Potential Inversion of the Two Dimensional Plasma Wave Equation

Hua Song
Guanquan Zhang

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Hua Song
Department of Mathematical Sciences
Rice University
Houston, TX 77251
USA

Guanquan Zhang
Computing Center
Chinese Academy of Sciences
Beijing 100080
China

Abstract

A layer-stripping type method is developed for solving an inverse problem for the two dimensional plasma wave equation where the object is to find the potential given Cauchy data on a time-like surface. The key point of the method is that we use one way wave approximation equations instead of the full wave equation for extrapolation of the wave field to avoid instability in numerical computation. Numerical experiments are performed to examine the effectiveness of the method. Numerical results show that this method works well for synthetic data.

1 Introduction

The object of this paper is to describe a numerical method for recovering the potential v in the two dimensional plasma wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z^2} + v(x,z)\right] u(x,z,t) = 0 \tag{1.1}$$

from the boundary response of the half plane medium $z > 0$ to an impulsive line source. One of the origins of this equation, which gives it the name we use, is that it governs a plasma electric field (see Balanis [1]). In one dimension, the inverse problems of the plasma wave equation have been studied extensively by many authors. Roughly speaking, there are three kinds of inverse problems related to this equation. The first one, usually with the name "inverse scattering," is to determine the potential from far field pattern of the wave field. There is a vast literature on inverse scattering problems, see, e.g., Morawetz [2], Cheney [3]. The second kind is determining the potential from some measurements of the responses of a half space medium to some excitements on its boundary, see [4–6]. The third one concerns the reduced form of this equation, i.e., the Schrödinger equation, where the task is to determine the potential from spectral functions of the differential operator, see Gel'fand and Levitan[7], which turns out to be the inspiration of many later reserches on inverse problems of wave equations. Some relations among the three kinds of problems can be found in [2, 5–6]. The first two kinds are in the center of interest of many recent researches, probably because they are closely linked to some very important and interesting real world problems, e.g., many kinds of nondestruction testing with the help of wave probing. We have not seen any work on the third kind of inverse problems for higher dimensional cases.

Our interest lies in constructing a stable numerical method for the second kind of inverse problem in higher dimensions. In one dimension, a lot of work has been done both theoretically [5, 8] and numerically [4, 6]. But in higher dimensions, like inverse problems of all other multidimensional hyperbolic equations, very little is known about the problem. Some partial theoretical results can be found in [3, 9]. We have not seen any numerical results on this problem. Yagle and Levy [10] suggested a layer-stripping type method for determining $v(x, z)$ numerically without any numerical results. The main difficulty, as pointed out by Morawetz and Kriegsmann[4] and Symes[5], is that the problem of extrapolating the wave field along a space-like direction is ill-posed. Hence any straightforward numerical scheme will be unstable.

Our approach aims to overcome this difficulty by splitting the wave field into upgoing and downgoing components. As proved in Song and Zhang [11], the one way wave equations are well-posed when the reference spatial direction is treated as evolution direction. Stable numerical algorithms for one way wave equations have also been studied by many people, e.g. Wang and Zhang [12]. Although the ideas of wave splitting and layer-stripping have been used by many people for a long time on many different occations, as far as the multidimensional inverse problems are concerned, our treatment is novel.

An outline of the idea is like this. First we translate the boundary measurements of the wave field $u(x, z, t)$ into conditions on up-going waves U and down-going waves D by using the definitions of U and D. Then we determine U, D and $v(x, z)$ alternatively layer by layer along the z direction, using the relation between the potential and the up-going waves on the characteristic surface $t = z$:

$$\frac{\partial U(x,z,z)}{\partial t} = -\frac{v(x,z)}{4}\ , \tag{1.2}$$

i.e., suppose we know U and D on layer i. we can get $v(x, z)$ on this layer by (1.2), then we extrapolate U and D along z-direction to the next layer — layer $(i + 1)$, see Figure 1.

This algorithm can be illustrated by the following loop:

Calculate U, D on layer $i = 0$ from boundary measurements;

While (layer i is not the maximal depth), do
  calculate v on layer i by (1.2);
  extrapolate U and D to layer $(i+1)$ using the
  one way wave system;
  $i = i + 1$;
endwhile.

In Section 2, we will describe the problem in more detail; in Section 3, we will use progressing wave expansion and singularity propagation arguments to reformulate the problem to a problem which does not involve singularities. Then we will present our numerical methods and experimental results in Section 4. Finally, we end the paper with some comments in Section 5.

2 The Problem

The following initial-boundary value problem

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + v(x,z)\right] u(x,z,t) = 0 \ , \quad -\infty < x < \infty, z > 0, t > 0, \tag{2.1}$$

$$u(x,0,t) = \delta(t)$$

$$u \equiv 0 \quad \text{for} \quad t < 0$$

is well-posed in the sense of distributions, given $v(x,z)$ smooth enough. This is the "direct problem." Its solution can be interpreted as the wave field in the half space $z > 0$ stimulated by an impulsive line source located on the x-axis. If $v \equiv 0$, the medium is not homogeneous, hence backscatterings will happen, and the quantity $\frac{\partial u}{\partial z}$ can be measured on the boundary $z = 0$. We call this measurement the "boundary responses." The inverse problem is determining the potential $v(x,z)$ from the boundary responses.

