A Global Convergence Theory for Arbitrary Norm Trust-Region Methods for Nonlinear Equations

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September 1993
(revised May 1995)

TR93-41
(replaces TR87-25)

A Global Convergence Theory for Arbitrary Norm Trust-Region Methods for Nonlinear Equations[1,2,3]

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Abstract. In this work we extend the Levenberg-Marquardt algorithm for approximating zeros of the nonlinear system $F(x) = 0$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. Instead of the ℓ_2 norm, arbitrary norms can be used in the trust-region objective function and in the trust-region constraint. The algorithm is shown to be globally convergent. This research was motivated by the recent work of Duff, Nocedal and Reid. A key point in our analysis is that the tools from nonsmooth analysis and the Zangwill convergence theory allow us to establish essentially the same properties for an arbitrary norm trust-region algorithm that have been established for the Levenberg-Marquardt algorithm using the tools from smooth optimization. It is shown that all members of this class of algorithms locally reduce to Newton's method and that the iteration sequence actually converges to a solution.

Key Words: trust region, Newton's method, global convergence, superlinear convergence, quadratic convergence, nonlinear systems.

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[1]Research sponsored by DOE DE-FG05-86ER25017, SIO/IST managed under ARO DAAG-03-86-K-0113, and AFOSR 85-0243.

[2]An earlier version of this paper was presented at the Second SIAM Meeting on Optimization, Houston, Texas, May 1987.

[3]The authors gratefully acknowledge discussions with J.E. Dennis, Jr.. They also thank Richard Byrd for pointing out a flaw in the original proof of Lemma 6.4. Finally, they thank Homer Walker for pointing out that the new Eisenstat-Walker theory could be used to prove convergence of the iteration sequence generated by the trust-region algorithm under consideration.

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1. Introduction

In this paper we consider the problem of solving the nonlinear system of equations

$$F(x) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. We will be concerned with the fact that the Jacobian of F at x, say $F'(x)$, may be sparse.

Locally, problem (1) is often solved by Newton's method. Globally difficulties arise when the Newton step, $s^N = -[F'(x_k)]^{-1}F(x_k)$, lies outside the region where the linear model $F(x_k) + F'(x_k)s$ is a good approximation to $F(x_k + s)$. One effective remedy when this occurs is to restrict the step s to a region where the linear model can be trusted. The classical approach for accomplishing this objective is the well-known Levenberg-Marquardt trust-region algorithms where the step s_k is the solution of the subproblem

$$\text{minimize} \quad \|F(x_k) + F'(x_k)s\|_2^2 \tag{2a}$$

$$\text{subject to} \quad \|s\|_2^2 \le \delta_k^2. \tag{2b}$$

The Karush-Kuhn-Tucker conditions for Problem (2) are equivalent to

$$s(\mu) = -\left[F'(x_k)^T F'(x_k) + \nu I\right]^{-1} F'(x_k)^T F(x_k) \tag{3a}$$

$$\nu \ge 0, \quad \|s(\mu)\|_2^2 \le \delta_k, \quad \text{and} \quad \left(\|s(\mu)\|_2^2 - \delta_k\right)\nu = 0 \tag{3b}$$

The solution of Problem (2) is $s(\nu_k)$ where ν_k satisfies $\|s(\nu_k)\|_2^2 = \delta_k$, unless $\|s(0)\|_2^2 \le \delta_k$, in which case $s(0) = s_k^N$, i.e. the Newton step is the solution of Problem (2). It can be obtained by the robust Hebden-Moré implementation of the Levenberg-Marquardt algorithm described in Moré [17]. (2), these conditions are both necessary and sufficient. However, for larger systems, this approach has the disadvantage that (3) has to be evaluated for several values of ν at each iteration. Also it is not obvious how one utilizes sparsity here; since multiplying a matrix by its transpose may destroy sparsity.

To avoid solving (3) at each iteration, the dogleg (Powell [18]) or the double dogleg (Dennis and Mei [5]) can be used to obtain a good approximation to the solution of Problem (2). However, we cannot expect the dogleg strategies to be as robust as the Levenberg-Marquardt algorithm. In fact, Reid [20] adapted the dogleg method to the sparse case, and reported finding examples for which the method did not converge, but the standard Levenberg-Marquardt method did converge.

Duff, Nocedal, and Reid [7] suggested replacing the square of the ℓ_2-norm in (2a) and (2b) with the ℓ_1-norm in (2a), and the ℓ_∞-norm in (2b). The resulting trust-region subproblem, in a standard manner, can be reformulated as a linear program, and hence, unlike the Levenberg-Marquardt approach, it is possible to take advantage of any sparsity patterns in the Jacobian $F'(x_k)$. Since $f = \|F\|_1$ is not differentiable, Duff, Nocedal and Reid use

$$\|F(x+s)\|_1 \le \|F(x)\|_1 - c_0\|F'(x)s\|_1. \tag{4}$$

as an acceptance criterion. It will be shown in Lemma 2.2, that

$$-\|F'(x)s\|_1 \le f'(x,s), \tag{5}$$

which implies that the descent condition (4) may be excessively conservative. Duff, Nocedal, and Reid do not include convergence results. However, they do give a detailed description of their algorithm and its implementation and point out that it is competitive with other methods.

The use of a different norm in (2a) instead of the square of the ℓ_2 norm and various alternatives to (2b), has been suggested and investigated by many authors. Madsen [15] uses the ℓ_∞-norm and considers the overdetermined system $F(x) = 0$ where $F : \mathbb{R}^n \to \mathbb{R}^m$, $n \le m$. Powell [19] also considered a trust-region algorithm for minimizing $h(F(x))$, where $F : \mathbb{R}^n \to \mathbb{R}^m$, $n \le m$, is continuously differentiable and h is any coercive continuous convex function. Both algorithms in Madsen [15] and in Powell [19] are demonstrated to be globally convergent in the sense that

$$\lim_{k \to +\infty} \psi(x_k) = 0$$

where

$$\psi(x) = h(F(x)) - \min\left\{h(F(x) + F'(x)s) \mid ||s|| \le 1\right\}.$$

In a similar approach to Powell [19], Yuan [21] and [22] uses the very simple descent condition

$$h(F(x_{k+1})) < h(F(x_k)).$$

He proves that $\liminf_{k \to +\infty} \psi(x_k) = 0$.

In the present work, we propose a class of globally convergent trust-region algorithms for approximating zeros of the square nonlinear system (1). At each iteration, we solve the following model trust-region problem:

$$\text{minimize} \quad m_k(s) = ||F(x_k) + F'(x_k)s||_a \tag{6a}$$

$$\text{subject to} \quad ||s||_b \le \delta_k, \tag{6b}$$

where $||\ ||_a$ and $||\ ||_b$ are two arbitrary but fixed norms on $\mathbb{R}^n$.

In Section 2 we compare differentiability properties of the function $f = ||F||$ and the local model $m_x(s) = ||F(x) + F'(x)s||$. We also derive a rather weak sufficient condition for stationary points to be solutions of the nonlinear system $F(x) = 0$. The General Trust-Region Algorithm is described in Section 3. In Section 4 we extend to arbitrary trust-region algorithms the well-known result that the solution of the Levenberg-Marquardt model trust-region problem (2) approaches a steepest descent direction as the trust-region radius approaches zero.

