On Motivating the Mitchell-Todd Modification of Karmarkar's Algorithm for LP Problems with Free Variables

H.J. Martínez

June, 1993

TR93-25

ON MOTIVATING THE MITCHELL-TODD MODIFICATION OF KARMARKAR'S ALGORITHM FOR LP PROBLEMS WITH FREE VARIABLES[1]

H.J. MARTÍNEZ[2]

Abstract. In this note, we first observe that the Morshedi-Tapia interpretation of the Karmarkar algorithm naturally offers an extension of the Karmarkar subproblem scaling to problems with free variables. We then note that this extended scaling is precisely the scaling suggested by Mitchell and Todd for problems with free variables. Mitchell and Todd gave no motivation for or justification of this extended scaling.

Key words. Linear programming, Karmarkar's algorithm, free variables.

1. Introduction

Morshedi and Tapia [Ref.1] argued that the Karmarkar algorithm [Ref.2] for linear programming can be viewed as an example of the steepest descent method applied to the equality constrained nonlinear program that results when the technique of squared-slack substitution is applied to the Karmarkar standard form linear program. In this standard form, the only inequality constraints are nonnegativity constraints on the variables. If some of the variables are allowed to be free, then the technique of squared-slack substitution dictates that they not be replaced by squared slacks.

The Morshedi-Tapia steepest descent interpretation of the Karmarkar algorithm naturally speaks to this more general problem. In fact, it says that the

[1]Research sponsored by NSF Cooperative Agreement No. CCR-88-09615, AFOSR Grant 89-0363, DOE Grant DEFG05-86-ER25017, and ARO Grant 9DAAL03-90-G-0093.

[2]Professor, Departamento de Matemáticas, Universidad del Valle, A.A. 25360, Cali, Colombia, and Visiting Scholar, Department of Computational & Applied Mathematics, Rice University, Houston, Texas 77251-1892. Partially supported by COLCIENCIAS CO: 1106-05-307-93

free variables should be assigned a scaling of one and the remaining variables be assigned the scale used in Karmakar's algorithm. This is precisely the extended scaling suggested by Mitchell and Todd [Ref.3] with no justification (see rescaled Problem $\overline{P}$, pg 32, [Ref.3]).

While this extension of the Karmarkar scaling to problems with free variables is not difficult to conjecture and some authors consider it natural without motivation, it is satisfying that we have motivated it in at least one manner.

2. The Morshedi–Tapia Equivalence Result

Consider the Karmarkar standard form linear program

$$\begin{array}{llcl} \underset{x}{\textit{minimize}} & c^T x & & \\ \textit{subject to} & Ax & = & 0 \\ & e^T x & = & 1 \\ & x & \geq & 0 \end{array} \tag{1}$$

where $c, e = (1, 1, \ldots, 1)^T \in \mathbf{R^n}$ and $A \in \mathbf{R^{m \times n}}$ $(m \leq n)$.

Morshedi and Tapia [Ref.1] use the technique of squared slack substitution (in this case $x = y^2$) to transform the linear program (1) into the equality constrained nonlinear program

$$\begin{array}{llcl} \underset{x,y}{\textit{minimize}} & c^T y^2 & & \\ \textit{subject to} & Ay^2 & = & 0 \\ & e^T y^2 & = & 1 \\ & x & = & y^2 \end{array} \tag{2}$$

where y^2 is shorthand notation for $(y_1^2, y_2^2, \ldots, y_n^2)^T$. Similarly, $\sqrt{x}$ will denote the corresponding expresion.

Then, they considered the method of weighted 2-norm steepest descent (weighted gradient) on problem (2). Specifically, for a given iterate x, strictly feasible with respect to problem (1), they let $y = \sqrt{x}$ and $Y = diag(y)$, and considered the following (weighted steepest descent) subproblem arising from problem (2)

$$\begin{array}{lrcl} \underset{s}{\textit{minimize}} & c^TYs & & \\ \textit{subject to} & AYs & = & 0 \\ & e^TY^{-1}s & = & 0 \\ & \|Y^{-1}s\|_2 & \leq & \delta, \end{array} \tag{3}$$

where δ is appropriately chosen to keep strictly positive iterates.

The subsequent y-iterate is obtained as $y_+ = y + \hat{s}$, where $\hat{s}$ is the solution of subproblem (3). The subsequent x-iterate is obtained according to the formula

$$x_+ = Yy_+/e^TYy_+. \tag{4}$$

The first two expressions in (3) represent Taylor series linearizations of the corresponding quantities in (2). The third one is also a linearization of the corresponding quantity in (2), but it is not a Taylor series linearization. The inequality in (3) is a weighted 2-norm steepest descent (weighted gradient) constraint. Morshedi and Tapia view (4) as a Taylor series linearization of the squared substitution defining relation $x = y^2$ followed by a normalization (projection) which closes (restores feasibility to) the constraint $e^Tx = 1$. This is important because any meaningful extension of the steepest descent method to problems with equality constraints will require feasible iterates.