One thing that can hardly be avoided by a large class of methods of solving an inverse problem like this is that one needs to extrapolate the wavefield along the z-axis, which is an initial-boundary value problem of the wave equation with Cauchy data given on the time-like surface $z = 0$. This is a well-known ill-posed problem, hence any straightforward numerical methods will be unstable. In order to overcome this difficulty, we approximate the original full wave equation by the following system of one way wave equations (see Song and Zhang [11]).

$$\begin{aligned} -\frac{\partial^2 U}{\partial t \partial z} + \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left[\sum_{k=1}^{n} a_k q_U(s_k)\right] & \\ + \frac{v}{2}\left\{\sum_{k=1}^{n} a_k \left[p_D(s_k) + p_U(s_k)\right] + D + U\right\} &= 0 \\ \frac{\partial^2 D}{\partial t \partial z} + \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left[\sum_{k=1}^{n} a_k q_D(s_k)\right] & \\ + \frac{v}{2}\left\{\sum_{k=1}^{n} a_k \left[p_D(s_k) + p_U(s_k)\right] + D + U\right\} &= 0 \end{aligned}$$

where q_U and q_D satisfy

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q_U(s_k) = \frac{\partial^2 U}{\partial x^2}$$
$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q_D(s_k) = \frac{\partial^2 D}{\partial x^2}$$

and $p_U(s_k)$ and $p_D(s_k)$ satisfy

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) p_U(s_k) - \sum_{k=1}^{n} a_k \frac{\partial^2 p_U(s_k)}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}$$
$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) p_D(s_k) - \sum_{k=1}^{n} a_k \frac{\partial^2 p_D(s_k)}{\partial x^2} = \frac{\partial^2 D}{\partial x^2} .$$

The up-going wave U and down-going wave D are related to the full wave u by the following equations:

$$\frac{\partial U}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{k=1}^{n} a_k q_u(s_k)\right] ,$$
$$\frac{\partial D}{\partial t} = \frac{1}{2}\left(-\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{k=1}^{n} a_k q_u(s_k)\right] ,$$

where $q_u(s_k)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q_u(s_k) = \frac{\partial^2 u}{\partial x^2} .$$

Finally, the coefficients a_k and s_k are

$$a_k = \frac{1}{n+1} \sin^2\left(\frac{k\pi}{n+1}\right) , \quad s_k = \cos\left(\frac{k\pi}{n+1}\right) , \quad k = 1, 2, \ldots, n .$$

For the sense in which this system is an approximation of the original wave equation, and other related discussions, see Song and Zhang [11].

The first thing we need to do is reformulating the original inverse problem in the language of one way waves. In order to make our idea clear, we will limit our discussion to $n = 1$. The reader will realize that this is not a restriction at all, and in many cases, e.g., for horizontally homogeneous media or media with slightly horizontal changes, this is good enough.

When $n = 1$, we have $s_1 = 0$, $a_1 = \frac{1}{2}$, and the one way wave equations become

$$\left(-\frac{\partial^2}{\partial t \partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) U + \frac{v}{2}\left(\frac{p_D + p_U}{2} + D + U\right) = 0 \quad (2.2)$$

$$\left(\frac{\partial^2}{\partial t \partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) D + \frac{v}{2}\left(\frac{p_D + p_U}{2} + D + U\right) = 0 \quad (2.3)$$

where p_U and p_D satisfy

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p_U = \frac{\partial^2 U}{\partial x^2} \quad (2.4)$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p_D = \frac{\partial^2 D}{\partial x^2} \quad (2.5)$$

and the relations between U, D and V are

$$\frac{\partial U}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{4}\frac{\partial q_u}{\partial t} \quad (2.6)$$

$$\frac{\partial D}{\partial t} = \frac{1}{2}\left(-\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{4}\frac{\partial q_u}{\partial t} \quad (2.7)$$

where q_u satisfies

$$\frac{\partial^2 q_u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

From Huyghen's principle, the wavefield u has δ singularity on the surfaces $t = z$, which is the envelope of the characteristic cones with vertices on the x-axis. We can also get this surface mathematically by solving the ray equations; see Courant and Hilbert [13]. From casuality we can write

$$u(x, z, t) = \delta(t - z) + a(x, z, t)H(t - z)$$

with some continuous function $a(x, z, t)$, where H is the Heaviside function. So we have

$$\begin{aligned}\frac{\partial u}{\partial z}(x, 0, t) &= -\delta'(t) - a(x, 0, t)\delta(t) + a_z(x, 0, t)H(t) \\ &= -\,\delta'(t) + a_z(x, 0, t)H(t)\end{aligned}$$

since $u(x,0,t) = \delta(t)$ and

$$\frac{\partial u}{\partial t}(x,0,t) = \delta'(t) \ .$$

From the definition of q_u and casuality, it is clear that $q_u(x,0,t) \equiv 0$. Thus we get

$$\begin{aligned} \frac{\partial U}{\partial t}(x,0,t) &= \frac{1}{2}\, a_z(x,0,t)H(t) \\ \frac{\partial D}{\partial t}(x,0,t) &= \delta'(t) - \frac{1}{2}\, a_z(x,0,t)H(t) \ , \end{aligned}$$

where $a_z(x,0,t)$ can be gotten from the boundary responses $\frac{\partial u}{\partial z}(x,0,t)$ as

$$a_z(z,0,t) = \frac{\partial u}{\partial z}\,(x,0,t) + \delta'(t) \ , \qquad -\infty < x < \infty \ , \quad 0 < t < \infty \ .$$

