Wavefield Splitting and Extrapolation of the Two Dimensional Plasma Wave Equation

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Abstract

Wave splitting is an important technique in a great variety of studies concerning wave propagation. It is a class of methods of splitting the wave equation under consideration into one way wave equations, or physically splitting the wave field into two components with opposite propagation directions. In this paper, a new approach for wave splitting is presented to get a coupled one way wave system for the two dimensional plasma wave equation by using the theory and techniques of pseudo-differential operators. The coupled system and the original equation are equivalent in the sense that they are the same for the singularities propagating in non-glancing directions. A localized approximation of the nonlocal system is given and energy estimates of some related wavefield extrapolation problems, corresponding to the migration problem and wavefield downward continuation problem in exploration geophysics respectively, are obtained. An application of the system to an inverse potential problem is outlined.

Key Words:

plasma wave equation, wave splitting, one way wave equations, wave field extrapolation, potential inversion

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Abbreviated Title: Wavefield Splitting and Extrapolation in Two Dimensions

1 Introduction

Wave splitting is an important method in a great variety of studies concerning wave propagations, e.g., constructing absorbing boundary conditions in numerical simulations of wave propagation[4,6,7], underwater acoustic calculations [8,13], geophysical migration of seismic waves [3,11], and solving inverse problems of partial differential equations [1,10,16]. There are different ways to accomplish the splitting, but the basic idea is always splitting the wave equation under consideration into one way wave equations (in the cases of heterogeneous media, they are coupled because of the existence of reflections, refractions, etc.), or physically splitting the wave field into two(even more, but we only consider two in this paper) components with different propagation directions, called down-going and up-going, left-going and right-going, or out-going and in-coming, etc.,waves, depending on the reference direction adopted.

In one dimension, wave splitting can always be done exactly, i.e. the system of one-way wave equations and the original wave equation are exactly equivalent. This is because waves traveling on a line can be classified into two groups according to their traveling directions without any ambiguity. But in higher dimensions, wave splitting can never be done exactly. For example, one cannot call the waves traveling horizontally either up-going or down-going. It is clear that no matter how many components we use to classify the waves, there are always some of them standing on the boundaries. We call them "glancing waves."

In some circumstances, these waves can be ignored (and this is just what people do in practice). Mathematically, this is where the microlocal analysis of pseudo-differential operators comes into play.

Our interest lies mainly in constructing a stable numerical method for solving inverse problems of higher dimensional wave equations. A main issue in computational inverse problem is the extrapolation of the wave field along a space-like direction. But as a well known fact, the boundary-initial value problem of the wave equation with a space-like evolution direction is not well-posed; hence one has no way to construct a stable numerical method to solve it. This is why we use one way wave equations instead of the full wave equation since in this way we can cure the problem. Hence we also view this treatment as a regularization of the ill-posed problem.

Our approach, inspired also by Zhang [15], uses the theory and techniques of the pseudo-differential operators (see, e.g. Nirenberg [9]) to get a coupled system of one-way wave equations for a model problem, the two dimensional plasma wave equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + v(x,z)\right] u(x,z,t) = 0 \qquad (1.1)$$

where $u(x,z,t)$ is the wavefield function, and $v(x,z)$ is potential. We get the decomposition in Section 2 in the form of a system of pseudo-differential equations. In Section 3, we give a localized approximation of the system and energy estimates for some related wavefield extrapolation problems, corresponding to the migration problem and wavefield downward continuation problem in exploration geophysics respectively, are obtained. We outline an application of the results to the inverse problem of recovering $v(x,z)$ from boundary responses of the medium to an impulsive line source in Section 4. Further research on this inverse problem and numerical results are reported in Song and Zhang [10]. Finally, we end the article with some discussions in Section 5.

2 Decomposition of the Wave Equation

In this section, we deduce a decomposition of the two dimensional plasma wave equation (1.1). First we rewrite the equation as follows:

$$\left[\left(\frac{\partial}{\partial z}+\lambda\right)\left(-\frac{\partial}{\partial z}+\lambda\right)+v\right]u=0 \tag{2.1}$$

where λ is a pseudo-differential operator of the class OP(1) with symbol

$$\sigma(\lambda)=i\,\sqrt{\tau^2-\xi^2}\,,\quad \tau^2>\xi^2\,. \tag{2.2}$$

At this point we need to explain our use of the terms "pseudo-differential operator" and "symbol". In fact, (2.2) is not a symbol in the traditional sense since it is not smoothly defined for all $(\xi,\tau)\in \mathbb{R}^2$. But we can use a nonnegative cut-off function $\rho(\xi,\tau)$ which is a smooth function of degree zero for $|\xi|+|\tau|$ large, with support in $\tau^2>\xi^2$, and identically one in $\tau^2>\xi^2+\epsilon$ for a fixed small $\epsilon>0$. Now $\rho\sigma(\lambda)$ is a symbol. We can identify λ with pseudo-differential operator

$$\int_{R^2} e^{i(\xi x+\tau t)}\, i\sqrt{\tau^2-\xi^2}\,\rho(\xi,\tau)\,\hat{u}(\xi,z,\tau)\,d\xi d\tau\;, \tag{2.3}$$

where $\hat{u}$ is the Fourier transform of u with respect to x and t, and view λ as an elliptic operator, in the directions which lie in the support of ρ. Microlocally, or more concretely, for non-glancing waves with directions away from the glancing directions with a certain angle (depending on the ϵ in $\tau^2>\xi^2+\epsilon$), one should get the same system of one-way equations. So for the sake of simplicity and clearness of the discussion, we will just use λ instead of (2.3) from now on, following a common practice (see [6,7]).

Now we define the up-going and down-going waves as follows.

$$\frac{\partial U}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial z}+\lambda\right)u\,, \tag{2.4}$$

$$\frac{\partial D}{\partial t} = \frac{1}{2}\left(-\frac{\partial}{\partial z}+\lambda\right)u\,. \tag{2.5}$$

We have immediately the following splitting of the wave field

$$u=\lambda^{-1}\,\frac{\partial}{\partial t}\,(D+U)\,, \tag{2.6}$$

where λ^{-1} is a paramatrix of λ, and the following lemma.

