n+1} + (\frac{1}{2} + \delta - 2\theta) f^n + (\frac{1}{2} - \delta + \theta) f^{n-1} \end{cases}$$

For the explicit scheme, we have $\delta = 1/2$ and $\theta = 0$, that is to say we have

$$\text{(44)} \qquad \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n \ = \ f^n$$

We define now the interpolation $\tilde{u}$ of $(u^n)_{n=1..N}$ such that

$$\text{(45)} \qquad \begin{cases} \tilde{u}(t^n) = u^n & \tilde{u}(t^{n+1}) = u^{n+1} \\ \dfrac{d\tilde{u}}{dt}(t^n) = \dot{u}^n & \dfrac{d\tilde{u}}{dt}(t^{n+1}) = \dot{u}^{n+1} \end{cases}$$

A simple computation shows that $\forall t \in [t^n, t^{n+1}]$ we have

$$\text{(46)} \qquad \begin{aligned} \tilde{u}(t^n) &= (1-\alpha)u^n + \alpha u^{n+1} + \tfrac{\Delta t^2}{4}(\alpha - 3\alpha^2 + 2\alpha^3)(\ddot{u}^{n+1} - \ddot{u}^n) \\ &- \tfrac{\Delta t^2}{4}(\alpha - \alpha^2)(\ddot{u}^{n+1} + \ddot{u}^n) \\ \alpha &= \alpha(t) = \frac{t - t^n}{\Delta t} \end{aligned}$$

whence for $t \in [t^n, t^{n+1}]$

$$\text{(47)} \qquad \begin{aligned} &\frac{d^2\tilde{u}}{dt^2}(t) + \omega^2 \tilde{u}(t) = 3(\alpha - 1/2)(\ddot{u}^{n+1} - \ddot{u}^n) + \frac{1}{2}(\ddot{u}^{n+1} + \ddot{u}^n) \\ &+\omega^2 \Big[(1-\alpha)u^n + \alpha u^{n+1} + \tfrac{\Delta t^2}{4}(\alpha - 3\alpha^2 + 2\alpha^3)(\ddot{u}^{n+1} - \ddot{u}^n) \\ &- \tfrac{\Delta t^2}{4}(\alpha - \alpha^2)(\ddot{u}^{n+1} + \ddot{u}^n) \Big] = \mu^n(t) \end{aligned}$$

Now we can write $\mu^n(t)$ as follows

$$\text{(48)} \qquad \begin{aligned} \mu^n(t) \quad &= \quad 3(\alpha - 1/2)(\ddot{u}^{n+1} - \ddot{u}^n) + \tfrac{1}{2}(f^{n+1} + f^n) + \omega^2 \left[(\alpha - 1/2)(u^{n+1} - u^n) \right] \\ + \quad \tfrac{\omega^2 \Delta t^2}{4} \quad &\left[(\alpha - 3\alpha^2 + 2\alpha^3)(\ddot{u}^{n+1} - \ddot{u}^n) + (\alpha - \alpha^2)(\ddot{u}^{n+1} + \ddot{u}^n) \right] \end{aligned}$$

Then we set

$$\varepsilon^n(t) = \mu^n(t) - \frac{1}{2}(f^{n+1} + f^n) \tag{49}$$

We are now ready to write down the equation verified by $u - \tilde{u}$. With

$$\begin{aligned} \varepsilon(t) &= \sum_{n=1}^{N} \varepsilon^n(t) 1_{[t^{n-1}, t^n]} \\ \tilde{f}(t) &= \frac{1}{2}\sum_{n=1}^{N} (f^{n+1} + f^n) 1_{[t^{n-1}, t^n]} \end{aligned} \tag{50}$$

one can write that $e(t) = \tilde{u}(t) - u(t)$ verifies

$$\left\{ \begin{array}{rcccl} \dfrac{d^2 e}{dt^2}(t) & + & \omega^2 e(t) & = & \varepsilon(t) + (\tilde{f} - f)(t) \\ e(0) & = & \dfrac{de}{dt}(0) & = & 0 \end{array} \right. \tag{51}$$

whence

$$e(t) = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\varepsilon(\tau)d\tau + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))(\tilde{f} - f)(\tau)d\tau \tag{52}$$