Now the inverse problem can be reformulated in terms of one way waves as determining $v(x,z)$ in the system (2.2)–(2.5) from the following boundary conditions,

$$\frac{\partial U}{\partial t}\,(x,0,t) = \frac{1}{2}\, u_z(x,0,t)H(t) \ , \tag{2.8}$$

$$\frac{\partial D}{\partial t}\,(x,0,t) = \delta'(t) - \frac{1}{2}\, u_z(x,0,t)H(t) \ , \tag{2.9}$$

where $u_z(x,0,t) = \frac{\partial}{\partial z}\, u(x,0,t) + \delta'(t)$ can be obtained from the boundary measurements, and initial conditions

$$U, D = 0 \quad \text{for} \quad t < 0 \ , \tag{2.10}$$

from the causality.

3 Propagation of Singularities

Using the well-known propagation of singularity arguments (see, e.g., Courant and Hilbert [13]),the forward problem (2.1) can be reformulated equivalently as

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + v(x,z)\right] u(x,z,t) = 0\,, \quad -\infty < x < \infty,\ z > 0,\ t > z, \tag{3.1}$$

$$u(x,0,t) = 0$$

$$u(x,z,z) = -\frac{1}{2}\int_0^z v(x,z)dz =: b(x,z)$$

$$u_t(x,z,z) = \frac{1}{2}\int_0^z (\Delta b - vb)dz$$

where Δ is the Laplacian in $x - z$ plane, for smooth $v(x,z)$. This problem is well-posed and the solution is smooth in the wedge-shaped region

$$\Omega = \{(x,z,t)\ :\ -\infty < x < \infty\,,\ \ z > 0\,,\ \ t > z\}\ .$$

The inverse problem is again determining $v(x,z)$ from $\frac{\partial u}{\partial z}(x,0,t)$.

The advantage of this new formulation is that it is suitable for numerical computations. We want to do the same reduction to the one way wave system (2.2)–(2.5), for which the behavior of singularity propagation is not as obvious as that of (2.1), where the characteristic cone is regular and well-known. First we have the following result.

Theorem 3.1 *Governed by system (2.2)–(2.5), the singularities generated by (2.9) that enter the medium $z > 0$ travel along the plane $t = z$.*

Proof. For the convenience of the proof, we rewrite (2.2) as follows:

$$W_t + AW_x + BWS_z + CW = 0 \tag{3.2}$$

where $W = \left(U, D, \frac{\partial U}{\partial x}, \frac{\partial D}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial D}{\partial t}\right)^T$, and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{v}{2}(1+\frac{H}{2}) & \frac{v}{2}(1+\frac{H}{2}) & 0 & 0 & 0 & 0 \\ \frac{v}{2}(1+\frac{H}{2}) & \frac{v}{2}(1+\frac{H}{2}) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where H is an operator of order zero satisfying

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) H = \frac{\partial^2}{\partial t^2} . \tag{3.3}$$

According to the general theory of wave propagation, the singularities generated at a single point (x_0, z_0) travel along a cone $t = \phi(x, z)$, where $\phi(x, z)$ satisfies

$$\begin{aligned} \det(I - A\phi_x - B\phi_z) &= 0 , \\ \phi(x_0, z_0) &= 0 , \end{aligned}$$

i.e.,

$$\begin{aligned} \phi_x^2 + \phi_z^2 - \frac{1}{4}\phi_x^4 &= 1 \\ \phi(x_0, z_0) &= 0 . \end{aligned} \tag{3.4}$$

This is the parabolic approximation of the corresponding equation for the original wave equation; see Fig. 2. Note, higher order approximations are no longer parabolic.

Equation (3.4) can be factored as

$$\left(\phi_z - 1 + \frac{1}{2}\phi_x^2\right)\left(\phi_z + 1 - \frac{1}{2}\phi_x^2\right) = 0 .$$

$\phi_z + 1 - \frac{1}{2}\phi_x^2 = 0$ is the left portion of the graph of (3.4), representing the waves traveling up into half space $z < 0$; $\phi_z - 1 + \frac{1}{2}\phi_x^2 = 0$ is the right portion of the graph of (3.4), representing the waves traveling down into the medium $z > 0$. According to Huyghen's principle, when the medium is excited by condition (2.9),the waves entering the medium $z > 0$ travel along the branch of the envelope of all the cones $t = \phi(x, z)$, as (x_0, t_0) run all over the x-axis, that lie in the medium $z > 0$. Clearly, this branch of the envelope is composed with all rays which enter the medium normally, and in fact, it is the plane $t = z$. This is actually true. We will verify this conclusion by solving the ray equations in the following manner. Let