Lemma 2.1 *The pseudo-differential operator* λ *has the following expression:*

$$\begin{aligned}\lambda &= (I-R)\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial t}(I-R) \qquad \textit{up to a smooth operator,}\end{aligned}$$

(we will ignore smooth operators throughout this paper) where I *is the identity operator, and* R *is the* $OP(0)$ *pseudo-differential operator defined by*

$$R = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\; Q(s)\,ds\,,$$

and $Q(s)$ *is an* $OP(0)$ *pseudo-differential operator with a parameter* $s \in [-1,1]$*, satisfying the following equation:*

$$\left(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2}\right) Q(s) = \frac{\partial^2}{\partial x^2}\,.$$

Proof. The symbol of $Q(s)$ is

$$\sigma(Q(s)) = \frac{\xi^2}{\tau^2 - s^2\xi^2}\,.$$

Using the following formula, which is easily verified by a trigonometric substitution,

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,\frac{\xi^2}{\tau^2-s^2\xi^2}\,ds = 1-\sqrt{1-\left(\frac{\xi}{\tau}\right)^2}\,,$$

we get the symbol of R as follows:

$$\begin{aligned}\sigma(R) &= \frac{1}{\pi}\int_{-2}^{1}\sqrt{1-s^2}\,\sigma\,(Q(s)\,ds \\ &= 1-\sqrt{1-\left(\frac{\xi}{\tau}\right)^2}\,.\end{aligned}$$

So the symbol of operator $[1-R]\frac{\partial}{\partial t}$ is

$$\begin{aligned}&\left[1-\left(1-\sqrt{1-\left(\frac{\xi}{\tau}\right)^2}\right)\right](i\tau) \\ &= i\sqrt{\tau^2-\xi^2} \\ &= \sigma(\lambda)\,.\end{aligned}$$

Thus, we have $\lambda = (I-R)\frac{\partial}{\partial t}$. Since $\sigma([I-R, \frac{\partial}{\partial t}]) = 0$, where $[I-R, \frac{\partial}{\partial t}]$ is the commutator of $I-R$ and $\frac{\partial}{\partial t}$, we also have $\lambda = \frac{\partial}{\partial t}(I-R)$.

Now we define another pseudo-differential operator H as follows:

$$H = \frac{\partial}{\partial t}\,\lambda^{-1}\ .$$

Regarding the representation of this paper, we have the following lemma.

Lemma 2.2 *H is defined by*

$$Hf = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,p_f(s)ds + f$$

where $p_f(s)$ satisfies the following intego-differential equation

$$\left(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2}\right)p_f(s) - \frac{\partial^2}{\partial x^2}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,p_f(s)ds\right) = \frac{\partial^2}{\partial x^2}f\ ,$$

for f in an appropriate space.

Proof.

Since

$$H = \frac{\partial}{\partial t}\,\lambda^{-1}$$

we have

$$\begin{aligned}(I-R)H &= (I-R)\frac{\partial}{\partial t}\,\lambda^{-1} \\ &= \left[(I-R)\,\frac{\partial}{\partial t}\right]\lambda^{-1} \\ &= \lambda\lambda^{-1} \\ &= I\ .\end{aligned}$$

Now, from $[I-R]Hf = f$, we get

$$Hf = (RH)f + f\ .$$

According to the definition of R (Lemma 2.1), we have

$$R(Hf) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,q_{Hf}(s)ds\ ,$$

where $q_{Hf}(s)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - s^2 \frac{\partial^2}{\partial x^2}\right) q_{Hf} = \frac{\partial^2}{\partial x^2}(Hf)$$
$$= \frac{\partial^2}{\partial x^2}(RHf) + \frac{\partial^2}{\partial x^2} f$$

i.e.

$$\left(\frac{\partial^2}{\partial t^2} - s^2 \frac{\partial^2}{\partial x^2}\right) q_{Hf}(s) - \frac{\partial^2}{\partial x^2}\left[\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-s^2}\, q_{Hf}(s) ds\right] = \frac{\partial^2}{\partial x^2} f \ .$$

Replacing $q_{Hf}(s)$ with $p_f(s)$ completes the proof. q.e.d.

Since $\sigma([\lambda^{-1}, \frac{\partial}{\partial t}]) = 0$, we also have

$$H = \lambda^{-1}\frac{\partial}{\partial t} \ .$$

So the relationship between one way waves U, D and the full wave field u can be written as

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{1}{2}\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial t}(1-R)\right] u \ , \\
\frac{\partial D}{\partial t} &= \frac{1}{2}\left[-\frac{\partial}{\partial z} + \frac{\partial}{\partial t}(1-R)\right] u \ , \\
u &= H(D+U) \ ,
\end{aligned}$$

and the plasma wave equation (1.1) can be splitted as

$$\begin{aligned}
\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial z} + \frac{\partial}{\partial t}(I-R)\right] U + \frac{v}{2} H(D+U) &= 0 \ , \\
\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial t}(I-R)\right] D + \frac{v}{2} H(D+U) &= 0 \ ,
\end{aligned}$$

where R and H are the pseudo-differential operators of order 0 described in Lemma 2.1 and Lemma 2.2. Using their representations given in the two lemmas, we get the following theorem:

Theorem 2.1 *For nonglancing waves the plasma wave equation can be written as the following coupled system of one way waves:*

$$\begin{aligned}
\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right) U-\frac{\partial^2}{\partial t^2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}q_U(s)ds\right] &+ \\
\frac{v}{2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,(p_D+p_U)ds+D+U\right] &= 0 \qquad (2.7)\\
\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right) D-\frac{\partial^2}{\partial t^2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}q_D(s)\,ds\right] &+ \\
\frac{v}{2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,(p_D+p_U)\,ds+D+U\right] &= 0 \qquad (2.8)
\end{aligned}$$

up to smoother errors, where q_U, q_D, p_U, p_D *satisfy the following equations respectively.*

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right) q_U=\frac{\partial^2 U}{\partial x^2}\,, \qquad (2.9)$$

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right) q_D=\frac{\partial^2 D}{\partial x^2}\,, \qquad (2.10)$$

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right) p_U-\frac{\partial^2}{\partial x^2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}p_U(s)ds\right]=\frac{\partial^2 U}{\partial x^2}\,, \qquad (2.11)$$

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right) p_D-\frac{\partial^2}{\partial x^2}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}p_D(s)ds\right]=\frac{\partial^2 D}{\partial x^2}\,, \qquad (2.12)$$

and U, D *and* u *are related by*

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{1}{2}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right) u-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\;q_u(s)ds\right]\\
\frac{\partial D}{\partial t} &= \frac{1}{2}\left(-\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right) u-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\;q_u(s)ds\right]
\end{aligned}$$

where $q_u(s)$ *satisfies*

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right) q_u(s)=\frac{\partial^2 u}{\partial x^2}\,.$$

Remarks.

1. We expressed the pseudo-differential operators by integral-differential operators. The nonlocalness of the system in variables x and t is revealed by the involved integrals with respect to the parameter s.

2. The system is usually coupled. If $v \equiv 0$, we will get a decoupled system of one way wave equations. It is clear that if $v \neq 0$ at some point in the medium, the up-going wave U and down-going wave D interact, and both back-scatterings and transmissions will happen. Since U appears in the equation of D, multiple back-scatterings are taken into account. This is crucial in simulating wave propagation and recovering mechanical parameters in the media which are complex and have large variations with good accuracy.

