

Dual Variable Schwarz Methods
for Mixed Finite Elements

Lawrence C. Cowsar

March, 1993

TR93-09

DUAL-VARIABLE SCHWARZ METHODS FOR MIXED FINITE ELEMENTS

LAWRENCE C. COWSAR

ABSTRACT. Schwarz methods for the mixed finite element discretization of second order elliptic problems are considered. By using an equivalence between mixed methods and conforming spaces first introduced in [13], it is shown that the condition number of the standard additive Schwarz method applied to the dual-variable system grows at worst like $O(1 + H/\delta)$ in both two and three dimensions and for elements of any order. Here, H is the size of the subdomains, and δ is a measure of the overlap. Numerical results are presented that verify the bound.

1. INTRODUCTION

We consider the convergence properties of Schwarz overlapping domain decomposition methods for the solution of the mixed finite element discretization of the following elliptic problem for p on the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with piecewise smooth boundary $\partial\Omega$:

$$(1.1) \quad -\nabla \cdot A \nabla p = f \quad \text{in } \Omega,$$

$$(1.2) \quad p = 0 \quad \text{on } \partial\Omega,$$

where A is a uniformly positive definite, bounded, symmetric tensor, and $f \in L^2(\Omega)$. The choice of homogeneous Dirichlet boundary conditions is merely for convenience. The extension to other boundary conditions is straightforward.

Mixed finite element methods have been used since the 1950's in their earliest incarnation as cell-centered finite differences, see [31]. Higher order mixed finite elements methods, as well as cell-centered finite differences, continue to be used in industrial problems (e.g. [19]) because of their inherent mass conservation properties and their high quality approximation of both the scalar variable and its flux.

In the mixed finite element method applied to (1.1)–(1.2), both the scalar variable p and its flux $-A\nabla p$ are approximated. Owing to our background in porous media

1991 *Mathematics Subject Classification.* 65N55.

Key words and phrases. Mixed Finite Elements, Domain Decomposition, Schwarz Methods, Second Order Elliptic Boundary Value Problems.

flow, we will refer to p as *pressure* and the flux as *velocity*. As is well known, the mixed formulation gives rise to a saddle point problem of the form

$$(1.3) \quad \begin{pmatrix} M & N \\ N^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix},$$

where P and U are the pressure and velocity unknowns, respectively. This problem is symmetric but obviously indefinite. However, the most effective domain decomposition techniques are best suited for symmetric, positive definite systems. Fortunately, by eliminating either the pressure or the velocity unknowns, one can reduce the mixed formulation to a problem that is both symmetric and positive definite. For instance, by first finding any U_F such that $N^T U_F = F$, one can reduce (1.3) to finding $U_0 = U - U_F$ that satisfies $N^T U_0 = 0$ and

$$(1.4) \quad V^T M U_0 = V^T M U_F, \quad \forall V \in \{N^T V = 0\}.$$

Alternatively, one can solve a reduced problem solely in terms of the pressure

$$(1.5) \quad -(\tilde{N}^T \tilde{M}^{-1} \tilde{N}) \tilde{P} = \tilde{F},$$

We add the tildes in (1.5) to allow for the possible introduction of additional pressure variables; e.g., the interelement multipliers discussed in Section 4. We will refer to standard domain decomposition methods applied to (1.4) as *primal-variable* methods, and those applied to (1.5) as *dual-variable* methods.

The first domain decomposition techniques for mixed finite elements were two substructuring methods formulated by Glowinski and Wheeler [23]. Glowinski and Wheeler's first method is a substructuring primal-variable method in which the normal component of velocity on the interface is the primary unknown. Primal-variable Schwarz methods were presented by Mathew [25, 26] and Ewing and Wang [20]. The convergence analysis of the primal-variable Schwarz methods in [27, 20] exploited the representation of divergence free vector fields as the curls of stream functions, and was thus limited to two dimensions.

The second method of Glowinski and Wheeler in [23] is an example of a dual-variable substructuring method in which additional pressure variables, traditionally called interelement multipliers, are introduced. Such methods have been investigated further in [14, 13]. Dual-variable Schwarz methods, in both two and three dimensions, are the subject of this paper.

In this paper, we show that the condition number of the dual-variable additive Schwarz method grows at most as $O(1 + H/\delta)$ in both two and three dimensions and for elements of any order. Here H is the size of the subdomains and δ is a measure of the overlap. This is the same bound derived by Dryja and Widlund [17] for the standard conforming linear Galerkin finite element method. If the overlap is "generous", i.e. δ is some fixed fraction of H , the condition number is bounded by a constant that is independent of both the subdomain size and the mesh size.

To this author's knowledge, this method is the first asymptotically optimal domain decomposition method for mixed finite elements in three dimensions.

The remainder of the paper is divided into seven sections. In the next section, we introduce some notation. The standard additive and multiplicative Schwarz methods are recalled in Section 3 along with their abstract convergence theory. In Section 4, we discuss briefly the mixed finite element approximation to (1.1)–(1.2), an equivalent hybrid method, and the reduced problem solely in terms of the dual variables. Readers familiar with these methods and results may wish simply to skim these sections in order to set some notation. The equivalence of the mixed method to a conforming problem is demonstrated in Section 5. In Section 6, the bound on the convergence rates of the Schwarz methods for the hybrid mixed finite element method are derived. Related results for the mixed method without hybridization are discussed in Section 7. The paper concludes with some numerical results that verify the estimates presented in this paper.

2. NOTATION AND PRELIMINARIES

Without loss of generality, we assume that Ω has unit diameter. We introduce two decompositions of Ω , a coarse partitioning into nonoverlapping subdomains Ω_i , and a further refinement into elements \mathcal{T} . We assume that the subdomains have a diameter that is $O(H)$ and are shape regular, see [11]. We assume that \mathcal{T} is quasi-regular with characteristic length h . For each subdomain Ω_i , extend it to a larger region Ω'_i such that Ω'_i is also the union of elements of \mathcal{T} . We characterize the extent of the overlap of the partition $\{\Omega'_i\}$ by δ , where

$$\delta = \min_{i=1,\dots,M} \text{dist}(\partial\Omega_i \setminus \partial\Omega, \partial\Omega'_i \setminus \partial\Omega).$$

Denote the internal interfaces by $\Gamma = \bigcup_i \partial\Omega_i \setminus \partial\Omega$, and let $\Gamma_{\delta,i} \subset \Omega_i$ be the set of points that are within a distance δ of Γ . We say that two subdomains Ω_i and Ω_j are δ -adjacent if $\Omega'_i \cap \Omega'_j$ is not empty.

Let dx denote the standard Lebesgue n -dimensional measure and ds the $(n-1)$ -dimensional surface measure. For a bounded open set $\omega \subseteq \mathbb{R}^n$, let $|\omega|$ denote the measure of the set, ν_ω denote its outward directed normal, and $L^2(\omega)$, $(L^2(\omega))^n$, $H^s(\omega)$, $(H^s(\omega))^n$ denote the standard Sobolev spaces of real-valued functions defined on ω (see, e.g., [1]). Let $\mathbf{H}(\omega; \text{div})$ be the subspace of $(L^2(\omega))^n$ that has divergences in $L^2(\omega)$, i.e.,

$$\mathbf{H}(\omega; \text{div}) = \{\mathbf{v} \in (L^2(\omega))^n \mid \nabla \cdot \mathbf{v} \in L^2(\omega)\}.$$

Following [4], define the scaled Sobolev norm for Ω_i of diameter H as

$$\|u\|_{1,\Omega_i}^2 = |u|_{1,\Omega_i}^2 + \frac{1}{H^2} \|u\|_{0,\Omega_i}^2,$$

where

$$\|u\|_{0,\Omega_i}^2 = \int_{\Omega_i} |u(x)|^2 dx, \quad |u|_{1,\Omega_i}^2 = \int_{\Omega_i} |\nabla u(x)|^2 dx.$$

We need to define local subspaces derived from a parent space. In general, having defined a parent space of functions $\mathcal{X}(\Omega)$ and a set $\omega \subset \Omega$, we will adopt the notation $\mathcal{X}(\omega)$ for the restriction of $\mathcal{X}(\Omega)$ to ω , i.e.

$$\mathcal{X}(\omega) = \{\phi|_\omega \mid \phi \in \mathcal{X}(\Omega)\}.$$

By an abuse of notation, we consider an element $\phi \in \mathcal{X}(\omega)$ to be an element also of $\mathcal{X}(\Omega)$, if the extension of ϕ by zero is in $\mathcal{X}(\Omega)$.

We say that two quadratic forms \mathcal{Q}_1 and \mathcal{Q}_2 with domain \mathcal{D} are *equivalent* and write $\mathcal{Q}_1 \simeq \mathcal{Q}_2$ if there exist constants $c, C > 0$ such that

$$c\mathcal{Q}_1(\phi, \phi) \leq \mathcal{Q}_2(\phi, \phi) \leq C\mathcal{Q}_1(\phi, \phi), \quad \forall \phi \in \mathcal{D}.$$

In what follows, the constants that appear in the equivalences are independent of h , δ , and H , but may depend on the coefficient A in (1.1), the degree of the mixed finite elements, the shape regularity of the subdomains, and the regularity of the triangulation \mathcal{T} .

3. SCHWARZ METHODS AND ABSTRACT THEORY

Following [17], we recall the abstract multiplicative and additive Schwarz methods for the following finite dimensional variational problem for $\hat{p} \in \mathcal{P}$:

$$(3.1) \quad d(\hat{p}, \hat{q}) = f(\hat{q}) \quad \forall \hat{q} \in \mathcal{P}.$$

We assume that the bilinear form $d(\cdot, \cdot)$ is selfadjoint, elliptic and bounded on $\mathcal{P} \times \mathcal{P}$.

