A New Formulation of Mixed Finite Element Methods for Second Order Elliptic Problems

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A NEW FORMULATION OF MIXED FINITE ELEMENT METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS*

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Abstract. In this paper we show that mixed finite element methods for a fairly general second order elliptic problem with variable coefficients can be given a nonmixed formulation. We define an approximation method by incorporating some projection operators within a standard Galerkin method, which we call a projection finite element method. It is shown that for a given mixed method, if the projection method's finite element space $\mathcal{M}_h$ satisfies two conditions, then the two approximation methods are equivalent. These two conditions can be simplified for a single element in the case of mixed spaces possessing the usual vector projection operator. For any such mixed spaces defined on a geometrically regular partition of the domain, we can then easily construct appropriate conforming spaces $\mathcal{M}_h$. We also present for several mixed methods alternative nonconforming spaces $\mathcal{M}_h$ that also satisfy the two conditions for equivalence.

Key words. finite element, mixed method, nonconforming method, projection finite element method

AMS(MOS) subject classifications. 65N30, 65N22

1. Introduction. In 1985, Arnold and Brezzi [1] showed that for a simple second order elliptic problem, the mixed finite element method for the lowest order Raviart-Thomas [10] space defined over triangles is equivalent to a modification of a standard nonconforming finite element method (see Section 5.1 below). This equivalence can be exploited to rearrange the computation of the mixed method solution. The nonconforming method yields a symmetric and positive definite problem (i.e., a minimization problem); whereas, the original mixed formulation is a saddle point problem.

Arnold and Brezzi considered only the highest order terms of the elliptic problem, and they required that the coefficients be piecewise constant over the finite element mesh. In this paper, we consider a large class of mixed methods for a more general second order differential problem with variable coefficients. We define for this problem a nonstandard Galerkin-like method, which we will call a projection finite element method, for some finite element space $\mathcal{M}_h$. Much of the paper is concerned with constructing appropriate $\mathcal{M}_h$ spaces; that is, ones which make this projection method equivalent to given mixed methods. Many of these are found, and they provide an alternate formulation for mixed methods. Our projection method is a Galerkin method with the addition of some projection operators. For the same approximate solution, it provides us a nonmixed, more standard formulation as an alternative to the usual mixed form. This not only illuminates the mixed method as an approximation technique, but it also has potential application in both computation and numerical analysis.

The differential problem and the mixed and projection finite element methods are defined below in this section. We develop in the next section two conditions on $\mathcal{M}_h$ that are sufficient to imply the equivalence of the two methods. In Section 3 we consider the problem of constructing finite element spaces that satisfy these two conditions. We derive a simple local criterion that guarantees the equivalence in the case of mixed spaces possessing the usual vector projection operator. With this

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criterion we can, in the last four sections, define some specific spaces for our projection finite element method that that give rise to equivalent mixed methods. In Section 4 we give a general theorem which defines a conforming space $\mathcal{M}_h$ for any mixed method that satisfies a few conditions and is defined on a geometrically regular partition of the domain. This theorem applies to the spaces of Raviart and Thomas [10], Nedelec [9], Brezzi, Douglas, and Marini [5], Brezzi, Douglas, Durán, and Fortin [3], Brezzi, Douglas, Fortin, and Marini [4], and Chen and Douglas [8] on simplexes, rectangular parallelepipeds, and prisms in two and three space dimensions. Finally, in Sections 5–7, we construct some alternate, nonconforming spaces which fulfill the same properties (thus more that one $\mathcal{M}_h$ may give rise to a given mixed method). We remark that Chen [7], independently of the author, has recently also derived some specific conforming and nonconforming methods that are equivalent to certain lower dimensional mixed methods.

In the remainder of the introduction we introduce the mathematical problem and the finite element methods. We consider the following elliptic problem for p on the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$:

$$-\nabla \cdot \big(a(\nabla p + bp - c)\big) + dp = f \quad \text{in } \Omega, \tag{1.1a}$$

$$p = -g \quad \text{on } \Gamma_1, \tag{1.1b}$$

$$-\big(a(\nabla p + bp - c)\big) \cdot \nu = 0 \quad \text{on } \Gamma_2, \tag{1.1c}$$