3. Since we obtained the one way wave system by ignoring the waves propagating transversally and smoothing operators, the system governs the propagation of the high frequency components of non-glancing waves in the same manner as the original plasma wave equation does. This means same wave fronts and same amplitudes. The latter should be understood in the sense that the amplitude of u equals a proper combination of the amplitudes of U and D. This kind of equivalence between the original wave equation and the one way wave system is the crucial idea underneath most applications of the one way wave equations, e.g., migration in seismology [3], and potential inversion of the plasma wave equation [10].

4. More accurate statement of the theorem should be in the language of microlocal analysis. See the discussion at the beginning of this section.

3 Approximation of the Coupled System and Wavefield Extrapolation

As pointed out in the last section, the one way wave system is nonlocal, hence not appropriate for numerical computation. This section is devoted to the localizationof the system. We realize this by using the Gaussian integration formula [5]

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\, f(s)ds = \sum_{k=1}^{n} a_k f(s_k)\,,$$

where

$$\begin{aligned} a_k &= \frac{1}{n+1}\sin^2\frac{k\pi}{n+1}\,, \\ s_k &= \cos\frac{k\pi}{n+1}\,, \quad k = 1, 2, \ldots, n\,. \end{aligned}$$

Applying this formula to the system (2.6)–(2.11), we get the following localized approximation:

$$\begin{aligned} &-\frac{\partial^2 U}{\partial t\partial z} + \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left[\sum_{k=1}^{n} a_k q_U(s_k)\right] \\ &\quad + \frac{v}{2}\left\{\sum_{k=1}^{n} a_k \left[p_D(s_k) + p_U(s_k)\right] + D + U\right\} = 0 \end{aligned} \qquad (3.1)$$

$$\begin{aligned} &\frac{\partial^2 D}{\partial t\partial z} + \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left[\sum_{k=1}^{n} a_k q_D(s_k)\right] \\ &\quad + \frac{v}{2}\left\{\sum_{k=1}^{n} a_k \left[p_D(s_k) + p_U(s_k)\right] + D + U\right\} = 0 \end{aligned} \qquad (3.2)$$

where $q_U(s_k), q_D(s_k), p_U(s_k)$ and $p_D(s_k)$ satisfy the following equations respectively.

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2\frac{\partial^2}{\partial x^2}\right) q_U(s_k) = \frac{\partial^2 U}{\partial x^2}\,, \qquad (3.3)$$

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2\frac{\partial^2}{\partial x^2}\right) q_D(s_k) = \frac{\partial^2 D}{\partial x^2}\,, \qquad (3.4)$$

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2\frac{\partial^2}{\partial x^2}\right) p_U(s_k) - \sum_{k=1}^{n} a_k \frac{\partial^2 p_U(s_k)}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}\,, \tag{3.5}$$

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2\frac{\partial^2}{\partial x^2}\right) p_D(s_k) - \sum_{k=1}^{n} a_k \frac{\partial^2 p_D(s_k)}{\partial x^2} = \frac{\partial^2 D}{\partial x^2}\,, \tag{3.6}$$

and the up-going and down-going waves are

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{k=1}^{n} a_k q_u(s_k)\right]\,,\\
\frac{\partial D}{\partial t} &= \frac{1}{2}\left(-\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right) u - \frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{k=1}^{n} a_k q_u(s_k)\right]\,,
\end{aligned}$$

where $q_u(s_k)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2\frac{\partial^2}{\partial x^2}\right) q_u(s_k) = \frac{\partial^2 u}{\partial x^2}\,.$$

If $v = 0$, $n = 1$, (3.1) and (3.2) will be

$$\left(-\frac{\partial^2}{\partial t\partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) U = 0 \tag{3.1$'$}$$

$$\left(\frac{\partial^2}{\partial t\partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) D = 0 \tag{3.2$'$}$$

which are the well-known 15^0 approximation equations of the one way waves. They got their names because they describe the one way waves fairly well as long as the interface in the medium is less than 15^0 away from the horizontal direction.

Taking $n > 1$, we can get large angle approximation equations for the one way waves. The advantage of the approximation in this form is that no matter what n we take, the system is always of order 2, unlike some other forms which involve higher order derivatives (see [14].)

The equations (3.3)–(3.6) control the growth of the solution in x direction. They play a crucial role in constructing stable numerical schemes for extrapolating the wave field from a time-like surface $z = 0$ along z direction, which is the main use of one way wave equations. This point is confirmed numerically in Song and Zhang[10].

Regarding the well-posedness of the system (3.1)–(3.6), we have the following results. Theorem 3.1 gives an energy estimate for an initial-boundary value problem of the down-going wave equations (3.2) and (3.4) ignoring the lower order term, i.e., taking $v = 0$. Similar results hold for the up-going wave eqautions (3.1) and (3.3) with "final" conditions, which is the well known migration problem in geophysics. We choose to work on equations (3.2) and (3.4) since the final value problem of (3.1) and (3.3) can be transformed to the same form of the initial value problem of (3.2) and (3.4) by a reverse time transform. Theorem 3.2 gives an energy estimate result of the initial-boundary value problem of the system (3.1)–(3.6) with $n = 1$. This problem is the one way wave version of the wavefield extrapolation problem along a space-like direction. Proposition 3.3 implies that the equation (3.6), hence (3.5), consists of a hyperbolic system of order 2.

First we consider the following initial-boundary value problem

$$\frac{\partial^2 D}{\partial t \partial z} + \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left(\sum_{k=1}^{n} a_k q_D(k)\right) = 0 \tag{3.7}$$

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q_D(k) = \frac{\partial^2 D}{\partial x^2} \tag{3.8}$$

with D satisfying the "initial" condition

$$D(x, 0, t) = \phi(x, t)$$

and the"boundary" condition

$$D(x, z, t) = 0 \quad \text{for} \quad t \leq 0$$

and $q_D(k)$ satisfying the initial condition

$$q_D(k) = 0 \quad \text{for} \quad t \leq 0 \, .$$

Notice D satisfies an initial-boundary value problem with z as the evolution direction, but $q_D(k)$ satisfies an initial-value problem with t as the evolution direction on a plane $z =$ constant. In order to simplify the proof, we integrate (3.7) with respect to t on interval $[0, t]$, and denote $q_D(k)$ by q_k. Then we get

$$\frac{\partial D}{\partial z} + \frac{\partial D}{\partial t} - \frac{\partial}{\partial t}\left(\sum_{k=1}^{n} a_k q_k\right) = 0 \tag{3.9}$$

$$\left(\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q_k = \frac{\partial^2 D}{\partial x^2} \tag{3.10}$$

$$
\begin{aligned}
D(x,0,t) &= \phi(x,t) && (3.11)\\
D(x,z,t) &= 0 \quad \text{for} \quad t \le 0 && (3.12)\\
q_k(x,z,t) &= 0 \quad \text{for} \quad t \le 0 \qquad k = 1,2,\ldots,n && (3.13)
\end{aligned}
$$