Let \mathcal{P}_i be subspaces of \mathcal{P} such that $\mathcal{P} = \mathcal{P}_0 + \dots + \mathcal{P}_M$. Let $e_i(\cdot, \cdot)$ be an inner product defined on $\mathcal{P}_i \times \mathcal{P}_i$ and assume that there is a constant C_1 such that

$$(3.2) \quad e_i(\hat{p}, \hat{p}) \leq C_1 d(\hat{p}, \hat{p}), \quad \forall \hat{p} \in \mathcal{P}_i.$$

For each subspace \mathcal{P}_i , define an operator $T_i : \mathcal{P} \rightarrow \mathcal{P}_i$ by

$$(3.3) \quad e_i(T_i \hat{p}, \hat{q}) = d(\hat{p}, \hat{q}) \quad \forall \hat{q} \in \mathcal{P}_i.$$

Remark 3.1. A natural choice for $e_i(\cdot, \cdot)$ is $d(\cdot, \cdot)$. In this case, $C_1 = 1$, and the operator T_i is orthogonal projection onto \mathcal{P}_i with respect to the d -inner product. We will refer to this choice as the Schwarz method with exact solves.

One iteration of the *multiplicative Schwarz algorithm* involves three steps. Given $\hat{p}^k \in \mathcal{P}$, define $\hat{p}^{k+1} \in \mathcal{P}$ by:

- (1) Set $\phi_{-1} = \hat{p}^k$;
- (2) For $i = 0, \dots, M$, define ϕ_i by

$$\phi_i = \phi_{i-1} + T_i(\hat{p} - \phi_{i-1});$$

(3) Set $\hat{p}^{k+1} = \phi_M$.

The calculation of $T_i \hat{p}$ in the second step presents no difficulty, since $e_i(T_i \hat{p}, \hat{q}) = f(\hat{q})$. It is easy to see that the error $\epsilon^k = \hat{p} - \hat{p}^k$ satisfies

$$(3.4) \quad \epsilon^{k+1} = E\epsilon^k, \quad E = (I - T_M)(I - T_{M-1}) \dots (I - T_0).$$

The *additive Schwarz algorithm* for (3.1) introduced by Dryja and Widlund [15] involves the solution of

$$(3.5) \quad T\hat{p} \equiv \sum_{i=0}^M T_i \hat{p} = \sum_{i=0}^M f_i,$$

where $f_i \in \mathcal{P}_i$ is defined by

$$e_i(f_i, \hat{q}) = f(\hat{q}) \quad \forall \hat{q} \in \mathcal{P}_i.$$

It is easy to see that the solutions to (3.1) and (3.5) are the same since

$$e_i(T_i \hat{p}, \hat{q}) = d(\hat{p}, \hat{q}) = f(\hat{q}).$$

For a symmetric, positive definite form d , the operator T is symmetric and positive definite with respect to the d -inner product; hence, conjugate gradients can be applied. Moreover, for well chosen subspaces \mathcal{P}_i , the condition number of T is much smaller than the one corresponding to (3.1) and so no further preconditioning is needed.

Abstract bounds on the norm of the error propagation operator E and bounds on the condition number of T have been derived in terms of the constant C_1 in (3.2) and two additional quantities, C_0 and the spectral radius of \mathcal{E} , which we now define. Let $C_0 > 0$ be a constant such that for every $\hat{p} \in \mathcal{P}$ there exists a representation $\hat{p} = \sum_{i=0}^M \hat{p}_i$ with $\hat{p}_i \in \mathcal{P}_i$ satisfying

$$(3.6) \quad \sum_{i=0}^M e_i(\hat{p}_i, \hat{p}_i) \leq C_0 d(\hat{p}, \hat{p}).$$

Let $\rho(\mathcal{E})$ denote the spectral radius of $\mathcal{E} = \{\epsilon_{ij}\}$, the matrix of strengthened Cauchy-Schwarz constants; that is, ϵ_{ij} is the smallest constant for which

$$(3.7) \quad |d(\hat{p}_i, \hat{p}_j)| \leq \epsilon_{ij} d(\hat{p}_i, \hat{p}_i)^{\frac{1}{2}} d(\hat{p}_j, \hat{p}_j)^{\frac{1}{2}} \quad \forall \hat{p}_i \in \mathcal{P}_i, \forall \hat{p}_j \in \mathcal{P}_j, i, j \geq 1.$$

The following theorem applies to the multiplicative Schwarz algorithm, and it is a variant of a result of Bramble, Pasciak, Wang and Xu [5]; see also [32].

Theorem 3.1. *The operator norm $\|E\|_d$ measured in the $d(\cdot, \cdot)$ inner product satisfies*

$$\|E\|_d \leq \sqrt{1 - \frac{(2 - C_1)}{(2C_1^2 \rho(\mathcal{E})^2 + 1)C_0}}.$$

For $C_1 \geq 2$ this bound gives us no information; however, by scaling the bilinear forms $e_i(\cdot, \cdot)$ in a suitable manner, a useful algorithm and bound can be obtained.

The next theorem, due to Dryja and Widlund [18], bounds the condition number of the additive Schwarz methods in terms of the same constants. It represents the continued refinement of results given in [29], [24] and [16].

Theorem 3.2. *The eigenvalues of T satisfy*

$$(3.8) \quad \lambda_{\min}(T) \geq C_0^{-1},$$

$$(3.9) \quad \lambda_{\max}(T) \leq C_1(\rho(\mathcal{E}) + 1).$$

Hence, the condition number $\kappa(T)$ of T satisfies

$$\kappa(T) = \lambda_{\max}(T)/\lambda_{\min}(T) \leq C_0 C_1(\rho(\mathcal{E}) + 1).$$

In this paper, we formulate our results in terms of the additive algorithms. Analogous statements for the multiplicative variants can be derived easily.

4. DUAL PROBLEM

In this section, we recall the mixed finite element method with and without hybridization in order to set some notation. We then formulate the dual problem analogous to (1.5). We refer the readers who are unfamiliar with the hybrid formulation to the expositions in [2, 10] and [13] for more detail.

4.1. Mixed Finite Element Formulation. Rewrite (1.1) as the first order system

$$(4.1) \quad A^{-1}\mathbf{u} = -\nabla p \quad \text{in } \Omega,$$

$$(4.2) \quad \nabla \cdot \mathbf{u} = f \quad \text{in } \Omega.$$

Multiplying by appropriate test functions, integrating (4.1) by parts, and using the boundary condition (1.2), we arrive at the following weak form: Find $\mathbf{u} \in \mathbf{H}(\Omega; \text{div})$ and $p \in L^2(\Omega)$ such that

$$(4.3) \quad \int_{\Omega} A^{-1}\mathbf{u} \cdot \mathbf{v} \, dx - \int_{\Omega} p \nabla \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\Omega; \text{div}),$$

$$(4.4) \quad \int_{\Omega} \nabla \cdot \mathbf{u} \, w \, dx = \int_{\Omega} f w \, dx, \quad \forall w \in L^2(\Omega).$$

Let $\mathbf{V}_h(\Omega) \subset \mathbf{H}(\Omega; \text{div})$ and $W_h(\Omega) \subset L^2(\Omega)$ be finite dimensional subspaces defined with respect to the triangulation \mathcal{T} . By the standard *mixed finite element approximation* to (4.3)–(4.4), we mean the pair $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h(\Omega) \times W_h(\Omega)$ satisfying

$$(4.5) \quad \int_{\Omega} A^{-1}\mathbf{u}_h \cdot \mathbf{v} \, dx - \int_{\Omega} p_h \nabla \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h(\Omega),$$

$$(4.6) \quad \int_{\Omega} \nabla \cdot \mathbf{u}_h \, w \, dx = \int_{\Omega} f w \, dx \quad \forall w \in W_h(\Omega).$$

For this problem to be well-posed, the spaces $\mathbf{V}_h(\Omega)$ and $W_h(\Omega)$ cannot be chosen arbitrarily. A well known sufficiency condition is the Babuska-Brezzi *inf-sup* condition [6]. There are many mixed finite element spaces defined in the literature for both two and three dimensions that satisfy this condition, including the Raviart-Thomas-Nedelec spaces [30, 28], the BDM spaces [9, 7], and the BDFM spaces [8]. These spaces also admit an equivalent hybrid form that we discuss below. The hybrid form will be the basis of our analysis and offers computational advantages for higher order elements. In Section 7, we will appeal to the analysis of the hybrid form to derive bounds for the non-hybridized form as well.

Let $\mathbf{V}_h^{-1}(\Omega)$ be the superset of $\mathbf{V}_h(\Omega)$ that is the tensor product of $\mathbf{V}_h(\Omega)$ restricted to elements of \mathcal{T} ; i.e.,

$$\mathbf{V}_h^{-1}(\Omega) = \{\mathbf{v} \mid \text{for each } \tau \in \mathcal{T}, \exists \mathbf{v}^\tau \in \mathbf{V}_h(\Omega) \text{ such that } \mathbf{v}|_\tau = \mathbf{v}^\tau|_\tau\}.$$

Let $\Lambda_h(\Omega)$ be the space of traces of the normal component of the flux on the element boundaries; that is,

$$\Lambda_h(\Omega) = \bigcup_{\tau \in \mathcal{T}} \{\mathbf{v} \cdot \boldsymbol{\nu}_\tau|_{\partial\tau} \mid \mathbf{v} \in \mathbf{V}_h(\tau)\}.$$

Let $\Lambda_h^0(\Omega)$ denote the set of elements in $\Lambda_h(\Omega)$ that vanish on the boundary of Ω .

The *hybrid mixed finite element approximation* to (4.3)–(4.4) is the triple

$$\{\mathbf{u}_h, p_h, \lambda_h\} \in \mathbf{V}_h^{-1}(\Omega) \times W_h(\Omega) \times \Lambda_h^0(\Omega)$$

satisfying

$$(4.7) \quad \sum_{\tau \in \mathcal{T}} \left(\int_\tau A^{-1} \mathbf{u}_h \cdot \mathbf{v} \, dx - \int_\tau p_h \nabla \cdot \mathbf{v} \, dx + \int_{\partial\tau} \lambda_h \mathbf{v} \cdot \boldsymbol{\nu}_\tau \, ds \right) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^{-1}(\Omega),$$

$$(4.8) \quad - \sum_{\tau \in \mathcal{T}} \int_\tau q \nabla \cdot \mathbf{u}_h \, dx = - \int_\Omega q f \, dx \quad \forall q \in W_h(\Omega),$$

$$(4.9) \quad \sum_{\tau \in \mathcal{T}} \int_{\partial\tau} \mu \mathbf{u}_h \cdot \boldsymbol{\nu}_\tau \, ds = 0 \quad \forall \mu \in \Lambda_h^0(\Omega).$$

Remark 4.1. The variable λ_h is traditionally called the interelement multiplier and admits a simple and important interpretation. If we consider the constitutive relationship (4.1) on a single element τ , then, after multiplying by a test function and integrating by parts, we find that

$$(4.10) \quad \int_\tau A^{-1} \mathbf{u} \cdot \mathbf{v} \, dx - \int_\tau p \nabla \cdot \mathbf{v} \, dx + \int_{\partial\tau} p \mathbf{v} \cdot \boldsymbol{\nu}_\tau \, ds = 0.$$

Comparing this with (4.7), we see that λ_h is naturally interpreted as an approximation to the trace of p on the boundaries of the elements.