where $a(x)$ is a uniformly positive definite, bounded, symmetric tensor, $b(x)$ and $c(x)$ are bounded vectors, $d(x) \geq 0$ is bounded, $f(x) \in L^2(\Omega)$, $g(x) \in H^1(\Omega)$ ($H^k(\Omega) = W^{k,2}(\Omega)$ is the Sobolev space of k differentiable functions in $L^2(\Omega)$), and ν is the outer unit normal to the domain. Let $(\,\cdot\,,\,\cdot\,)_S$ denote the $L^2(S)$ inner product (we omit S if $S = \Omega$). We assume that our problem is coercive in the sense that there is a positive constant κ such that for any $v \in \big(L^2(\Omega)\big)^n$ and $w \in L^2(\Omega)$,

$$(a^{-1}v, v) + (bw, v) + (dw, w) \geq \kappa\{\|v\|^2_{(L^2(\Omega))^n} + (dw, w)\}, \tag{1.2}$$

and also that if $\Gamma_1 = \emptyset$, $d(x) > 0$ on some set of positive measure.

Problem (1.1) is recast in mixed form as follows. Let

$$\begin{aligned} H(\mathrm{div};\Omega) &= \{v \in \big(L^2(\Omega)\big)^n : \nabla \cdot v \in L^2(\Omega)\}, \\ \mathcal{V} &= \{v \in H(\mathrm{div};\Omega) : v \cdot \nu = 0 \text{ on } \Gamma_2\}, \\ \mathcal{W} &= L^2(\Omega). \end{aligned}$$

Then the mixed form of (1.1) for the pair $(u, p) \in \mathcal{V} \times \mathcal{W}$ is

$$(a^{-1}u, v) - (p, \nabla \cdot v) + (bp, v) = (c, v) + (g, v \cdot \nu)_{\Gamma_1} \quad \text{for all } v \in \mathcal{V}, \tag{1.3a}$$

$$(\nabla \cdot u, w) + (dp, w) = (f, w) \quad \text{for all } w \in \mathcal{W}. \tag{1.3b}$$

Note that (1.3a) implies that $u = -a(\nabla p + bp - c)$.

To define a finite element method, we need a partition $\mathcal{T}_h$ of Ω into, say, simplexes, rectangular parallelepipeds, and/or prisms, where only edges or faces on $\partial\Omega$ may be curved. In $\mathcal{T}_h$, we also need that adjacent elements completely share their common edge or face; let $\mathcal{E}_h$ denote the set of all interior edges ($n = 2$) or faces ($n = 3$) of $\mathcal{T}_h$. We tacitly assume that $\mathcal{E}_h \neq \emptyset$. Finally, each exterior edge or face has imposed on it either Dirichlet or Neumann conditions, but not both.

Let $\mathcal{V}_h \times \mathcal{W}_h \subset \mathcal{V} \times \mathcal{W}$ denote some standard mixed finite element space (for second order elliptic problems) defined over $\mathcal{T}_h$ such that $\nabla \cdot \mathcal{V}_h = \mathcal{W}_h$ (see, e.g., [3], [4], [5], [8], [9], and [10]). This space is finite dimensional and defined locally on each element $T \in \mathcal{T}_h$, so let $\mathcal{V}_h(T) = \mathcal{V}_h|_T$ and $\mathcal{W}_h(T) = \mathcal{W}_h|_T$. The constraint $\mathcal{V}_h \subset \mathcal{V}$ says that the normal component of the members of $\mathcal{V}_h$ are continuous across the interior boundaries in $\mathcal{E}_h$. Following [1], we relax this constraint on $\mathcal{V}_h$ by defining

$$\tilde{\mathcal{V}}_h = \{v \in L^2(\Omega) : v|_T \in \mathcal{V}_h(T) \text{ for each } T \in \mathcal{T}_h\}.$$

We then need to introduce Lagrange multipliers to enforce the required continuity on $\tilde{\mathcal{V}}_h$, so define

$$\mathcal{L}_h = \left\{\lambda \in L^2\biggl(\bigcup_{e\in\mathcal{E}_h} e\biggr) : \lambda|_e \in \mathcal{V}_h \cdot \nu|_e \text{ for each } e \in \mathcal{E}_h\right\}.$$

The mixed finite element solution of (1.3) is $(U, P) \in \mathcal{V}_h \times \mathcal{W}_h$ satisfying

$$(a^{-1}U, v) - (P, \nabla \cdot v) + (bP, v) = (c, v) + (g, v \cdot \nu)_{\Gamma_1} \quad \text{for all } v \in \mathcal{V}_h, \tag{1.4a}$$

$$(\nabla \cdot U, w) + (dP, w) = (f, w) \quad \text{for all } w \in \mathcal{W}_h. \tag{1.4b}$$