Theorem 3.1 *If $D, q_1, \cdots, q_n$ satisfy the equations (3.9)–(3.13) and they are smooth enough, and $\frac{\partial}{\partial t}D$, $\frac{\partial}{\partial x}D$, $\frac{\partial}{\partial t}q_k$, $\frac{\partial}{\partial x}q_k$, $k = 1,2,\ldots,n$, are square integrable with respect to $x \in (-\infty, \infty)$, then for any $z > 0$ we have the following identities:*

$$
\int_0^T dt \int dx \left(\frac{\partial D}{\partial t}\right)^2 (x,Z,t) + \int_0^Z dz \int dx \left[\left(\frac{\partial D}{\partial t}\right)^2 + \sum_{k=1}^n a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + \sum_{k=1}^n a_k \left(\frac{\partial D}{\partial x} + s_k^2 \frac{\partial}{\partial x} q_k\right)^2\right](x,z,T) = \int_0^T dt \int dx \left(\frac{\partial \phi}{\partial t}\right)^2 (x,t) \qquad (3.14)
$$

$$
\int_0^T dt \int dx \left(\frac{\partial D}{\partial x}\right)^2 (x,Z,t) + \int_0^Z dz \int dx \left[\left(\frac{\partial D}{\partial x}\right)^2 + \sum_{k=1}^n a_k \left(\frac{\partial q_k}{\partial x}\right)^2 + \sum_{k=1}^n a_k s_k^2 \left(\frac{\partial q_k}{\partial x}\right)^2\right](x,z,T) = \int_0^T dt \int dx \left(\frac{\partial \phi}{\partial x}\right)^2 (x,t) \qquad (3.15)
$$

where the integration with respect to x is from $-\infty$ to $+\infty$ (we will follow this convention from now on).

Proof. Apply $\frac{\partial}{\partial t}$ on (3.9) and multiply it by $2\frac{\partial D}{\partial t}$, then integrate with respect to $x \in (-\infty, \infty)$, we get

$$
\begin{aligned}
0 &= \int dx\, 2\,\frac{\partial D}{\partial t}\left[\frac{\partial^2 D}{\partial t \partial z} + \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left(\sum_{k=1}^n a_k q_k\right)\right]\\
&= \frac{\partial}{\partial z}\int dx \left(\frac{\partial D}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \left(\frac{\partial D}{\partial t}\right)^2 - 2\int dx \frac{\partial D}{\partial t}\,\frac{\partial^2}{\partial t^2}\left(\sum_{k=1}^n a_k q_k\right) \qquad (3.16)
\end{aligned}
$$

Using (3.10), the third integral in (3.16) becomes

$$
-2\int dx\, \frac{\partial D}{\partial t}\sum_{k=1}^n a_k \left(\frac{\partial^2 q_k}{\partial t^2}\right)
$$

$$
\begin{aligned}
&= -2\int dx\, \frac{\partial D}{\partial t}\sum_{k=1}^{n} a_k \left(s_k^2 \frac{\partial^2 q_k}{\partial x^2} + \frac{\partial^2 D}{\partial x^2}\right) \\
&= 2\int dx\, \frac{\partial^2 D}{\partial t \partial x}\sum_{k=1}^{n} a_k \left(s_k^2 \frac{\partial q_k}{\partial x} + \frac{\partial D}{\partial x}\right) \\
&= 2\int dx\, \frac{\partial^2 D}{\partial t \partial x}\, \frac{\partial D}{\partial x} + 2\int dx \frac{\partial}{\partial t}\left[\frac{\partial D}{\partial x}\sum_{k=1}^{n}\left(a_k s_k^2 \, \frac{\partial q_k}{\partial x}\right)\right] \\
&\qquad - 2\int dx\, \frac{\partial D}{\partial x}\sum_{k=1}^{n} a_k s_k^2 \, \frac{\partial^2 q_k}{\partial t \partial x} \\
&= \int dx \frac{\partial}{\partial t}\left[\sum_{k=1}^{n} a_k \left(\frac{\partial D}{\partial x}\right)^2 \right] + 2\,\frac{\partial}{\partial t}\int dx \frac{\partial D}{\partial x}\sum_{k=1}^{n} a_k s_k^2 \frac{\partial q_k}{\partial x} \\
&\qquad + 2\int dx \frac{\partial^2 D}{\partial x^2}\sum_{k=1}^{n}\left(a_k s_k^2 \frac{\partial q_k}{\partial t}\right)\,, \qquad (3.17)
\end{aligned}
$$

where we integrated by parts with respect to x and used the assumption that $\frac{\partial D}{\partial x}(x,\cdot\,,\cdot) \in L^2(-\infty,\infty)$. We will do similar things repeatedly without pointing out. Now the third integral in (3.17) can be reformed further as follows:

$$
\begin{aligned}
& 2\int dx\, \frac{\partial^2 D}{\partial x^2}\sum_{k=1}^{n} a_k s_k^2 \frac{\partial q_k}{\partial t} \\
&= 2\int dx \sum_{k=1}^{n} a_k s_k^2 \, \frac{\partial^2 D}{\partial x^2}\, \frac{\partial q_k}{\partial t} \\
&= 2\int dx \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial^2 q_k}{\partial t^2} - s_k^2 \, \frac{\partial^2 q_k}{\partial x^2}\right) \frac{\partial q_k}{\partial t} \\
&= 2\int dx \sum_{k=1}^{n} a_k s_k^2 \, \frac{\partial^2 q_k}{\partial t^2}\, \frac{\partial q_k}{\partial t} - 2\int dx \sum_{k=1}^{n} a_k s_k^4 \, \frac{\partial^2 q_k}{\partial x^2}\, \frac{\partial q_k}{\partial t} \\
&= \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + 2\int dx \sum_{k=1}^{n} a_k s_k^4 \, \frac{\partial q_k}{\partial x}\, \frac{\partial^2 q_k}{\partial t \partial x}
\end{aligned}
$$

$$= \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^4 \left(\frac{\partial q_k}{\partial x}\right)^2 \qquad (3.18)$$

Substituting (3.18) into (3.17), we get

$$\begin{aligned} & -2\int dx\, \frac{\partial D}{\partial x}\sum_{k=1}^{n} a_k \left(\frac{\partial^2 q_k}{\partial t^2}\right) \\ = \; & \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k \left(\frac{\partial D}{\partial x}\right)^2 + 2\,\frac{\partial}{\partial t}\int dx\, \frac{\partial D}{\partial x}\sum_{k=1}^{n} a_k s_k^2 \frac{\partial q_k}{\partial x} \\ & + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^4 \left(\frac{\partial q_k}{\partial x}\right)^2 \\ = \; & \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k \left(\frac{\partial D}{\partial x} + s_k^2 \frac{\partial q_k}{\partial x}\right)^2 \qquad (3.19) \end{aligned}$$

Substituting (3.19) into (3.16), we get

$$0 = \frac{\partial}{\partial z}\int dx \left(\frac{\partial D}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \left[\left(\frac{\partial D}{\partial t}\right)^2 + \sum_{k=1}^{n} a_k s_k^2 \left(\frac{\partial q_k}{\partial t}\right)^2 + \sum_{k=1}^{n} a_k \left(\frac{\partial D}{\partial x} + s_k^2 \frac{\partial q_k}{\partial x}\right)^2\right].$$

Integrate this identity on $(z,t) \in [0,Z] \times [0,T]$, we get (3.14).