Working on this expression will give us the a posteriori estimate. Let us treat the first integral.

$$I_1(t) = \int_0^t \sin(\omega(t-\tau))\varepsilon(\tau)d\tau \tag{53}$$

$$|I_1(t)| \le \int_0^t |\varepsilon(\tau)|d\tau \tag{54}$$

Decomposing $\varepsilon^k(\tau)$ as

$$\begin{aligned} \varepsilon^k(\tau) &= m^k(\tau) + p^k(\tau) \\ m^k(\tau) &= 3(\alpha - 1/2)(\ddot{u}^{k+1} - \ddot{u}^k) + \omega^2\left[(\alpha - 1/2)(u^{k+1} - u^k)\right] \\ &\quad + \frac{\omega^2 \Delta t^2}{4}(\alpha - 3\alpha^2 + 2\alpha^3)(\ddot{u}^{k+1} - \ddot{u}^k) \\ p^k(\tau) &= \frac{\omega^2 \Delta t^2}{4}(\ddot{u}^{k+1} + \ddot{u}^k)(\alpha - \alpha^2) \end{aligned} \tag{55}$$

Then

$$\begin{aligned} \int_0^t |\varepsilon(\tau)|d\tau &\le \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} |\varepsilon^k(\tau)|d\tau \\ &\le \sum_{n=1}^{N} \left(\int_{t^{n-1}}^{t^n} |m^{n-1}(\tau)|d\tau + \int_{t^{n-1}}^{t^n} |p^{n-1}(\tau)|d\tau \right) \end{aligned} \tag{56}$$

Now it is easy to see that

$$(57)\quad \int_{t^{n-1}}^{t^n} |m^{n-1}(\tau)|d\tau \leq \frac{\Delta t}{4}\left(3|\ddot{u}^n - \ddot{u}^{n-1}| + \omega^2|u^n - u^{n-1}| + \frac{\omega^2\Delta t^2}{16}|\ddot{u}^n - \ddot{u}^{n-1}|\right)$$

and

$$(58)\quad \int_{t^{n-1}}^{t^n} |p^{n-1}(\tau)|d\tau \leq \frac{\Delta t}{4}\frac{\omega^2\Delta t^2}{6}|\ddot{u}^n + \ddot{u}^{n-1}|$$

Thus

$$(59)\quad \begin{aligned} |I_1(t)| \leq\ & \frac{\Delta t}{4}\left[\sum_{n=1}^{N} 3|\ddot{u}^n - \ddot{u}^{n-1}| + \omega^2|u^n - u^{n-1}| + \frac{\omega^2\Delta t^2}{16}|\ddot{u}^n - \ddot{u}^{n-1}|\right. \\ & \left. + \frac{\omega^2\Delta t^2}{6}|\ddot{u}^n + \ddot{u}^{n-1}|\right] \end{aligned}$$

We have now to treat the second integral.

$$(60)\quad I_2(t) = \int_0^t \sin(\omega(t-\tau))\left(\tilde{f}(\tau) - f(\tau)\right)d\tau$$

whence

$$(61)\quad \begin{aligned} |I_2(t)| &= \int_0^t |\tilde{f} - f|(\tau)d\tau \\ \int_0^t |\tilde{f} - f|(s)ds &\leq \sum_{n=1}^{N}\int_{t^{n-1}}^{t^n} |\tilde{f} - f|(s)ds \leq \Delta t\frac{T}{2}||f'||_\infty \end{aligned}$$

We can now state the main result of this section

PROPOSITION 3. *Let $u(t)$ be the solution of the problem (1) and $\tilde{u}(t)$ the interpolation of $(u^n)_{n=1..N}$ solution of (44). Assuming that $\forall k = 1..N \quad f^k = f(t^k)$, for $t \in [t^{N-1}, t^N]$, we have :*

$$(62)\quad \begin{aligned} |e(t)| \leq\ & \Delta t\frac{T}{2\omega}||f'||_\infty + \frac{3\Delta t}{4\omega}\sum_{n=1}^{N}|\frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t}|\Delta t \\ & + \Delta t\omega\left(\sum_{n=1}^{N}|\frac{u^n - u^{n-1}}{\Delta t}|\Delta t + \frac{\Delta t^2}{16}|\frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t}|\Delta t + \frac{\Delta t^2}{6}|\ddot{u}^n + \ddot{u}^{n-1}|\right) \end{aligned}$$