The analysis presented in the subsequent sections is applicable to a large number of mixed finite element spaces since it makes use only of properties that most mixed finite element spaces share. In particular, the spaces listed above share the necessary properties. We make explicit here in the form of assumptions the properties of the mixed spaces that are used in the analysis.

We assume that $\mathbf{V}_h^{-1}(\Omega)$ and $W_h(\Omega)$ are constructed from disjoint local subspaces defined on each element. In particular, for $\tau \in \mathcal{T}$, we let $\mathbf{V}_h(\tau) \subset \mathbf{H}(\tau; \text{div})$, $W_h(\tau) \subset L^2(\tau)$, and we assume that

$$W_h(\Omega) = \bigoplus_{\tau \in \mathcal{T}} W_h(\tau), \quad \mathbf{V}_h^{-1}(\Omega) \equiv \bigoplus_{\tau \in \mathcal{T}} \mathbf{V}_h(\tau).$$

We assume that $\mathbf{V}_h(\Omega)$ is the subspace of functions in $\mathbf{V}_h^{-1}(\Omega)$ whose normal components are continuous across the edges (faces in 3 dimensions, though we henceforth neglect the distinction) of the elements in the triangulation. In particular, $\mathbf{v} \in \mathbf{V}_h^{-1}(\Omega)$ is in $\mathbf{V}_h(\Omega)$ if and only if

$$\sum_{\tau \in \mathcal{T}} \int_{\partial\tau} \lambda \mathbf{v} \cdot \nu_\tau = 0 \quad \forall \lambda \in \Lambda_h^0(\Omega).$$

We also assume that

$$\text{div}(\mathbf{V}_h(\Omega)) \subseteq W_h(\Omega),$$

and that the standard mixed finite element projection

$$\Pi_h : \mathbf{H}(\Omega; \text{div}) \cap \{\mathbf{v} \cdot \nu_\tau \in L^2(\partial\tau), \tau \in \mathcal{T}\} \rightarrow \mathbf{V}_h(\Omega)$$

exists and satisfies, among other properties, that for every $\tau \in \mathcal{T}$ and edge e of the boundary of τ that

$$(4.11) \quad \int_e (\Pi_h \mathbf{u} - \mathbf{u}) \cdot \nu_\tau \lambda \, ds = 0 \quad \forall \lambda \in \Lambda_h(e),$$

$$(4.12) \quad \int_\tau \nabla \cdot (\Pi_h \mathbf{u} - \mathbf{u}) q \, dx = 0 \quad \forall q \in W_h(\tau).$$

In particular, (4.11) and (4.12) imply that certain moments of the normal component of velocity on each edge and certain moments of the divergence of the velocity are independent degrees of freedom for $\mathbf{V}_h(\tau)$.

4.2. The Dual Problem. Henceforth, we shall only be concerned with the solution of the finite dimensional problem (4.7)–(4.9) (and (4.5)–(4.6) in Section 7). Consequently, we will drop the “h” subscript from \mathbf{u}_h , p_h and λ_h .

We parameterize the space $W_h(\Omega)$ element-wise by using a local nodal basis with nodes in the interior of elements. Likewise, we parameterize $\Lambda_h(\Omega)$ by a nodal basis defined on the edges of elements of the triangulation. Let N denote the set of the

nodal points in Ω corresponding to the degrees of freedom of $W_h(\Omega) \times \Lambda_h(\Omega)$, and let $\mathcal{P}(\Omega)$ be the set of real valued functions defined on N , i.e.,

$$\mathcal{P}(\Omega) = \{\hat{p} : N \rightarrow \mathbb{R}\}.$$

For $\hat{p} \in \mathcal{P}(\Omega)$ and $[p, \lambda] \in W_h(\Omega) \times \Lambda_h(\Omega)$, we write (by an abuse of notation) $\hat{p} = [p, \lambda]$ if \hat{p} and $[p, \lambda]$ agree at all nodal points; moreover, we consider $[p, \lambda]$ as elements of $\mathcal{P}(\Omega)$ and $\hat{p} \in W_h(\Omega) \times \Lambda_h(\Omega)$. In view of Remark 4.1, $\mathcal{P}(\Omega)$ has the natural interpretation as the space of pressure values on all nodes, in the interior as well as on the edges of the elements. For $[p, \lambda] \in W_h(\Omega) \times \Lambda_h(\Omega)$ and $\tau \in \mathcal{T}$, we write $[p_\tau, \lambda_\tau]$ for the restriction $[p, \lambda]|_\tau$.

In a variational framework, one can eliminate the velocity in (4.7)–(4.9) by introducing a discretization of the flux operator $A\nabla$ denoted by

$$\nabla_h^A : W_h(\Omega) \times \Lambda_h(\Omega) \rightarrow \mathbf{V}_h^{-1}(\Omega),$$

and defined by

$$(4.13) \quad \sum_{\tau \in \mathcal{T}} \int_{\tau} A^{-1} \nabla_h^A [q, \mu] \cdot \mathbf{v} \, dx = \sum_{\tau \in \mathcal{T}} \left(- \int_{\tau} q \nabla \cdot \mathbf{v} \, dx + \int_{\partial\tau} \mu \mathbf{v} \cdot \nu_{\tau} \, ds \right) \quad \forall \mathbf{v} \in \mathbf{V}_h^{-1}(\Omega).$$

Since $\mathbf{V}_h^{-1}(\Omega)$ is the disjoint union of local spaces, we note that the restriction of $\nabla_h^A [q, \mu]$ to an element $\tau \in \mathcal{T}$ is determined by the restriction of q and μ to τ . Also, by comparing with (4.7), we see that for p and λ satisfying (4.7)–(4.9), $\mathbf{u} = -\nabla_h^A [p, \lambda]$.

It is easy to show, see [2], that a problem equivalent to (4.7)–(4.9) is finding the pair

$$[p, \lambda] \in W_h(\Omega) \times \Lambda_h^0(\Omega)$$

satisfying

$$(4.14) \quad d([p, \lambda], [q, \mu]) = \int_{\Omega} f q \, dx \quad \forall [q, \mu] \in W_h(\Omega) \times \Lambda_h^0(\Omega),$$

where

$$d([p, \lambda], [q, \mu]) \equiv \sum_{\tau \in \mathcal{T}} \int_{\tau} A^{-1} \nabla_h^A [p, \lambda] \cdot \nabla_h^A [q, \mu] \, dx.$$

The velocity can be recovered as $\mathbf{u} = -\nabla_h^A [p, \lambda]$. The bilinear form d is obviously positive semi-definite. The strict positivity is a simple corollary to Lemma 4.1 below.

Note that we can decompose d locally as

$$d([p, \lambda], [q, \mu]) = \sum_{\tau \in \mathcal{T}} d_{\tau}([p_{\tau}, \lambda_{\tau}], [q_{\tau}, \mu_{\tau}]),$$

where, more generally for $\Omega' \subseteq \Omega$ composed of elements of the triangulation \mathcal{T} , we define

$$(4.15) \quad d_{\Omega'}([p, \lambda], [q, \mu]) \equiv \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'} \int_{\tau} A^{-1} \nabla_h^A [p, \lambda] \cdot \nabla_h^A [q, \mu] dx.$$

Remark 4.2. In practice, the bilinear form $d_{\Omega'}([p, \lambda], [q, \mu])$ is more easily evaluated as

$$d_{\Omega'}([p, \lambda], [q, \mu]) = \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'} \left(- \int_{\tau} q \nabla \cdot (\nabla_h^A [p, \lambda]) dx + \int_{\partial \tau} \mu (\nabla_h^A [p, \lambda]) \cdot \nu_{\tau} ds \right),$$

which can be seen to be equivalent to (4.15) by using (4.13).

We now recall an important characterization of the dual problem that was proven in [13].

Lemma 4.1. *Let $\hat{p} = [p, \lambda] \in W_h(\Omega') \times \Lambda_h(\Omega')$ for $\Omega' \subseteq \Omega$ composed of elements of the triangulation \mathcal{T} . Then*

$$(4.16) \quad d_{\Omega'}(\hat{p}, \hat{p}) \simeq \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'} |\tau|^{1-2/n} \sum_{\substack{\text{nodes:} \\ n_i, n_j \in \tau}} (\hat{p}_{\tau}(n_i) - \hat{p}_{\tau}(n_j))^2$$

with the bounds in the equivalence being independent of both h and $|\Omega'|$.

Remark 4.3. The proof of Lemma 4.1 follows directly from the characterization of the local kernel of $d(\cdot, \cdot)$ as the space of constant functions, see [13]. Hence, the theory presented in this paper is applicable to a large class of non-conforming Lagrange elements that satisfy this property. See [12] for more details.

5. A CONFORMING EQUIVALENCE

In Lemma 4.1, we see that the mixed finite element discretization of (1.1)–(1.2) over a region Ω' gives rise to a quadratic form similar to a discretization of $\int_{\Omega'} |\nabla p|^2 dx$. In this section, we show that $W_h(\Omega') \times \Lambda_h(\Omega')$ with the norm induced by $d_{\Omega'}(\cdot, \cdot)$ is isomorphic to a conforming space of piecewise linear functions with the H^1 -seminorm. This isomorphism enables us in Section 6 to analyze the mixed method using the domain decomposition theory for the conforming case with only a few changes.