Now, differentiate (3.9) with respect to x, multiply it by $2\frac{\partial D}{\partial x}$, then integrate with respect to $x \in (-\infty, \infty)$ we get

$$\begin{aligned} 0 \;=\; & \int dx\, 2\,\frac{\partial D}{\partial x}\left[\frac{\partial^2 D}{\partial z \partial x} + \frac{\partial^2 D}{\partial t \partial x} - \frac{\partial^2}{\partial t \partial x}\left(\sum_{n=1}^{n} a_k q_k\right)\right] \\ =\; & \frac{\partial}{\partial z}\int dx \left(\frac{\partial D}{\partial x}\right)^2 + \frac{\partial}{\partial t}\int dx \left(\frac{\partial D}{\partial x}\right)^2 - 2\int dx\, \frac{\partial D}{\partial x}\sum_{k=1}^{n}\left[a_k\, \frac{\partial^2 q_k}{\partial t \partial x}\right]. \qquad (3.20) \end{aligned}$$

Multiply (3.10) by $2\frac{\partial q_k}{\partial t}$, then integrate with respect to x on $(-\infty, \infty)$, we get

$$0 \;=\; \int dx \left[\left(\frac{\partial^2}{\partial t^2} - s_k^2\, \frac{\partial^2}{\partial x^2}\right) q_k - \frac{\partial^2 D}{\partial x^2}\right] 2\, \frac{\partial q_k}{\partial t}$$

$$= \frac{\partial}{\partial t}\int dx \left(\frac{\partial q_k}{\partial t}\right)^2 + 2\int dx \frac{\partial q_k}{\partial x}\frac{\partial^2 q_k}{\partial t \partial x} + 2\int dx \frac{\partial D}{\partial x}\frac{\partial^2 q_k}{\partial t \partial x}$$

$$= \frac{\partial}{\partial t}\int dx \left(\frac{\partial q_k}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \left(\frac{\partial q_k}{\partial x}\right)^2 + 2\int dx \frac{\partial D}{\partial x}\frac{\partial^2 q_k}{\partial t \partial x} . \quad (3.21)$$

From (3.20), (3.21) we have

$$0 = \frac{\partial}{\partial z}\int dx \left(\frac{\partial D}{\partial x}\right)^2 + \frac{\partial}{\partial t}\int dx \left(\frac{\partial D}{\partial x}\right)^2 + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k \left(\frac{\partial q_k}{\partial t}\right)^2 + \frac{\partial}{\partial t}\int dx \sum_{k=1}^{n} a_k \left(\frac{\partial q_k}{\partial x}\right)^2 .$$

Integrate this identity on $(z,t) \in [0,Z] \times [0,T]$, we get (3.15).

q.e.d.

Zhang [14] has a similar estimate for a slightly different system in moving coordinates. Bamberger *et al* [2] has a similar estimate for the case $n = 1$.

Now we will consider an initial-boundary value problem of system (3.1)–(3.6) with $n = 1$, with z acting as the evolution direction. We will assume for simplicity of the proof that $v(x,z)$ depends on z only in the following theorem(Theorem 3.2). We claim that similar results are also true for general $v(x,z)$ under appropriate assumptions on its transversal variations, but we are not going to prove them here since our purpose is to show how this problem might be stable and we want to make the proof simple.

Theorem 3.2 *Suppose U, D, p_U and p_D satisfy the initial-boundary value problem*

$$\left(-\frac{\partial^2}{\partial t \partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) U + \frac{v}{2}\left(\frac{p_D + p_U}{2} + D + U\right) = 0 \quad (3.22)$$

$$\left(\frac{\partial^2}{\partial t \partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) D + \frac{v}{2}\left(\frac{p_D + p_U}{2} + D + U\right) = 0 \quad (3.23)$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p_U - \frac{\partial^2 U}{\partial x^2} = 0 \quad (3.24)$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p_D - \frac{\partial^2 D}{\partial x^2} = 0 \quad (3.25)$$

$$U = p_U = 0 \quad for \quad t \le 0$$

$$D = p_D = 0 \quad for \quad t \ge T > 0$$

$$U(x,0,t) = \psi(x,t)$$

$$D(x,0,t) = \phi(x,t)$$

and smooth enough, and $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial t \partial x}$, $\frac{\partial^2 U}{\partial x^2}$, $\frac{\partial D}{\partial t}$, $\frac{\partial D}{\partial x}$, $\frac{\partial^2 D}{\partial t \partial x}$, $\frac{\partial^2 D}{\partial x^2}$, $\frac{\partial p_U}{\partial t}$, $\frac{\partial p_U}{\partial x}$, $\frac{\partial p_D}{\partial t}$, $\frac{\partial p_D}{\partial x}$ *are square integrable on* $x \in (-\infty, \infty)$. *Suppose* v *and the initial data satisfy the following conditions:*

$$\frac{\int_0^T dt \int dx \left(\frac{v(x,0)}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_\psi}{\partial x} + \frac{\partial \psi}{\partial x}\right)^2 (x,t)}{\int dx \frac{v(x,0)}{2} \left(\frac{1}{2}\frac{\partial p_\psi}{\partial x} + \frac{\partial \psi}{\partial x}\right)^2 (x,0)} < 1$$

$$\frac{\int_0^T dt \int dx \left(\frac{v(x,0)}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_\phi}{\partial x} + \frac{\partial \phi}{\partial x}\right)^2 (x,t)}{\int dx \frac{v(x,0)}{2} \left(\frac{1}{2}\frac{\partial p_\phi}{\partial x} + \frac{\partial \phi}{\partial x}\right)^2 (x,0)} < 1$$

where the notations p_ψ *and* p_ϕ *have obvious meanings. Then there exists* $\bar{Z} > 0$ *such that for any* Z, $0 < Z < \bar{Z}$,

$$\begin{aligned} &\int_0^T dt \int dx \left[\left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 + \left(\frac{\partial^2 D}{\partial t \partial x}\right)^2\right](x, Z, t) \\ &\le \; e^Z \int_0^T dt \int dx \left[\left(\frac{\partial^2 \psi}{\partial t \partial x}\right)^2 + \left(\frac{\partial^2 \phi}{\partial t \partial x}\right)^2\right](x,t)\,. \end{aligned}$$

