

Analysis of Explicit/Implicit, Block Centered
Finite Difference Domain Decomposition
Procedures for Parabolic Problems

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Abstract: Domain decomposition procedures for solving parabolic equations on rectangular meshes are considered. The underlying discretization is block-centered finite differences. In these procedures, fluxes at subdomain interfaces are calculated from the solution at the previous time level. These fluxes serve as Neumann boundary data for implicit, block-centered discretizations in the subdomains. The procedures are globally conservative for homogeneous Neumann boundary conditions. *A priori* error estimates for first and second order time discretizations are derived, and numerical results examining the stability and accuracy of the scheme are presented.

Keywords: Domain decomposition, parabolic equations, block-centered finite differences, mixed finite elements, parallel computing.

AMS(MOS) Subject Classification: 65M60, 65P05

1 Introduction

In this paper, we present a domain decomposition procedure for parabolic partial differential equations, where the underlying discretization is block-centered finite dif-

ferences. Block-centered finite differences are useful for problems in rectangular geometries where approximations to both solution values and fluxes are desired. Unlike point-centered finite differences, the block-centered method gives solution values and fluxes to the same order of accuracy.

For parabolic equations, it is often useful to use implicit time-stepping, due to the severe time step constraint which arises from explicit time-stepping. Thus, a global system of equations must be solved at each time step. Domain decomposition can be used to divide the global problem into smaller subdomain problems, which can be solved in parallel. The difficulty lies in piecing the subdomain solutions together into a reasonable approximation to the true solution.

In the block-centered finite difference approach, solution values are approximated by constants at grid block centers, and fluxes are approximated on the block edges. In the domain decomposition procedure presented here, the interfaces between subdomains coincide with certain grid block edges. At the beginning of each time step, fluxes on these edges are calculated using the solution values from the previous time step. These fluxes then serve as Neumann boundary data for implicit subdomain problems. The explicit interface flux calculations are local and inexpensive. Moreover, the communication cost of passing the flux to the adjacent subdomains should also be inexpensive. Hence, this domain decomposition approach should be nearly optimal, in the sense that the speed-up obtained is roughly equal to the number of subdomains. Furthermore, the method is easy to incorporate into existing implicit codes. The major questions associated with this method have more to do with ap-

proximation properties; i.e., how do we calculate fluxes explicitly so as to maintain stability and accuracy? Thus, the focus of this paper will be on the accuracy of such a procedure, which we shall examine both theoretically and numerically. As we will see, the explicit nature of the flux calculation gives rise to a stability constraint on the time step; however, this constraint is much less severe than the constraint needed for a fully explicit scheme.

The procedures defined here are similar to the Galerkin domain decomposition procedures developed in [1, 2, 3]. In [1, 2], finite element and point-centered finite difference domain decomposition procedures were analyzed, which used Dirichlet boundary data on subdomain interfaces. These approaches are of optimal order in error but have the disadvantage that they are not globally conservative. In [3], we defined conservative Galerkin procedures which use Neumann boundary data at interfaces. These procedures give flexibility in geometry and approximating spaces; however, they do not necessarily give accurate fluxes.

In [4], the scheme presented here is analyzed in the case of one space dimension. In this case, the arguments can be simplified, and a slightly better rate of convergence proven.

In the next section, we state the problem of interest, and in Section 3, we analyze block-centered finite difference-domain decomposition (BCFD-DD) procedures in the case of two subdomains in two space dimensions. Second and fourth order interface flux approximations are presented. We also analyze a second order in time method. In Section 4, we examine the stability and accuracy of the scheme on certain test

problems. The numerical rates of convergence for these problems are slightly better than what is proved here, and in fact, agree with the one-dimensional results in [4]. Finally, in an appendix, the extension of the ideas to multiple rectangular subdomains is discussed.

2 Preliminaries

Let $\Omega = (0, 1) \times (0, 1)$. We will concentrate on two space dimensions. Assume that u^0 , a , and b are smooth, real-valued functions on $\bar{\Omega}$, with $a = \text{diag}(a^x, a^y)$. Assume b is nonnegative, and positive constants a_0^x , a_1^x , a_0^y , and a_1^y exist such that

$$a_0^x \leq a^x(x, y) \leq a_1^x,$$

and

$$a_0^y \leq a^y(x, y) \leq a_1^y.$$

For some $T > 0$, assume the function $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} - \nabla \cdot (a \nabla u) + bu = 0, \quad \text{on } \Omega \times (0, T], \quad (2.1)$$

$$\frac{\partial u}{\partial n_\Omega} = 0, \quad \text{on } \partial\Omega \times (0, T], \quad (2.2)$$

$$u(x, 0) = u^0(x), \quad \text{on } \Omega, \quad (2.3)$$

where n_Ω is the outward normal to $\partial\Omega$. Let q denote the diffusive flux,

$$q = (q^x, q^y) = -(a^x(x, y)u_x, a^y(x, y)u_y). \quad (2.4)$$

Then, by (2.1),

$$\frac{\partial u}{\partial t} + \nabla \cdot q + bu = 0, \quad \text{on } \Omega \times (0, T], \quad (2.5)$$

and, by (2.2),

$$q \cdot n_\Omega = 0, \quad \text{on } \partial\Omega \times (0, T]. \quad (2.6)$$

In the calculation of the flux on an interface between subdomains, we integrate the approximate solution at the previous time step against a one-dimensional function.

For future reference we define two special functions ϕ_2 and ϕ_4 as follows:

$$\phi_2(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1, \\ x + 1, & -1 \leq x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_4(x) = \begin{cases} (x - 2)/12, & 1 \leq x \leq 2, \\ -5x/4 + 7/6, & 0 \leq x \leq 1, \\ 5x/4 + 7/6, & -1 \leq x \leq 0, \\ -(x + 2)/12, & -2 \leq x \leq -1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $p(x)$ is a polynomial of degree at most one then

$$\int_{\mathbf{R}} p(x)\phi_2(x)dx = p(0),$$

and if p is a polynomial of degree at most three

$$\int_{\mathbf{R}} p(x)\phi_4(x)dx = p(0).$$

3 BCFD-DD methods

In this section, we define BCFD-DD methods in the case of two subdomains. For simplicity, take $b = 1$ in (2.1). The case $b > 0$ and bounded above is a trivial extension and only involves the adjustment of some constants in our arguments.

We first establish some notation. Let δ_x and δ_y be partitions of $(0, 1)$:

$$\delta_x : 0 = x_{1/2} < x_{3/2} < \dots < x_{I+1/2} = 1,$$

$$\delta_y : 0 = y_{1/2} < y_{3/2} < \dots < y_{J+1/2} = 1,$$

and set, for $i = 1, \dots, I$,

$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2},$$

$$h_i^x = x_{i+1/2} - x_{i-1/2},$$

$$h_{i+1/2}^x = x_{i+1} - x_i,$$

$$\Omega_i^x = [x_{i-1/2}, x_{i+1/2}],$$

with similar definitions for y_j , h_j^y , $h_{j+1/2}^y$, and Ω_j^y , $j = 1, \dots, J$. Let $h^x = \max_i h_i^x$, $h^y = \max_j h_j^y$, $h = \max(h^x, h^y)$, and $\Omega_{i,j} = \Omega_i^x \times \Omega_j^y$.

For $f(x, y)$ and $g(x, y)$, let $f_{i,j} = f(x_i, y_j)$, $f_{i+1/2,j} = f(x_{i+1/2}, y_j)$ and $f_{i,j+1/2} = f(x_i, y_{j+1/2})$. Define the discrete inner products:

$$\begin{aligned} \langle f, g \rangle_{x, a^x} &= \sum_{j=1}^J \sum_{i=1}^{I-1} \frac{1}{a_{i+1/2, j}^x} f_{i+1/2, j} g_{i+1/2, j} h_{i+1/2}^x h_j^y, \\ \langle f, g \rangle_{y, a^y} &= \sum_{j=1}^{J-1} \sum_{i=1}^I \frac{1}{a_{i, j+1/2}^y} f_{i, j+1/2} g_{i, j+1/2} h_i^x h_{j+1/2}^y \end{aligned}$$

and the corresponding seminorms

$$|||f|||_{x, a^x}^2 = \langle f, f \rangle_{x, a^x},$$

$$\|f\|_{y,a^y}^2 = \langle f, f \rangle_{y,a^y}.$$

Let $\langle f, g \rangle_x$, $\|f\|_x$ denote $\langle f, g \rangle_{x,a^x}$ and $\|f\|_{x,a^x}$ with $a^x \equiv 1$, similarly for $\langle f, g \rangle_y$ and $\|f\|_y$. Note that $\|f\|_x$ ($\|f\|_y$) and $\|f\|_{x,a^x}$ ($\|f\|_{y,a^y}$) are equivalent:

$$a_0^x \|f\|_{x,a^x} \leq \|f\|_x \leq a_1^x \|f\|_{x,a^x}. \quad (3.1)$$

Let $(\cdot, \cdot)_\Sigma$ denote the $L^2(\Sigma)$ inner product. When $\Sigma = \Omega$, we will omit the subscript. Let $\|\cdot\|$ denote the L^2 norm on Ω .

It has been noted in [6, 7] that block-centered finite differences can be derived from the mixed finite element method using the lowest-order Raviart-Thomas approximating spaces [5] and special quadrature rules. Denote by $\mathcal{M}_{-1}(d; h^x)$ the finite dimensional subspace of $L^2(0, 1)$ consisting of all functions which are polynomials of degree d on each interval Ω_i^x . For $r \geq 0$, let

$$\mathcal{M}_r(d; h^x) = \mathcal{M}_{-1}(d; h^x) \cap C^r(0, 1),$$

and

$$\mathcal{M}_r^0(d; h^x) = \mathcal{M}_r(d; h^x) \cap \{v(x) \mid v(0) = v(1) = 0\}.$$

Define $\mathcal{M}_{-1}(d; h^y)$, $\mathcal{M}_r(d; h^y)$, and $\mathcal{M}_r^0(d; h^y)$ similarly. Let $(\mathcal{Q}, \mathcal{U})$ be the lowest order Raviart-Thomas approximating spaces; i.e., $\mathcal{Q} = \mathcal{Q}^x \times \mathcal{Q}^y$, where

$$\mathcal{Q}^x = \mathcal{M}_0^0(1; h^x) \otimes \mathcal{M}_{-1}(0; h^y),$$

$$\mathcal{Q}^y = \mathcal{M}_{-1}(0; h^x) \otimes \mathcal{M}_0^0(1; h^y),$$

and

$$\mathcal{U} = \mathcal{M}_{-1}(0; h^x) \otimes \mathcal{M}_{-1}(0; h^y).$$

Note that the dimensions of \mathcal{Q}^x , \mathcal{Q}^y , and \mathcal{U} are $(I-1)J$, $I(J-1)$, and $I \cdot J$, respectively. By using a standard nodal basis, namely, the linear “chapeau” functions in x tensored with piecewise constants in y , a function $v^x \in \mathcal{Q}^x$ is determined by its values at the points $(x_{i+1/2}, y_j)$, $i = 1, \dots, I-1$, $j = 1, \dots, J$. Similarly, a function $v^y \in \mathcal{Q}^y$ is determined by its values at $(x_i, y_{j+1/2})$, $i = 1, \dots, I$, $j = 1, \dots, J-1$. A function $w \in \mathcal{U}$ is piecewise constant on $\Omega_{i,j}$, $i = 1, \dots, I$, $j = 1, \dots, J$. Denote this constant by $w_{i,j}$.