We first construct a special subtriangulation $\hat{\mathcal{T}}$ of \mathcal{T} . Given an element $\tau \in \mathcal{T}$, let $\hat{\mathcal{T}}_{\tau}$ be a subtriangulation of τ such that the vertices of the subtriangulation include the vertices of τ and the nodal points in τ pertaining to the degrees of freedom of $W_h(\tau) \times \Lambda_h(\tau)$. Moreover, every element in the new triangulation should have at least one vertex that corresponds to a nodal point of $W_h(\tau)$. The subtriangulations should be constructed in such a way that the union of subtriangulations gives rise to a refined triangulation of Ω which we denote by

$$\hat{\mathcal{T}} \equiv \bigcup_{\tau \in \mathcal{T}} \hat{\mathcal{T}}_{\tau}.$$

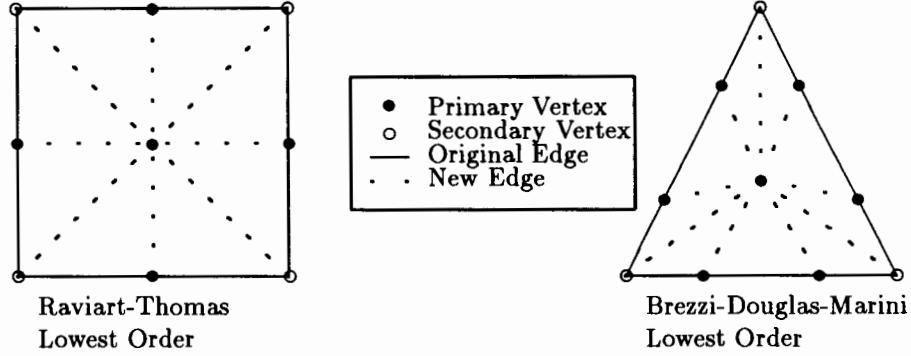


FIGURE 1. Examples of subtriangulations of two commonly used elements

Furthermore, we assume that the regularity of the refined mesh $\hat{\mathcal{T}}$ is a function only of the regularity of the original mesh \mathcal{T} and the degree of the mixed finite element space. A vertex of $\hat{\mathcal{T}}$ will be called *primary* if it was a nodal point corresponding to a degree of freedom of $W_h(\Omega) \times \Lambda_h(\Omega)$; otherwise, we call the vertex *secondary*. We say that two vertices of the triangulation $\hat{\mathcal{T}}$ are *adjacent* if there exists an edge of $\hat{\mathcal{T}}$ connecting the vertices. Several examples are given in Figures 1 and 2.

Let $U_h(\Omega)$ be the space of continuous piecewise linear functions subordinate to the triangulation $\hat{\mathcal{T}}$. For $\Omega' \subset \Omega$, a union of elements, define $U_h(\Omega')$ by restriction, i.e.

$$U_h(\Omega') = \{u|_{\Omega'} \mid u \in U_h(\Omega)\}.$$

Since the functions in $U_h(\Omega')$ are naturally parameterized by the values they attain at the vertices, we define a mapping $I_h^{\Omega'}$ into $U_h(\Omega')$ for any function ϕ defined at the primary vertices contained in Ω' by

$$(5.1) \quad I_h^{\Omega'} \phi(x) = \begin{cases} \phi(x), & \text{if } x \text{ is a primary vertex;} \\ \text{The average of all adjacent primary vertices on the boundary} \\ \quad \text{of } \Omega', & \text{if } x \text{ is a secondary vertex on the boundary of } \Omega'; \\ \text{The average of all adjacent primary vertices, if } x \text{ is a secondary} \\ \quad \text{vertex in the interior of } \Omega'; \\ \text{The continuous piecewise linear interpolant of the above vertex} \\ \quad \text{values, if } x \text{ is not a vertex of } \hat{\mathcal{T}}. \end{cases}$$

Since $I_h^{\Omega'}$ is defined for any function defined at primary vertices, by an abuse of

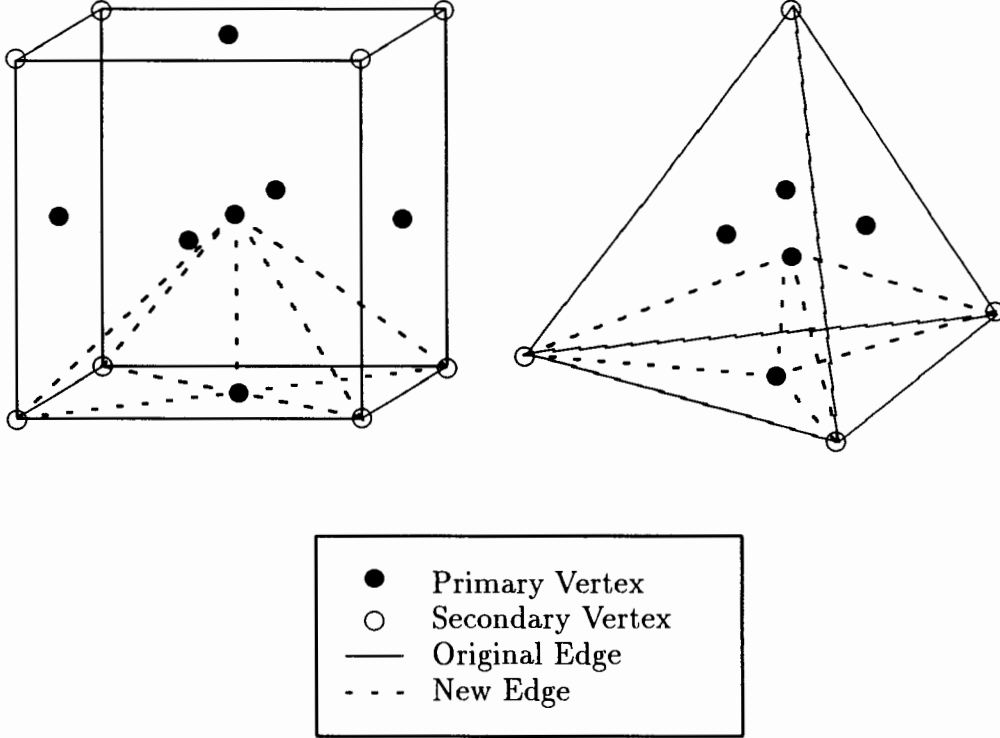


FIGURE 2. Partial subtriangulation of the lowest order Raviart-Thomas-Nedelec elements

notation, we can understand $I_h^{\Omega'}$ as a map from $W_h(\Omega') \times \Lambda_h(\Omega')$ into $U_h(\Omega')$, a map from $\mathcal{P}(\Omega')$ into $U_h(\Omega')$, and a map from $U_h(\Omega')$ into $U_h(\Omega')$. Let $\tilde{U}_h(\Omega') \subset U_h(\Omega')$ be the range of $I_h^{\Omega'}$; that is,

$$\tilde{U}_h(\Omega') = \{\psi \in U_h(\Omega') \mid \psi = I_h^{\Omega'} \phi, \phi \in U_h(\Omega')\}.$$

Recall that for $\phi \in U_h(\Omega')$,

$$(5.2) \quad \|\phi\|_{0,\Omega'}^2 \simeq \sum_{\tau \in \hat{\mathcal{T}}, \tau \subset \Omega'} |\tau| \sum_{\substack{\text{vertices:} \\ v_i \in \tau}} \phi(v_i)^2,$$

$$(5.3) \quad |\phi|_{1,\Omega'}^2 \simeq \sum_{\tau \in \hat{\mathcal{T}}, \tau \subset \Omega'} |\tau|^{1-2/n} \sum_{\substack{\text{vertices:} \\ v_i, v_j \in \tau}} (\phi(v_i) - \phi(v_j))^2.$$

The next lemma tabulates some properties of the $I_h^{\Omega'}$ -projection that we need.

Lemma 5.1. *There exists a constant $C > 0$ independent of h and $|\Omega'|$ such that*

$$(5.4) \quad \|I_h^{\Omega'} \phi\|_{0,\Omega'} \leq C \|\phi\|_{0,\Omega'} \quad \forall \phi \in U_h(\Omega'),$$

$$(5.5) \quad |I_h^{\Omega'} \phi|_{1,\Omega'} \leq C |\phi|_{1,\Omega'} \quad \forall \phi \in U_h(\Omega'),$$

$$(5.6) \quad \|\phi - I_h^{\Omega'} \phi\|_{0,\Omega'} \leq Ch |\phi|_{1,\Omega'} \quad \forall \phi \in U_h(\Omega').$$

For a subset $\omega \subset \Omega'$ composed of elements of the triangulation,

$$(5.7) \quad \sum_{\tau \in \mathcal{T}, \tau \subset \omega} \|I_h^\tau \phi\|_{0,\tau}^2 \leq C \|I_h^{\Omega'} \phi\|_{0,\omega}^2 \quad \forall \phi \in U_h(\Omega').$$

Proof. The stability with respect to the L^2 -norm and H^1 -norm was proven in Lemma 6.1 of [13]. We now verify (5.6) and (5.7).