Since in our analysis we will consider iterates having the form (x_k, δ_k) where x_k and δ_k will not be uniquely specified, we choose to model our algorithm with a point-to-set map. Therefore in Section 5 we review some properties of point-to-set maps and Zangwill's convergence theorem [23].

The bulk of our analysis is contained in Section 6 where we demonstrate that the General Trust-Region Algorithm is globally convergent. In Section 7 we establish that either all accumulation points of the sequence generated by the General Trust-Region Algorithm are solutions of $F(x) = 0$ or no linear system $F(x_*) + F'(x_*)s = 0$, where x_* is arbitrary accumulation point of the sequence, has a solution. We then use a theorem of Eisenstat and Walker [8] to show that the General Trust Region Algorithm converges to a solution of $F(x) = 0$ whenever the iteration sequence has an accumulation point x_* such that $F'(x_*)$ is nonsingular. The q-quadratic convergence of the algorithm is demonstrated in Section 8 by proving that the General Trust-Region Algorithm reduces to Newton's method after a finite number of steps. Finally, in Section 9 we present a summary and some concluding remarks.

2. Differentiability of $f = ||F||$ and Optimality Conditions

In this section, we present subdifferentiability properties of $f = ||F||$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. These properties are needed to derive the optimality conditions and to characterize the solutions of $F(x) = 0$.

The locally Lipshitz function f is regular, i.e. its *one-sided directional* and *generalized directional derivatives* at $x \in \mathbb{R}^n$ in the direction $s \in \mathbb{R}^n$, denoted $f'(x;s)$ and $f^0(x;s)$ respectively, exist and are equal They are defined respectively by

$$f'(x;s) = \lim_{t \downarrow 0} \frac{f(x+ts) - f(x)}{t}. \tag{7}$$

and

$$f^0(x;s) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+ts) - f(y)}{t}. \tag{8}$$

Moreover the *generalized gradient* of f at x, denoted $\partial f(x)$, is the subset of $\mathbb{R}^n$ defined by

$$\partial f(x) = \left\{ g \in \mathbb{R}^n | f^0(x;s) \geq g^T s, \quad \forall s \in \mathbb{R}^n \right\}. \tag{9}$$

We refer to [3] for more details about subdifferentiability properties.

The following lemma shows that the local model m_x and the function f have the same descent directions. This is important from an algorithmic point of view.

Lemma 2.1. *Let x and s be any points in $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function at x, and $f = \|F\|$. Then*

$$f'(x;s) = m'_x(0;s), \tag{10}$$

where

$$m_x(s) = \|F(x) + F'(x)s\|. \tag{11}$$

Proof. Because F is differentiable at x, we have

$$F(x+ts) = F(x) + tF'(x)s + o(t)$$

where

$$\lim_{t \to 0} \frac{o(t)}{t} = 0.$$

Using the triangle inequality, we establish that

$$\frac{m_x(ts) - f(x)}{t} - \frac{o(t)}{t} \leq \frac{f(x+ts) - f(x)}{t} \leq \frac{m_x(ts) - f(x)}{t} + \frac{o(t)}{t}$$

and by taking the limit as t decreases to zero, we obtain (10). □

The following lemmas suggest that an approximation of the directional derivative, say γ, that can be used in a relaxed descent condition test should satisfy

$$\max \left\{ f'(x;s), \ \frac{c_1}{c_0} \left[m_x(s) - f(x) \right] \right\} \leq \gamma(x,s). \tag{12}$$

They also demonstrate the conservatism of the choice (4).

Lemma 2.2. *Let x be any point in $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function at x, and $f = \|F\|$. Then*

$$f'(x;s) \leq \|F(x) + F'(x)s\| - \|F(x)\| \tag{13a}$$

and

$$-\|F'(x)s\| \leq f'(x;s) \tag{13b}$$

for all $s \in \mathbb{R}^n$. Moreover if the linear system

$$F(x) + F'(x)s = 0 \tag{14a}$$

has a solution s_*, then

$$f'(x, s_*) = -\|F(x)\|. \tag{14b}$$

Proof. The inequality (13a) is a consequence of the convexity of the function m_x and Lemma 2.1. On the other hand, we have for all $s \in \mathbb{R}^n$ and for all positive t

$$|\, \|F(x) + tF'(x)s\| - \|F(x)\| \,| \leq t\, \|F'(x)s\|\,,$$

which, together with the definition of m_x, implies

$$-\|F'(x)s\| \leq \frac{m_x(ts) - m_x(0)}{t} \quad \forall\, t > 0.$$

By passing to the limit as t decreases to zero and using Lemma 2.1 we obtain (13b). We now suppose that the linear system (14a) has a solution s_*. Then the inequalities (13a) and (13b) become

$$f'(x; s_*) \leq -\|F(x)\| \quad \text{and} \quad -\|F'(x)s_*\| \leq f'(x; s_*).$$

The result now follows from the equality $\|F(x)\| = \|F'(x)s_*\|$. □

The property in equality (14b) is also proved in Burdakov [2] for the special case where s_* is the Newton direction, i.e. $s_* = -[F'(x)]^{-1}F(x)$ and also for any norm.

Lemma 2.3. *Assume the hypotheses of Lemma 2.2. Let $\{s_k \neq 0, k \in \mathbb{N}\}$ be a sequence that converges to zero. If d is an accumulation point of $\{d_k = s_k/\|s_k\|, k \in \mathbb{N}\}$ such that $f'(x,d) < 0$ and $0 < c_0 < c_1 < 1$, then*

$$\frac{c_1}{c_0}\,[\, m_x(s_k) - f(x)\,] \quad \leq \quad f'(x, s_k), \tag{15a}$$

holds for sufficiently large $k \in N$.

Proof. From the Lipschitz continuity of m_x and Lemma 2.1 we obtain

$$\lim_{k \in N \to +\infty} \frac{m_x(s_k) - f(x)}{\|s_k\|} = f'(x, d). \tag{15b}$$

Since $f'(x,d) < 0$ and $0 < c_0 < c_1$, the continuity of $f'(x, \cdot)$ and (15b) imply that

$$f'(x, s_k) > \frac{c_1}{c_0}[m_x(s_k) - f(x)]. \tag{15c}$$

The algorithmic implication of the following lemma is very important as will be seen in Lemma 6.1. Observe that the choice (4) would not allow us to establish this result.

Lemma 2.4. *Assume the hypotheses of Lemma 2.2. Let $\{s_k \neq 0, k \in \mathbb{N}\}$ be a sequence that converges to zero and satisfies*

$$f(x + s_k) > f(x) + c_0\gamma(x, s_k) \tag{16}$$

where $0 < c_0 < 1$ and γ satisfies (12). Then

$$f'(x, d) \geq 0 \tag{17}$$

holds for any accumulation point d of the sequence $\{d_k = s_k/||s_k||, k \in \mathbb{N}\}$.