To see that the process described above is equivalent to the Karmarkar algorithm, we need only observe that the change of variables $s = Ys'$ in (3) and (4) leads to

$$\begin{array}{lrcl} \underset{s'}{\textit{minimize}} & c^TDs' & & \\ \textit{subject to} & ADs' & = & 0 \\ & e^Ts' & = & 0 \\ & \|s'\|_2 & \leq & \delta. \end{array} \tag{5}$$

and

$$x_+ = (x + \hat{s}')/e^T(x + \hat{s}') \tag{6}$$

where $D = diag(x)$ and $\hat{s}'$ is the optimal solution of (5). Clearly, these are the defining relations for the iterates in the Karmarkar algorithm.

3. Natural Scaling for Problem with Free Variables

Consider the following free variable extension of problem (1)

$$\begin{array}{lrcl} \underset{x_p, x_f}{\textit{minimize}} & c_p^T x_p + c_f^T x_f & & \\ \textit{subject to} & A x_p + F x_f & = & 0 \\ & e_p^T x_p + e_f^T x_f & = & 1 \\ & x_p & \geq & 0, \end{array} \tag{7}$$

where $c_p, x_p, e_p = (1, 1, \ldots, 1)^T \in \mathbf{R}^\mathbf{n}$, $c_f, x_f, e_f = (1, 1, \ldots, 1)^T \in \mathbf{R}^\mathbf{r}$, $A \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ and $F \in \mathbf{R}^{\mathbf{m}\times\mathbf{r}}$.

The Morshedi-Tapia approach gives as counterpart to (2)

$$\begin{array}{lrcl} \underset{y, x_p, x_f}{\textit{minimize}} & c_p^T y^2 + c_f^T x_f & & \\ \textit{subject to} & A y^2 + F x_f & = & 0 \\ & e_p^T y^2 + e_f^T x_f & = & 1 \\ & x_p & = & y^2; \end{array} \tag{8}$$

and as counterpart to (3)

$$\begin{array}{lrcl} \underset{s_p, s_f}{\textit{minimize}} & c_p^T Y s_p + c_f^T s_f & & \\ \textit{subject to} & A Y s_p + F s_f & = & 0 \\ & e_p^T Y^{-1} s_p + e_f^T s_f & = & 0 \\ & \|Y^{-1} s_p\|_2^2 + \|s_f\|_2^2 & \leq & \delta^2. \end{array} \tag{9}$$

As was the case in Section 2, it is possible to write the subproblem (9) in a form which maintains y, the slack variable, in squared form only. In fact, if we make the transformation $s_p = Y s_p'$ and write $D = diag(x_p)$, (9) becomes

$$\begin{array}{lrcl} \underset{s_p', s_f}{\textit{minimize}} & c_p^T D s_p' + c_f^T s_f & & \\ \textit{subject to} & A D s_p' + F s_f & = & 0 \\ & e_p^T s_p' + e_f^T s_f & = & 0 \\ & \|s_p'\|_2^2 + \|s_f\|_2^2 & \leq & \delta^2 \end{array} \tag{10}$$

which says that the free variables should be assigned a scaling of one and the remaining variables be assigned the scale used in Karmakar's algorithm. This is precisely the scaling used by Mitchell and Todd in their modified Karmarkar's algorithm for LP problems with free variables (see rescaled Problem $\overline{P}$, pg 32, [Ref.3]).

While Mitchell and Todd used the extended scaling motivated above, they actually worked with a slight modification of problem (10). The direction obtained from their modified subproblem has nice properties, but technically it is only equivalent to the extended Karmarkar direction given by the solution of (10) in the weak sense that they give the same value to the Karmarkar potential function. For more details on this weak equivalence see Gonzaga [Ref.4].

REFERENCES

1. MORSHEDI, A.M. AND TAPIA, R.A., *Karmarkar as a classical method,* Rice University, Mathematical Sciences Technical Report No.87–7, 1987.
2. KARMARKAR, N., *A new polynomial algorithm for linear programming,* Combinatorica, Vol 4, pp.373–395, 1984.
3. MITCHELL, J.E. AND TODD, M., *A variant of Karmarkar's linear programming algorithm for problems with some unrestricted variables,* SIAM J. Matrix Analysis and Applications, Vol 10, No. 1, pp. 30–38, 1989.
4. GONZAGA, G., *A conical projection algorithm for linear programming,* Mathematical Programming, Vol. 43, pp.151-173, 1989.