Let v_{n+1} be a secondary vertex with adjacent primary vertices v_1, \dots, v_n . Denoting $\phi_j = \phi(v_j)$, then

$$(5.8) \quad (\phi_{n+1} - (I_h^{\Omega'} \phi)(v_{n+1}))^2 = \left(\phi_{n+1} - \frac{\sum_{j=1}^n \phi_j}{n} \right)^2 \leq \frac{1}{n} \sum_{j=1}^n (\phi_{n+1} - \phi_j)^2,$$

by the Cauchy-Schwarz inequality. By construction, ϕ and $I_h^{\Omega'} \phi$ agree at primary vertices; hence, using (5.2), we see that

$$(5.9) \quad \|\phi - I_h^{\Omega'} \phi\|_{0,\Omega'}^2 \leq C \sum_{\tau \in \widehat{\mathcal{T}}, \tau \subset \Omega'} |\tau| \sum_{\substack{\text{secondary} \\ \text{vertices :} \\ v_i \in \tau}} (\phi(v_i) - I_h^{\Omega'} \phi(v_i))^2.$$

Using (5.8) we can bound the differences at secondary vertices by the sum of differences at primary vertices. Since the mesh is regular, there is an a priori bound on the maximum number of secondary vertices per element and the number of possible adjacent primary vertices to a secondary vertex. The proof of (5.6) now follows using (5.3), since

$$(5.10) \quad \begin{aligned} \|\phi - I_h^{\Omega'} \phi\|_{0,\Omega'}^2 &\leq C \sum_{\tau \in \widehat{\mathcal{T}}, \tau \subset \Omega'} |\tau| \sum_{\substack{\text{primary} \\ \text{vertices :} \\ v_i, v_j \in \tau}} (\phi(v_i) - \phi(v_j))^2 \\ &\leq C \sum_{\tau \in \widehat{\mathcal{T}}, \tau \subset \Omega'} |\tau|^{2/n} |\tau|^{1-2/n} \sum_{\substack{\text{vertices :} \\ v_i, v_j \in \tau}} (\phi(v_i) - \phi(v_j))^2 \\ &\leq Ch^2 |\phi|_{1,\Omega'}^2. \end{aligned}$$

Verification of (5.7) follows simply from the equivalence

$$\|I_h^\tau \phi\|_{0,\tau}^2 \simeq |\tau| \sum_{\substack{\text{primary} \\ \text{vertices :} \\ v_i \in \tau}} \phi(v_i)^2,$$

and the the fact that $I_h^\tau \phi$ and $I_h^{\Omega'} \phi$ agree on primary nodes. \square

The following theorem makes explicit the relationship between the bilinear form d and a discretization of $\int_\Omega |\nabla p|^2 dx$. In particular, it proves that the mapping $I_h^{\Omega'}$ preserves the norms on $\mathcal{P}(\Omega')$ and $\tilde{U}_h(\Omega')$; hence, $I_h^{\Omega'} : \mathcal{P}(\Omega') \rightarrow \tilde{U}_h(\Omega')$ is an isomorphism.

Theorem 5.2. *For Ω' a union of elements of \mathcal{T} , the quadratic form $d_{\Omega'}(\cdot, \cdot)$ is equivalent to the H^1 -seminorm of the $I_h^{\Omega'}$ -interpolant; that is, there exist constants $c, C > 0$, independent of h and $|\Omega'|$ such that*

$$(5.11) \quad c|I_h^{\Omega'} \hat{p}|_{1,\Omega'}^2 \leq d_{\Omega'}(\hat{p}, \hat{p}) \leq C|I_h^{\Omega'} \hat{p}|_{1,\Omega'}^2, \quad \forall \hat{p} \in \mathcal{P}(\Omega').$$

Proof. Since vertices of $\hat{\mathcal{T}}_\tau$ contain the nodal points of τ , and $\hat{p} = I_h^{\Omega'} \hat{p}$ at these points, it is easy to show that

$$\sum_{\substack{\text{nodes :} \\ n_i, n_j \in \tau}} (\hat{p}(n_i) - \hat{p}(n_j))^2 \leq C \sum_{\hat{\tau} \in \hat{\mathcal{T}}_\tau} \sum_{\substack{\text{vertices :} \\ v_i, v_j \in \hat{\tau}}} (I_h^{\Omega'} \hat{p}(v_i) - I_h^{\Omega'} \hat{p}(v_j))^2,$$

where the constant is controlled by the regularity of the subtriangulation. The upper bound follows by summing over the elements in Ω' and using Lemma 4.1 and (5.3).

To prove the lower bound, we note that the regularity of the mesh implies an a priori maximum number of adjacent elements that can share a secondary point. The bound then follows from Lemma 4.1 and (5.3), since the differences between secondary and primary points can be bounded as in (5.8). \square

6. A BOUND ON THE CONDITION NUMBER

In this section, we first define the set of subspaces that we will use in the Schwarz algorithms and collect a few technical tools. We then prove the main result of this paper, a bound on the condition number of the additive Schwarz method introduced in Section 3.

6.1. Decomposition of $\mathcal{P}(\Omega)$. For convenience, we will assume that Ω is partitioned into non-overlapping subdomains $\{\Omega_i\}$ that are triangular or rectangular (tetrahedral or rectangular solids in 3D). Let $U_H(\Omega)$ be the space of continuous functions that are linear, bilinear, or trilinear as appropriate, on each Ω_i . Let

$$\mathcal{P}_0 = \{\hat{p} \in \mathcal{P}(\Omega) \mid \hat{p} = \mathcal{I}^N \phi, \phi \in U_H(\Omega)\},$$

where \mathcal{I}^N is interpolation at the primary vertices. For $i = 1, \dots, M$, let $\mathcal{P}_i \subset \mathcal{P}(\Omega)$ be such that $\hat{p} \in \mathcal{P}_i$ is zero at all nodes on the boundary and outside of Ω'_i . In terms of the mixed method spaces, \mathcal{P}_i is the space of nodal values of functions in $W_h(\Omega'_i) \times \Lambda_h^0(\Omega'_i)$ extended by zero at nodes outside of Ω'_i .

6.2. Technical Tools. In the subsequent analysis, we will need a mapping into \mathcal{P}_0 , so define $\hat{Q}_H : \mathcal{P}(\Omega) \rightarrow \mathcal{P}_0$ by

$$(6.1) \quad \hat{Q}_H = \mathcal{I}^N I_h^\Omega Q_H I_h^\Omega,$$

where Q_H is standard L^2 -projection onto $U_H(\Omega)$.

Lemma 6.1. *There exists a constant $C > 0$, independent of h and H , such that*

$$(6.2) \quad \|I_h^\Omega \hat{Q}_H \hat{p}\|_{1,\Omega} \leq C \|I_h^\Omega \hat{p}\|_{1,\Omega} \quad \forall \hat{p} \in \mathcal{P}(\Omega),$$

$$(6.3) \quad \|I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega} \leq CH |I_h^\Omega \hat{p}|_{1,\Omega} \quad \forall \hat{p} \in \mathcal{P}(\Omega).$$

Proof. Since $I_h^\Omega \hat{Q}_H \hat{p} = I_h^\Omega Q_H I_h^\Omega \hat{p}$, the H^1 -stability of \hat{Q}_H follows from the H^1 -stability of I_h^Ω proven in Lemma 5.1 and the H^1 -stability of L^2 projection.

To prove (6.3), it is enough to bound $\|Q_H I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega}$, since by the triangle inequality and the standard approximation estimate for L^2 -projection we have

$$\begin{aligned} \|I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega} &\leq \|I_h^\Omega \hat{p} - Q_H I_h^\Omega \hat{p}\|_{0,\Omega} + \|Q_H I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega} \\ &\leq CH |I_h^\Omega \hat{p}|_{1,\Omega} + \|Q_H I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega}. \end{aligned}$$

Using (5.6) of Lemma 5.1 and the H^1 -stability of L^2 -projection, we have

$$\|Q_H I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H \hat{p}\|_{0,\Omega} \leq Ch |Q_H I_h^\Omega \hat{p}|_{1,\Omega} \leq Ch |I_h^\Omega \hat{p}|_{1,\Omega},$$

and the lemma is proven. \square

Using the notation introduced in Section 2, we recall the follow lemma due to Dryja and Widlund [17].

Lemma 6.2. *There exists a constant $C > 0$ such that for any $u \in H^1(\Omega_i)$,*

$$\|u\|_{0,\Gamma_{\delta,i}}^2 \leq C \delta^2 \left(\left(1 + \frac{H}{\delta}\right) |u|_{1,\Omega_i}^2 + \frac{1}{H\delta} \|u\|_{0,\Omega_i}^2 \right).$$

6.3. The Additive Schwarz Condition Number Bound. Exploiting the isomorphism between $\mathcal{P}(\Omega)$ and $\tilde{U}_h(\Omega)$ introduced in Section 5, we now derive a bound for the condition number of the additive Schwarz method.

Theorem 6.3. *For the decomposition given in Section 6.1, the condition number of the additive Schwarz method with exact subdomain solves satisfies*

$$\kappa(T) \leq C(1 + H/\delta).$$

The constant C is independent of the parameters H , h and δ , but may depend on the order of the mixed finite element space and the regularity of the mesh.

Proof. The proof of this theorem is patterned after the proof of the analogous theorem of Dryja and Widlund in [17] for the conforming Galerkin finite element method with piecewise linear elements. We first give a direct bound on the largest eigenvalue of T , and then estimate the constant C_0 in (3.6).

As noted in Remark 3.1, for exact solvers, the T_i 's are orthogonal projection onto \mathcal{P}_i and $C_1 = 1$. Hence

$$d(T_i \hat{p}, \hat{p}) \leq d_{\Omega'_i}(\hat{p}, \hat{p}), \quad i \geq 1.$$

By construction of the subspaces, there is an a priori maximum number of subspaces N_{\max} to which each element can belong. Hence,

$$(6.4) \quad \sum_{i=0}^M d(T_i \hat{p}, \hat{p}) \leq d(T_0 \hat{p}, \hat{p}) + \sum_{i=1}^M d_{\Omega'_i}(\hat{p}, \hat{p}) \leq (1 + N_{\max})d(\hat{p}, \hat{p}),$$

which provides an upper bound on $\lambda_{\max}(T)$.

Let $\{\theta_i\}$ be a partition of unity subordinate to $\{\Omega'_i\}_{i=1}^M$; that is, $0 \leq \theta_i(x) \leq 1$, $\text{supp } \theta_i \subset \Omega'_i$ and $\sum_{i=1}^M \theta_i(x) = 1$, $\forall x \in \Omega$. On $\Omega_i \setminus \Gamma_{\delta,i}$, $\theta_i = 1$, and θ_i decreases to zero over a distance proportional to δ . It is easy to construct such a partition also satisfying

$$|\nabla \theta_i| \leq C/\delta.$$

For $\hat{p} \in \mathcal{P}(\Omega)$, let $\hat{p}_0 = \hat{Q}_H \hat{p}$, $\hat{q} = \hat{p} - \hat{p}_0$, and $\hat{p}_i = \mathcal{I}^N(\theta_i \hat{q})$, where \hat{Q}_H is defined in (6.1) and \mathcal{I}^N is interpolation at the primary vertices; then

$$\hat{p} = \hat{p}_0 + \sum_{i=1}^M \hat{p}_i.$$

We now derive a bound on the lower eigenvalue of T by demonstrating that

$$(6.5) \quad \sum_{i=0}^M d(\hat{p}_i, \hat{p}_i) \leq C(1 + H/\delta)d(\hat{p}, \hat{p}).$$

We first estimate $d(\hat{p}_i, \hat{p}_i)$ in terms of $d(\hat{q}, \hat{q})$.