Proof. We denote $\frac{\partial U}{\partial x}$ by $\dot{U}$ for simplicity. Differentiate (3.22) with respect to x, multiply it by $2\frac{\partial \dot{U}}{\partial t}$, then integrate with respect to x in (∞, ∞), we get

$$0 \;=\; \int dx\, 2\, \frac{\partial \dot{U}}{\partial t}\left[-\frac{\partial^2 \dot{U}}{\partial t \partial z} + \frac{\partial^2 \dot{U}}{\partial t^2} - \frac{1}{2}\frac{\partial^2 \dot{U}}{\partial x^2}\right.$$

$$+\frac{v}{2}\left(\frac{\frac{\partial p_D}{\partial x}+\frac{\partial p_U}{\partial x}}{2}+\frac{\partial D}{\partial x}+\frac{\partial U}{\partial x}\right)\Bigg]$$

$$= \int dx\left[-\frac{\partial}{\partial z}\left(\frac{\partial \dot{U}}{\partial t}\right)^2+\frac{\partial}{\partial t}\left(\frac{\partial \dot{U}}{\partial t}\right)^2+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \dot{U}}{\partial x}\right)^2\right.$$

$$\left.+\frac{\partial \dot{U}}{\partial t}\,v\left(\frac{\frac{\partial p_D}{\partial x}+\frac{\partial p_U}{\partial x}}{2}+\frac{\partial D}{\partial x}+\frac{\partial U}{\partial x}\right)\right] \quad (3.26)$$

where we did integration by parts with respect to variable x to the third term, and used the assumption that $\frac{\partial^2 U}{\partial x^2}(\cdot\,,z,t)\in L^2(-\infty,\infty)$. We will do these repeatedly without pointing out. Now we work on the last term on the right-hand side of (3.26). First, it is easy to get

$$\int dx\,\frac{\partial \dot{U}}{\partial t}\,\frac{\partial U}{\partial x}=\int dx\,\frac{\partial^2 U}{\partial t\partial x}\,\frac{\partial U}{\partial x}=\frac{1}{2}\int dx\,\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x}\right)^2. \quad (3.27)$$

Using, (3.24) we have

$$\begin{aligned}0 &= \int dx\,2\,\frac{\partial p_U}{\partial t}\left(\frac{\partial^2 p_U}{\partial t^2}-\frac{1}{2}\frac{\partial^2 p_U}{\partial x^2}-\frac{\partial^2 U}{\partial x^2}\right)\\ &= \int dx\,\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial t}\right)^2+\frac{\partial^2 p_U}{\partial t\partial x}\,\frac{\partial p_U}{\partial x}+2\,\frac{\partial^2 p_U}{\partial t\partial x}\,\frac{\partial U}{\partial x}\\ &= \int dx\,\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\,p_U\right)^2+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial x}\right)^2+2\,\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial x}\,\frac{\partial U}{\partial x}\right)-2\,\frac{\partial p_U}{\partial x}\,\frac{\partial^2 U}{\partial t\partial x}\end{aligned}$$

therefore

$$\int dx\frac{1}{2}\frac{\partial \dot{U}}{\partial t}\,\frac{\partial p_U}{\partial x}=\int dx\,\frac{1}{4}\,\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial t}\right)^2+\frac{1}{8}\,\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial x}\right)^2+\frac{1}{2}\,\frac{\partial}{\partial t}\left(\frac{\partial p_U}{\partial x}\,\frac{\partial U}{\partial x}\right) \quad (3.28)$$

Adding (3.27) and (3.28), we get

$$\int dx\,\frac{\partial \dot{U}}{\partial t}\left(\frac{1}{2}\,\frac{\partial p_U}{\partial x}+\frac{\partial U}{\partial x}\right)$$

$$= \int dx \, \frac{1}{4} \frac{\partial}{\partial t} \left(\frac{\partial p_U}{\partial t} \right)^2 + \frac{1}{2} \left[\frac{1}{4} \frac{\partial}{\partial t} \left(\frac{\partial p_U}{\partial x} \right)^2 + \frac{\partial}{\partial t} \left(\frac{\partial p_U}{\partial x} \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial t} \left(\frac{\partial U}{\partial x} \right)^2 \right]$$

$$= \int dx \, \frac{1}{4} \frac{\partial}{\partial t} \left(\frac{\partial p_U}{\partial t} \right)^2 + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{1}{2} \frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x} \right)^2 .$$

From (3.26) and the above identity we have

$$\int dx \left[\frac{\partial}{\partial z} \left(\frac{\partial \dot{U}}{\partial t} \right)^2 - \frac{\partial}{\partial t} \left(\frac{\partial \dot{U}}{\partial t} \right)^2 - \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \dot{U}}{\partial x} \right)^2 - \frac{v}{4} \frac{\partial}{\partial t} \left(\frac{\partial p_U}{\partial t} \right)^2 - \frac{v}{2} \frac{\partial}{\partial t} \left(\frac{1}{2} \frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x} \right)^2 \right]$$

$$= \int dx \, \frac{\partial \dot{U}}{\partial t} \, v \left(\frac{1}{2} \frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x} \right) .$$

Integrating this equality on $(z, t) \in [0, Z] \times [0, T]$ and using the initial-boundary conditions, we get

$$\int_0^T dt \int dx \left(\frac{\partial \dot{U}}{\partial t} \right)^2 (x, Z, t)$$

$$+ \int_0^Z dz \int dx \left[\left(\frac{\partial \dot{U}}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \dot{U}}{\partial x} \right)^2 + \frac{v}{4} \left(\frac{\partial p_U}{\partial t} \right)^2 \right] (x, z, 0)$$

$$+ \int_0^Z dz \int dx \, \frac{v}{2} \left(\frac{1}{2} \frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x} \right)^2 (x, z, 0)$$

$$= \int_0^T dt \int dx \left(\frac{\partial^2 \psi}{\partial t \partial x} \right)^2 (x, t) + \int_0^Z dz \int_0^T dt \int dx \, \frac{\partial \dot{U}}{\partial t} \, v \left(\frac{1}{2} \frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x} \right) (x, z, t) ,$$

hence,

$$\int_0^T dt \int dx \left(\frac{\partial \dot{U}}{\partial t} \right)^2 (x, Z, t) + \int_0^Z dz \int dx \, \frac{v}{2} \left(\frac{1}{2} \frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x} \right)^2 (x, z, 0)$$

$$\leq \int_0^T dt \int dx \left(\frac{\partial^2 \psi}{\partial t \partial x} \right)^2 (x, t) + \int_0^Z dz \int_0^T dt \int dx \left(\frac{\partial \dot{U}}{\partial t} \right)^2 (x, z, t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,t) \ .$$