Proof. Let $t_k = ||s_k||$ and $d_k = s_k/||s_k||$. Let d be any accumulation point of $\{d_k, k \in \mathbb{N}\}$. From (16) and (12) we obtain

$$\frac{f(x + t_k d_k) - f(x)}{t_k} > c_0 f'(x; d_k)$$

which implies (17), since f is Lipschitz near x and $0 < c_0 < 1$. □

The standard definition of a stationary point x_* of a real-valued function f in unconstrained nonsmooth optimization is that $0 \in \partial f(x_*)$. In our case, the function f is regular, therefore this characterization is equivalent to

$$f'(x_*; s) \geq 0 \tag{18}$$

for all s in $\mathbb{R}^n$ (see (9)). The following proposition relates the definition of stationarity to the set of minimizers of the local model.

Proposition 2.1. *Let $f = ||F||$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Then $x_* \in \mathbb{R}^n$ is a stationary point of f if and only if for all $s \in \mathbb{R}^n$*

$$||F(x_*)|| \leq ||F(x_*) + F'(x_*)s|| \tag{19}$$

or equivalently $m_{x_*}(0) \leq m_{x_*}(s)$ for all $s \in \mathbb{R}^n$ where m_x is given in (10).

Proof. Suppose that x_* is a stationary point of f, i.e. $f'(x_*; s) \geq 0$ for all $s \in \mathbb{R}^n$. By Lemma 2.2, we have

$$f'(x_*; s) \leq m_{x_*}(s) - m_{x_*}(0)$$

for all $s \in \mathbb{R}^n$. This, together with (18), implies (19). Now assume that (19) holds, and let s be any point of $\mathbb{R}^n$. Then, we have

$$\frac{m_{x_*}(ts) - m_{x_*}(0)}{t} \geq 0, \quad \forall\, t > 0.$$

This, together with Lemma 2.1 implies that $f'(x_*; s) \geq 0$. □

From Proposition 2.1 it is obvious that any solution of the nonlinear system (1) is a stationary point of $f = ||F||$. In the following theorem, we establish a sufficient condition for a stationary point x_* to be a solution of to Problem (1).

Theorem 2.1. *Let x_* be a stationary point of $f = ||F||$. Then either $F(x_*) = 0$, or the linear system*

$$F(x_*) + F'(x_*)s = 0 \tag{20}$$

does not have a solution.

Proof. Assume that $F(x_*) \neq 0$ and consider a solution s_* of the linear system , i.e

$$F(x_*) + F'(x_*)s_* = 0.$$

From Lemma 2.2, we conclude that

$$f'(x_*; s_*) = -||F(x_*)||.$$

This contradicts the hypothesis that x_* is a stationary point of f.□

3. The General Trust-Region Algorithm

In this section we define our general trust-region algorithm for approximating a solution of the nondifferentiable optimization problem

$$\text{minimize}_{x\in \mathbb{R}^n}\ f(x) = ||F(x)||$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

Let c_i, $i = 0, \ldots, 5$ be positive scalars such that

$$0 < c_0 < 1 \qquad 0 < c_1 < c_2 < 1 \leq c_3 \qquad 0 < c_4 < c_5 < 1,$$

Also let δ_{min} be any arbitrary small positive scalar, let x_0 be any point in $\mathbb{R}^n$, let δ_0 be any positive scalar, and let $|| \ ||_a$ and $|| \ ||_b$ be any two norms on $\mathbb{R}^n$. Consider a real valued upper semi-continuous function γ defined on

$$\mathcal{D} = \{(x, s) \in \mathbb{R}^n \times \mathbb{R}^n \ \mid \ m_x(s) - f(x) < 0\} \tag{21a}$$

and satisfying

$$\max\{\, f'(x; s)\,,\ \frac{c_1}{c_0}\,[\, m_x(s) - f(x)\,]\,\} \quad \leq \quad \gamma(x, s) \quad < 0. \tag{21b}$$

Suppose that x_k and δ_k are the iterate and the trust-region radius determined by the algorithm at the k^{th} iteration. The algorithm determines x_{k+1} and δ_{k+1} in the following manner:

STEP 1. Set $\mu_k = \delta_k$.

STEP 2. Obtain s_k as a solution of the model trust-region subproblem (6)

STEP 3. If $f(x_k + s_k) \leq f(x_k) + c_0\gamma(x_k, s_k)$ set $x_{k+1} = x_k + s_k$, and go to STEP 4,
Else choose μ_k such that $c_4||s_k||_b \leq \mu_k \leq c_5||s_k||_b$ and go to STEP 2;

STEP 4. If $f(x_k + s_k) < f(x_k) + c_2[m_k(s_k) - f(x_k)]$
choose δ_{k+1} so that $||s_k||_b \leq \delta_{k+1} \leq \max(\mu_k, c_3||s_k||_b)$,
Else if $f(x_k + s_k) > f(x_k) + c_2[m_k(s_k) - f(x_k)]$
choose δ_{k+1} such that $c_4||s_k||_b \leq \delta_{k+1} \leq c_5||s_k||_b$;
Else choose δ_{k+1} so that $c_4||s_k||_b \leq \delta_{k+1} \leq \max(\mu_k, c_3||s_k||_b)$;

STEP 5. Set $\delta_{k+1} = \max(\delta_{k+1}, \delta_{min})$.

Definition 3.1. The scalar μ_k for which the test in STEP 3 of the algorithm is satisfied will be said to determine an acceptable step with respect to (x_k, δ_k). (observe that it is not an arbitrary μ_k in $(0, \delta_k]$)

Remark. If x_k is not a stationary point of $f = ||F||$ and $\delta_k > 0$, we obtain from Lemma 6.1 that $(x_k, s_k) \in \mathcal{D}$ defined in (21a) with $m_{x_k} = m_k$. Therefore inequality (21b) is consistent and STEP 3 is well defined.

Possible choices for the function γ used in STEP 3 of the algorithm are

$$\gamma(x, s) = \frac{c_1}{c_0}\,[\, m_x(s) - f(x)\,] \tag{22a}$$

for $c_1 \leq c_0$, or

$$\gamma(x, s) = f'(x; s). \tag{22b}$$

for $c_0 << c_1$ and sufficiently small s. In the choice (22b), γ is upper semi-continuous (see [2, pp.25-26]. In the choice (22a) γ is obviously continuous. Because of (25b) and Lemmas 2.3 and 2.4, our theory does not allow the Duff, Nocedal and Reid choice [7], i.e.

$$\gamma(x, s) = -||F'(x)s||_1 \tag{22c}$$

(see (4)) unless, by Lemma 2.2, we have the extreme case

$$f'(x_k; s_k) = -\|F'(x_k)s_k\|_1 \qquad \forall k \in \mathbb{N}.$$

Near a solution we expect s_k to be the Newton step, and in this case the choices (22a), (22b), and (22c) are equivalent (see Lemma 2.2). It follows that the asymptotic properties of the respective algorithms would be the same.

4. A Fundamental Property of Trust-Region Algorithms

In this section we will demonstrate that trust-region algorithms enjoy the satisfying property that as the radius of the trust region approaches zero the solutions of the model trust-region problem approach directions of steepest descent of f. For the case where this norm is $\| \|_2^2$, this result is well-known and is often used as a theoretical tool. This result will play an important role in the convergence analysis developed in a later section.