Assume a decomposition of Ω into two strips, $\Omega_1 = (0, \bar{x}) \times (0, 1)$ and $\Omega_2 = (\bar{x}, 1) \times (0, 1)$, where $\bar{x} = x_{\bar{k}+1/2} \in \delta_x$ for some integer \bar{k} , $0 < \bar{k} < I$. Let $0 < H \leq \min(\bar{x}, 1-\bar{x})$, and assume $\bar{x} - H$, $\bar{x} + H$ are also in δ_x .

For a given smooth function ψ , let $B(\psi)(\bar{x}, y)$ denote an approximation to $\psi_x(\bar{x}, y)$, determined by

$$B(\psi)(\bar{x}, y) = -\frac{1}{H} \int_0^1 \phi'(x) \psi(x, y) dx, \quad (3.2)$$

where $\phi = \phi_2((x - \bar{x})/H)$. We note that, for ψ thrice differentiable in x ,

$$|\psi_x(\bar{x}, y) - B(\psi)(\bar{x}, y)| \leq CH^2. \quad (3.3)$$

A BCFD-DD procedure can be defined as follows. Let $0 = t^0 < t^1 < \dots < t^M = T$ be a given sequence, $\Delta t^n = t^n - t^{n-1}$, and for $f = f(t)$, let

$$\partial_t f^n = \frac{f^n - f^{n-1}}{\Delta t^n}.$$

Assume $U^{n-1} \in \mathcal{U}$ is given. Let $Q^n = (Q^{x,n}, Q^{y,n}) \in \mathcal{Q}$. First, approximate $q^{x,n}(\bar{x}, y_j)$ by $Q^{x,n}(\bar{x}, y_j)$:

$$Q^{x,n}(\bar{x}, y_j) \equiv Q_{\bar{k}+1/2,j}^{x,n} = -\bar{a}_j^x B(U^{n-1})(\bar{x}, y_j), \quad j = 1, \dots, J. \quad (3.4)$$

The term \bar{a}_j^x can be considered in two ways. From the finite difference viewpoint, $\bar{a}_j^x = a^x(\bar{x}, y_j)$. From the mixed finite element viewpoint,

$$\frac{1}{\bar{a}_j^x} \approx \frac{1}{2H} \int_0^1 \frac{\phi(x)}{a^x(x, y_j)} dx.$$

Here we take the latter approach. We will consider the former viewpoint later. Thus, set

$$\frac{1}{\bar{a}_j^x} = \frac{1}{2H} \sum_i h_{i+1/2}^x \frac{\phi_{i+1/2}}{a_{i+1/2,j}^x}. \quad (3.5)$$

Enforce the boundary condition (2.6) by setting

$$Q_{1/2,j}^{x,n} = Q_{I+1/2,j}^{x,n} = 0, \quad j = 1, \dots, J, \quad (3.6)$$

and

$$Q_{i,1/2}^{y,n} = Q_{i,J+1/2}^{y,n} = 0, \quad i = 1, \dots, I. \quad (3.7)$$

For $j = 1, \dots, J$, $1 \leq i \leq I-1$, $i \neq \bar{k}$, approximate $q_{i+1/2,j}^{x,n}$ by

$$Q_{i+1/2,j}^{x,n} = -a_{i+1/2,j}^x \frac{U_{i+1,j}^n - U_{i,j}^n}{h_{i+1/2}^x}. \quad (3.8)$$

For $i = 1, \dots, I$, $j = 1, \dots, J-1$, approximate $q_{i,j+1/2}^{y,n}$ by

$$Q_{i,j+1/2}^{y,n} = -a_{i,j+1/2}^y \frac{U_{i,j+1}^n - U_{i,j}^n}{h_{j+1/2}^y}, \quad (3.9)$$

and approximate $u_{i,j}^n$ by $U_{i,j}^n$, where

$$\partial_i U_{i,j}^n + \frac{Q_{i+1/2,j}^{x,n} - Q_{i-1/2,j}^{x,n}}{h_i^x} + \frac{Q_{i,j+1/2}^{y,n} - Q_{i,j-1/2}^{y,n}}{h_j^y} + U_{i,j}^n = 0. \quad (3.10)$$

Note that (3.8) and (3.9) are finite difference versions of (2.4), and (3.10) is a finite difference discretization of (2.5). For nonuniform mesh, the truncation error associated

with (3.10) is $\mathcal{O}(1)$; however, the method is convergent, as we will show. Substituting (3.4), (3.6), (3.7), (3.8), and (3.9) into (3.10) where appropriate, we obtain a positive definite, symmetric system of equations for determining U^n . Since the flux $Q_{\bar{k}+1/2,j}^{x,n}$ is determined independently of U^n , the resulting system of equations decouples into two disjoint sets of equations, corresponding to the subdomains Ω_1 and Ω_2 . These systems can be solved simultaneously.

Note that this procedure is conservative in the following sense. If $b = 0$ in (2.5) and the analogue of (3.10) is integrated over $\Omega_{i,j}$ and the result summed on i and j , then applying (3.6) and (3.7), $(U^n, 1) = (U^{n-1}, 1)$.

The initial condition can be enforced by setting

$$U^0 = \tilde{u}^0. \quad (3.11)$$

where \tilde{u}^0 is the L^2 -projection of u^0 into \mathcal{U} ; i.e.,

$$(u^0 - \tilde{u}^0, w) = 0, \quad w \in \mathcal{U}. \quad (3.12)$$

Thus, determining U^0 is a completely local calculation.

Note that (3.8)-(3.10) is equivalent to the following system of equations:

$$\langle Q^{x,n}, v^x \rangle_{x,a^x} + \langle Q^{y,n}, v^y \rangle_{y,a^y} - (U^n, \nabla \cdot v) = 0, \quad v \in \bar{\mathcal{Q}}, \quad (3.13)$$

$$(\partial_t U^n, w) + (\nabla \cdot Q^n, w) + (U^n, w) = 0, \quad w \in \mathcal{U}, \quad (3.14)$$

where

$$\bar{\mathcal{Q}}^x = \mathcal{Q}^x \cap \{v^x \mid v^x(\bar{x}, y) = 0\},$$

$$\bar{\mathcal{Q}} = \bar{\mathcal{Q}}^x \times \mathcal{Q}^y.$$

Note also that $\phi \in \mathcal{M}_0^0(1; h^x)$.

In order to state and prove an error estimate for the scheme given by (3.4)-(3.11), define $\bar{Q} \in \mathcal{Q}$, $\bar{U} \in \mathcal{U}$ to be the elliptic block-centered finite difference approximations to q and u ; that is, for each $t \in [0, T]$,

$$\langle \bar{Q}^x, v^x \rangle_{x, a^x} + \langle \bar{Q}^y, v^y \rangle_{y, a^y} - (\bar{U}, \nabla \cdot v) = 0, \quad v \in \mathcal{Q}, \quad (3.15)$$

$$(\nabla \cdot \bar{Q}, w) + (\bar{U}, w) = (\nabla \cdot q, w) + (u, w) = -(u_t, w), \quad w \in \mathcal{U}. \quad (3.16)$$

Then, by trivial extensions of Theorems 4.1, 4.2, and (5.27) in Weiser and Wheeler [7] to the case $b > 0$,

$$\left\| \bar{Q}^x - q^x \right\|_x + \left\| \bar{Q}^y - q^y \right\|_y + \|\bar{U} - \tilde{u}\| + \|\partial_t(\bar{U}^n - \tilde{u}^n)\| \leq Ch^2, \quad (3.17)$$

where \tilde{u} is the L^2 -projection of u into \mathcal{U} ; i.e.,

$$(u - \tilde{u}, w) = 0, \quad w \in \mathcal{U}.$$

Let $\mu = Q - \bar{Q}$, $\nu = U - \bar{U}$, $\kappa = q - \bar{Q}$, and $\eta = \tilde{u} - \bar{U}$. In the remainder of the paper, C will represent a generic constant, independent of the discretization.

Theorem 1 *Assume u , a^x , a^y , b , and u^0 are sufficiently smooth, and*

$$\frac{\Delta t}{H^2} \max_j \bar{a}_j^x \leq \frac{1}{2} - \sigma' \quad (3.18)$$

where σ' is a small positive constant and $\Delta t = \max_n \Delta t^n$. Then, there exists a constant C , independent of h , Δt , and H , but dependent on $(\sigma')^{-1}$, such that,

$$\begin{aligned} & \left(\sum_{n=1}^M \left[\left\| q^{x,n} - Q^{x,n} \right\|_x^2 + \left\| q^{y,n} - Q^{y,n} \right\|_y^2 \right] \Delta t^n \right)^{1/2} + \max_n \|\tilde{u}^n - U^n\| \\ & \leq C \left(\Delta t H^{1/2} + H^{2.5} + H^{-1/2} h^2 + H \left[\sum_{n=1}^M \sum_{j=1}^J |\kappa_{\bar{k}+1/2, j}^{x,n}|^2 h_j^y \Delta t^n \right]^{1/2} \right) \\ & \quad + C(h^2 + \Delta t). \end{aligned} \quad (3.19)$$

In certain cases the $H^{-1/2}h^2$ term is improved to $H^{1/2}h^2$, depending on the mesh. In particular, this is the case if the mesh is symmetric about \bar{x} in the interval $[\bar{x} - H, \bar{x} + H]$, or if it is “translation invariant in H ” in this interval. By “translation invariant in H ” we mean that the mesh in the interval $[\bar{x}, \bar{x} + H]$ can be obtained by adding H to each mesh point in the interval $[\bar{x} - H, \bar{x}]$. The fourth summand on the right side of (3.19) is at worst $\mathcal{O}(Hh^{3/2})$, by (3.17).