For τ an element of the triangulation contained in Ω'_i , either $\tau \subset \Omega_i \setminus \Gamma_{\delta,i}$, $\tau \subset \Gamma_{\delta,i}$, or $\tau \subset \Gamma_{\delta,j}$ for some subdomain Ω_j that is δ -adjacent to Ω_i . If $\tau \subset \Omega_i \setminus \Gamma_{\delta,i}$, then trivially

$$(6.6) \quad d_\tau(\widehat{p}_i, \widehat{p}_i) = d_\tau(\widehat{q}, \widehat{q}).$$

Assume $\tau \subset \Gamma_{\delta,i}$ and let $\bar{\theta}$ denote the average value of θ_i over τ . Then, we have

$$(6.7) \quad \begin{aligned} d_\tau(\widehat{p}_i, \widehat{p}_i) &= d_\tau(\mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q}) + \bar{\theta}\widehat{q}, \mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q}) + \bar{\theta}\widehat{q}) \\ &\leq 2d_\tau(\mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q}), \mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q})) + 2d_\tau(\widehat{q}, \widehat{q}). \end{aligned}$$

By Theorem 5.2, the standard inverse inequality for linear functions, the bound on the gradient of θ_i , and since $I_h^\tau((\theta_i - \bar{\theta})\widehat{q}) = I_h^\tau((\theta_i - \bar{\theta})I_h^\tau\widehat{q})$, we see that

$$(6.8) \quad \begin{aligned} d_\tau(\mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q}), \mathcal{I}^N((\theta_i - \bar{\theta})\widehat{q})) &\leq C|I_h^\tau((\theta_i - \bar{\theta})\widehat{q})|_{1,\tau}^2 \\ &\leq Ch^{-2}\|I_h^\tau((\theta_i - \bar{\theta})\widehat{q})\|_{0,\tau}^2 \leq C\delta^{-2}\|I_h^\tau\widehat{q}\|_{0,\tau}^2 \leq C\delta^{-2}\|I_h^\Omega\widehat{q}\|_{0,\tau}^2, \end{aligned}$$

where the last inequality follows from (5.7) of Lemma 5.1. Summing (6.7) over elements in $\Gamma_{\delta,i}$ and using (6.8), we see that

$$(6.9) \quad d_{\Gamma_{\delta,i}}(\widehat{p}_i, \widehat{p}_i) \leq C(d_{\Gamma_{\delta,i}}(\widehat{q}, \widehat{q}) + \delta^{-2}\|I_h^\Omega\widehat{q}\|_{0,\Gamma_{\delta,i}}^2).$$

By an application of Lemma 6.2, we conclude that

$$(6.10) \quad d_{\Gamma_{\delta,i}}(\widehat{p}_i, \widehat{p}_i) \leq C \left(d_{\Gamma_{\delta,i}}(\widehat{q}, \widehat{q}) + \left(1 + \frac{H}{\delta}\right) |I_h^\Omega\widehat{q}|_{1,\Omega_i}^2 + \frac{1}{H\delta} \|I_h^\Omega\widehat{q}\|_{0,\Omega_i}^2 \right).$$

Likewise for those $\tau \subset \Gamma_{\delta,j}$, we find that

$$(6.11) \quad d_{\Gamma_{\delta,j}}(\widehat{p}_i, \widehat{p}_i) \leq C \left(d_{\Gamma_{\delta,j}}(\widehat{q}, \widehat{q}) + \left(1 + \frac{H}{\delta}\right) |I_h^\Omega\widehat{q}|_{1,\Omega_j}^2 + \frac{1}{H\delta} \|I_h^\Omega\widehat{q}\|_{0,\Omega_j}^2 \right).$$

Let Ω_i^E be a union of δ -adjacent subdomains of Ω_i that cover Ω'_i . Then combining (6.6), (6.10) and (6.11), as well as the fact that each element in Ω'_i is covered by a fixed maximum number of δ -adjacent subdomains, we conclude that

$$(6.12) \quad d_{\Omega'_i}(\widehat{p}_i, \widehat{p}_i) \leq C \left(d_{\Omega'_i}(\widehat{q}, \widehat{q}) + \left(1 + \frac{H}{\delta}\right) |I_h^\Omega\widehat{q}|_{1,\Omega_i^E}^2 + \frac{1}{H\delta} \|I_h^\Omega\widehat{q}\|_{0,\Omega_i^E}^2 \right).$$

Again, since each element is contained a fixed maximum number of times in $\bigcup_{i=1}^M \Omega_i^E$, we have that

$$(6.13) \quad \sum_{i=1}^M d_{\Omega'_i}(\widehat{p}_i, \widehat{p}_i) \leq C \left(d(\widehat{q}, \widehat{q}) + \left(1 + \frac{H}{\delta}\right) |I_h^\Omega\widehat{q}|_{1,\Omega}^2 + \frac{1}{H\delta} \|I_h^\Omega\widehat{q}\|_{0,\Omega}^2 \right).$$

Since $I_h^\Omega \hat{q} = I_h^\Omega \hat{p} - I_h^\Omega \hat{Q}_H I_h^\Omega \hat{p}$, and by combining (6.3) of Lemma 6.1 and Theorem 5.2, the third term on the right is bounded by $C(H/\delta) d(\hat{p}, \hat{p})$. Applying Theorem 5.2 to the second term on the right we conclude that

$$(6.14) \quad \begin{aligned} \sum_{i=1}^M d_{\Omega'_i}(\hat{p}_i, \hat{p}_i) &\leq C \left(\left(1 + \frac{H}{\delta}\right) d(\hat{q}, \hat{q}) + \frac{H}{\delta} d(\hat{p}, \hat{p}) \right) \\ &\leq C \left(1 + \frac{H}{\delta}\right) (d(\hat{p}, \hat{p}) + d(\hat{p}_0, \hat{p}_0)), \end{aligned}$$

since

$$d(\hat{q}, \hat{q}) \leq 2d(\hat{p}, \hat{p}) + 2d(\hat{p}_0, \hat{p}_0).$$

Finally, $d(\hat{p}_0, \hat{p}_0)$ can be bounded by

$$d(\hat{p}_0, \hat{p}_0) \leq C |I_h^\Omega \hat{Q}_H \hat{p}|_{1,\Omega}^2 \leq C |Q_H I_h^\Omega \hat{p}|_{1,\Omega}^2 \leq C d(\hat{p}, \hat{p}),$$

and hence, (6.5) is verified.

The proof is completed by applying Theorem 3.2 with the largest eigenvalue bounded in (6.4) and $C_0 = C(1 + H/\delta)$. \square

7. SCHWARZ METHODS WITHOUT HYBRIDIZATION

In this section, we consider the application of the Schwarz method for the solution of (4.5)–(4.6), the mixed finite element approximation without hybridization. Analogous to (4.13), define a discrete gradient operator $\widetilde{\nabla}_h^A : W_h(\Omega) \rightarrow \mathbf{V}_h(\Omega)$ by

$$(7.1) \quad \int_{\Omega} A^{-1}(\widetilde{\nabla}_h^A[q]) \cdot \mathbf{v} \, dx = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, dx \quad \forall \mathbf{v} \in \mathbf{V}_h(\Omega).$$

A problem, defined solely in terms of the dual variable p , equivalent to (4.5)–(4.6) is to find $p \in W_h(\Omega)$ such that

$$(7.2) \quad \hat{d}_{\Omega}(p, q) = \int_{\Omega} f q \, dx \quad \forall q \in W_h(\Omega),$$

where

$$\hat{d}_{\Omega}(p, q) \equiv \int_{\Omega} A^{-1} \widetilde{\nabla}_h^A[p] \cdot \widetilde{\nabla}_h^A[q] \, dx = - \int_{\Omega} q \nabla \cdot (\widetilde{\nabla}_h^A[p]) \, dx.$$

As before, the velocity \mathbf{u} can be recovered by setting $\mathbf{u} = -\widetilde{\nabla}_h^A[p]$.

Consider a decomposition of $W_h(\Omega) = \mathcal{W}_0 + \dots + \mathcal{W}_M$. For $i \geq 1$, let \mathcal{W}_i be the set of functions in $W_h(\Omega)$ that vanish outside of Ω'_i . Making use of the natural isomorphism between functions in $W_h(\Omega)$ and the values they attain at the nodal points, let

$$\mathcal{W}_0 = \{p \in W_h(\Omega) \mid p = \mathcal{I}^{\mathcal{W}} \phi, \phi \in U_H(\Omega)\},$$

where $\mathcal{I}^{\mathcal{W}}$ is interpolation at the nodes.

Note that unlike the hybrid case, the support of $\widetilde{\nabla}_h^A[p]$ is not in general contained in the support of p since the velocity space $\mathbf{V}_h(\Omega)$ has continuity constraints across

elements. If one used exact solves on the subdomains, the calculation of the projections T_i would still involve the calculation of the discrete gradient in all of $\mathbf{V}_h(\Omega)$; therefore, the application of the Schwarz method using exact solves is impractical. We are led to consider the following local, approximate solve. Let $e_0(p, q) = \widehat{d}_\Omega(p, q)$, and for each \mathcal{W}_i , $i = 1, \dots, M$, define

$$(7.3) \quad e_i(p, q) = \int_{\Omega'_i} A^{-1} \nabla_i^A [p] \cdot \nabla_i^A [q] dx,$$

where $\nabla_i^A : \mathcal{W} \rightarrow \mathbf{V}_h(\Omega'_i)$ is defined by $\nabla_i^A [p] \in \mathbf{V}_h(\Omega'_i)$ satisfying

$$(7.4) \quad \int_{\Omega'_i} A^{-1} \nabla_i^A [p] \cdot \mathbf{v} = - \int_{\Omega'_i} p \nabla \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbf{V}_h(\Omega'_i).$$

Let $\widehat{T} \equiv \widehat{T}_0 + \widehat{T}_1 + \dots + \widehat{T}_M$, where for $i = 0, \dots, M$, $\widehat{T}_i : \mathcal{W}_i \rightarrow \mathcal{W}_i$ is defined by

$$(7.5) \quad e_i(\widehat{T}_i p, q) = \widehat{d}_\Omega(p, q), \quad \forall q \in \mathcal{W}_i.$$

The following lemma relates the non-hybridized bilinear form \widehat{d} and the inner products e_i to the hybridized bilinear form d .