Theorem 4.1. *Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and let $x \in \mathbb{R}^n$ be such that the one-sided directional derivative $\omega'(x; s)$ exists for all $s \in \mathbb{R}^n$. Also let $\{\delta_k, k \in \mathbb{N}\}$ be a sequence of real numbers decreasing to 0. Consider a sequence $\{s_k, k \in \mathbb{N}\}$, where s_k is a solution of the problem*

$$\begin{array}{ll} \text{minimize} & \omega(x+s) \\ \text{subject to} & \|s\| \le \delta_k. \end{array}$$

If $s_k \neq 0$ for all $k \in \mathbb{N}$, then any accumulation point d_* of $\{d_k = s_k/\|s_k\|,\ k \in \mathbb{N}\}$ is a steepest descent direction for ω at x with respect to the norm $\| \|$.

Proof. Let s be any vector of norm one, and let d_* be any accumulation point of $\{d_k, k \in \mathbb{N}\}$. By choosing a subsequence, if needed, we can assume without loss of generality that $\{d_k, k \in \mathbb{N}\}$ converges to d_*. We have

$$\frac{1}{\|s_k\|}\left[\omega(x+s_k) - \omega(x)\right] \le \frac{1}{\|s_k\|}\left[\omega(x+\|s_k\|s) - \omega(x)\right]. \tag{23}$$

By using the quantities $d_k = s_k/\|s_k\|$ and $t_k = \|s_k\|$ in (23) we obtain

$$\frac{\omega(x+t_k d_*) - \omega(x)}{t_k} + \frac{\omega(x+t_k d_k) - \omega(x+t_k d_*)}{t_k} \le \frac{\omega(x+t_k s) - \omega(x)}{t_k}$$

which implies, because ω is locally Lipschitz, that

$$\omega'(x; d_*) \le \omega'(x; s)\ .$$

This inequality means that d_* is a steepest descent direction for ω at x with respect to the norm $\| \|$. □

Remark. In our application the function ω can represent either f or m_x (see Lemma 2.1).

5. Zangwill's Global Convergence Theory

In numerical optimization, most algorithms are iterative. Namely, given a point $z_0 \in \mathbb{R}^n$, a sequence of points $\{z_k, k \in \mathbb{N}\}$ is generated recursively according to the defining relation $z_{k+1} \in A(z_k)$ where A is a point-to-set map and any point in the set $A(z_k)$ is an acceptable successor point of z_k.

Notice that the model does not specify the type of problem we are solving. We refer to the set of solutions as the solution set P. For a specific application, A and P must be defined.

Our motivation for using point-to-set maps to model our algorithm stems from the following theorem due to Zangwill [23]. We present the theorem as stated in Huard [14]. We first need the following definitions.

Definition 5.1. The point-to-set map A is said to be upper-continuous at $x \in \mathbb{R}^n$ if $\{x_k, k \in \mathbb{N}\}$ converges to x and $\{y_k \in A(x_k), k \in \mathbb{N}\}$ converges to y implies that $y \in A(x)$.

Definition 5.2. The point-to-set map A is said to be lower-continuous at $x \in \mathbb{R}^n$ if for any sequence $\{x_k, k \in \mathbb{N}\}$ converging to x and for any $y \in A(x)$, there exist a sequence $\{y_k, k \in \mathbb{N}\}$ converging to y and an integer $\bar{k}$ such that $y_k \in A(x_k)$ for $k \geq \bar{k}$.

Definition 5.3. The point-to-set map A is said to be continuous at $x \in \mathbb{R}^n$ if it is both upper-continuous and lower-continuous at x.

Theorem 5.1. *Consider a compact set $E \subset \mathbb{R}^n$, a solution set $P \subset E$, a point-to-set-map $A : E \to 2^E$, and a continuous function $h : E \to \mathbb{R}$. Assume that for any $z \in E$ and $z \notin P$ we have*

(i) $A(z) \neq \emptyset$

(ii) $h(z') < h(z)$ *for any* $z' \in A(z)$.

(iii) A *is upper-continuous at* z.

Assume further that a sequence $\{z_k, k \in \mathbb{N}\}$ has been obtained by the following recursion relation: let z_0 be any point in E, if $z_k \notin P$ then $z_{k+1} \in A(z_k)$, otherwise $z_{k+1} = z_k$. Then any accumulation point z_* of $\{z_k, k \in \mathbb{N}\}$ is contained in P.

Proof. This theorem is Convergence Theorem A in [23] or is a consequence of Corollary 3 and Remark 6 in [14]. □

More details regarding point-to-set maps can be found in Berge [1], Denel [4], Hogan [11], Huard [12], Huard [13] and Huard [14], and Meyer [16].

6. Global Convergence of the General Trust-Region Algorithm

We will establish global convergence of the General Trust-Region Algorithm described in Section 3 by modeling it by a point-to-set map A which satisfies the hypotheses of Zangwill's theorem (Theorem 5.1).

If we considered only $c_0 = c_1$ and γ given by (22a) (Powell's choice in [19], then we could obtain global convergence of our arbitrary norm trust-region algorithm from the global convergence theory developed by Powell [19]. However, even for this special case the results established in Sections 4, 7, and 8 would be new and important contributions.

In order to apply Theorem 5.1, we need the following lemma whose proof will be given later.

Lemma 6.1. *Let x_0 be any point $\in \mathbb{R}^n$. If the subset of $\mathbb{R}^n$ $X_0 = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is bounded, then there exists a positive scalar δ_{max} such that the trust-region radius δ_k satisfies*

$$0 < \delta_k \leq \delta_{max} \quad \forall k \in \mathbb{N}. \tag{24}$$

We will define the compact set of Theorem 5.1 as

$$E = X_0 \times [\delta_{min}, \quad \delta_{max}], \tag{25}$$

the solution set P will consist of the points $(x, \delta) \in E$ such that x is a stationary point of f, and the merit function h will be

$$h(x, \delta) = f(x). \tag{26}$$

Finally, the point-to-set map A will be defined as follows:

Definition of the point-to-set A. For $z \in P$, we set $A(z) = \{z\}$, and for $z = (x, \delta) \notin P$ we say that $z' = (x', \delta') \in A(z)$ if the scalar μ that determines, with respect to (x, δ), an acceptable step s and $x' = x + s$ satisfy the following five conditions

$$
\begin{array}{llll}
(a) & & & s = s(\mu) \in \text{argmin}\{m_x(s) \mid \|s\|_b \le \mu\}, \\
(b) & & & f(x') \le f(x) + c_0\gamma(x, s); \\
(c) & \text{if} & & f(x') < f(x) + c_2[m_x(s) - f(x)], \\
& & \text{then} & \|s\|_b \le \delta' \le \max(\delta, c_3\|s\|_b); \\
& & \text{else if} & f(x') > f(x) + c_2[m_x(s) - f(x)], \\
& & \quad\text{then} & c_4\|s\|_b \le \delta' \le c_5\|s\|_b; \\
& & \quad\text{else} & c_4\|s\|_b \le \delta' \le \max(\delta, c_3\|s\|_b); \\
\text{and} & & & \\
(d) & & & \delta' = \max(\delta', \delta_{min}),
\end{array} \tag{27}
$$

We now state our global convergence theorem.