Proof of Theorem 1: Subtract (3.15) from (3.13) and (3.16) from (3.14), to obtain

$$\langle \mu^{x,n}, v^x \rangle_{x,ax} + \langle \mu^{y,n}, v^y \rangle_{y,ay} - (\nu^n, \nabla \cdot v) = 0, \quad v \in \bar{\mathcal{Q}}, \quad (3.20)$$

$$(\partial_t \nu^n, w) + (\nabla \cdot \mu^n, w) + (\nu^n, w) = (u_t^n - \partial_t u^n, w) + (\partial_t \eta^n, w), \quad w \in \mathcal{U}. \quad (3.21)$$

Use (3.4), (3.2), and (3.15) to find

$$\begin{aligned} \frac{H}{\bar{a}_j^x} \mu_{\bar{k}+1/2,j}^{x,n} &\equiv \frac{H}{\bar{a}_j^x} \left(Q_{\bar{k}+1/2,j}^{x,n} - \bar{Q}_{\bar{k}+1/2,j}^{x,n} \right) \\ &= (\nu^{n-1}(\cdot, y_j), \phi_x)_{[0,1]} - \sum_i \frac{\kappa_{i+1/2,j}^{x,n-1}}{\bar{a}_{i+1/2,j}^x} \phi_{i+1/2} h_{i+1/2}^x \\ &\quad + \left[\sum_i \frac{q_{i+1/2,j}^{x,n-1}}{\bar{a}_{i+1/2,j}^x} \phi_{i+1/2} h_{i+1/2}^x - \frac{H}{\bar{a}_j^x} q_{\bar{k}+1/2,j}^{x,n-1} \right] \\ &\quad + \frac{H}{\bar{a}_j^x} \left[q_{\bar{k}+1/2,j}^{x,n-1} - q_{\bar{k}+1/2,j}^{x,n} \right] + \frac{H}{\bar{a}_j^x} \kappa_{\bar{k}+1/2,j}^{x,n}. \end{aligned} \quad (3.22)$$

Define

$$\begin{aligned} \langle f, g \rangle_{H, \bar{a}^x} &= \sum_j \frac{H}{\bar{a}_j^x} f_{\bar{k}+1/2,j} g_{\bar{k}+1/2,j} h_j^y, \\ |||f|||_{H, \bar{a}^x}^2 &= \langle f, f \rangle_{H, \bar{a}^x}. \end{aligned}$$

In the arguments below, we will frequently use the following inequality, with

various values of ϵ , a , and b :

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbf{R}, \quad \epsilon > 0. \quad (3.23)$$

Let $\hat{\mu}(x, y) = (\mu^x(\bar{x}, y)\phi(x), 0) \equiv (\hat{\mu}^x, 0) \in \mathcal{Q}$. Then, set $v = \mu^n - \hat{\mu}^n \in \bar{\mathcal{Q}}$ and $w = \nu^n$ in (3.20) and (3.21), multiply (3.22) by $\mu_{\bar{k}+1/2, j}^{x, n} h_j^y$ and sum on j , and add the resulting equations, to obtain

$$\begin{aligned} & |||\mu^{x, n}|||_{x, a^x}^2 + |||\mu^{y, n}|||_{y, a^y}^2 + (\partial_t \nu^n, \nu^n) + |||\hat{\mu}^{x, n}|||_{H, \bar{a}^x}^2 + \|\nu^n\|^2 \\ &= \langle \mu^{x, n}, \hat{\mu}^{x, n} \rangle_{x, a^x} + (\nu^{n-1} - \nu^n, \hat{\mu}_x^{x, n}) - \langle \kappa^{x, n-1}, \hat{\mu}^{x, n} \rangle_{x, a^x} \\ &+ \left[\langle q^{x, n-1}, \hat{\mu}^{x, n} \rangle_{x, a^x} - \langle q^{x, n-1}, \hat{\mu}^{x, n} \rangle_{H, \bar{a}^x} \right] \\ &+ \langle q^{x, n-1} - q^{x, n}, \hat{\mu}^{x, n} \rangle_{H, \bar{a}^x} + \langle \kappa^{x, n}, \hat{\mu}^{x, n} \rangle_{H, \bar{a}^x} \\ &+ (u_t^n - \partial_t u^n, \nu^n) + (\partial_t \eta^n, \nu^n) \\ &\equiv \sum_{l=1}^8 R_l. \end{aligned} \quad (3.24)$$

Use (3.23) with $\epsilon = 1 - \sigma$, for σ a small positive constant, to see

$$R_1 \leq (1 - \sigma) |||\mu^{x, n}|||_{x, a^x}^2 + \frac{1}{4(1 - \sigma)} |||\hat{\mu}^{x, n}|||_{x, a^x}^2. \quad (3.25)$$

By (3.5), and since $0 \leq \phi \leq 1$,

$$|||\hat{\mu}^{x, n}|||_{x, a^x}^2 \leq 2 |||\hat{\mu}^{x, n}|||_{H, \bar{a}^x}^2. \quad (3.26)$$

Thus,

$$R_1 \leq (1 - \sigma) |||\mu^{x, n}|||_{x, a^x}^2 + \frac{1}{2(1 - \sigma)} |||\hat{\mu}^{x, n}|||_{H, \bar{a}^x}^2. \quad (3.27)$$

Use the fact that

$$\|\hat{\mu}_x^{x, n}\|^2 \leq \frac{2}{H^2} \max_j \bar{a}_j^x |||\hat{\mu}^{x, n}|||_{H, \bar{a}^x}^2, \quad (3.28)$$

to find

$$\begin{aligned}
R_2 &\leq \frac{1}{2\Delta t^n} \|\nu^n - \nu^{n-1}\|^2 + \frac{\Delta t^n}{2} \|\hat{\mu}_x^{x,n}\|^2 \\
&\leq \frac{1}{2\Delta t^n} \|\nu^n - \nu^{n-1}\|^2 + \frac{\Delta t^n}{H^2} \max_j \bar{a}_j^x \|\hat{\mu}^{x,n}\|_{H, \bar{a}^x}^2.
\end{aligned} \tag{3.29}$$

By (3.1), (3.26), and (3.17),

$$\begin{aligned}
R_3 &\leq \frac{1}{2\sigma} \|\kappa^{x,n-1}\|_{x, a^x}^2 + \frac{\sigma}{2} \|\hat{\mu}^{x,n}\|_{x, a^x}^2 \\
&\leq C(\sigma^{-1})h^4 + \sigma \|\hat{\mu}^{x,n}\|_{H, \bar{a}^x}^2.
\end{aligned} \tag{3.30}$$

For the next term, R_4 , we have

$$\begin{aligned}
R_4 &= \sum_j \mu_{\bar{k}+1/2, j}^{x,n} h_j^y \\
&\quad \cdot \left[\sum_i \frac{q_{i+1/2, j}^{x, n-1}}{a_{i+1/2, j}^x} \phi_{i+1/2} h_{i+1/2}^x - \left(\frac{q_{\cdot, j}^{x, n-1}}{a_{\cdot, j}^x}, \phi \right)_{[0,1]} + \left(\frac{q_{\cdot, j}^{x, n-1}}{a_{\cdot, j}^x}, \phi \right)_{[0,1]} - H \frac{q_{\bar{k}+1/2, j}^{x, n-1}}{\bar{a}_j^x} \right] \\
&\equiv R_{4,1} + R_{4,2}.
\end{aligned} \tag{3.31}$$

Let $c(x)$ be a twice differentiable function. By error analysis for the trapezoidal rule of integration,

$$\begin{aligned}
E_T &\equiv \sum_i c_{i+1/2} \phi_{i+1/2} h_{i+1/2}^x - (c, \phi)_{[0,1]} \\
&= \sum_i \int_{\Omega_i^x} (c\phi)_{xx} K_i(x) dx
\end{aligned} \tag{3.32}$$

where $K_i(x)|_{\Omega_i^x} = (x - x_{i-1/2})(x_{i+1/2} - x)/2$. Thus, since $\phi_{xx} = 0$ on Ω_i^x , $|\phi_x| = H^{-1}$, and the measure of the support of ϕ is $2H$,

$$\begin{aligned}
E_T &= \sum_i \left[\int_{\Omega_i^x} c_x \phi_x K_i(x) dx + \int_{\Omega_i^x} c_{xx} \phi K_i(x) dx \right] \\
&\leq C \left(\|c\|_{1, \infty} h^2 + \|c\|_{2, \infty} H h^2 \right).
\end{aligned} \tag{3.33}$$

If the mesh is translation invariant in H in the interval $[\bar{x} - H, \bar{x} + H]$, then $K_i(x) = K_i(x - H)$ for $x \in [\bar{x}, \bar{x} + H]$, and (3.32) becomes

$$\sum_{\{i | \bar{x} - H \leq x_{i+1/2} \leq \bar{x}\}} \int_{\Omega_i^x} \frac{c'(x) - c'(x + H)}{H} K_i(x) dx + \mathcal{O}(Hh^2) = \mathcal{O}(Hh^2).$$

If the mesh is symmetric about \bar{x} in the interval $[\bar{x} - H, \bar{x} + H]$, then $K_i(x) = K_i(2\bar{x} - x)$ for $x \in [\bar{x}, \bar{x} + H]$, and (3.32) becomes

$$\sum_{\{i | \bar{x} - H \leq x_{i+1/2} \leq \bar{x}\}} \int_{\Omega_i^x} \frac{c'(x) - c'(2\bar{x} - x)}{H} K_i(x) dx + \mathcal{O}(Hh^2) = \mathcal{O}(Hh^2).$$

Apply (3.33) with $c(x) = q^{x,n-1}(x, y_j)/a^x(x, y_j)$ to see

$$\begin{aligned} |R_{4,1}| &\leq C \max_j \bar{a}_j^x \|(a^x)^{-1} q^{x,n-1}\|_{2,\infty} h^2 \cdot \sum_j \frac{\mu_{\bar{k}+1/2,j}^{x,n}}{\bar{a}_j^x} h_j^y \\ &\leq C(\sigma^{-1}) h^4 H^{-1} + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2. \end{aligned} \quad (3.34)$$

For the estimate of $R_{4,2}$, let $c(x)$ again denote a smooth function, let $K_H(x)|_{[\bar{x}-H,\bar{x}]} = (\bar{x} - x)(x - \bar{x} + H)/2$, and note that

$$\begin{aligned} Hc_{\bar{k}+1/2} - \int_{\bar{x}-H}^{\bar{x}+H} \phi(x)c(x)dx &= \int_{\bar{x}-H}^{\bar{x}} (\phi c)_{xx} K_H(x) dx + \int_{\bar{x}}^{\bar{x}+H} (\phi c)_{xx} K_H(x - H) dx \\ &= \int_{\bar{x}-H}^{\bar{x}} \frac{c'(x) - c'(x + H)}{H} K_H(x) dx + \mathcal{O}(H^3) \\ &= \mathcal{O}(H^3). \end{aligned} \quad (3.35)$$