3.2. A Posteriori Error Estimate in Time.

3.2.1. L^2 Norm Estimate. We know that $u_h(t^n)$ is the solution of

$$(63)\quad \begin{cases} \dfrac{\partial^2}{\partial t^2}(u_h(t^n), \dfrac{v_h}{K}) \;+\; a(u_h(t^n), v_h) \;=\; (f(t^n), v_h) & \forall v_h \in V_h \\ u_h(x,0) = 0 & \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

and that u_h^n is solution of

$$(64)\quad \begin{cases} (\dfrac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, \dfrac{v_h}{K}) \;+\; a(u_h^n, v_h) \;=\; (f^n, v_h) & \forall v_h \in V_h \\ u_h^0(x) = 0 & u_h^1(x) = 0 \end{cases}$$

we can use the basis of V_h, $(v_{i,h})_{i=1..I}$ orthonormal in $L^2(\Omega)$ associated to the eigenvalues $(\lambda_i)_{i=1..I}$, such that

$$(65)\quad \begin{array}{lll} \forall v_h \in V_h \quad a(v_h, v_{i,h}) & = \lambda_i (v_h, v_{i,h})_{0,\Omega} & 1 \le i \le I \\ \qquad\qquad ||v_{i,h}||_{0,\Omega} & = 1 & 1 \le i \le I \end{array}$$

Writing u_h et u_h^n on this basis, we get

$$(66)\quad \begin{array}{llll} u_h & = \displaystyle\sum_{i=1}^{I} u_i v_{i,h} & u_{i,h} = (u_h, v_{i,h})_{0,\Omega} & ||u_h||_{0,\Omega}^2 = \displaystyle\sum_{i=1}^{I} |u_i|^2 \\ u_h^n & = \displaystyle\sum_{i=1}^{I} u_i^n v_{i,h} & u_i^n = (u_h^n, v_{i,h})_{0,\Omega} & ||u_h^n||_{0,\Omega}^2 = \displaystyle\sum_{i=1}^{I} |u_i^n|^2 \end{array}$$

With $v_h = w_j$ in (63) we find that u_h is solution of the following ordinary differential equation:

$$(67)\quad \begin{cases} \dfrac{d^2 u_i}{dt^2}(t)||v_{i,h}||_K^2 + \lambda_i u_i(t) \;=\; f_i(t) & i = 1..I \\ u_i(0) = 0 & \dfrac{du_i}{dt}(0) = 0 \end{cases}$$

and u_h^n is solution of

$$(68)\quad \begin{cases} \dfrac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}||v_{i,h}||_K^2 + \lambda_i u_i^n \;=\; f_i^n & i = 1..I \\ u_j^0(0) = 0 & u_j^1(0) = 0 \end{cases}$$

With $\omega_i^2 = \lambda_i/||v_{i,h}||_K^2$ and $(u_i^n)_{n=1..N}$ we construct $\tilde{u}_i(t)$ defined previously. $\tilde{u}_i(t)$ is the solution of :

$$(69)\quad \begin{cases} \dfrac{d^2\tilde{u}_i}{dt^2}(t) + \omega_i^2 \tilde{u}_i(t) \;=\; \tilde{f}_i(t) \\ \tilde{u}_i(0) = 0 & \dfrac{d\tilde{u}_i}{dt}(0) = 0 \end{cases}$$

It is easily seen that the error $e_i(t) = \tilde{u}_i(t) - u_i(t)$ verifies

$$(70)\qquad \begin{cases} \dfrac{d^2e_i}{dt^2}(t) + \omega_i^2 e_i(t) = \varepsilon_i(t) + \frac{1}{\|v_{i,h}\|_K^2}(\tilde{f}_i - f_i)(t) \\ e_i(0) = \dfrac{de_i}{dt}(0) = 0 \end{cases}$$