Theorem 6.1. *Consider a continuously differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $\|\ \|_a$ and $\|\ \|_b$ be arbitrary norms on $\mathbb{R}^n$, let x_0 be an arbitrary point in $\mathbb{R}^n$, let $f(x) = \|F(x)\|_a$, and let $\mathcal{D}$ be defined in (21a). Assume that the level set $X_0 = \{x \in \mathbb{R}^n \mid f(x) \le f(x_0)\}$ is bounded and that the function $\gamma : \mathcal{D} \to \mathbb{R}$ used in Step 3 of the General Trust Region Algorithm is upper semi-continuous and satisfies (21b). Then any accumulation point of the sequence $\{x_k, k \in \mathbb{N}\}$ generated by the General Trust-Region Algorithm presented in Section 3 using x_0 as initial point is a stationary point of f.*

The proof of the theorem will require the use of Lemma 6.1 and the following lemmas whose proofs will be given shortly. We will use $\|\ \|$ for either norm $\|\ \|_a$ or $\|\ \|_b$ since their use will be clear from the context.

Lemma 6.2. *Consider (x, δ) where $\delta > 0$ and x is not a stationary point of f. Then the General Trust-region Algorithm cannot loop infinitely often between STEP 3 and STEP 2.*

Lemma 6.3. *The point-to-set map A is upper-continuous at any $(x, \delta) \in E - P$.*

Proof of Theorem 6.1. It is sufficient to prove that the conditions of Theorem 5.1 hold. Because of Lemma 6.1 the subset E of $\mathbb{R}^n \times \mathbb{R}$ defined by (25) is compact. We also have that the function h defined on E by (26) is continuous. Let us show that conditions (i), (ii), and (iii) of Theorem 5.1 hold. First, let $z = (x, \delta) \notin P$. Then, by (25) it is obvious that $\delta > 0$, and by Lemma 6.2, there exists $\mu \in (0, \delta]$ and $s \in \text{argmin}\{m_x(s) \mid \|s\| \le \mu\}$ such that $x' = x + s$ satisfies

$$f(x') \le f(x) + c_0\gamma(x, s).$$

The existence of δ' such that $z' = (x', \delta') \in A(z)$ is obvious. Therefore, property (i) of Zangwill's Theorem 5.1 holds. Secondly, if $(x', \delta') \in A(x, \delta)$ it is straightforward from (26), (27c), and the fact that γ satisfies (21b), that $h(x', \delta') < h(x, \delta)$. This is condition (ii) of Theorem 5.1. The third condition (iii) follows from Lemma 6.3. Therefore, our theorem is a consequence of Zangwill's Theorem 5.1 and Lemmas 6.1, 6.2, and 6.3. □

We now return to the proofs of Lemmas 6.1, 6.2, and 6.3

Proof of Lemma 6.1. The sequence $\{s_k = x_{k+1} - x_k, k \in \mathbb{N}\}$ is bounded, say by M. Since $\delta_{k+1} \le max(\mu_k, c_2\|s_k\|), \mu_k \le \delta_k$, and $\|s_k\| \le M$, we obtain

$$\delta_{k+1} \le max(\delta_k, c_2 M). \tag{28}$$

Assume that there exists a subsequence $\{\delta_k, k \in N' \subset \mathbb{N}\}$ diverging to $+\infty$. Let $k_* \in N'$ be the smallest integer such that $\delta_{k_*} > c_2 M$. Then we obtain $\delta_j \le \delta_{k_*} \ \forall\, j \ge k_*, j \in \mathbb{N}$. This contradicts the divergence hypothesis of $\{\delta_k, k \in N' \subset \mathbb{N}\}$. Consequently there exists a positive scalar δ_{max} such that (24) holds.□

Proof of Lemma 6.2. We prove the contrapositive. Suppose that the algorithm loops indefinitely. Let $\{x_j, j \in \mathbb{N}\}$ be the sequence generated by letting $x_j = x + s_j$ where s_j is a solution of the following model trust-region problem

$$\begin{array}{ll} \text{minimize} & m_x(s) = \|F(x) + F'(x)s\| \\ \text{subject to} & \|s\| \le \mu_j . \end{array}$$

Observe that $\|s_{j+1}\| \le \mu_{j+1} \le c_4 \|s_j\|$ for all $j \in \mathbb{N}$ and that $0 < c_4 < 1$, so the sequence $\{\|s_j\|, j \in \mathbb{N}\}$ is decreasing to $s = 0$. Under our hypothesis the test in Step 3 fails for all $j \in \mathbb{N}$, thus, since $\gamma(x, s_j) \ge f'(x; s_j)$, we have

$$f(x + s_j) > f(x) + c_0 f'(x; s_j). \tag{29}$$

Therefore, from Lemma 2.4 we obtain

$$f'(x; d_*) \ge 0$$

where d_* is any accumulation point of $\{d_j, j \in \mathbb{N}\}$. But from Theorem 4.1 and Lemma 2.1 we obtain that d_* is a steepest descent direction for f at x. Consequently, for all $d \in \mathbb{R}^n$ with norm one, we have

$$f'(x; d) \ge 0,$$

which implies that x is a stationary point of f. □

To prove Lemma 6.3 we need the following lemma.

Lemma 6.4. *Suppose that the sequence $\{(x_k, \delta_k) \notin P, k \in \mathbb{N}\}$ converges to some $(x, \delta) \notin P$. If μ_k is a scalar that determines an acceptable step with respect to (x_k, δ_k), then any accumulation point of $\{\mu_k, k \in \mathbb{N}\}$, say μ, satisfies the inequality*

$$\mu \ge 0 . \tag{30}$$

Proof of Lemma 6.3. Let $\{(x_k, \delta_k), k \in \mathbb{N}\}$ be a sequence that converges to (x_*, δ_*) and let $\{(x'_k, \delta'_k) \in A(x_k, \delta_k), k \in \mathbb{N}\}$ be a sequence that converges to some (x'_*, δ'_*). We want to establish that $(x'_*, \delta'_*) \in A(x_*, \delta_*)$. By the definition of A, $(x'_k, \delta'_k) \in A(x_k, \delta_k)$ implies that there exists a positive scalar μ_k determining an acceptable step s_k such for $x'_k = s_k + x_k$, the following conditions hold.

$$0 < \mu_k \le \delta_k \tag{31a}$$
$$s_k \in \operatorname{argmin}\{m_k(s) \mid \|s\| \le \mu_k\}, \tag{31b}$$
$$f(x'_k) \le f(x_k) + c_0 \gamma(x_k, s_k). \tag{31c}$$