Similarly, we have

$$\int_0^T dt \int dx \left(\frac{\partial \dot{D}}{\partial t}\right)^2 (x,Z,t) \ + \ \int_0^Z dz \int dx \frac{v}{2} \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,T)$$

$$\leq \int_0^T dt \int dx \left(\frac{\partial^2 \phi}{\partial t \partial x}\right)^2 (x,t) + \int_0^Z dz \int_0^T dt \int dx \left(\frac{\partial \dot{D}}{\partial t}\right)^2 (x,z,t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,t) \ .$$

Adding the above two inequalities, we get

$$\int_0^T dt \int dx \left[\left(\frac{\partial \dot{U}}{\partial t}\right)^2 + \left(\frac{\partial \dot{D}}{\partial t}\right)^2\right] (x,Z,t)$$

$$+ \int_0^Z dz \int dx \frac{v}{2} \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,T)$$

$$+ \int_0^Z dz \int dx \ \frac{v}{2} \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,0)$$

$$\leq \int_0^T dt \int dx \left[\left(\frac{\partial^2 \psi}{\partial t \partial x}\right)^2 + \left(\frac{\partial^2 \phi}{\partial t \partial x}\right)^2\right] (x,t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left[\left(\frac{\partial \dot{U}}{\partial t}\right)^2 + \left(\frac{\partial \dot{D}}{\partial t}\right)^2\right] (x,z,t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,t) \ . \qquad (3.29)$$

Now using L'Hospital's rule we have

$$\frac{\int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,t)}{\int_0^Z dz \int dx \left(\frac{v}{2}\right) \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,0)}$$

$$\longrightarrow \quad \frac{\int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,0,t)}{\int dx \left(\frac{v}{2}\right) \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,0,0)}$$

$$= \frac{\int_0^T dt \int dx \left(\frac{v(x,0)}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_\psi}{\partial x} + \frac{\partial \psi}{\partial x}\right)^2 (x,t)}{\int dx \frac{v(x,0)}{2} \left(\frac{1}{2}\frac{\partial p_\psi}{\partial x} + \frac{\partial \psi}{\partial x}\right)^2 (x,0)} \qquad \text{as} \quad Z \to 0 \; .$$

Under the assumption of the theorem, we know that there exists a $Z_1 > 0$ such that for $0 < Z < Z_1$,

$$\begin{aligned} &\int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,t) \\ &\leq \int_0^Z dz \int dx \left(\frac{v}{2}\right) \left(\frac{1}{2}\frac{\partial p_U}{\partial x} + \frac{\partial U}{\partial x}\right)^2 (x,z,0) \; . \end{aligned} \tag{3.30}$$

Similarly, there exists $Z_2 > 0$ such that for $0 < Z < Z_2$,

$$\begin{aligned} &\int_0^Z dz \int_0^T dt \int dx \left(\frac{v}{2}\right)^2 \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,t) \\ &\leq \int_0^Z dz \int dx \left(\frac{v}{2}\right) \left(\frac{1}{2}\frac{\partial p_D}{\partial x} + \frac{\partial D}{\partial x}\right)^2 (x,z,T) \; . \end{aligned} \tag{3.31}$$

Let $\bar{Z} = \min\{Z_1, Z_2\}$, and apply inequalities (3.30), (3.31) to (3.29), we get

$$\int_0^T dt \int dx \left[\left(\frac{\partial \dot{U}}{\partial t}\right)^2 + \left(\frac{\partial \dot{D}}{\partial t}\right)^2\right] (x,Z,t)$$

$$\leq \int_0^T dt \int dx \left[\left(\frac{\partial^2 \psi}{\partial t \partial x} \right)^2 + \left(\frac{\partial^2 \phi}{\partial t \partial x} \right)^2 \right] (x,t)$$

$$+ \int_0^Z dz \int_0^T dt \int dx \left[\left(\frac{\partial \dot{U}}{\partial t} \right)^2 + \left(\frac{\partial \dot{D}}{\partial t} \right)^2 \right] (x,z,t) .$$

Now a simple application of the Gronwall's inequality completes the proof.
q.e.d.

The proof above applies to the situations where the "boundary values" are nonzero and/or the "boundaries" $t = 0$ and $t = T$ are replaced by the characteristic boundaries $t = z$ and $t = 2T - z$, respectively, with little change. It also applies to the pure "initial value" problem with z as the evolution variable with appropriate decay assumptions in variable t. We notice that the conditions Theorem 3.2 are consistent in spirit with Symes' observation(see [12]) that in order to make the initial value problem of a wave equation with Cauchy data given on a time-like surface $z = 0$ well-posed, one needs to require more smoothness in the x direction – the transversal direction.

Now we turn to the discussion of equations (3.6). They can be written as the following matrix form

$$\frac{\partial^2}{\partial t^2} W = A \frac{\partial^2}{\partial x^2} W + F$$

where $W = (p_D(s_1), \ldots, p_D(s_n))^T$, $F = \left(\frac{\partial^2}{\partial x^2} D, \ldots, \frac{\partial^2}{\partial x^2} D \right)^T$, and

$$A = \begin{pmatrix} s_1^2 & & & \\ & s_2^2 & & \\ & & \ddots & \\ & & & s_n^2 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} . \qquad (3.32)$$

Proposition 3.3 *All eigenvalues of matrix A are positive and real.*

Proof. Denote

$$m_i = \sin^2 \frac{i\pi}{n+1}$$

and

$$M = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix} .$$

We get $A = I_n - \frac{1}{n+1} D$, where $D = HM$, and

$$H = (n+1)I_n - \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} .$$

Since M is positive definite, it is invertible. So we have

$$D = HM = M^{-\frac{1}{2}} \left(M^{\frac{1}{2}} H M^{\frac{1}{2}} \right) M^{\frac{1}{2}} .$$

Since $M^{\frac{1}{2}} H M^{\frac{1}{2}}$ is symmetric, its eigenvalues are real, hence the eigenvalues of D are real since D and $M^{\frac{1}{2}} H M^{\frac{1}{2}}$ are similar. According to Hamilton's theorem, we have

$$\lambda_i(A) = 1 - \frac{1}{n+1} \lambda_i(D) , \qquad i = 1, 2, \ldots n .$$

Thus, the eigenvalues of A are real.