Lemma 7.1. *For all $q \in W_h(\Omega)$,*

$$(7.6) \quad \widehat{d}_\Omega(q, q) = \inf_{\mu \in \Lambda_h^0(\Omega)} d([q, \mu], [q, \mu]).$$

Likewise, for $i \geq 1$, and for all $q_i \in \mathcal{W}_i$,

$$(7.7) \quad e_i(q_i, q_i) = \inf_{\mu_i \in \Lambda_h^0(\Omega'_i)} d_{\Omega'_i}([q_i, \mu_i], [q_i, \mu_i]).$$

Proof. We prove (7.6); (7.7) is proven analogously. Since $\mathbf{v} \in \mathbf{V}_h^{-1}(\Omega)$ is in $\mathbf{V}_h(\Omega)$ if and only if

$$\sum_{\tau \in \mathcal{T}} \int_{\partial\tau} \lambda \mathbf{v} \cdot \nu_\tau = 0 \quad \forall \lambda \in \Lambda_h^0(\Omega),$$

we see that the definition of the discrete gradient operator in (7.1) is equivalent to finding $\widetilde{\nabla}_h^A [q] \in \mathbf{V}_h^{-1}(\Omega)$ and $\mu[q] \in \Lambda_h^0(\Omega)$ satisfying

$$(7.8) \quad \sum_{\tau \in \mathcal{T}} \left(\int_\tau A^{-1} \widetilde{\nabla}_h^A [q] \cdot \mathbf{v} dx - \int_{\partial\tau} \mu[q] \mathbf{v} \cdot \nu_\tau ds \right) = - \sum_{\tau \in \mathcal{T}} \int_\tau q \nabla \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbf{V}_h^{-1}(\Omega),$$

$$(7.9) \quad - \sum_{\tau \in \mathcal{T}} \int_{\partial\tau} \lambda (\nabla_h^A [q]) \cdot \nu_\tau = 0 \quad \forall \lambda \in \Lambda_h^0(\Omega).$$

Hence, $\widetilde{\nabla}_h^A [q] = \nabla_h^A [q, \mu[q]]$.

We now show that the infimum on the right hand side of (7.6) is attained at $\mu[q]$. Since the right hand side of (7.6) is the minimization of a nonnegative quadratic

function over a finite dimensional convex set, the infimum is attained for some unique choice for μ , call it μ^* . By direct computation using (4.13), one can show that

$$\frac{d}{dt} (d([q, \mu^* + t\hat{\mu}], [q, \mu^* + t\hat{\mu}]))|_{t=0} = 2 \sum_{\tau \in \mathcal{T}} \int_{\partial\tau} \hat{\mu} \nabla_h^A [q, \mu^*] \cdot \nu_\tau ds.$$

Since μ^* is the unique μ for which the directional derivatives vanish for all $\hat{\mu} \in \Lambda_h^0(\Omega)$, we conclude from (7.9) that $\mu^* = \mu[p]$. Hence, $\widetilde{\nabla}_h^A [q] = \nabla_h^A [q, \mu^*]$ and

$$\widehat{d}_\Omega(q, q) = d([q, \mu^*], [q, \mu^*]),$$

which proves the lemma. \square

In order to prove a bound on the condition number of \widehat{T} analogous to Theorem 6.3, we will need an assumption on the decay of $\widetilde{\nabla}_h^A [p]$ away from the support of p . We formulate this assumption in terms of the strengthened Cauchy-Schwarz inequalities.

Assumption 7.1. Let $\Delta\mathcal{E} = \{\epsilon_{ij}\}$, $1 \leq i, j \leq M$, be the matrix with $\epsilon_{ij} = 0$, if Ω_i and Ω_j are δ -adjacent (i.e. $\Omega'_i \cap \Omega'_j$ is empty); otherwise, ϵ_{ij} is the smallest constant for which

$$|\widehat{d}_\Omega(p_i, p_j)| \leq \epsilon_{ij} \widehat{d}_\Omega(p_i, p_i)^{\frac{1}{2}} \widehat{d}_\Omega(p_j, p_j)^{\frac{1}{2}} \quad \forall p_i \in \mathcal{W}_i, \forall p_j \in \mathcal{W}_j.$$

We assume that there exists a constant C_2 , independent of h , H , and δ , such that the spectral radius of $\Delta\mathcal{E}$ satisfies

$$(7.10) \quad \rho(\Delta\mathcal{E}) \leq C_2.$$

Remark 7.1. In most computationally feasible methods, the integrals in (7.1) are evaluated by a clever choice of quadrature rules so that a compact difference stencil results (e.g., cell-centered finite differences [31]). In this case, the support of $\widetilde{\nabla}_h^A [p]$ is within $O(h)$ of the support of p , and Assumption 7.1 is trivially satisfied. If such a choice of quadrature rules is not available, then the right hand side of (7.5) involves the computation of $\widetilde{\nabla}_h^A [p]$ by the solution of a non-local linear system. Most likely, the hybrid method in which there are no non-local problems would offer computational advantages.

Theorem 7.2. *If Assumption 7.1 is satisfied, then the condition number of the additive Schwarz operator \widehat{T} for the non-hybridized mixed method defined in (7.5) satisfies*

$$\kappa(\widehat{T}) \leq C(1 + H/\delta).$$

The constant C is independent of the parameters H , h and δ , but can depend on the order of the mixed method space and the regularity of the mesh.

Proof. The proof is very similar to the proof of Theorem 6.3, so we comment only on the differences due to the non-hybrid form and the inexact solves.

Define a mapping $L : W_h(\Omega) \rightarrow \Lambda_h^0(\Omega)$ such that for each nodal point $n_i \in N$ that corresponds to a degree of freedom of $\Lambda_h^0(\Omega)$ in the interior of Ω , $Lq(n_i)$ is the average of q at all adjacent nodal points of pressure in the $\hat{\mathcal{T}}$ triangulation of Ω . By the construction of $\hat{\mathcal{T}}$, there is at least one adjacent pressure node. Using Lemma 4.1, one can show that

$$(7.11) \quad \inf_{\lambda \in \Lambda_h^0(\Omega)} d([p, \lambda], [p, \lambda]) \simeq d([p, Lp], [p, Lp]),$$

by treating the nodes corresponding to degrees of freedom of $\Lambda_h^0(\Omega)$ in the same manner as secondary vertices were treated in the proof of Lemma 5.1. Therefore, by Lemma 7.1, we have

$$(7.12) \quad \hat{d}_\Omega(p, p) \simeq d([p, Lp], [p, Lp]).$$

Furthermore, by using Lemma 4.1 on $d([p, Lp], [p, Lp])$, one can deduce that for $p \in \mathcal{W}_i$, $i \geq 1$,

$$(7.13) \quad \begin{aligned} \hat{d}_\Omega(p, p) \simeq & \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'_i} |\tau|^{1-\frac{2}{n}} \sum_{\substack{\text{pres. nodes :} \\ n_i, n_j \in \text{adj}(\tau)}} (p(n_i) - p(n_j))^2 \\ & + \sum_{\substack{\tau \in \mathcal{T}, \tau \subset \Omega'_i \\ \partial\tau \cap \partial\Omega \neq \emptyset}} |\tau|^{1-\frac{2}{n}} \sum_{\substack{\text{pres. nodes :} \\ n_i \in \tau}} p(n_i)^2, \end{aligned}$$

where

$$\text{adj}(\tau) = \bigcup_{\substack{\tau_i \in \mathcal{T} \\ \partial\tau \cap \partial\tau_i \neq \emptyset}} \tau_i.$$

The second term in (7.13) arises from the fact that $[p, Lp](n_i) = 0$ for those nodes on the boundary of Ω . Note also that $\text{adj}(\tau)$ can include elements outside Ω'_i .

A similar expression can be derived for $e_i(p, p)$, namely,

$$(7.14) \quad \begin{aligned} e_i(p, p) \simeq & \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'_i} |\tau|^{1-\frac{2}{n}} \sum_{\substack{\text{pres. nodes :} \\ n_i, n_j \in (\text{adj}(\tau) \cap \Omega'_i)}} (p(n_i) - p(n_j))^2 \\ & + \sum_{\substack{\tau \in \mathcal{T}, \tau \subset \Omega'_i \\ \partial\tau \cap \partial\Omega'_i \neq \emptyset}} |\tau|^{1-\frac{2}{n}} \sum_{\substack{\text{pres. nodes :} \\ n_i \in \tau}} p(n_i)^2. \end{aligned}$$

Since $p \in \mathcal{W}_i$ is zero at nodes outside Ω'_i , using (7.13) and (7.14), we conclude that there is a constant C_1 , independent of h , δ , and H , such that

$$(7.15) \quad \hat{d}_\Omega(p, p) \leq C_1 e_i(p, p) \quad \forall p \in \mathcal{W}_i.$$

Using (3.9), it is easy to show that

$$\lambda_{\max}(\hat{T}) \leq C_1(1 + \rho(\Delta\mathcal{E}) + N_{\max}),$$

where N_{\max} is the a priori maximum number of δ -adjacent subdomains, and $\Delta\mathcal{E}$ is defined in Assumption 7.1. Hence, under Assumption 7.1, the largest eigenvalue of \hat{T} is bounded by a constant independent of H , h , and δ .

The proof of the bound on the lower eigenvalue follows the proof of Theorem 6.3 by exploiting the equivalence in (7.12) and the bound in (7.15). \square

Remark 7.2. Note that (7.13) is the analogue of Lemma 4.1 for the mixed method without hybridization. Using (7.13), one can show that the bilinear form $\hat{d}_\Omega(\cdot, \cdot)$ gives rise a norm on $W_h(\Omega)$ equivalent to $|I_h^\Omega([p, Lp])|_{1,\Omega}$. Alternately, one can simply construct a mesh associated with the pressure nodes without regard to the interelement multipliers and prove a similar equivalence.