The sequence $\{s_k, k \in \mathbb{N}\}$ converges to s_* such that $x'_* = x_* + s_*$. Let μ_* be any accumulation point of the sequence $\{\mu_k, k \in \mathbb{N}\}$. Since $(x_*, \delta_*) \notin S$, i.e. x_* is not a stationary point of f, by Lemma 6.4 and (31a) we obtain that

$$0 < \mu_* \le \delta_* . \tag{32}$$

Now we establish that

$$s_* \in \operatorname{argmin}\{m_x(s) \mid \|s\| \le \mu_*\}. \tag{33}$$

We can rewrite (31b) as

$$s_k \in \operatorname{argmin}\{\phi(s, x_k, \mu_k) \mid s \in T(x_k, \mu_k)\}.$$

where $\phi(s,x,\mu) = m_x(s)$ and $T(x,\mu) = \{s \in \mathbb{R}^n \mid \|s\| \le \mu\}$, The point-to-set map T is the composition of the projection $\Pi(y,r) = r$ which is continuous and the point-to-set map $B(r) = \{s \in \mathbb{R}^n \mid r \ge \|s\|\}$ which is continuous by Theorem A.10 of [13]. Consequently, by Theorem A.6 of [13], the point-to-set map $T = B \circ \Pi$ is continuous. Therefore, since the function ϕ is also continuous, we obtain from Theorem A.15 of [13] that the point-to-set map

$$\psi : (x,\mu) \longrightarrow \text{argmin}\,\{\phi(s,x,\mu) \mid s \in T(x,\mu)\}$$

is upper-continuous. Because $\{(x_k,\mu_k), k \in N' \subset \mathbb{N}\}$ converges to (x_*,μ_*), $\{s_k \in \psi(x_k,\mu_k), k \in N'\}$ converges to s_*, the upper-continuity of ψ implies (33). The upper semi-continuity of the real-valued function γ implies that the function g defined by

$$g(x,s) = f(x+s) - f(x) - c_0\gamma(x,s)$$

is lower semi-continuous. This implies, because of (31c), that

$$f(x_* + s_*) \le f(x_*) + c_0\gamma(x_*,s_*). \tag{34}$$

Properties (32), (33) and (34) establish the first three properties needed to conclude that (x'_*, δ'_*) belongs to $A(x_*,\delta_*)$, i.e. (27a), (27b), and (27c). Let us establish the fourth property (27d). Suppose that

$$f(x_* + s_*) - f(x_*) - c_2\,\{m_{x_*}(s_*) - m_{x_*}(0)\} < 0. \tag{35a}$$

The sequence $\{(x_k,\delta_k), k \in \mathbb{N}\}$ converges to (x_*,δ_*), so for all large $k \in \mathbb{N}$ we have

$$f(x_k + s_k) - f(x_k) - c_2\,\{m_k(s_k) - m_k(0)\} < 0,$$

which gives

$$\|s_k\| \le \delta'_k \le c_3 \max(\delta_k, c_3\|s_k\|),$$

and consequently, we obtain

$$\|s_*\| \le \delta'_* \le c_3 \max(\delta_*, c_3\|s_*\|). \tag{35b}$$

Now suppose that

$$f(x_* + s_*) - f(x_*) - c_2\,\{m_{x_*}(s_*) - m_{x_*}(0)\} > 0. \tag{36a}$$

We establish in the same way as (35b) that

$$c_4\|s_*\| \le \delta'_* \le c_5\|s_*\|. \tag{36b}$$

Finally, if neither (35a) nor (36a) holds, then necessarily we have

$$f(x_* + s_*) - f(x_*) - c_2\,\{m_{x_*}(s_*) - m_{x_*}(0)\} = 0, \tag{37a}$$

and it is obvious that

$$c_4\|s_*\| \le \delta'_* \max(\delta_*, c_3\|s_*\|) \tag{37b}$$

holds. Properties (35), (36), and (37) establish (27d). The fifth property

$$\delta'_* = \max(\delta'_*, \delta_{min})$$

is obvious. And we conclude that $(x'_*, \delta'_*) \in A(x_*, \delta_*)$, and the map A is upper-continuous. $\square$

Now we prove Lemma 6.4.

Proof of Lemma 6.4. Let μ be any accumulation point of $\{\mu_k, k \in \mathbb{N}\}$. Without loss of generality, we can assume that $\{\mu_k, k \in \mathbb{N}\}$ converges to μ. It follows that $\mu_k \leq \delta_k$. We consider two cases:

Case i). We suppose that there exists a subsequence of $\{\mu_k, k \in N' \subset \mathbb{N}\}$ such that $\mu_k = \delta_k$ in which case we have $\mu = \delta$. Consequently we obtain (30) because $\delta \geq \delta_{min}$.

Case ii). Suppose that $\mu_k < \delta_k$ for all sufficiently large $k \in \mathbb{N}$. Therefore δ_k never gives an acceptable step. Let $\bar{s}_k$ be the last non-acceptable step obtained by decreasing δ_k. Since $\delta_k > 0$ and x_k is not a stationary point of f we have, by Lemma 6.2, that $\bar{s}_k \neq 0$ and $\mu_k > 0$. Also we have for large $k \in \mathbb{N}$

$$\mu_k = c_4 ||\bar{s}_k||. \tag{38}$$

Assume that $\mu = 0$. From inequality (38) we obtain that $\{\bar{s}_k \mid k \in \mathbb{N}\}$ converges to zero.

Let $s^*_k \in \operatorname{argmin}\{m_x(s) \mid ||s|| \leq \mu_k\}$, and let d^* be any accumulation point of $\{d^*_k = s^*_k/||s^*_k||, k \in \mathbb{N}\}$. Since $\{\mu_k > 0, k \in \mathbb{N}\}$ converges to zero, we obtain from Theorem 4.1 and Lemma 2.1 that d^* is a steepest descent direction of f at x. Consider a subsequence $\{d^*_k, k \in N \subset \mathbb{N}\}$ that converges to d^*, and let α_k be a positive scalar such that $||\alpha_k s^*_k|| = ||\bar{s}_k||$. Then we have for all sufficiently large $k \in N$

$$\frac{m_k(\bar{s}_k) - f(x_k)}{||\bar{s}_k||} \leq \frac{m_k(\alpha_k s^*_k) - f(x_k)}{||\alpha_k s^*_k||}. \tag{39}$$

Let us set $t_k = ||\bar{s}_k|| = ||\alpha_k s^*_k||$, $y^*_k = \alpha_k s^*_k$ and

$$\bar{d}_k = \frac{\bar{s}_k}{||\bar{s}_k||}, \quad d^*_k = \frac{y^*_k}{||y^*_k||} = \frac{s^*_k}{||s^*_k||}.$$

Therefore (39) becomes

$$\frac{m_k(t_k \bar{d}_k) - f(x_k)}{t_k} \leq \frac{m_k(t_k d^*_k) - f(x_k)}{t_k};$$

which implies, since $||\bar{d}_k|| = ||d^*_k|| = 1$,

$$\frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} \leq \frac{f(x_k + t_k d^*_k) - f(x_k)}{t_k} + \frac{o(t_k)}{t_k}.$$

Therefore we obtain

$$\limsup_{k \in N \to +\infty} \frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} \leq \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td^*) - f(y)}{t},$$

or, since f is regular,

$$\limsup_{k \in N \to +\infty} \frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} \leq f'(x; d^*). \tag{40a}$$

Moreover, since $\bar{s}_k$ is not acceptable, we have

$$f(x_k + \bar{s}_k) > f(x_k) + c_0 \gamma(x_k, \bar{s}_k)$$

which implies, together with (21b), that for sufficiently large $k \in N$

$$\frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} > c_1 \frac{m_k(t_k \bar{d}_k) - f(x_k)}{t_k}.$$