It is easy to see by Hamilton's theorem that the eigenvalues of H are as follows:

$$\begin{aligned} &\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = n+1 \\ &\lambda_n = 1 , \end{aligned}$$

so H is symetric and positive definite. According to the Courant-Fischer Minimax Theorem for the eigenvalues of symmetric matrices, we have

$$\begin{aligned} \lambda_i(D) &= \lambda_i \left(M^{\frac{1}{2}} H M^{\frac{1}{2}} \right) \\ &\leq \lambda_{\max}(M)\, \lambda_i(H) = \max_{1 \leq j \leq n} \left\{ \sin^2 \left(\frac{j\pi}{n+1} \right) \right\} \lambda_i(H) . \end{aligned}$$

Hence we get

$$\begin{aligned} \lambda_i(A) &\geq 1 - \max_{1 \leq j \leq n} \left\{ \sin^2 \left(\frac{j\pi}{n+1} \right) \right\} , \qquad n = 1, 2, \ldots, n-1 \\ \lambda_n(A) &\geq 1 - \frac{1}{n+1} \max_{1 \leq j \leq n} \left\{ \sin^2 \left(\frac{j\pi}{n+1} \right) \right\} . \end{aligned}$$

Clearly, when n is even, $\lambda_i > 0$ for all $i = 1, 2, \ldots, n$. If n is odd, then we have

$$\lambda_i(A) \geq 0\,, \qquad i = 1, 2, \ldots, n-1 \tag{3.33}$$

$$\lambda_n(A) \geq \frac{n}{n+1}\,. \tag{3.34}$$

We will prove that the eigenvalues of A cannot be zero by proving that A is not singular. Let $k = \frac{n+1}{2}$, then

$$\begin{aligned} s_k &= 0 \\ s_i &\neq 0 \qquad \text{for } i \neq k \\ a_k &= \frac{1}{n+1}\,. \end{aligned}$$

Interchange first row and k^{th} row, then first column and k^{th} column of matrix A in (3.32), we get

$\tilde{A} = PAP^T =$

$$\begin{pmatrix} 0 & & & & & & & \\ & s_2^2 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-1}^2 & & & & \\ & & & & s_1^2 & & & \\ & & & & & s_k^2 & & \\ & & & & & & \ddots & \\ & & & & & & & s_n^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{n+1} & a_2 & \cdots & a_{k-1}, & a_1, & a_k, & \cdots & a_n \\ \frac{1}{n+1} & a_2 & \cdots & a_{k-1}, & a_1, & a_k, & \cdots & a_n \\ & & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & \\ \frac{1}{n+1} & a_2 & \cdots & a_{k-1}, & a_1, & a_k, & \cdots & a_n \end{pmatrix}$$

where P is a basic row transformation matrix. Now let

$$L = \begin{pmatrix} 1 & 0 \\ -e_{n-1} & I_{n-1} \end{pmatrix}$$

where $e_n = (1, \ldots, 1)^T \in \mathbb{R}^{n-1}$. Then we have

$L\tilde{A} =$

$$\begin{pmatrix} 0 & & & & & & & \\ & s_2^2 & & & & & & \\ & & \ddots & & & & & \\ & & & s_{k-1}^2 & & & & \\ & & & & s_1^2 & & & \\ & & & & & s_k^2 & & \\ & & & & & & \ddots & \\ & & & & & & & s_n^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{n+1} & a_2 & \cdots & a_{k-1}, & a_1, & a_k, & \cdots & a_n \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

Now it is clear that

$$\det(A) = \det(\tilde{A}) = \det(L\tilde{A}) = \frac{1}{n+1} \sum_{\substack{i=1 \\ i \neq k}}^{n} s_i^2 > 0 .$$

Hence A is nonsingular, and from (3.33) and (3.34) we can conclude that all eigenvalues of A are positive.

q.e.d.

4 Applications to Inverse Potential Problems

The following initial-boundary value problem

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + v(x,z)\right] u(x,z,t) = 0\,, \quad -\infty < x < \infty\,, \;\; t > 0\,, z > 0$$

$$u(x,0,t) = \delta(t)$$

$$u \equiv 0 \quad \text{for} \quad t < 0$$

is well-posed in the sense of distributions, given $v(x,z)$ is smooth enough. This is the "direct problem." Its solution can be interpreted as the wave field in the half space $z > 0$ stimulated by an impulsive line source located on the x-axis. If $v \not\equiv 0$, the medium is heterogeneous; thus backscatterings will happen, and the quantity $\frac{\partial u}{\partial z}$ can be measured on the surface $z = 0$. We call these measurements "boundary responses." The inverse problem is to determine the potential $v(x,z)$ from the boundary responses.

An important ingredient of the solution of this inverse problem is to extrapolate the wave field from the boundary measurements along the z-direction. This is a Cauchy problem of the wave equation with a space-like direction as the evolution direction, and, as a well-known fact, it is ill-posed. In order to extrapolate the wave field stably along z-direction, we use the one way wave equations instead of the original full wave equation. First we translate the boundary responses into conditions on up-going wave U and down-going wave D on the surface $z = 0$. Then we determine U, D and v alternatively layer by layer, using a relation between the potential and the up-going waves on the characteristiic surface $t = z$:

$$\frac{\partial U(x,z,z)}{\partial t} = -\frac{v(x,z)}{4}\,, \tag{4.1}$$

i.e., suppose we know U and D on layer i, we can get v on this layer by (4.1), then we extrapolate U and D along z-direction to layer $(i+1)$. This algorithm can be illustrated by the following loop:

Calculate U, D on layer $i = 0$ from boundary measurements;
While (layer i is not the maximal depth), do
  Calculate v on layer i by (4.1);

extrapolate U and D to layer $(i+1)$ using the one way wave system;
$i = i + 1$;
endwhile.

For the derivation of (4.1), and a more detailed discussion of the inverse potential problem and its numerical computation, the reader is referred to Song and Zhang [10]. A slight variation of this algorithm can be used to perform wavefiled extrapolation, where v is known, even when the boundary condition is not impulsive. The reason is that the impulsive boundary condition is neccessary only in potential inversion where progressing wave expansion in geometric optics in used to get the identity (4.1).

5 Conclusions

A coupled system of one way wave equations for the two dimensional plasma wave equation has been obtained. A localized approximation of the system has also been presented, which has the advantage that it always involves only second order partial differential operators, even in the case of higher order approximations. This feature is very useful in constructing numerical schemes for the equations. We also discussed two related initial-boundary value problems, which have seismic migration and wavefield extrapolation along a space-like direction as backgrounds. The results in this paper can be used to construct stable numerical methods for extrapolating the wavefield along a space-like direction and for recovering the potential $v(x, z)$ from the boundary responses of the medium to an impulsive line source, as outlined in this paper. Further studies in this direction and numerical results are reported in Song and Zhang [10].

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