$$\phi(x, z, t) = \phi(x, z) - t ,$$

then the characteristic form of equation (3.2) is

$$Q = \phi_t^2\phi_x^2 + \phi_t^2\phi_z^2 - \phi_t^4 - \frac{1}{4}\,\phi_x^4 .$$

Let $p = \phi_x$, $q = \phi_q$, $r = \phi_t = -1$, then the ray equations are

$$\begin{aligned}
\frac{dx}{ds} &= Q_p = 2r^2p - p^3 \\
\frac{dz}{ds} &= Q_q = 2r^2q \\
\frac{dt}{ds} &= Q_r = 2\,(p^2 + q^2)\,r - 4r^3 \\
\frac{dp}{ds} &= -Q_x = 0 \\
\frac{dq}{ds} &= -Q_z = 0 \\
\frac{dr}{ds} &= -Q_t = 0
\end{aligned}$$

where s is a parameter along a ray. Keeping the facts $p^2 + q^2 = 1 + \frac{p^4}{4}$ and $r = -1$ in mind, we get

$$ds = \left(2 - \frac{1}{2}\,p^4\right)^{-1} dt ,$$

and the ray equations for a ray starting at the point $(x_0, 0)$, entering the medium normally, are reparameterized as

$$\begin{aligned}
\frac{dx}{dt} &= \frac{2p}{2+p^2} , & x(0) &= x_0 \\
\frac{dz}{dt} &= \frac{2}{2+p^2} , & z(0) &= 0 \\
\frac{dp}{dt} &= 0 , & p(0) &= 0 \\
\frac{dq}{dt} &= 0 , & q(0) &= 1 .
\end{aligned}$$

Solving this initial value problem, we get the ray

$$\begin{cases} x = x_0 \\ z = t \end{cases}$$

which is a straight line. When $(x_0, 0)$ runs through the x-axis, we get the plane $t = z$.

q.e.d.

In the rest of this section, we will use the progressing wave expansion to discuss the solution of (2.2)–(2.5) with the initial-boundary conditions (2.8)–(2.10) near the characteristic surface $t = z$. As a result we will get a relation between $v(x, z)$ and the upgoing wave $\frac{\partial U}{\partial t}$, which is crucial in solving the inverse problem, on the characteristic surface $t = z$, and a new formulation of the problem as we did for the original wave equation at the beginning of this section. We still use the first order system (3.2) with (3.3) to do calculations, for convenience, and let $p_U = HU$, $p_D = HD$. We expand $W, p_U, p_D, \frac{\partial p_U}{\partial t}, \frac{\partial p_D}{\partial t}$ as follows:

$$\begin{aligned}
W &= g_0\delta' + g_1\delta + g_2 H + R_g , \\
p_U &= a_0\delta' + a_1\delta + a_2 H + R_a , \\
p_D &= b_0\delta' + b_1\delta + b_2 H + R_b , \\
\frac{\partial p_U}{\partial t} &= \alpha_0\delta' + \alpha_1\delta + \alpha_2 H + R_\alpha , \\
\frac{\partial p_D}{\partial t} &= \beta_0\delta' + \beta_1\delta + \beta_2 H + R_\beta ,
\end{aligned}$$

where $g_i = \left(g_i^{(1)}, g_i^{(2)}, \cdots, g_i^{(6)}\right)^T$, $i = 0, 1, 2$, are smooth vector-valued functions of x and z, a_i, b_i, γ_i and β_i, $i = 0, 1, 2$, are smooth functions of x and z, the remainders R_g, R_a, R_b, R_α, and R_β are continuous functions of x, z and t, and δ', δ and H are functions of $t - z$.

In order to get the coefficients g_i, a_i, b_i, α_i and β_i, $i = 0, 1, 2$, we substitute the expansions into (3.2)–(3.3) and compare the like terms with some singularities. We will omit these routine calculations and refer the reader to Courant and Hilbert [13] for a general setting of the method of progressing wave expansions. After all, we get

$$W = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \delta'(t-z) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ g_1^{(6)} \end{pmatrix} \delta(t-z) + \begin{pmatrix} 0 \\ g_1^{(6)} \\ 0 \\ g_{1x}^{(6)} \\ -\frac{v}{4}(x,z) \\ g_2^{(6)} \end{pmatrix} H(t-z) + R_g \quad (3.5)$$

$$\begin{aligned} p_U &= R_a \\ p_D &= R_b \\ \frac{\partial p_U}{\partial t} &= R_\alpha \\ \frac{\partial p_D}{\partial t} &= R_\beta \end{aligned}$$

where

$$g_1^{(6)}(x,z) = -\frac{1}{2}\int_0^z v(x,z)dz \ ,$$

$$g_2^{(6)}(x,z) = \frac{1}{2}\int_0^z [g_{1xx}^{(6)}(x,z) - v(x,z)g_1^{(6)}(x,z)]dz \ .$$