But since F is continuously differentiable (see (23)), this implies

$$\frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} > c_1 \frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} + \frac{o(t_k)}{t_k},$$

and because $0 < c_1 < 1$

$$\limsup_{k \in N \to +\infty} \frac{f(x_k + t_k \bar{d}_k) - f(x_k)}{t_k} \geq 0. \tag{40b}$$

From (40a) and (40b), we obtain

$$f'(x; d^*) \geq 0, \tag{41a}$$

and since d_* is a steepest descent direction of f at x, this implies that

$$f'(x; s) \geq 0. \tag{41b}$$

for all $s \in \mathbb{R}^n$, which contradicts the hypothesis that x is not a stationary point of f. Therefore any accumulation point of the sequence $\{\mu_k \mid k \in \mathbb{N}\}$, say μ, satisfies (30). □

7. Convergence to a Solution of $F(x) = 0$

In this section we establish a mild condition which guarantees that any accumulation point of the sequence $\{x_k, k \in \mathbb{N}\}$ generated by the General Trust-Region Algorithm is actually a solution of the nonlinear system $F(x) = 0$. We then demonstrate that if an accumulation point x_* is such that $F'(x_*)$ is nonsingular, then the iteration sequence actually converges to x_*.

Theorem 7.1. *Let S be the set of accumulation points of the sequence $\{x_k, k \in \mathbb{N}\}$ generated by the General Trust-Region Algorithm. Under the assumptions of Theorem 6.1, one of the following holds:*

(i) *all accumulation points are solutions of the nonlinear system, i.e*

$$F(x_*) = 0 \quad \forall x_* \in S \tag{42a}$$

(ii) *for all $x_* \in S$, the linear system*

$$F(x_*) + F'(x_*)s = 0 \tag{42b}$$

does not have a solution.

To prove this theorem we will need the following lemma.

Lemma 7.1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuous. Also let $\{z_k, k \in \mathbb{N}\}$ be a bounded sequence such that the sequence $\{h(z_k), k \in \mathbb{N}\}$ is decreasing. Then the function h is constant on the set of accumulation points of $\{z_k, k \in \mathbb{N}\}$.*

Proof . Let z_* and z'_* be two accumulation points of the sequence $\{z_k, k \in \mathbb{N}\}$. Then, there exist two subsequences $\{z_k, k \in N\}$ and $\{z_k, k \in N'\}$ that converge respectively to z_* and z'_*. We have that for every j in N, there exists k_j in N' such that

$$h(z_{k_j}) \leq h(z_j), \quad k_j \geq j. \tag{33}$$

From the continuity of h and (33) we obtain

$$h(z'_*) \leq h(z_*). \tag{44}$$

Since the roles of z'_* and z_* in establishing (44) are symmetric we conclude that

$$h(z_*) = h(z'_*),$$

which establishes the lemma. □

Proof of Theorem 7.1. The function $f = ||F||_a$ is continuous, the sequence $\{x_k, k \in \mathbb{N}\}$ is bounded and the sequence $\{f(x_k), k \in \mathbb{N}\}$ is decreasing. Therefore, by Lemma 7.1, $f = ||F||_a$ is constant on the set S of accumulation points of $\{x_k, k \in \mathbb{N}\}$. By Theorem 6.1, any $x_* \in S$ is a stationary point of f. Therefore, by Theorem 2.1, either any $x_* \in S$ solves the nonlinear system (1) or no linear system (42b) has a solution. □

Corollary 7.1. *Under the assumptions of Theorem 6.1, if the sequence $\{x_k, k \in \mathbb{N}\}$ generated by the General Trust-Region Algorithm has an accumulation point, say x_*, such that $F'(x_*)$ is nonsingular, then $F(x_*) = 0$ and $\{x_k, k \in \mathbb{N}\}$ converges to x_*.*

Proof . By Theorem 6.1, the accumulation point x_* is a stationary point of $f = ||F||$. Since $F'(x_*)$ is nonsingular, the linear system (42b) has a solution. Therefore, by Theorem 7.1, we obtain

$$F(x_*) = 0.$$

Now the convergence of the sequence $\{x_k, k \in\}$ to x_* follows from Theorem 3.3 of Eisenstat and Walker [8].□

Remark. Homer Walker pointed out to the authors that the new Eisenstat and Walker theory [8] could be used to actually demonstrate convergence of the sequence $\{x_k, k \in \mathbb{N}\}$ as stated in Corollary 7.1.

8. Q-quadratic Convergence of the General Trust-Region Algorithm

Corollary 7.1 shows that the algorithm, under mild assumptions, generates a sequence $\{x_k, k \in \mathbb{N}\}$ which converges to a nonsingular solution. Under the same assumptions, we prove that, for large k, the General Trust-Region Algorithm reduces to Newton's method and consequently the convergence of $\{x_k, k \in \mathbb{N}\}$ to x_* is q-quadratic.

Theorem 8.1. *Assume that the hypotheses of Theorem 6.1 hold. Also assume that the sequence $\{x_k, k \in \mathbb{N}\}$ generated by the General Trust-Region Algorithm has an accumulation point, say x_*, such that $F'(x_*)$ is nonsingular and F' is Lipschitz near x_*. Then for sufficiently large k, x_k is the Newton iterate for the nonlinear equation $F(x) = 0$, and consequently the convergence of $\{x_k, k \in \mathbb{N}\}$ to x_* is q-quadratic.*

Proof. By Corollary 7.1, the iteration sequence converges to x_*. To prove that the algorithm, for large k, is equivalent to Newton's method, first we establish that the test

$$f(x_{k+1}) \leq f(x_k) + c_2\,[m_k(s_k) - m_k(0)] \tag{45}$$

is satisfied for large k. Since F is continuously differentiable we have

$$f(x_k + s_k) = ||F(x_k) + F'(x_k)s_k + o(||s_k||)||,$$

and therefore

$$f(x_k) - f(x_k + s_k) \geq f(x_k) - [m_k(s_k) + ||o(||s_k||)||]\,.$$

Because $f(x_k) - m_k(s_k) > 0$ this implies that

$$\frac{f(x_k) - f(x_k + s_k)}{f(x_k) - m_k(s_k)} \geq 1 - ||\frac{o(||s_k||)}{||s_k||}|| \times \frac{||s_k||}{f(x_k) - m_k(s_k)}. \tag{46}$$

Let us show that the ratio

$$\frac{f(x_k) - m_k(s_k)}{||s_k||}$$

is bounded away from zero. Since $\{x_k, k \in \mathbb{N}\}$ converges to x_*, $F'(x_*)$ is nonsingular and F is continuously differentiable, there exists $k_* \in \mathbb{N}$ and $0 < \lambda_*$ such that $F'(x_k)$ is nonsingular for all $k \geq k_*$ and

$$\|F'(x_k)d\| \geq \lambda_* \|d\| \quad \forall d \in \mathbb{R}^n \quad \text{and} \quad \forall k \geq k_*. \tag{47}$$