With proposition 3, we have

$$(71)\qquad \begin{aligned} |e_i(t)| &\leq T\frac{\Delta t}{2\omega_i\|v_{i,h}\|_K^2}\left(\|f_i'\|_\infty + \Delta t\|f_i''\|_\infty\right) + \frac{3\Delta t}{4\omega_i}\sum_{n=1}^{N}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t \\ &+ \Delta t\omega_i\left(\sum_{n=1}^{N}\left|\frac{u_i^n - u_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{16}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{6}|\ddot{u}_i^n + \ddot{u}_i^{n-1}|\right) \end{aligned}$$

Now with this estimate of $e_i(t)$, we can deduce an estimate of $e(t) = \tilde{u}_h(t) - u_h(t)$ in $L^2(\Omega)$ norm. In fact

$$(72)\qquad \|e(t)\|_{0,\Omega}^2 = \sum_{i=1}^{+\infty}|e_i(t)|^2$$

but we know

$$(73)\qquad \omega_i^2 = \lambda_i/\|v_{i,h}\|_K^2$$

therefore since eigenvalue sequence λ_i is increasing, we get

$$(74)\qquad \frac{1}{\omega_i} \leq \frac{1}{\sqrt{\lambda_1 K_m}}$$

therefore

$$(75)\qquad \begin{aligned} |e_i(t)| &\leq \frac{T\Delta t}{2\sqrt{\lambda_1 K_m}}\left(\|f_i'\|_\infty + \Delta t\|f_i''\|_\infty\right) + \frac{3\Delta t\sqrt{K_M}}{4\sqrt{\lambda_1}}\sum_{n=1}^{N}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t \\ &+ \Delta t\omega_i\left(\sum_{n=1}^{N}\left|\frac{u_i^n - u_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{16}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{6}\ddot{u}_i^n + \ddot{u}_i^{n-1}|\right) \end{aligned}$$

whence

$$(76)\qquad \begin{aligned} |e_i(t)|^2 &\leq \Delta t^2\left[\frac{3T^2}{2\lambda_1 K_m}\|f_i'\|_\infty^2 + \Delta t\|f_i''\|_\infty^2 + \frac{27K_M}{16\lambda_1}\left(\sum_{n=1}^{N}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t\right)^2\right. \\ &\left.+ 3\omega_i^2\left(\sum_{n=1}^{N}\left|\frac{u_i^n - u_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{16}\left|\frac{\ddot{u}_i^n - \ddot{u}_i^{n-1}}{\Delta t}\right|\Delta t + \frac{\Delta t^2}{6}|\ddot{u}_i^n + \ddot{u}_i^{n-1}|\right)^2\right] \end{aligned}$$

We can now state

PROPOSITION 4. *Let $u_h(t)$ be the solution of (63) and $\tilde{u}(t)$ the interpolation of $(u_h^n)_{n=1..N}$ solution of (64). Assuming that $\forall k = 1..N \quad f^k = f(t^k)$, for $t \in [t^{N-1}, t^N]$ we have the following estimate*

$$
\begin{aligned}
||e(t)||^2_{0,\Omega} \leq \; & \Delta t^2 \Bigg[\frac{3T^2}{2\lambda_1 K_m} (||f'||^2_{L^\infty(0,T;L^2(\Omega))} + \Delta t ||f''||^2_{L^\infty(0,T;L^2(\Omega))}) \\
& + \frac{27K_M}{16\lambda_1} || \sum_{n=1}^{N} \frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t} \Delta t ||^2_{0,\Omega} + 9K_M || \sum_{n=1}^{N} (\frac{u_i^n - u_i^{n-1}}{\Delta t}) \Delta t ||^2_E \\
& + \frac{9K_M \Delta t^2}{16} || \sum_{n=1}^{N} (\frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t}) \Delta t ||^2_E + \frac{9K_M \Delta t^2}{6} || \sum_{n=1}^{N} \ddot{u}^n + \ddot{u}^{n-1} ||^2_E \Bigg]
\end{aligned} \tag{77}
$$