Since the singularity occurs only on $t = z$, by a limit argument, we know that the smooth parts of the functions, still denoted by their old notations, satisfy the following initial-boundary value problem in the wedge shaped region Ω,

$$\left(-\frac{\partial^2}{\partial t \partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) U + \frac{v}{2}\left(\frac{p_D + p_U}{2} + D + U\right) = 0 \quad \text{in } \Omega \ ,$$

$$\left(\frac{\partial^2}{\partial t\partial z}+\frac{\partial^2}{\partial t^2}-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)D+\frac{V}{2}\left(\frac{p_D+p_U}{2}+D+U\right)=0 \quad \text{in } \Omega\ ,$$

$$\frac{\partial U}{\partial t}(x,0,t)=\frac{1}{2}u_z(x,0,t)H(t)\ ,$$

$$\frac{\partial D}{\partial t}(x,0,t)=-\frac{1}{2}u_z(x,0,t)H(t)\ ,$$

where p_D and p_U satisfy

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)p_D=\frac{\partial^2 D}{\partial x^2}\ ,$$

$$p_D=\frac{\partial p_D}{\partial t}=0 \qquad \text{for } t=z\ ,$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)p_U=\frac{\partial^2 U}{\partial x^2}\ ,$$

$$p_U=\frac{\partial p_U}{\partial t}=0 \qquad \text{for } t=z\ .$$

Notice that this is an initial value problem for U and D, with the z-axis as the evolution direction, while an initial value problem for p_U and p_D, with t-axis as evolution direction, on a plane $z=$ constant. The relation between potential $v(x,z)$ and up-going wave U is

$$\frac{\partial U}{\partial t}(x,z,z)=-\frac{v(x,z)}{4}\ .$$

4 Numerical Methods and Results

In this section we describe the numerical methods we use to solve the inverse potential problem and some numerical experiments.

First we choose a potential function $v(x,z)$ and solve the forward problem (3.1) to generate synthetic data $\frac{\partial u}{\partial z}(x,0,t)$. The usual explicit finite difference scheme for two dimensional wave equations is used. We use zero boundary conditions on x, but we take the domain wide enough to reduce reflections.

Then we use the synthetic data to recover potential $v(x,z)$, and compare it with the real potential. We follow the outline in the Introduction section to recover the wave field U, D and potential v alternatively layer by layer. As it is well known, there are some implicit finite difference schemes of solving the one way wave equations along z-direction. But it seems that there is no simple explicit scheme. In order to avoid implicit schemes, we use the following method. We add conditions on U and D in t:

$$\begin{aligned} D(x,z,z) &= -\frac{1}{2}\int_0^z v(x,z)dz \\ U(x,z,T) &= 0 \end{aligned}$$

for some large T. The condition on D is from the expansian (3.5). The condition on U is just approximate. It can be justified by the fact that, if the support of $v(x,z)$ is compact in z, then the "energy"

$$\int_0^t dz \int_{-\infty}^{\infty} dx \left(u_t^2 + u_x^2 + u_z^2 + vu^2\right)(x,z,t)$$

is constant for large t, so if T is much larger than the maximal depth of inversion, we can consider u, hence U, almost zero. Now we use the stencil (a) in Fig. 3 to discrete the equation for U backward in t, the stencil (b) in Fig. 3 to discrete the equation for D forward in t, using v on the previous layer, and an obvious Piccard type iteration to deal with the interactions between U, D and v .

We take $T = 5$, the maximal depth for inversion equals 1, and the width of the computation domain in x direction equals 10 in all the exambles.

Example 1. We take a lateral model

$$v(x,z) = \begin{cases} 0.5exp\left(\frac{1}{(x-0.3)^2-0.2^2} + \frac{1}{0.2^2}\right), & 0.2 < x < 0.5 \\ 0.5exp\left(\frac{1}{(x-0.7)^2-0.2^2} + \frac{1}{0.2^2}\right), & 0.5 < x < 0.9 \\ 0 & \text{otherwise .} \end{cases}$$

The computed potential is almost the same as the real potential. They have at least 4 decimal digits in common at each grid point. See Fig. 4. The solid curve is the computed potential and the dashed curve is the computed potential. The difference between two potentials should be invisible. The inconsistency in the curves is caused by the coarse grid used in computing the inverse problem.

Example 2. In this example, the model changes in x direction:

$$v(x,z) = \begin{cases} 0.5exp\left(\frac{1}{[z-(0.2x+0.5)]^2-0.2^2}+\frac{1}{0.2^2}\right), & \text{if} \quad |z-(0.2x+0.5)|^2 < 0.2 \\ 0, & \text{otherwise}. \end{cases}$$

The true model is illustrated in Fig. 5. The computed potential is illustrated in Fig. 6.

Example 2′. An interesting thing is observed when we change the sign of $v(x,z)$ in Example 2 to get a negative potential. The computed potential(Fig. 7) is very close to the real potential. This is something remarkable noticing that almost all work on inverse potential problems make the assumption that $v(x,z)$ is nonnegative to avoid bound states.

In the above two examples, the maxima of the absolute values of the computed potentials are about 0.601. One definite source of the error is that we let $U = 0$ at $t = T$, which is a good approximation only for very large T. The unfavorable twists are due mainly to the non-absorbing boundary conditions we adopted in x direction.

Example 3. In this example, we work with the model

$$v(x,z) = 0.2\ exp\left(-100\left(\frac{x}{4}\right)^2 - a\ z^2\right)$$

where $a = 150$ or 300. Some of our observations are as follows:

1. For$a = 300$, see Fig. 8 for the true model and Fig. 9 for the computed potential. The peak height of the computed potential is 0.1689.