Let us set

$$\alpha_k = \frac{\|s_k\|}{\|s_k^N\|} \qquad\qquad y_k = \alpha_k s_k^N \tag{48}$$

where s_k^N is the Newton step, i.e.

$$F'(x_k)s_k^N + F(x_k) = 0. \tag{49}$$

The definition of s_k and the nonsingularity of $F'(x_k)$ imply that either $s_k = s_k^N$ or $\|s_k\| < \|s_k^N$. therefore the inequality $\|s_k\| \leq \|s_k^N\|$ holds for all sufficiently large k, which shows that $\alpha_k \in (0, 1]$. We have, since $\|y_k\| = \|s_k\|$,

$$f(x_k) - m_k(y_k) \leq f(x_k) - m_k(s_k). \tag{50}$$

From (48), (49), (50) and $\|s_k\| = \|y_k\|$ we obtain

$$\frac{\|F'(x_k)s_k^N\|}{\|s_k^N\|} \leq \frac{f(x_k) - m_k(s_k)}{\|s_k\|}.$$

Using inequality (47) we get

$$0 < \lambda_* \leq \frac{f(x_k) - m_k(s_k)}{\|s_k\|} \tag{51}$$

for all $k \geq k_*$. Property (51) and inequality (46) imply that for $k \geq k_*$ we have

$$\frac{f(x_k) - f(x_k + s_k)}{f(x_k) - m_k(s)} \geq 1 - \frac{1}{\lambda_*}\|\frac{o(\|s_k\|)}{\|s_k\|}\|.$$

On the other hand, there exists an integer, say k_*, such that

$$1 - \frac{1}{\lambda_*}\|\frac{o(\|s_k\|)}{\|s_k\|}\| \geq c_2$$

for all $k > k_*$. Consequently, inequality (45) holds for $k \geq k_*$. Furthermore the trust-region radius is updated according to the rule

$$\|s_k\| \leq \delta_{k+1} \leq \max(\mu_k, c_3\|s_k\|). \tag{52}$$

Also, since $0 < c_1 < c_2$, we obtain from (21b) that

$$c_0\gamma(x_k, s_k) \geq c_2\,[m_k(s_k) - m_k(0)]$$

which, together with (45), implies that δ_k determines an acceptable step with respect to (x_k, δ_k), i.e. $\mu_k = \delta_k$ for all $k \geq k_*$. Therefore, for $k \geq k_*$, the trust-region radius δ_k is updated according to the rule

$$\|s_k\| \leq \delta_{k+1} \leq \max(\delta_k, c_3\|s_k\|). \tag{53}$$

Suppose that there exists an integer $k_1 \geq k_*$ such that $s_k \neq s_k^N$ for all $k \geq k_1$. This implies that $\|s_k\| = \delta_k$ for all $k \geq k_1$ and by (52) $\|s_k\| \leq \|s_{k+1}\|$ for all $k \geq k_1$. This contradicts the hypothesis that $\{s_k = x_{k+1} - x_k, k \in \mathbb{N}\}$ converges to zero. Therefore for all $j \geq k_*$ there exists an integer $k_j \geq j$ such that

$$s_{k_j} = s_{k_j}^N. \tag{54}$$

Let $\mathcal{N}(x_*)$ be a sufficiently small neighborhood of x_* where the local q-quadratic convergence occurs, (see [6]). Let j be the smallest integer such that $x_{k_j} \in \mathcal{N}(x_*)$. Newton steps in $\mathcal{N}(x_*)$ verify

$$||s^N_{k_j+1}|| \leq ||s^N_{k_j}||,$$

which, by (53) and (54), imply

$$||s^N_{k_j+1}|| \leq \delta_{k_j+1},$$

and consequently

$$s_{k_j+1} = s^N_{k_j+1},$$

and $k_j + 1 = k_{j+1}$. By induction, we establish that

$$s_k = s^N_k$$

holds for all sufficiently large k, say $k \geq k'$. Consequently the sequence $\{x_k, k \geq k'\}$ generated by the local version of the General Trust-Region Algorithm is q-quadratically convergent to x_*. □

9. Summary and Concluding Remarks

A very successful trust-region algorithm for approximating the solution of the square nonlinear system of equations $F(x) = 0$ is the well-known Levenberg-Marquardt trust-region algorithm. The model trust-region problem in the Levenberg-Marquardt algorithm has the form

$$\text{minimize} \quad ||F(x) + F'(x)s||_2^2 \tag{55a}$$
$$\text{subject to} \quad ||s||_2^2 \leq \delta. \tag{55b}$$

where $|| \ ||_2$ denotes the ℓ_2-norm on $\mathbb{R}^n$.

Recently Duff, Nocedal and Reid [7] suggested a trust-region algorithm where the Levenberg-Marquardt model trust-region problem (55) is replaced with the model trust-region problem

$$\text{minimize} \quad ||F(x) + F'(x)s||_1 \tag{56a}$$
$$\text{subject to} \quad ||s||_\infty \leq \delta. \tag{56b}$$

In (56a) $|| \ ||_1$ denotes the ℓ_1-norm on $\mathbb{R}^n$ and in (56b) $|| \ ||_\infty$ denotes the ℓ_∞ norm on $\mathbb{R}^n$. The subproblem (56) can be solved using linear programming techniques and allows one to take advantage of sparsity in $F'(x)$. Duff, Nocedal and Reid [7] gave no convergence analysis, but included convincing numerical experimentation.

Motivated by the work of Duff, Nocedal and Reid, in this paper we have presented a General Trust-Region Algorithm where the model trust-region problem has the form

$$\text{minimize} \quad ||F(x) + F'(x)s||_a \tag{75a}$$
$$\text{subject to} \quad ||s||_b \leq \delta. \tag{75b}$$

where $|| \ ||_a$ and $|| \ ||_b$ are arbitrary but fixed norms on $\mathbb{R}^n$. Levenberg-Marquardt and Duff-Nocedal-Reid are special cases of our General Trust-Region Algorithm.

Using the tools from convex analysis, nonsmooth optimization and the Zangwill convergence theorem we have established an effective global convergence theory for our General Trust-Region Algorithm. The specialization of our theory to the case when $|| \ ||_a = || \ ||_b = || \ ||_2$, i.e., Levenberg-Marquardt gives a global convergence theorem which is competitive with the standard result.

Our global convergence theory indicates that the choice Duff, Nocedal and Reid made for the descent condition (criterion for accepting the solution of the model trust-region problem) can be improved and we suggest alternative choices. Using these choices our global theory applies to the algorithm suggested by Duff, Nocedal and Reid. Moreover, using the new Eisenstat and Walker theory, we have been able to show that the iteration sequence actually converges to a solution of the nonlinear system.

It is satisfying to us that we have been able to demonstrate that our General Trust-Region Algorithm reduces to Newton's method after a finite number of steps and consequently the convergence of the algorithm is q-quadratic.

It is also satisfying that we have been able to demonstrate, for the General Trust-Region Algorithm, an analog of the well-known result that the solution of the Levenberg-Marquardt model trust-region problem approaches a steepest descent direction as the trust-region radius approaches zero.

While we have stated our algorithm for functions of the form $f = \|F\|$, a significant amount of our formulation and theory applies to more general functions, e.g., regular or locally Lipschitz f.

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