<u>Remark</u> In homogeneous media with $K(x) = 1$ et $\rho = 1$ we have

$$A = -\Delta \qquad \lambda_1 = \frac{\pi^2}{L^2}$$

and the error estimate

$$
\begin{aligned}
||e(t)||^2_{0,\Omega} \leq \; & \Delta t^2 \Bigg[\frac{3T^2 L^2}{2\pi^2} (||f'||^2_{L^\infty(0,T;L^2(\Omega))} + \Delta t ||f''||^2_{L^\infty(0,T;L^2(\Omega))}) \\
& + \frac{27L^2}{16\pi^2} || \sum_{n=1}^{N} \frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t} \Delta t ||^2_{0,\Omega} + 9 || \sum_{n=1}^{N} (\frac{u_i^n - u_i^{n-1}}{\Delta t}) \Delta t ||^2_E \\
& + \frac{9\Delta t^2}{16} || \sum_{n=1}^{N} (\frac{\ddot{u}^n - \ddot{u}^{n-1}}{\Delta t}) \Delta t ||^2_E + \frac{9\Delta t^2}{6} || \sum_{n=1}^{N} \ddot{u}^n + \ddot{u}^{n-1} ||^2_E \Bigg]
\end{aligned} \tag{78}
$$

Like the spatial error estimate we have

$$||u_h(t) - \tilde{u}(t)||_{0,\Omega} = O(\Delta t) \tag{79}$$

4. Total A Posteriori Estimate.

4.1. L^2 Norm Estimate. We can apply the preceeding error estimates, and deduce an a posteriori error estimate in $L^2(\Omega)$ norm, between the exact solution and the discrete solution interpolated in time. More precisely with

$$||u(t) - \tilde{u}(t)||_{0,\Omega} \leq ||u(t) - u_h(t)||_{0,\Omega} + ||u_h(t) - \tilde{u}(t)||_{0,\Omega} \tag{80}$$

we have the following proposition

PROPOSITION 5. *Let $u(t)$ the solution of (1) and $\tilde{u}(t)$ the interpolation of $(u_h^n)_{n=1..N}$ solution of (44) Assuming that $\forall k = 1..N \quad f^k = f(t^k)$, for $t \in [t^{N-1}, t^N]$ we have*

$$
\begin{aligned}
\|u(t)-\tilde u(t)\|_{0,\Omega}^2 \le\ & C^2(h)t\sum_{i=1}^{I-1}\lambda_{i,min}^{-1}\int_0^t\Big(\int_{x_i}^{x_{i+1}}|r_i(x,\tau)|^2dx\Big)d\tau \\
& \Delta t^2\left[\frac{3T^2}{2\lambda_1 K_m}(\|f'\|^2_{L^\infty(0,T;L^2(\Omega))}+\Delta t\|f''\|^2_{L^\infty(0,T;L^2(\Omega))})\right. \\
+\ & \frac{27K_M}{16\lambda_1}\|\sum_{n=1}^{N}\frac{\ddot u^n-\ddot u^{n-1}}{\Delta t}\Delta t\|^2_{0,\Omega}+9K_M\|\sum_{n=1}^{N}(\frac{u_i^n-u_i^{n-1}}{\Delta t})\Delta t\|_E^2 \\
+\ & \left.\frac{9K_M\Delta t^2}{16}\|\sum_{n=1}^{N}(\frac{\ddot u^n-\ddot u^{n-1}}{\Delta t})\Delta t\|_E^2+\frac{9K_M\Delta t^2}{6}\|\sum_{n=1}^{N}\ddot u^n+\ddot u^{n-1}\|_E^2\right]
\end{aligned}
\tag{81}
$$

with

$$
\begin{aligned}
r_i(x,t) &= f(x,t)-Au_h(x,t)-\tfrac{1}{K}\tfrac{\partial^2 u_h}{\partial t^2}(x,t) \qquad x\in]x_i,x_{i+1}[\\
T &= N\,\Delta t
\end{aligned}
\tag{82}
$$