2. For $a = 150$, the peak height of the computed potential is 0.1575.

This example shows that appropriately more roughness in the vertical direction hopes to improve the resolution of the inversion, probably because the up-going waves carry more information of the medium.

Another important feature of this method is observed when we perform the computation without using the auxiliary functions p_U and p_D . The computation crashes down. A typical graph is Fig. 10. This verified numerically our prediction in Song and Zhang[10] that p_U and p_D (and q_U and q_D in large angle approximations, i.e. $n > 1$) play a very important role in controlling the transversal growth of the reconstructed wavefield.

5 Discussions

We presented a stable numerical method for solving an inverse potential problem of the two dimensional plasma wave equation. Using this method, one does not have the trouble of choosing initial guess or reference model as in some iterative or linearized inversion algorithms. This method takes multiple backscatterings into account. The stability of the method is verified numerically considering that we have made three major approximations: the data, the one way wave equations, and the up-going waves at $t = T$. We also found the important role played by the auxiliary functions in stablizing the method. It works reasonably well for our simple models. For more complicated models, large angle approximations of the one way wave equations and more sophisticated implementations of the method might be neccessary.

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Figuer Captions

1. Layer-stripping

2. Parabolic approximation

3. Stencil (a) is for up-going waves; Stencil (b) is for down-going waves.

4. Example 1. The solid curve is the computed potential; The dashed line is the true model.

5. Example 2. True model.

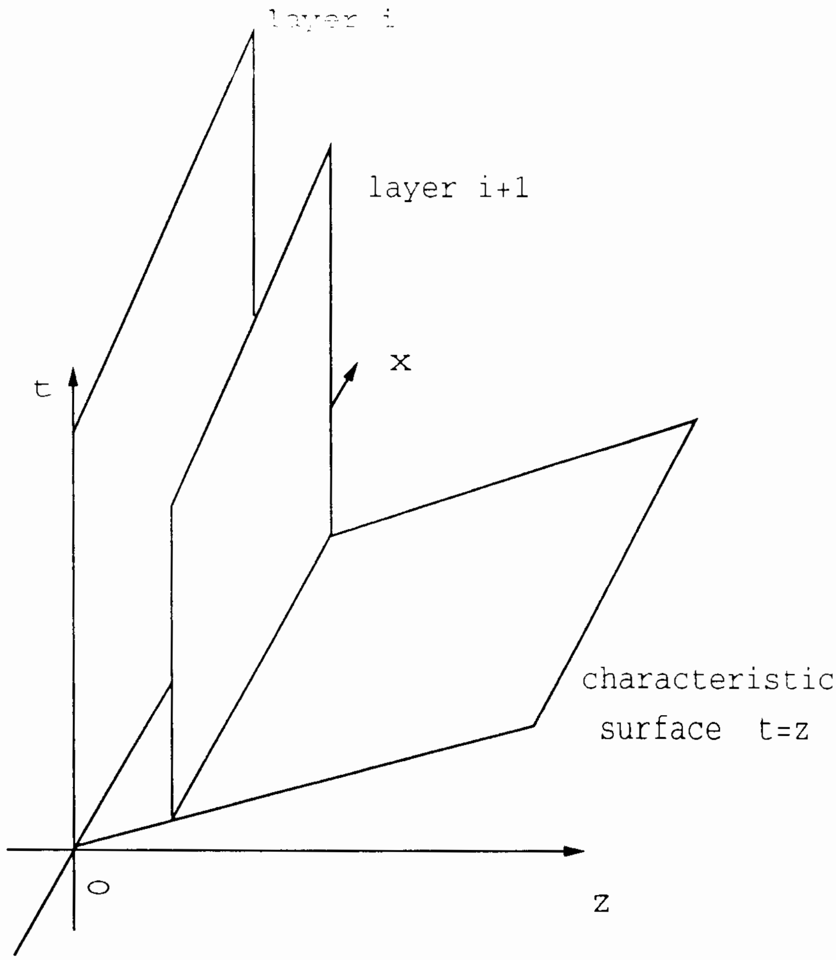
6. Example 2. Computed potential.