Letting $c(x) = q^{x,n-1}(x, y_j)/a^x(x, y_j)$,

$$\begin{aligned} R_{4,2} &= \sum_j \mu_{\bar{k}+1/2,j}^{x,n} h_j^y \left[(c, \phi)_{[0,1]} - Hc_{\bar{k}+1/2} \right] \\ &\quad + H \sum_j \mu_{\bar{k}+1/2,j}^{x,n} h_j^y q_{\bar{k}+1/2,j}^{x,n-1} \left[\frac{1}{a_{\bar{k}+1/2,j}^x} - \frac{1}{\bar{a}_j^x} \right]. \end{aligned} \quad (3.36)$$

The first term on the right side of (3.36) is bounded by

$$C(\sigma^{-1})H^5 + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2$$

by (3.35). Another way of deriving this bound is to note that, by integration by parts, the definition of q^x , and (3.3),

$$\begin{aligned}
\left(\frac{q_{\cdot,j}^{x,n-1}}{a_{\cdot,j}^x}, \phi \right)_{[0,1]} - H \frac{q_{\bar{k}+1/2,j}^{x,n-1}}{a_{\bar{k}+1/2,j}^x} &= (u_{\cdot,j}^{n-1}, \phi_x)_{[0,1]} - H \frac{q_{\bar{k}+1/2,j}^{x,n-1}}{a_{\bar{k}+1/2,j}^x} \\
&= H \left[B(u^{n-1})(\bar{x}, y_j) - u_x(\bar{x}, y_j) \right] \\
&= \mathcal{O}(H^3).
\end{aligned} \tag{3.37}$$

For the second difference on the right side of (3.36), add and subtract $\left((a_{\cdot,j}^x)^{-1}, \phi \right)_{[0,1]}$ and recall the definition of \bar{a}_j^x , i.e., (3.5), and apply both (3.35) and (3.33) to conclude that this term is bounded by

$$C(\sigma^{-1})(h^4 H^{-1} + H^5) + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2.$$

Thus,

$$R_4 \leq C(\sigma^{-1})(h^4 H^{-1} + H^5) + 3\sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2. \tag{3.38}$$

Continuing,

$$R_5 \leq C(\sigma^{-1})\Delta t^2 H \|q_t^x\|_{L^\infty(L^\infty)} + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2, \tag{3.39}$$

and

$$R_6 + R_8 \leq C(\sigma^{-1})H \sum_j |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y + \frac{1}{2} \|\partial_t \eta^n\|^2 + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 + \frac{1}{2} \|\nu^n\|^2. \tag{3.40}$$

Finally, use time truncation error analysis to see that

$$R_7 \leq C\Delta t \|u_{tt}\|_{L^2(t^{n-1}, t^n; L^2)}^2 + \frac{1}{2} \|\nu^n\|^2. \tag{3.41}$$

Combine (3.27)-(3.30), (3.38)-(3.41) with (3.24), and apply (3.17) to see that

$$\begin{aligned}
& |||\mu^{x,n}|||_{x,a^x}^2 + |||\mu^{y,n}|||_{y,a^y}^2 + (\partial_t \nu^n, \nu^n) + |||\hat{\mu}^{x,n}|||_{H,\bar{a}^x}^2 + \|\nu^n\|^2 \\
& \leq (1 - \sigma) |||\mu^{x,n}|||_{x,a^x}^2 + \left(\frac{1}{2(1 - \sigma)} + \frac{\Delta t^n}{H^2} \max_j \bar{a}_j^x + 6\sigma \right) |||\hat{\mu}^{x,n}|||_{H,\bar{a}^x}^2 \\
& \quad + \frac{1}{2\Delta t^n} \|\nu^n - \nu^{n-1}\|^2 + C(\sigma^{-1}) \left(h^4 H^{-1} + H^5 + \Delta t^2 H + H \sum_j |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y \right) \\
& \quad + Ch^4 + C\Delta t \|u_{tt}\|_{L^2(t^{n-1}, t^n; L^2)}^2 + \|\nu^n\|^2. \tag{3.42}
\end{aligned}$$

Set

$$\sigma' = -\frac{1}{2} + \frac{1}{2(1 - \sigma)} + 6\sigma.$$

Choosing σ small, the first and second terms on the right side of (3.42) can be hidden, using the constraint (3.18). Use the fact that

$$(\partial_t \nu^n, \nu^n) = \frac{1}{2\Delta t^n} \left[\|\nu^n\|^2 - \|\nu^{n-1}\|^2 + \|\nu^n - \nu^{n-1}\|^2 \right],$$

multiply (3.42) by $2\Delta t^n$, and sum on n , to obtain

$$\begin{aligned}
& 2\sigma \sum_{n=0}^m \Delta t^n \left[|||\mu^{x,n}|||_{x,a^x}^2 + |||\mu^{y,n}|||_{y,a^y}^2 \right] + \|\nu^m\|^2 \\
& \leq \|\nu^0\|^2 + C(\sigma^{-1}) \left(h^4 H^{-1} + H^5 + \Delta t^2 H + H \sum_{n=1}^M \sum_j |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y \Delta t^n \right) \\
& \quad + C(\Delta t^2 + h^4). \tag{3.43}
\end{aligned}$$

Since $\|\nu^0\|^2 \leq Ch^4$ by (3.11) and (3.17), applying the triangle inequality completes the proof of Theorem 1. //

Extension to higher-order in H . Suppose $q^{x,n}(\bar{x}, y_j)$ is approximated by $Q^{x,n}(\bar{x}, y_j)$, with

$$Q^{x,n}(\bar{x}, y_j) = -\bar{a}_j^x B(U^{n-1})(\bar{x}, y_j), \tag{3.44}$$

where $\bar{a}_j^x = a^x(\bar{x}, y_j)$, and

$$B(\psi)(\bar{x}, y) = -\frac{1}{H} \int_0^1 \phi'(x) \psi(x, y) dx. \quad (3.45)$$

Assume ϕ and δ_x are such that the measure of the support of ϕ is $\mathcal{O}(H)$,

$$\phi \in \mathcal{M}_0^0(1; h^x), \quad (3.46)$$

$$\phi(\bar{x}) \geq 1, \quad (3.47)$$

$$|||\phi|||_x^2 \leq C_4 H, \quad (3.48)$$

$$||\phi_x||^2 \leq C_5 H^{-1}, \quad (3.49)$$

and for ψ sufficiently smooth,

$$B(\psi)(\bar{x}, y) = \psi_x(\bar{x}, y) + \mathcal{O}(H^k). \quad (3.50)$$

Here C_4 and C_5 are positive constants, and we require

$$C_4 \leq 2. \quad (3.51)$$

Let

$$\alpha = \frac{1}{\phi(\bar{x})} \leq 1. \quad (3.52)$$

Note that for $\phi = \phi_4((\bar{x} - H)/H)$, $\alpha = 6/7$, $k = 4$, $||\phi_x||^2 \approx 3.14H^{-1}$, and $||\phi||^2 \approx .85H$. Thus, for reasonable meshes, we expect C_4 to be close to .85 in this case.

Set $\nu = \mu^n - \alpha \hat{\mu}^n$, $w = \nu^n$, in (3.20) and (3.21), respectively, multiply (3.22) by $\alpha \mu_{\bar{k}+1/2, j}^{x, n} h_j^y$ and sum on j , to obtain

$$\begin{aligned} & |||\mu^{x, n}|||_{x, a^x}^2 + |||\mu^{y, n}|||_{y, a^y}^2 + (\partial_t \nu^n, \nu^n) + \alpha |||\hat{\mu}^{x, n}|||_{H, \bar{a}^x}^2 + ||\nu^n||^2 \\ & = \alpha \sum_{l=1}^6 R_l + R_7 + R_8, \end{aligned} \quad (3.53)$$

where $R_1 - R_8$ are given by (3.24).

Again let σ be a small positive constant, which we shall choose explicitly below.

The estimates for $R_3, R_5 - R_8$ are essentially unchanged:

$$\begin{aligned}
& \alpha(R_3 + R_5 + R_6) + R_7 + R_8 \\
& \leq C(\sigma^{-1}) \left(h^4 H^{-1} + \Delta t^2 H + H \sum_j |\kappa_{k+1/2,j}^{x,n}|^2 h_j^y \right) + C h^4 \\
& \quad + 3\sigma\alpha \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 + C\Delta t \|u_{tt}\|_{L^2(t^{n-1},t^n;L^2)}^2 + \|\nu^n\|^2. \tag{3.54}
\end{aligned}$$

For R_1 , choose $\epsilon = 1/6$ in (3.23) to obtain

$$\alpha R_1 \leq \frac{5}{6} \|\mu^{x,n}\|_{x,a^x}^2 + \frac{3\alpha^2}{10} \|\hat{\mu}^{x,n}\|_{x,a^x}^2. \tag{3.55}$$

Now

$$\begin{aligned}
\|\hat{\mu}^{x,n}\|_{x,a^x}^2 &= \sum_j \sum_i \frac{h_{i+1/2}^x}{a_{i+1/2,j}^x} |\mu_{k+1/2,j}^{x,n}|^2 |\phi_{i+1/2}|^2 h_j^y \\
&= \left[\frac{1}{H} \sum_i |\phi_{i+1/2}|^2 h_{i+1/2}^x \right] \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 \\
& \quad + \sum_j \sum_i \left[\frac{1}{a_{i+1/2,j}^x} - \frac{1}{\bar{a}_j^x} \right] |\mu_{k+1/2,j}^{x,n}|^2 |\phi_{i+1/2}|^2 h_{i+1/2}^x h_j^y, \tag{3.56}
\end{aligned}$$

and for H sufficiently small,

$$\begin{aligned}
\sum_i \left| \frac{1}{a_{i+1/2,j}^x} - \frac{1}{\bar{a}_j^x} \right| |\phi_{i+1/2}|^2 h_{i+1/2}^x &\leq H \|(a^x)^{-1}\|_\infty C_4 H \\
&\leq \frac{10\sigma}{3\alpha} \frac{H}{\bar{a}_j^x}. \tag{3.57}
\end{aligned}$$

Thus,

$$\begin{aligned}
\alpha R_1 &\leq \frac{5}{6} \|\hat{\mu}^{x,n}\|_{x,a^x}^2 + \alpha \left[\frac{3\alpha}{10H} \|\phi\|_x^2 + \sigma \right] \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 \\
&\leq \frac{5}{6} \|\mu^{x,n}\|_{x,a^x}^2 + \alpha \left[\frac{3\alpha C_4}{10} + \sigma \right] \|\mu^{x,n}\|_{H,\bar{a}^x}^2. \tag{3.58}
\end{aligned}$$

By (3.49),

$$\alpha R_2 \leq \frac{1}{2\Delta t} \|\nu^n - \nu^{n-1}\|^2 + \frac{\alpha^2 \Delta t C_5}{H^2} \max_j \bar{a}_j^x \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2. \quad (3.59)$$

We again have

$$R_4 = R_{4,1} + R_{4,2}, \quad (3.60)$$

where $R_{4,1}$ and $R_{4,2}$ are given by (3.31). Since $\phi_{xx} = 0$ on Ω_i^x by (3.46), $|\phi_x| = \mathcal{O}(H^{-1})$, and ϕ has support on an interval of length $\mathcal{O}(H)$, use (3.33) to conclude

$$R_{4,1} \leq C(\sigma^{-1})h^4H^{-1} + \alpha\sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2. \quad (3.61)$$

Applying the definition of q^x and integrating by parts as in (3.37),

$$\begin{aligned} R_{4,2} &= \sum_j \mu_{\bar{k}+1/2,j}^{x,n} h_j^y \left[B(u^{n-1})(\bar{x}, y_j) - u_x^{n-1}(\bar{x}, y_j) \right] H \\ &\leq C(\sigma^{-1})H^{2k+1} + \alpha\sigma \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2, \end{aligned} \quad (3.62)$$

by (3.50).