<u>Remark</u> In homogeneous media with $K(x)=1$ et $\rho=1$ we have

$$\|u(t)-\tilde u(t)\|_{0,\Omega} = o(h)+O(\Delta t) \tag{83}$$

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Appendix A. If $1/\rho\in C^1(\Omega)$ then there exists a constante $C=C(h)$ such that

$$\inf_{\{v_h\in V_h\ /\ \Pi_h v_h=\Pi_h z\}}\|z-v_h\|_E \le C(h)\|z\|_E \tag{84}$$

We define the space

$$W = \{v\in V\cap C^0(\Omega)\ /\ \Pi_h v=0\}$$

and set $w=z-v_h$. Since $\Pi_h v_h=\Pi_h z$ we have $w\in W$. Then

$$
\begin{aligned}
a(z,z) &= a(w+v_h,w+v_h) \\
&= a(w,w)+a(v_h,v_h)+2\,a(w,v_h)
\end{aligned}
$$

For the term $a(w,v_h)$, we can write :

$$
\begin{aligned}
a(w,v_h) &= \int_\Omega \frac{1}{\rho}\nabla w\nabla v_h dx \\
&= \sum_{i=1}^{I-1}\int_{x_i}^{x_{i+1}}\frac{1}{\rho}\nabla w\nabla v_h dx \\
&= \sum_{i=1}^{I-1}\int_{x_i}^{x_{i+1}}(\frac{1}{\rho(x)}-\frac{1}{\rho_{i+1/2}})\nabla w\nabla v_h dx+\int_{x_i}^{x_{i+1}}\frac{1}{\rho_{i+1/2}}\nabla w\nabla v_h dx
\end{aligned}
$$

where

$$\rho_{i+1/2} = \rho((x_i + x_{i+1})/2)$$

With $1/\rho(x) \in C^1(\Omega)$ we have

$$|\frac{1}{\rho(x)} - \frac{1}{\rho_{i+1/2}}| \leq Ch$$

whence

$$\int_{x_i}^{x_{i+1}} (\frac{1}{\rho(x)} - \frac{1}{\rho_{i+1/2}})\nabla w \nabla v_h dx \leq Ch \int_{x_i}^{x_{i+1}} \nabla w \nabla v_h dx$$

For the second term, we have :

$$\int_{x_i}^{x_{i+1}} \frac{1}{\rho_{i+1/2}} \nabla w \nabla v_h dx = -\int_{x_i}^{x_{i+1}} \frac{1}{\rho_{i+1/2}} w \Delta v_h dx + [\frac{1}{\rho_{i+1/2}} \frac{\partial v_h}{\partial x} w]_{x_i}^{x_{i+1}}$$

Since $v_{h/]xi,x_{i+1}[} \in P_1(]xi, x_{i+1}[)$ we have $\Delta v_h = 0$.
Plus we have :$\Pi_h w = 0$ therefore $\forall j = 1..I-1 \quad w(x_j) = 0$ which implies

$$\int_{x_i}^{x_{i+1}} \frac{1}{\rho_{i+1/2}} \nabla w \nabla v_h dx = 0$$

Finaly

$$\begin{aligned} a(w, v_h) &\leq Ch|w|_{1,\Omega}|v_h|_{1,\Omega} \\ &\leq Ch\ ||w||_E||v_h||_E \\ &\leq Ch\ \{||w||_E^2 + ||v_h||_E^2\} \end{aligned}$$

Therefore we can write :

$$||z||_E^2 \leq (1 + Ch)\ \{||w||_E^2 + ||v_h||_E^2\}$$

which can be written as :

$$||z||_E^2 = (1 + O(h))\ \{||w||_E^2 + ||v_h||_E^2\}$$

whence

$$||w||_E^2 = (1 + O(h))^{-1}||z||_E^2 - ||v_h||_E^2$$

whence

$$||w||_E^2 \leq (1 + O(h))||z||_E^2$$

that is to say

$$||z - v_h||_E^2 \leq (1 + O(h))||z||_E^2$$

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