7. Example 2′. Computed potential.

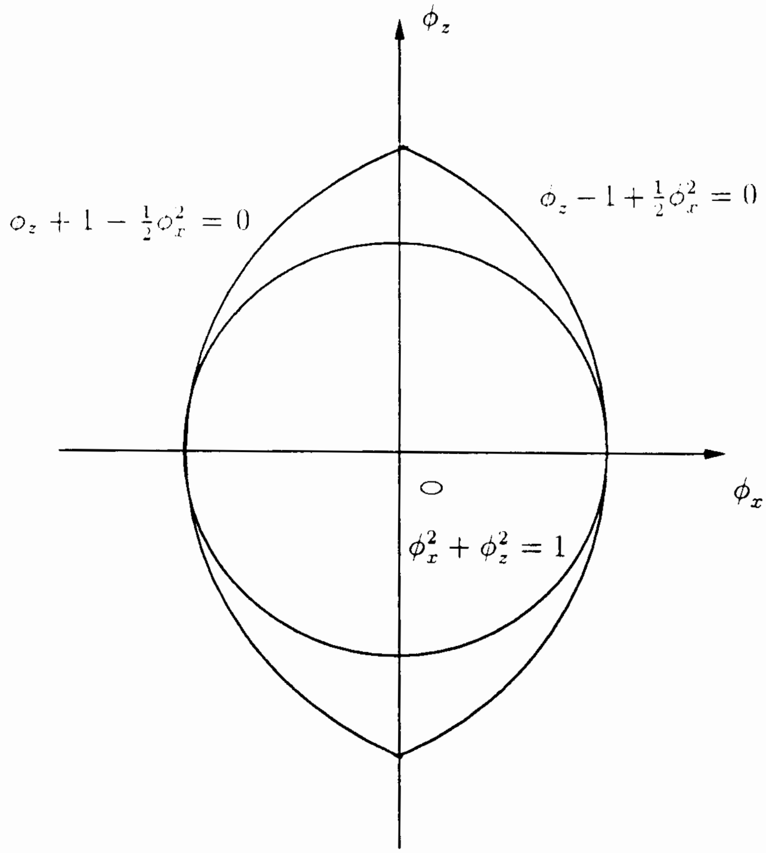
8. Example 3. True model with a=300.

9. Examble 3. Computed potential with a=300.

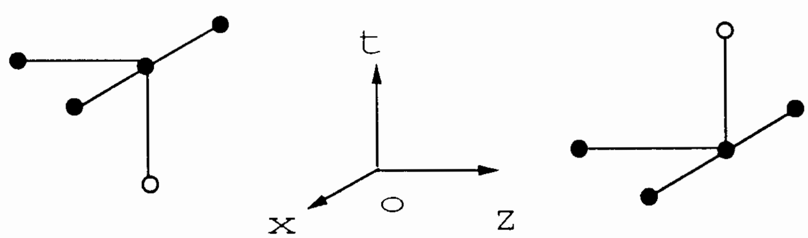
10. Instability happens if the auxiliary functions p_U and p_D are not used.



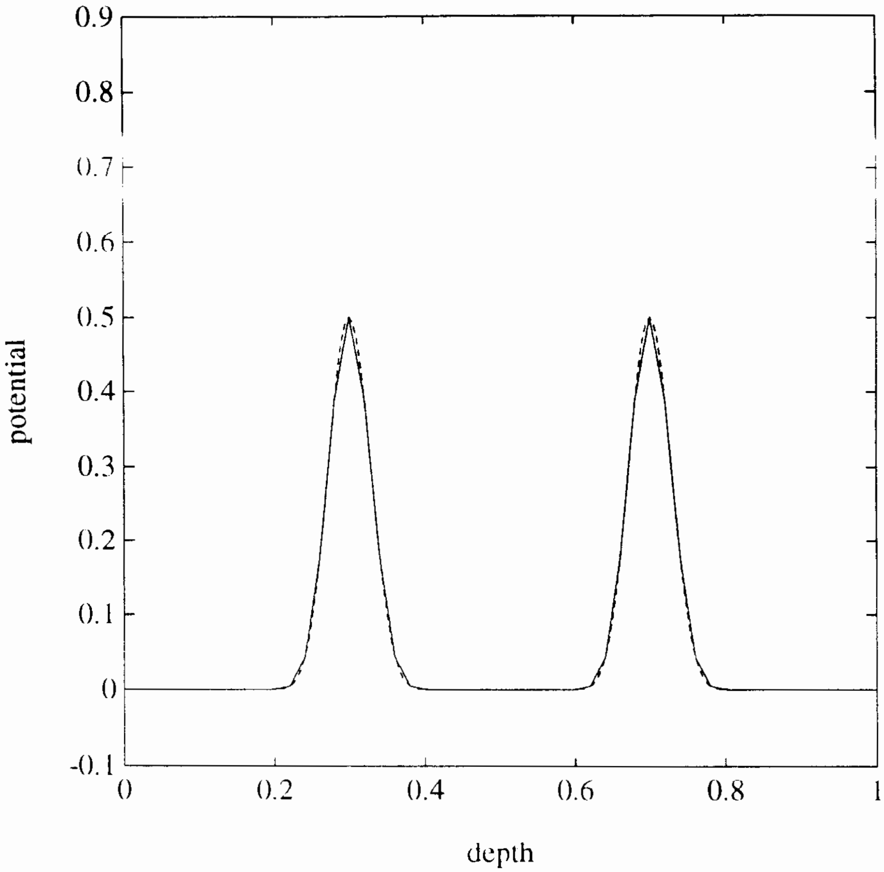
layer i
layer i+1
t
x
characteristic
surface t=z
o
z