Combine (3.53)-(3.62) to obtain

$$\begin{aligned} &\|\mu^{x,n}\|_{x,a^x}^2 + \|\mu^{y,n}\|_{y,a^y}^2 + (\partial_t \nu^n, \nu^n) + \alpha \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 + \|\nu^n\|^2 \\ &\leq \frac{5}{6} \|\mu^{x,n}\|_{x,a^x}^2 + \alpha \left[\frac{3\alpha C_4}{10} + \alpha C_5 \frac{\Delta t}{H^2} \max_j \bar{a}_j^x + 6\sigma \right] \|\hat{\mu}^{x,n}\|_{H,\bar{a}^x}^2 \\ &\quad + C(\sigma^{-1}) \left(h^4 H^{-1} + \Delta t^2 H + H^{2k+1} + H \sum_j |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y \right) \\ &\quad + C(\Delta t \|u_{tt}\|_{L^2(t^{n-1}, t^n; L^2)}^2 + h^4) + \|\nu^n\|^2. \end{aligned} \quad (3.63)$$

For concreteness, choose $\sigma = 1/36$, and assume

$$\alpha \frac{\Delta t}{H^2} \max_j \bar{a}_j^x \leq \frac{1}{5C_5}. \quad (3.64)$$

Then, since $C_4 \leq 2$,

$$\frac{3\alpha C_4}{10} + \alpha C_5 \frac{\Delta t}{H^2} \max_j \bar{a}_j^x + 6\sigma \leq 1. \quad (3.65)$$

Hide terms in (3.63), multiply by $2\Delta t^n$, sum on n , and apply the triangle inequality and (3.17) to obtain:

Theorem 2 *Assume u and a^x are sufficiently smooth, and (3.46)-(3.50) hold. Then, for h , Δt , and H sufficiently small, there exists a constant C , independent of h , Δt , and H , such that,*

$$\begin{aligned} & \left(\sum_{n=1}^M \left[\| \|q^{x,n} - Q^{x,n}\| \|_{x}^2 + \| \|q^{y,n} - Q^{y,n}\| \|_{y}^2 \right] \Delta t^n \right)^{1/2} + \max_n \| \tilde{u}^n - U^n \| \\ & \leq C \left(\Delta t H^{1/2} + H^{k+1/2} + H^{-1/2} h^2 + H \left[\sum_{n=1}^M \sum_{j=1}^J |\kappa_{k+1/2,j}^{x,n}|^2 h_j^y \Delta t^n \right]^{1/2} \right) \\ & \quad + C(h^2 + \Delta t), \end{aligned} \quad (3.66)$$

provided (3.64) holds.

Suppose ϕ is constructed from ϕ_4 . As before, the $\mathcal{O}(H^{-1/2}h^2)$ term above can be improved to $\mathcal{O}(H^{1/2}h^2)$ depending on the partition δ_x in the region where ϕ is nonzero. For example, this is the case if the partition is symmetric about \bar{x} in this region. Moreover, by (3.63), assuming σ is small, and $C_4 \approx .85$, the constraint on Δt and H becomes

$$\max_j \bar{a}_j^x \Delta t < H^2 \frac{1 - .3(C_4)}{C_5} \approx .24H^2. \quad (3.67)$$

A second order in time scheme. Similar to the Galerkin domain decomposition procedure defined in [3], we can define a second order in time backward difference scheme for the block centered approach discussed here.

Assume $\Delta t^n \equiv \Delta t$. We will consider for simplicity the context of Theorem 1, with ϕ constructed from ϕ_2 .

Let

$$\delta_2 U^n = \Delta t^{-1} \left(\frac{3}{2} U^n - 2U^{n-1} + \frac{1}{2} U^{n-2} \right) = \partial_t U^n + \frac{1}{2} (\partial_t U^n - \partial_t U^{n-1}). \quad (3.68)$$

Assume U^0, U^1 in \mathcal{U} , and Q^0, Q^1 in \mathcal{Q} are known. Define $U^n \in \mathcal{U}$, $Q^n \in \mathcal{Q}$ for $n = 2, 3, \dots$ by the following procedure. First, the interface flux $Q_{\bar{k}+1/2,j}^{x,n}$ is given by

$$Q^{x,n}(\bar{x}, y_j) \equiv Q_{\bar{k}+1/2,j}^{x,n} = -\bar{a}_j^x B(2U^{n-1} - U^{n-2})(\bar{x}, y_j), \quad j = 1, \dots, J. \quad (3.69)$$

Define $Q^{x,n}$ and $Q^{y,n}$ for the interior edges by (3.8) and (3.9), and instead of (3.10), use

$$\delta_2 U_{i,j}^n + \frac{Q_{i+1/2,j}^{x,n} - Q_{i-1/2,j}^{x,n}}{h_i^x} + \frac{Q_{i,j+1/2}^{y,n} - Q_{i,j-1/2}^{y,n}}{h_j^y} + U_{i,j}^n = 0. \quad (3.70)$$

Since the flux at the interface is calculated explicitly by (3.69), (3.70) decouples into two equations, which can be solved simultaneously.

We have the following error estimate:

Theorem 3 *Assume u and a^x are sufficiently smooth and*

$$\frac{\Delta t}{H^2} \max_j \bar{a}_j^x \leq \frac{1}{4} - \sigma' \quad (3.71)$$

where σ' is a small positive constant. Assume U^0 and U^1 are such that

$$\|\bar{U}^0 - U^0\| + \|\bar{U}^1 - U^1\| = \mathcal{O}(h^2 + \Delta t^2). \quad (3.72)$$

Then, there exists a constant C , independent of h , Δt , and H , but dependent on $(\sigma')^{-1}$, such that,

$$\left(\sum_{n=1}^M \left[\|q^{x,n} - Q^{x,n}\|_x^2 + \|q^{y,n} - Q^{y,n}\|_y^2 \right] \Delta t^n \right)^{1/2} + \max_n \|\tilde{u}^n - U^n\|$$

$$\leq C \left(\Delta t^2 H^{1/2} + H^{2.5} + H^{-1/2} h^2 + H \left[\sum_{n=1}^M \sum_{j=1}^J |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y \Delta t^n \right]^{1/2} \right) + C(h^2 + \Delta t^2). \quad (3.73)$$

Proof: Let R_1 and R_6 be defined as in (3.24). Then, the analogue of (3.24) in this case is

$$\begin{aligned} & |||\mu^{x,n}|||_{x,\alpha^x}^2 + |||\mu^{y,n}|||_{y,\alpha^y}^2 + (\delta_2 \nu^n, \nu^n) + |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2 + ||\nu^n||^2 \\ &= R_1 + (2\nu^{n-1} - \nu^n - \nu^{n-2}, \hat{\mu}_x^{x,n}) - \langle 2\kappa^{x,n-1} - \kappa^{x,n-2}, \hat{\mu}^{x,n} \rangle_{x,\alpha^x} \\ &+ \left[\langle 2q^{x,n-1} - q^{x,n-2}, \hat{\mu}^{x,n} \rangle_{x,\alpha^x} - \langle 2q^{x,n-1} - q^{x,n-2}, \hat{\mu}^{x,n} \rangle_{H,\bar{\alpha}^x} \right] \\ &+ \langle 2q^{x,n-1} - q^{x,n} - q^{x,n-2}, \hat{\mu}^{x,n} \rangle_{H,\bar{\alpha}^x} + R_6 \\ &+ (u_i^n - \delta_2 u^n, \nu^n) + (\delta_2 \eta^n, \nu^n) \\ &\equiv \sum_{l=1}^8 S_l. \end{aligned} \quad (3.74)$$

By previous arguments,

$$\begin{aligned} S_1 + S_3 + S_4 + S_6 &\leq (1 - \sigma) |||\mu^{x,n}|||_{x,\alpha^x}^2 + \frac{1}{2(1 - \sigma)} |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2 \\ &+ C(\sigma^{-1})h^4 + \sigma |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2 \\ &+ C(\sigma^{-1})(h^4 H^{-1} + H^5) + 3\sigma |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2 \\ &+ C(\sigma^{-1})H \sum_j |\kappa_{\bar{k}+1/2,j}^{x,n}|^2 h_j^y + Ch^4 \\ &+ \sigma |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2 + \frac{1}{2} ||\nu^n||^2. \end{aligned} \quad (3.75)$$

Analogous to the argument used to bound R_2 above,

$$\begin{aligned} S_2 &\leq \frac{1}{4\Delta t} ||2\nu^{n-1} - \nu^n - \nu^{n-2}||^2 + \frac{\Delta t}{H^2} ||\hat{\mu}_x^{x,n}||^2 \\ &\leq \frac{1}{4\Delta t} ||2\nu^{n-1} - \nu^n - \nu^{n-2}||^2 + \frac{2\Delta t}{H^2} \max_j \bar{\alpha}_j^x |||\hat{\mu}^{x,n}|||_{H,\bar{\alpha}^x}^2. \end{aligned} \quad (3.76)$$

Next,

$$\begin{aligned}
S_5 &= \left\langle 2q^{x,n-1} - q^{x,n} - q^{x,n-2}, \hat{\mu}^{x,n} \right\rangle_{H,\bar{\alpha}^x} \\
&\leq C(\sigma^{-1})\Delta t^4 H \|q_{tt}^x\|_{L^\infty(L^\infty)} + \sigma \|\hat{\mu}^{x,n}\|_{H,\bar{\alpha}^x}^2
\end{aligned} \tag{3.77}$$

Multiplying by Δt and summing on n , and applying time truncation error analysis and (3.17),

$$\sum_{n=2}^m [S_7 + S_8] \Delta t \leq C(\Delta t^4 + h^4) + C\|\nu^n\|^2. \tag{3.78}$$

Use

$$(\delta_2 \nu^n, \nu^n) = \frac{1}{4} \left[\partial_t \left(\|\nu^n\|^2 + \|2\nu^n - \nu^{n-1}\|^2 \right) + \Delta t^{-1} \|\nu^n - 2\nu^{n-1} + \nu^{n-2}\|^2 \right], \tag{3.79}$$

multiply (3.74) by Δt and sum on n , $n = 2, \dots, M$, and apply (3.75)-(3.78) and (3.72) to obtain the theorem.