8. NUMERICAL EXPERIMENTS

In this section, we describe the results of some numerical experiments in which the additive Schwarz method was applied to the non-hybridized mixed finite element discretization using the local solves described in Section 7. To that end, we consider the following elliptic problem on the domain $\Omega = (0, 1)^3$ with boundary $\partial\Omega$:

$$(8.1) \quad -\Delta p = 0 \quad \text{in } \Omega,$$

$$(8.2) \quad p = g_D \quad \text{on } \partial\Omega.$$

The boundary condition g_D was chosen so that the solution is

$$p(x, y, z) = (\cosh(\pi(1 - y)) - \tanh(\pi) \sinh(\pi(1 - y))) \cos(\pi x).$$

Equations (8.1)–(8.2) were discretized using cell-centered finite differences; this is equivalent to using the lowest-order Raviart-Thomas-Nedelec space defined on rectangular solids in 3D with special quadrature rules [31]. In all experiments, the mesh spacing in each coordinate direction was uniform. The initial guess of the solution was zero, and conjugate gradient iterations were continued until a reduction of 10^{-6} was achieved in the relative residual as measured in the energy norm. Estimates of the condition number were obtained by exploiting the similarity between conjugate gradients and Lanczos' method for finding eigenvalues using the code of Ashby, Manteuffel and Joubert [3].

The experiments test two aspects of the condition number bound given in Theorem 7.2. In the first experiment, we took a generous overlap of the subdomains, $\delta = H/4$, and verified that the condition number is uniformly bounded as predicted. In the second experiment, we used a minimal overlap of $\delta = h$. The results of the experiments are presented in Tables 1 and 2, respectively. The results of these experiments, along with some additional decompositions, are displayed in Figure 3.

h	num. of domains	decomposition	A. Schwarz	
			cond.	iter.
1/16	8	2x2x2	8.60	18
1/16	64	4x4x4	7.97	17
1/24	8	2x2x2	8.79	18
1/24	27	3x3x3	9.15	20
1/32	8	2x2x2	8.86	18
1/32	64	4x4x4	9.67	20
1/36	27	4x4x4	9.35	20
1/48	8	2x2x2	9.05	20
1/48	27	3x3x3	9.47	20
1/48	64	4x4x4	9.92	21

TABLE 1. Laplace's Equation with generous overlap, $\delta = H/4$

h	num. of domains	decomposition	A. Schwarz	
			cond.	iter.
1/36	8	2x2x2	24.20	23
1/36	27	3x3x3	16.15	23
1/36	64	4x4x4	14.61	23
1/36	216	6x6x6	10.48	20
1/48	8	2x2x2	31.91	26
1/48	27	3x3x3	21.10	27
1/48	64	4x4x4	19.05	26

TABLE 2. Laplace's Equation with minimal overlap, $\delta = h$

They clearly illustrate that the condition number depends linearly on the ratio of subdomain size to overlap.

9. ACKNOWLEDGMENTS

The author would like to thank his advisor Dr. Mary Wheeler, Dr. Todd Arbogast, and Dr. Joseph Pasciak for their insightful discussions and observations concerning the material in this paper. The author was supported by NSF grant DMS-9112847 during the time of this research.

REFERENCES

1. R. ADAMS, *Sobolev Spaces*, Academic Press, 1975.
2. D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates*, Math. Model. and Numer. Anal., 19 (1985), pp. 7–32.
3. S. F. ASHBY, T. A. MANTEUFFEL, AND P. E. SAYLOR, *A taxonomy for conjugate gradient methods*, SIAM J. Numer. Anal., 27 (1990), pp. 1542–1568.

Summary of Numerical Experiments

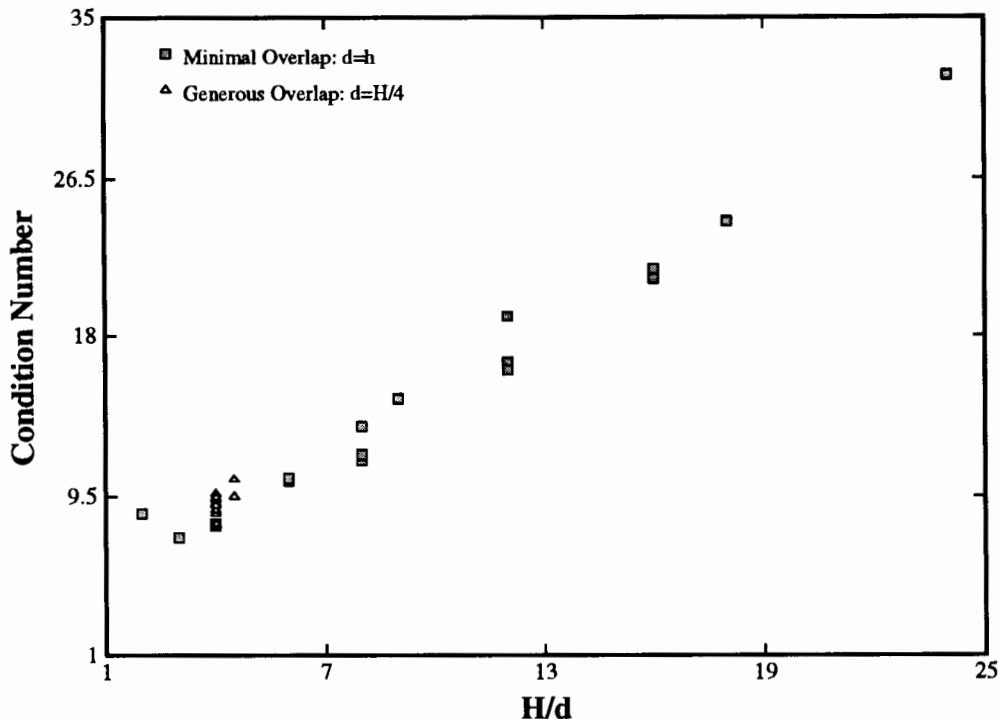


FIGURE 3. Scatter plot of results from Experiment I and II

4. J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. of Comp., 46 (1986), pp. 361–369.
5. J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. of Comp., 57 (1991), pp. 1–21.
6. F. BREZZI, *On the existence, uniqueness and approximation of saddle point problems arising for lagrangian multipliers*, R.A.I.R.O. Anal. Numer., 8 (1974), pp. 129–151.
7. F. BREZZI, J. DOUGLAS, JR, R. DURÁN, AND M. FORTIN, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.
8. F. BREZZI, J. DOUGLAS, JR, M. FORTIN, AND L. D. MARINI, *Efficient rectangular mixed finite elements in two and three space variables*, Math. Modelling and Num. Ana., 21 (1987), pp. 581–604.
9. F. BREZZI, J. DOUGLAS, JR, AND L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.
10. F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.
11. P. G. CIARLET, *The finite element method for elliptic problems*, vol. 4 of Studies in mathematics and its applications, North-Holland Pub. Co., New York, 1978.
12. L. C. COWSAR, *A note on domain decomposition methods for nonconforming finite element spaces of Lagrange-type*. To appear in Proceedings of the Sixth Copper Mountain Conference on Multigrid Methods.
13. L. C. COWSAR, J. MANDEL, AND M. F. WHEELER, *Balancing domain decomposition for*

- mized finite elements*, Dept. Comp. and Appl. Math. TR93-08, Rice University, Mar. 1993.
14. L. C. COWSAR AND M. F. WHEELER, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, in Glowinski et al. [22].
 15. M. DRYJA AND O. B. WIDLUND, *An additive variant of the Schwarz alternating method for the case of many subregions*, Tech. Rep. 339, Courant Institute of Mathematical Sciences, 1987.
 16. ———, *Some domain decomposition algorithms for elliptic problems*, in *Iterative Methods for Large Linear Systems*, D. R. Kincaid and L. J. Hayes, eds., Academic Press, 1989, ch. 16, pp. 273–291.
 17. ———, *Domain decomposition algorithms with small overlap*, Tech. Rep. 606, Department of Computer Science, Courant Institute, May 1992.
 18. ———, *The Neumann-Neumann method as an additive Schwarz method for finite element elliptic problems*, tech. rep., Department of Computer Science, Courant Institute, 1993. In preparation.
 19. L. J. DURLOFSKY AND M. C. H. CHIEN, *Development of a mixed finite-element based compositional reservoir simulator*, in *Proceedings of 12th SPE Symposium on Reservoir Simulation*, SPE, Inc., 1993, Society of Petroleum Engineers, pp. 221–231.
 20. R. E. EWING AND J. WANG, *Analysis of the Schwarz algorithm for mixed finite element methods*, R.A.I.R.O., Modél. Math. Anal. Numér., 26 (1992), pp. 739–756.
 21. R. GLOWINSKI ET AL., eds., *Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, Jan. 1987.
 22. ———, eds., *Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, 1991.
 23. R. GLOWINSKI AND M. F. WHEELER, *Domain decomposition and mixed finite element methods for elliptic problems*, in Glowinski et al. [21], pp. 144–172.
 24. P. L. LIONS, *On the Schwarz alternation method, I*, in Glowinski et al. [21], pp. 1–42.
 25. T. P. MATHEW, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretisations of Elliptic Problems*, PhD thesis, Courant Institute of Mathematical Sciences, Sept. 1989. Also Computer Science Tech. Report 463.
 26. ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and numerical results*. Dec. 1991.
 27. ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Convergence theory*. Dec. 1991.
 28. J. NEDELEC, *Mixed finite elements in \mathbb{R}^3* , Numer. Math., 35 (1980), pp. 315–341.
 29. S. V. NEPOMNYASCHIKH, *Domain decomposition and Schwarz methods in a subspace for the approximate solution of elliptic boundary value problems*, PhD thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, USSR, 1986.
 30. P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for second order elliptic problems*, in *Mathematical Aspects of Finite Element Methods: Lecture Notes in Mathematics 606*, I. Galligani and E. Magenes, eds., Berlin, 1977, Springer-Verlag, pp. 292–315.
 31. A. WEISER AND M. WHEELER, *On convergence of block-centered finite differences for elliptic problems*, SIAM J. Numer. Anal., 25 (1988), pp. 351–357.
 32. J. XU, *Iterative methods by space decomposition and subspace correction*, SIAM Review, 34 (1992), pp. 581–613.

DEPARTMENT OF COMPUTATIONAL AND APPLIED MATHEMATICS, RICE UNIVERSITY, HOUSTON, TX 77251-1892

E-mail address: cowsar@rice.edu