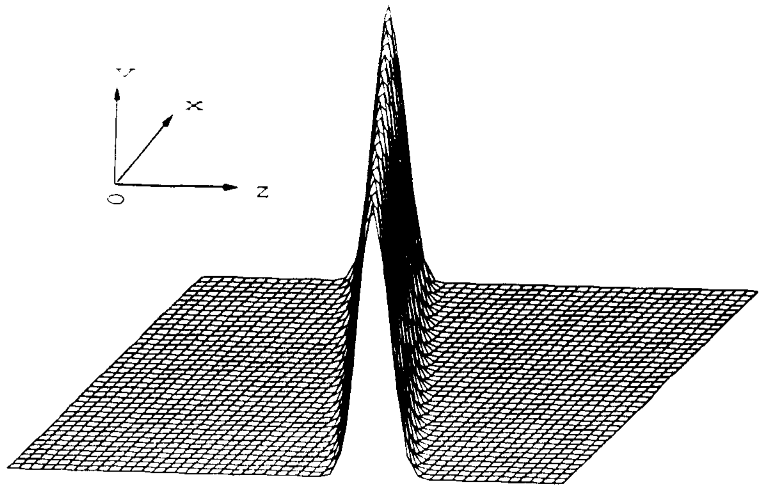
ϕ_z
$\phi_z + 1 - \frac{1}{2}\phi_x^2 = 0$
$\phi_z - 1 + \frac{1}{2}\phi_x^2 = 0$
O
ϕ_x
$\phi_x^2 + \phi_z^2 = 1$



(a) (b)



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
potential
0
0.2
0.4
0.6
0.8
1
depth



y
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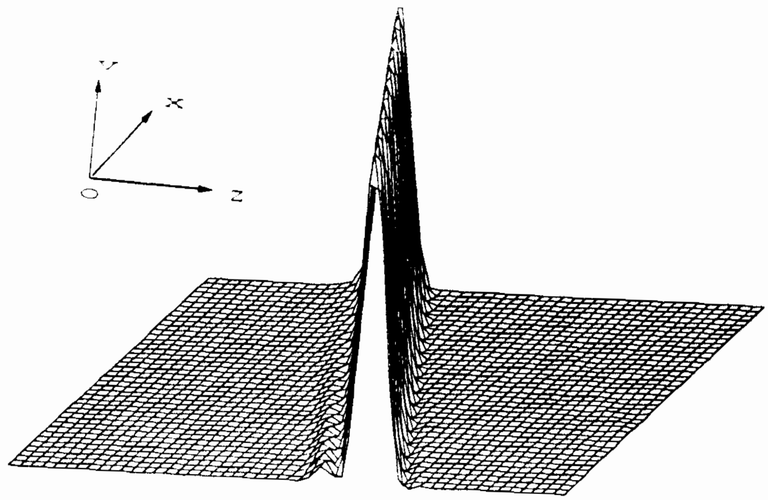
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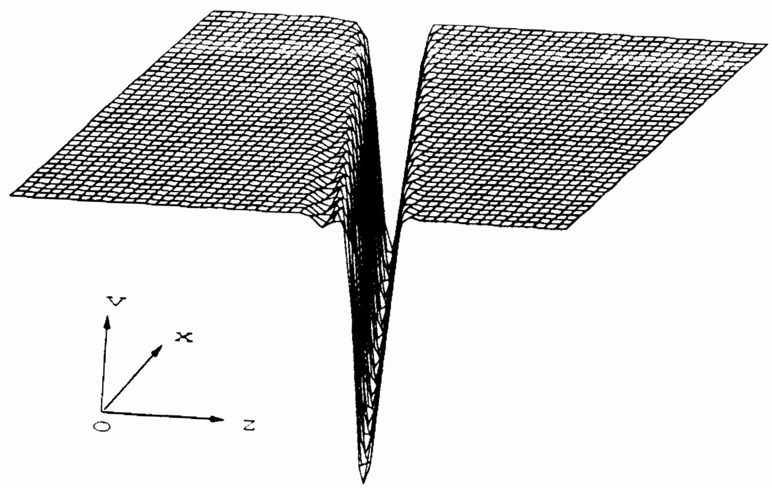

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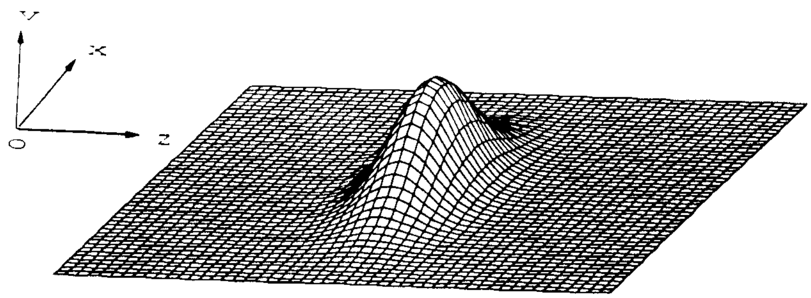
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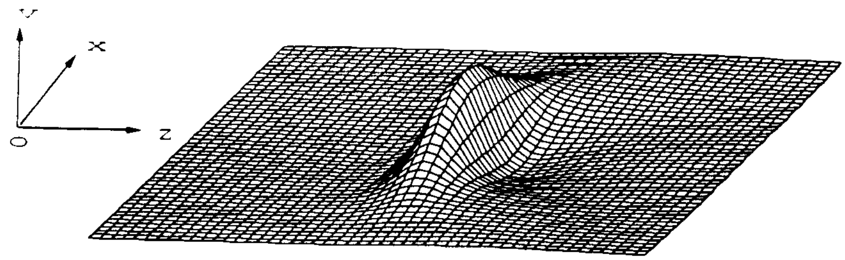
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