The assumption (3.72) requires a second order in time approximation at the first time step. Thus, for the first time step, a global Crank-Nicholson scheme could be used. After this step, all other time steps can be done in parallel.

Extension to multiple subdomains. The extension of the algorithms described above to multiple, rectangular subdomains is discussed in the Appendix.

4 Numerical results

In this section, we present numerical results examining the stability and convergence of the schemes analyzed above.

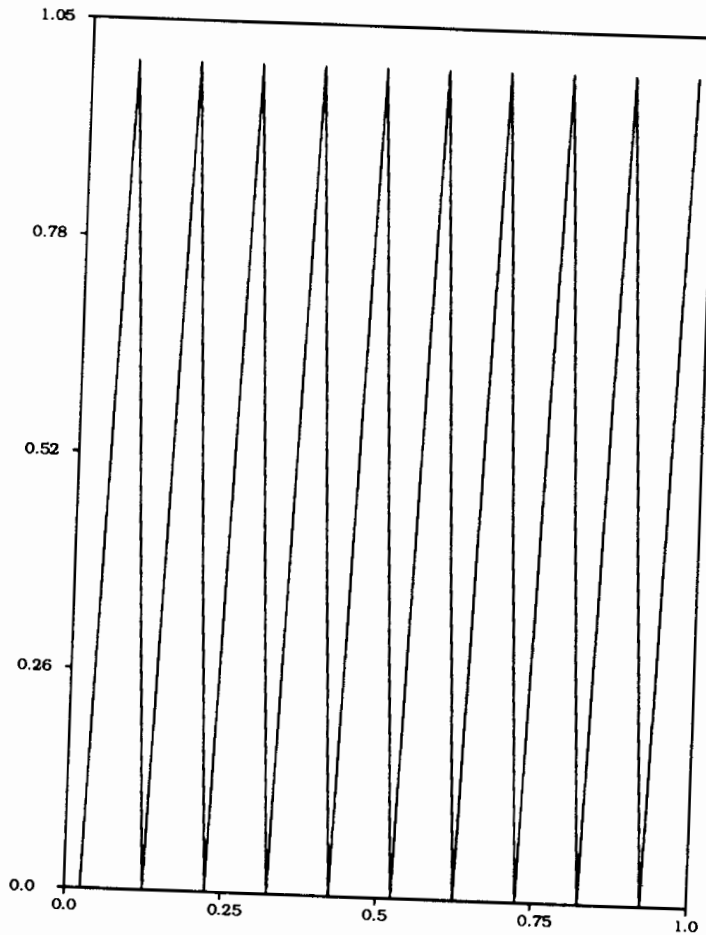


Figure 1: Initial condition for stability test

First, we examine the stability of the scheme as it relates to the time step constraint (3.18). We consider the algorithm (3.4)-(3.10) with $B(\psi)$ determined by ϕ_2 . Take $a = \text{diag}(1, 1)$, and $b = 0$, with initial data given in Figure 1. In Figure 2, we plot $\|U(\cdot, t)\|$ versus t for different values of $\Delta t/H^2$. As can be seen in this figure, when the constraint is violated by as much as a factor of 3, the L^2 norm of U blows up with time.

Next, we examine the errors in the solution and the diffusive flux for two test

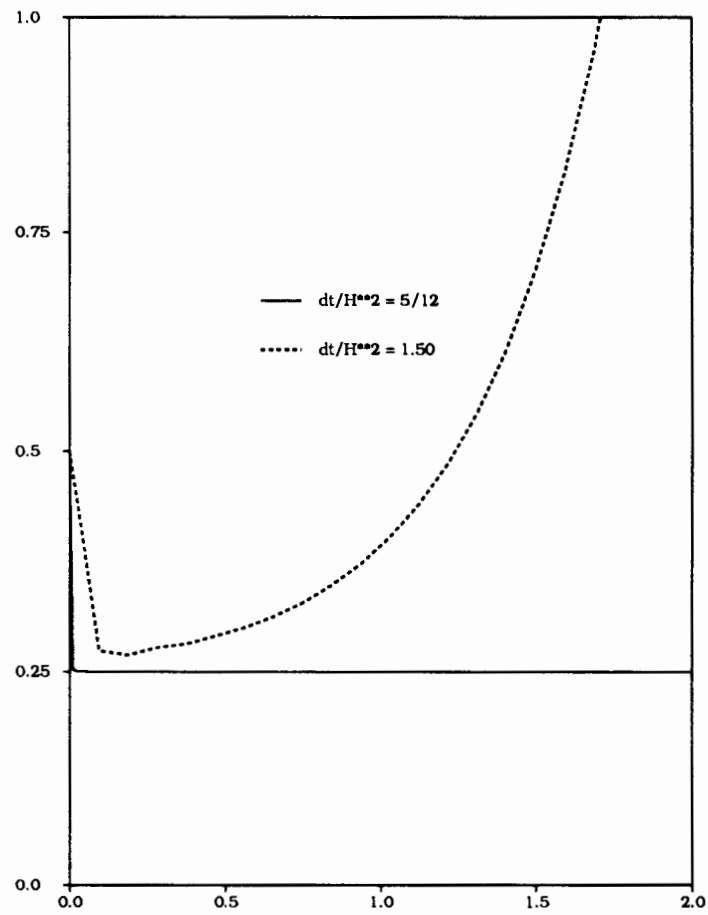


Figure 2: Plot of $\|U\|$ versus time

problems. First, we consider the problem

$$u_t - \Delta u = 0,$$

$$u^0(x, y) = \cos(2\pi x) \cos(\pi y),$$

which has true solution $u(x, y, t) \equiv u_1(x, y, t) = e^{-5\pi^2 t} \cos(2\pi x) \cos(\pi y)$. We will look at three scenarios. Scenario 1 is fully implicit block-centered finite differences (no domain decomposition). Scenarios 2 and 3 involve domain decomposition with uniform and nonuniform mesh. We give the details of each case below.

Scenarios:

1. Fully implicit block-centered finite differences on uniform mesh h ; $\Delta t = 4h^2$.
2. Global uniform mesh; two subdomains $\Omega_1 = (0, .5) \times (0, 1)$, $\Omega_2 = (.5, 1) \times (0, 1)$ with $H = 3h$, $\Delta t = 4h^2$.
3. Two subdomains $\Omega_1 = (0, .4) \times (0, 1)$, $\Omega_2 = (.4, 1) \times (0, 1)$. The coarsest mesh on Ω_1 consists of 5 blocks in the x -direction and 20 blocks in the y -direction. On Ω_2 , the coarsest mesh has 15 blocks in the x -direction and 20 in the y -direction. All subsequent mesh refinements are obtained by halving this mesh. Moreover, defining h_i^x to be the mesh spacing in the x -direction for subdomain i , $i = 1, 2$, then $H = 3h_1^x$ and $\Delta t = 4(h_1^x)^2$.

The errors in the solution for Scenarios 1-3 for this problem are compared in Table

1. Here

$$e_h = \|(U - \tilde{u})(\cdot, .1)\|.$$

Scenario 1			Scenario 2		Scenario 3	
h^{-1} ,	$e_h * 10^4$	Rate	$e_h * 10^4$	Rate	$e_h * 10^4$	Rate
20	56.7	–	65.1	–	55.8	–
40	12.0	–	11.97	–	11.78	–
80	2.87	2.15	2.73	2.28	2.8	2.16

Table 1: Convergence of solution: $u(x, t) = u_1(x, y, t)$

Scenario 1			Scenario 2		Scenario 3	
h^{-1} ,	$\gamma_h * 10^3$	Rate	$\gamma_h * 10^3$	Rate	$\gamma_h * 10^3$	Rate
20	54.9	–	51.9	–	67.0	–
40	15.0	–	14.2	–	17.5	–
80	3.87	1.91	3.63	1.92	4.30	1.98

Table 2: Convergence of diffusive flux: $u(x, t) = u_1(x, y, t)$

Three mesh refinements were used, and an experimental rate of convergence was calculated using a least squares fit of the data. As can be seen in this table, the errors for each scenario are roughly of the same order of magnitude, and the errors appear to be $\mathcal{O}(h^2)$ in each case. This can be predicted from Theorem 1 in Scenario 2, but is better than what is predicted for Scenario 3, where nonuniform mesh is used.

The errors in the diffusive flux are given in Table 2. Here

$$\gamma_h = \left(\sum_n \left[\|q^{x,n} - Q^{x,n}\|_x^2 + \|q^{y,n} - Q^{y,n}\|_y^2 \right] \Delta t \right)^{\frac{1}{2}}.$$

We see virtually the same phenomena in Table 2 as in Table 1. The errors are all of roughly the same magnitude, and converge like h^2 .

Next, consider

$$u_t - \Delta u = f,$$

Scenario 1			Scenario 2		Scenario 3		Scenario 4	
h^{-1} ,	$e_h * 10^4$	Rate	$e_h * 10^4$	Rate	$e_h * 10^4$	Rate	$e_h * 10^4$	Rate
20	11.8	-	24.2	-	65.8	-	10.6	-
40	2.85	-	4.45	-	10.9	-	2.56	-
80	.738	-	.891	-	1.73	-	.661	-
160	.183	2.00	.20	2.31	.29	2.16	.17	2.00

Table 3: Convergence of solution: $u(x, t) = u_2(x, y, t)$

$$u^0(x, y) = 0,$$

with f chosen so that $u(x, y, t) = u_2(x, y, t) = 100tx^3(1-x)^2 \cos(2\pi y)$. We consider the Scenarios 1-3 above, as well as a fourth scenario, which is simply the fully implicit scheme on the same grid used in Scenario 3. The errors in solution values and the diffusive flux are given in Tables 3 and 4, respectively. Here we see that on the coarser grids, the fully implicit solutions (Scenarios 1 and 4) have smaller error than the domain decomposition solutions (Scenarios 2 and 3). As the mesh is refined, the errors in the domain decomposition solutions drop dramatically, so that on the finest mesh, these errors are comparable in size to the fully implicit errors. This is not surprising; heuristically, one would expect the domain decomposition solution to approach the fully implicit solution as h , H , and Δt approach zero. Because the errors decrease so rapidly in the domain decomposition cases, the experimental rates of convergence are substantially higher than predicted by Theorem 1. This simply indicates that we haven't quite reached the asymptotic range in h where the theorem holds.

Scenario 1			Scenario 2		Scenario 3		Scenario 4	
h^{-1} ,	$\gamma_h * 10^3$	Rate	$\gamma_h * 10^3$	Rate	$\gamma_h * 10^3$	Rate	$\gamma_h * 10^3$	Rate
20	.777	–	7.02	–	16.9	–	.870	–
40	.171	–	1.03	–	2.76	–	.194	–
80	.044	–	.138	–	.393	–	.050	–
160	.011	2.04	.019	2.84	.053	2.78	.012	2.04

Table 4: Convergence of diffusive flux: $u(x, t) = u_2(x, y, t)$

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Appendix. In this appendix, we discuss the extension of the block-centered finite difference-domain decomposition procedure presented in Section 3 to the case of multiple rectangular subdomains. We will focus on the case where the flux is calculated using ϕ_2 .

Let

$$0 = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_{L+1} = 1, \quad (4.1)$$

$$0 = \bar{y}_0 < \bar{y}_1 < \dots < \bar{y}_{K+1} = 1 \quad (4.2)$$

denote interface points between subdomains. Thus, the subdomains are the rectangular regions $(\bar{x}_{l-1}, \bar{x}_l) \times (\bar{y}_{k-1}, \bar{y}_k)$. Similar to H above, associate a parameter $H_l^x > 0$ with \bar{x}_l , and associate $H_k^y > 0$ with \bar{y}_k . Assume \bar{x}_l , H_l^x , \bar{y}_k , and H_k^y satisfy the following:

$$\bar{x}_l, \bar{x}_l - H_l^x, \bar{x}_l + H_l^x \in \delta_x, \quad l = 1, \dots, L, \quad (4.3)$$

$$\bar{y}_k, \bar{y}_k - H_k^y, \bar{y}_k + H_k^y \in \delta_y, \quad k = 1, \dots, K, \quad (4.4)$$

$$\bar{x}_{l-1} + H_{l-1}^x \leq \bar{x}_l, \quad l = 2, \dots, L + 1, \quad (4.5)$$

$$\bar{x}_l - H_l^x \geq \bar{x}_{l-1}, \quad l = 1, \dots, L; \quad (4.6)$$

and

$$\bar{y}_{k-1} + H_{k-1}^y \leq \bar{y}_k, \quad k = 2, \dots, K + 1, \quad (4.7)$$

$$\bar{y}_k - H_k^y \geq \bar{y}_{k-1}, \quad k = 1, \dots, K. \quad (4.8)$$

Take $\phi^{l,x} = \phi_2((x - \bar{x}_l)/H_l^x)$, and note that by (4.5)-(4.6)

$$\text{supp } \phi^{l,x} \subset [\bar{x}_{l-1}, \bar{x}_{l+1}], \quad l = 1, \dots, L. \quad (4.9)$$

If $\phi^{l,x} = \phi_4((x - \bar{x}_l)/H_l^x)$, then (4.5), (4.6) must be changed to

$$\bar{x}_{l-1} + 2H_{l-1}^x \leq \bar{x}_l,$$

$$\bar{x}_l - 2H_l^x \geq \bar{x}_{l-1}.$$

so that (4.9) will hold. In general, (4.9) must hold for the arguments given in Section 3 to carry through. Define $\phi^{k,y} = \phi_2((y - \bar{y}_k)/H_k^y)$, and note

$$\text{supp } \phi^{k,y} \subset [\bar{y}_{k-1}, \bar{y}_{k+1}]. \quad (4.10)$$

Assume $U^{n-1} \in \mathcal{U}$ is given. At each point (\bar{x}_l, y_j) , $l = 1, \dots, L$, $j = 1, \dots, J$, set

$$Q^{x,n}(\bar{x}_l, y_j) = -a^x(\bar{x}_l, y_j) B_l^x(U^{n-1})(\bar{x}_l, y_j) \quad (4.11)$$

where

$$B_l^x(\psi)(\bar{x}_l, y) = -\frac{1}{H_l^x} \int_0^1 \frac{d\phi^{l,x}(x)}{dx} \psi(x, y) dx. \quad (4.12)$$

At each (x_i, \bar{y}_k) , $i = 1, \dots, I$, $k = 1, \dots, K$, set

$$Q^{y,n}(x_i, \bar{y}_k) = -a^y(x_i, \bar{y}_k) B_k^y(U^{n-1})(x_i, \bar{y}_k) \quad (4.13)$$

where

$$B_k^y(\psi)(x_i, \bar{y}_k) = -\frac{1}{H_k^y} \int_0^1 \frac{d\phi^{k,y}(y)}{dy} \psi(x, y) dy. \quad (4.14)$$

The boundary condition (2.6) is again enforced by (3.6)-(3.7). At points $(x_{i+1/2}, y_j)$ where $x_{i+1/2} \neq \bar{x}_l$, $Q_{i+1/2,j}^{x,n}$ is given by (3.8). At points $(x_i, y_{j+1/2})$, $y_{j+1/2} \neq \bar{y}_k$, $Q_{i,j+1/2}^{y,n}$ is given by (3.9), and $U_{i,j}^n$ is given by (3.10). The initial condition is enforced by (3.11).

In this case, (3.8)-(3.10) is equivalent to:

$$\langle Q^{x,n}, v^x \rangle_{x,a^x} + \langle Q^{y,n}, v^y \rangle_{y,a^y} - (U^n, \nabla \cdot v) = 0, \quad v \in \bar{Q}, \quad (4.15)$$

$$(\partial_t U^n, w) + (\nabla \cdot Q^n, w) + (U^n, w) = 0, \quad w \in \mathcal{U}, \quad (4.16)$$

where

$$\bar{Q}^x = Q^x \cap \{v^x \mid v^x(\bar{x}_l, y) = 0, \quad l = 1, \dots, L\},$$

$$\bar{Q}^y = Q^y \cap \{v^y \mid v^y(x, \bar{y}_k) = 0, \quad k = 1, \dots, K\},$$

$$\bar{Q} = \bar{Q}^x \times \bar{Q}^y.$$

Assume δ_x and δ_y are sufficiently “regular” so that

$$\|\|\phi^{l,x}\|\|_x^2 \leq C_0 H_l^x, \quad (4.17)$$

$$\|\|\phi^{k,y}\|\|_y^2 \leq C_0 H_k^y, \quad (4.18)$$

where C_0 is a positive constant less than 3/2. Note that

$$\|\|\phi^{l,x}\|\|^2 = \frac{2}{3} H_l^x,$$

so in this case C_0 should be close to 2/3.

We have the following result:

Theorem 4 *Assume u , a^x , and a^y are sufficiently smooth, and h^x , h^y , Δt , H_l^x , and H_k^y are sufficiently small. Then*

$$\max_n \|\nu^n\|^2 + \delta \left(\sum_{n=1}^M \left[\|\|q^{x,n} - Q^{x,n}\|\|_x^2 + \|\|q^{y,n} - Q^{y,n}\|\|_y^2 \right] \Delta t^n \right)^{1/2} \quad (4.19)$$

$$\leq C(\delta^{-1})(L + K) (h^4 H^{-1} + H^5 + \Delta t^2 H)$$

$$+ C(\delta^{-1}) \sum_n \Delta t^n \left[\sum_l H_l^x \sum_j |\kappa^{x,n}(\bar{x}_l, y_j)|^2 h_j^y + \sum_k H_k^y \sum_i |\kappa^{y,n}(x_i, \bar{y}_k)|^2 h_i^x \right]$$

$$+ C(\Delta t^2 + h^4), \quad (4.20)$$

provided for some sufficiently small constant δ ,

$$\frac{\Delta t}{(\bar{H}_x)^2} \|a^x\|_\infty \leq \frac{1}{4} \left(1 - \frac{C_0 + \delta}{2(1 - \delta)} - 4\delta \right) \quad (4.21)$$

$$\frac{\Delta t}{(\bar{H}_y)^2} \|a^y\|_\infty \leq \frac{1}{4} \left(1 - \frac{C_0 + \delta}{2(1 - \delta)} - 5\delta \right). \quad (4.22)$$

Here $H = \max_{l,k}(H_l^x, H_k^y)$, $\bar{H}_x = \min_l H_l^x$, and $\bar{H}_y = \min_k H_k^y$.

Proof of Theorem 5. Take \bar{U} and \bar{Q}) as in (3.15)-(3.16), and let $\mu = Q - \bar{Q}$, $\nu = U - \bar{U}$, $\kappa = q - \bar{Q}$, and $\eta = \tilde{u} - \bar{U}$. Set $\hat{\mu}^n = (\sum_{l=1}^L \hat{\mu}_l^{x,n}, \sum_{k=1}^K \hat{\mu}_k^{y,n})$, where $\hat{\mu}_l^{x,n}(x, y) = \mu^{x,n}(\bar{x}_l, y) \phi^{l,x}(x)$, and $\hat{\mu}_k^{y,n}(x, y) = \mu^{y,n}(x, \bar{y}_k) \phi^{k,y}(y)$. Then setting $v = \mu^n - \hat{\mu}^n$, and $w = \nu^n$, the analogue of (3.24) is

$$\begin{aligned} & |||\mu^{x,n}|||_{x,a^x}^2 + |||\mu^{y,n}|||_{y,a^y}^2 + (\partial_t \nu^n, \nu^n) + \sum_{l=1}^L |||\hat{\mu}_l^{x,n}|||_{H_l^x, a^x}^2 + \sum_{k=1}^K |||\hat{\mu}_k^{y,n}|||_{H_k^y, a^y}^2 + \|\nu^n\|^2 \\ &= \left\langle \mu^{x,n}, \sum_l \hat{\mu}_l^{x,n} \right\rangle_{x,a^x} + \left\langle \mu^{y,n}, \sum_k \hat{\mu}_k^{y,n} \right\rangle_{y,a^y} + (\nu^{n-1} - \nu^n, \nabla \cdot \hat{\mu}^n) + (\partial_t \eta^n, \nu^n) \\ &+ (u_t^n - \partial_t u^n, \nu^n) + \sum_{l=1}^L E_l^x + \sum_{k=1}^K E_k^y \\ &= R'_1 + R'_2 + R'_3 + R'_4 + R'_5 + \sum_{l=1}^L E_l^x + \sum_{k=1}^K E_k^y. \end{aligned} \quad (4.23)$$

Here

$$\langle f, g \rangle_{H_l^x, a^x} = \sum_j \frac{H_l^x}{a^x(\bar{x}_l, y_j)} f(\bar{x}_l, y_j) g(\bar{x}_l, y_j) h_j^y, \quad (4.24)$$

$$\langle f, g \rangle_{H_k^y, a^y} = \sum_i \frac{H_k^y}{a^y(x_i, \bar{y}_k)} f(x_i, \bar{y}_k) g(x_i, \bar{y}_k) h_i^x, \quad (4.25)$$

$$|||f|||_{H_l^x, a^x}^2 = \langle f, f \rangle_{H_l^x, a^x}, \quad (4.26)$$

$$|||f|||_{H_k^y, a^y}^2 = \langle f, f \rangle_{H_k^y, a^y}, \quad (4.27)$$

and

$$E_l^x = -\langle \kappa^{x,n-1}, \hat{\mu}_l^{x,n} \rangle_{x,a^x} + \left[\langle q^{x,n-1}, \hat{\mu}_l^{x,n} \rangle_{x,a^x} - \langle q^{x,n-1}, \hat{\mu}_l^{x,n} \rangle_{H_l^x, a^x} \right]$$

$$\begin{aligned}
& + \langle q^{x,n-1} - q^{x,n}, \hat{\mu}_l^{x,n} \rangle_{H_l^x, a^x} + \langle \kappa^{x,n}, \hat{\mu}_l^{x,n} \rangle_{H_l^x, a^x} \\
& \equiv \sum_{m=1}^4 E_{l,m}^x;
\end{aligned} \tag{4.28}$$

E_l^y is defined similarly to (4.28) with x replaced by y and l replaced by k .

Before estimating the right side of (4.23), note that

$$\begin{aligned}
\left\| \sum_l \hat{\mu}_l^{x,n} \right\|_{x, a^x}^2 &= \sum_j \sum_i \frac{h_{i+1/2}^x}{a_{i+1/2, j}^x} \left[\sum_l \mu^{x,n}(\bar{x}_l, y_j) \phi_{i+1/2}^{l,x} \right]^2 h_j^y \\
&= \sum_j h_j^y \sum_i \frac{h_{i+1/2}^x}{a_{i+1/2, j}^x} \left\{ \sum_l |\mu^{x,n}(\bar{x}_l, y_j)|^2 |\phi_{i+1/2}^{l,x}|^2 \right. \\
&\quad \left. + 2 \sum_{l=2}^L \mu^{x,n}(\bar{x}_l, y_j) \mu^{x,n}(\bar{x}_{l-1}, y_j) \phi_{i+1/2}^{l,x} \phi_{i+1/2}^{l-1,x} \right\} \\
&\equiv T_1 + T_2.
\end{aligned} \tag{4.29}$$

For a^x smooth and H sufficiently small, follow the argument in (3.56)-(3.58) to find

$$T_1 \leq (C_0 + \delta) \sum_{l=1}^L \left\| \hat{\mu}_l^{x,n} \right\|_{H_l^x, a^x}^2. \tag{4.30}$$

Next, let $I^l = \{i \mid \bar{x}_{l-1} \leq x_{i+1/2} \leq \bar{x}_l\}$, then, dropping y -dependence momentarily,

$$\begin{aligned}
2 \sum_{l=2}^L \sum_{I^l} \frac{h_{i+1/2}^x}{a_{i+1/2}^x} \mu^{x,n}(\bar{x}_l) \mu^{x,n}(\bar{x}_{l-1}) \phi_{i+1/2}^{l,x} \phi_{i+1/2}^{l-1,x} &\leq \sum_{l=2}^L \left[\sum_{I^l} \frac{h_{i+1/2}^x}{a_{i+1/2}^x} |\mu^{x,n}(\bar{x}_l)|^2 |\phi_{i+1/2}^{l,x}|^2 \right. \\
&\quad \left. + \sum_{I^l} \frac{h_{i+1/2}^x}{a_{i+1/2}^x} |\mu^{x,n}(\bar{x}_{l-1})|^2 |\phi_{i+1/2}^{l-1,x}|^2 \right] \\
&\leq (C_0 + \delta) \sum_l |\mu^{x,n}(\bar{x}_l)|^2 H_l^x.
\end{aligned} \tag{4.31}$$

Thus, multiplying by h_j^y and summing on j ,

$$T_2 \leq (C_0 + \delta) \sum_l \left\| \hat{\mu}_l^{x,n} \right\|_{H_l^x, a^x}^2. \tag{4.32}$$

Combine these bounds to find

$$\left\| \sum_l \hat{\mu}_l^{x,n} \right\|_{x, a^x}^2 \leq 2(C_0 + \delta) \sum_l \left\| \hat{\mu}_l^{x,n} \right\|_{H_l^x, a^x}^2. \tag{4.33}$$

By (4.33),

$$\begin{aligned}
R'_1 &\leq (1 - \delta) \|\mu^{x,n}\|_{x,a^x}^2 + \frac{1}{4(1 - \delta)} \left\| \sum_l \hat{\mu}_l^{x,n} \right\|_{x,a^x}^2 \\
&\leq (1 - \delta) \|\mu^{x,n}\|_{x,a^x}^2 + \frac{1}{2(1 - \delta)} (C_0 + \delta) \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^x, a^x}^2. \quad (4.34)
\end{aligned}$$

Similarly

$$R'_2 \leq (1 - \delta) \|\mu^{y,n}\|_{y,a^y}^2 + \frac{1}{2(1 - \delta)} (C_0 + \delta) \sum_k \|\hat{\mu}_k^{y,n}\|_{H_k^y, a^y}^2. \quad (4.35)$$

For R'_3 , note that

$$\left\| \sum_l (\hat{\mu}_l^{x,n})_x \right\|^2 = \sum_j h_j^y \int_0^1 \left| \sum_l \mu^{x,n}(\bar{x}_l, y_j) \phi_x^{l,x}(x) \right|^2 dx. \quad (4.36)$$

Consider

$$\begin{aligned}
\int_0^1 \left| \sum_l \mu^{x,n}(\bar{x}_l, y_j) \phi_x^{l,x}(x) \right|^2 dx &= \sum_l |\mu^{x,n}(\bar{x}_l)|^2 \int_0^1 |\phi_x^{l,x}(x)|^2 dx \\
&\quad + 2 \sum_l \mu^{x,n}(\bar{x}_l) \mu^{x,n}(\bar{x}_{l-1}) \int_0^1 \phi_x^{l,x}(x) \phi_x^{l-1,x}(x) dx \\
&= 2 \sum_l \frac{|\mu^{x,n}(\bar{x}_l)|^2}{|H_l^x|^2} H_l^x \\
&\quad + 2 \sum_l \frac{\mu^{x,n}(\bar{x}_l) \mu^{x,n}(\bar{x}_{l-1})}{H_l^x H_{l-1}^x} \min(H_l^x, H_{l-1}^x). \quad (4.37)
\end{aligned}$$

Thus

$$\left\| \sum_l (\hat{\mu}_l^{x,n})_x \right\|^2 \leq \frac{4}{\bar{H}_x^2} \|a^x\|_\infty \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^x, a^x}^2. \quad (4.38)$$

Similarly,

$$\left\| \sum_k (\hat{\mu}_k^{y,n})_y \right\|^2 \leq \frac{4}{\bar{H}_y^2} \|a^y\|_\infty \sum_k \|\hat{\mu}_k^{y,n}\|_{H_k^y, a^y}^2. \quad (4.39)$$

Thus,

$$\begin{aligned}
R'_3 &\leq \frac{1}{2\Delta t} \|\nu^{n-1} - \nu^n\|^2 + \frac{\Delta t}{2} \|\nabla \cdot \hat{\mu}^n\|^2 \\
&\leq \frac{1}{2\Delta t} \|\nu^{n-1} - \nu^n\|^2 + \frac{4\Delta t}{(\bar{H}_x)^2} \|a^x\|_\infty \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2 \\
&\quad + \frac{4\Delta t}{(\bar{H}_y)^2} \|a^y\|_\infty \sum_k \|\hat{\mu}_k^{y,n}\|_{H_k^{y,a^y}}^2.
\end{aligned} \tag{4.40}$$

As we have seen before,

$$R'_4 + R'_5 \leq C(h^4 + \Delta t \|u_{tt}\|_{L^2(t^{n-1}, t^n; L^2)}^2) + \|\nu^n\|^2. \tag{4.41}$$

Next, consider $\sum_l E_l^x$. Using (4.33),

$$\begin{aligned}
\sum_l E_{l,1}^x &\equiv \left\langle \kappa^{x,n}, \sum_l \hat{\mu}_l^{x,n} \right\rangle_{x,a^x} \\
&\leq C(\delta^{-1})h^4 + \delta \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2.
\end{aligned} \tag{4.42}$$

Use the estimate for R_4 above, i.e., (3.38), and sum on l to see that

$$\sum_l E_{l,2}^x \leq C(\delta^{-1} [h^4 LH^{-1} + (LH)H^4] + \delta \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2). \tag{4.43}$$

Use the estimate for R_5 , i.e., (3.39), and sum on l to see that

$$\sum_l E_{l,3}^x \leq C(\delta^{-1})\Delta t^2(LH) + \delta \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2. \tag{4.44}$$

Use the estimate for R_6 ((3.40)) to see that

$$\sum_l E_{l,4}^x \leq C(\delta^{-1}) \sum_l H_l^x \sum_j |\kappa^{x,n}(\bar{x}_l, y_j)|^2 h_j^y + \delta \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2. \tag{4.45}$$

Derive similar estimates for $\sum_k E_k^y$, and combine the results with (4.23)-(4.45) to obtain

$$\mathcal{E} \leq \left(\frac{C_0 + \delta}{2(1 - \delta)} + \frac{4\Delta t}{\bar{H}_x^2} \|a^x\|_\infty + 4\delta \right) \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,a^x}}^2$$

$$\begin{aligned}
& + \left(\frac{C_0 + \delta}{2(1-\delta)} + \frac{4\Delta t}{H_y^2} \|a^y\|_\infty + 4\delta \right) \sum_k \|\hat{\mu}_k^{y,n}\|_{H_k^{y,ay}}^2 \\
& + C(\Delta t \|u_{tt}\|_{L^2(t^n, t^{n-1}; L^2)}^2 + h^4) + C(\delta^{-1})(L+K)(h^4 H^{-1} + H^5 + H\Delta t^2) \\
& + C(\delta^{-1}) \left[\sum_l H_l^x \sum_j |\kappa^{x,n}(\bar{x}_l, y_j)|^2 h_j^y + \sum_k H_k^y \sum_i |\kappa^{y,n}(x_i, \bar{y}_k)|^2 h_i^x \right], \quad (4.46)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E} & = \frac{1}{2\Delta t^n} \left[\|\nu^n\|^2 - \|\nu^{n-1}\|^2 \right] + \delta \|\mu^{x,n}\|_{x,ax}^2 + \delta \|\mu^{y,n}\|_{y,ay}^2 \\
& \quad + \sum_l \|\hat{\mu}_l^{x,n}\|_{H_l^{x,ax}}^2 + \sum_k \|\hat{\mu}_k^{y,n}\|_{H_k^{y,ay}}^2.
\end{aligned}$$

Invoke the constraints (4.21) and (4.22) to hide the first two terms on the right side above, multiply above by $2\Delta t^n$, and sum on n , and invoke the triangle inequality and (3.17) to complete the proof.