The unconstrained problem is to find $(U, P, \lambda) \in \mathcal{V}_h \times \mathcal{W}_h \times \mathcal{L}_h$ such that

$$\begin{aligned}(a^{-1}U, v) - \sum_{T\in\mathcal{T}_h} \left[(P, \nabla \cdot v)_T - (\lambda, v \cdot \nu_T)_{\partial T\setminus\partial\Omega}\right] + (bP, v)& \\ = (c, v) + (g, v \cdot \nu)_{\Gamma_1} \quad \text{for all } v \in \tilde{\mathcal{V}}_h,&\end{aligned} \tag{1.5a}$$

$$\sum_{T\in\mathcal{T}_h} (\nabla \cdot U, w)_T + (dP, w) = (f, w) \quad \text{for all } w \in \mathcal{W}_h, \tag{1.5b}$$

$$\sum_{T\in\mathcal{T}_h} (U \cdot \nu_T, \mu)_{\partial T\setminus\partial\Omega} = 0 \quad \text{for all } \mu \in \mathcal{L}_h. \tag{1.5c}$$

Note that U and P are identical in the two formulations, since (1.5c) forces U to be in $\mathcal{V}_h$.

We now define the projection operators that we need to define our new new method. Let $\mathcal{P}_{\mathcal{W}_h} : L^2(\Omega) \to \mathcal{W}_h$ denote $L^2(\Omega)$-projection: For $\psi \in L^2(\Omega)$,

$$(\psi - \mathcal{P}_{\mathcal{W}_h}\psi, w) = 0 \quad \text{for all } w \in \mathcal{W}_h. \tag{1.6}$$

Similarly define $\mathcal{P}_{\mathcal{L}_h} : L^2(\cup_{e\in\mathcal{E}_h} e) \to \mathcal{L}_h$ to be $L^2(\cup_{e\in\mathcal{E}_h} e)$-projection. To handle variable $a(x)$, we introduce the weighted $\left(L^2(\Omega)\right)^n$-projection $\tilde{\mathcal{P}} : \left(L^2(\Omega)\right)^n \to \tilde{\mathcal{V}}_h$ defined by

$$\left(a^{-1}(\psi - \tilde{\mathcal{P}}\psi), v\right) = 0 \quad \text{for all } v \in \tilde{\mathcal{V}}_h. \tag{1.7}$$

Note that each of these operators is defined locally on each $T \in \mathcal{T}_h$ or on each $e \in \mathcal{E}_h$, since only $\mathcal{V}_h$ has a continuity constraint.

Now we can define abstractly our projection finite element method. Let $\mathcal{M}_h$ denote some as yet unspecified finite dimensional finite element space defined over $\mathcal{T}_h$ such that $\mathcal{M}_h|_{\Gamma_1} = 0$. Then we seek $Q \in \mathcal{M}_h - g$ satisfying

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \left(\tilde{\mathcal{P}}[a(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q - c)], \nabla\xi\right)_T + (d\mathcal{P}_{\mathcal{W}_h} Q, \mathcal{P}_{\mathcal{W}_h}\xi)& \\ = (f, \mathcal{P}_{\mathcal{W}_h}\xi) \quad \text{for all } \xi \in \mathcal{M}_h.&\end{aligned} \tag{1.8}$$

Our goal is to define $\mathcal{M}_h$ so that

$$U = \tilde{\mathcal{P}}[a(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q - c)], \tag{1.9a}$$

$$P = \mathcal{P}_{\mathcal{W}_h} Q, \tag{1.9b}$$

$$\lambda = \mathcal{P}_{\mathcal{L}_h} Q. \tag{1.9c}$$

In the next section we derive two conditions on $\mathcal{M}_h$ which give (1.9).

2. Two conditions for the equivalence of the methods. The first thing that we require of the space $\mathcal{M}_h$ is that it give rise to a legitimate finite element method defined by (1.8); hence, we require that there exists a unique solution to the problem. Since (1.8) is a square linear system, uniqueness implies existence. For uniqueness, if $Q \in \mathcal{M}_h$ satisfies

$$\sum_{T\in\mathcal{T}_h} \left(\tilde{\mathcal{P}}[a(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q)], \nabla\xi\right)_T + (d\mathcal{P}_{\mathcal{W}_h} Q, \mathcal{P}_{\mathcal{W}_h}\xi) = 0 \quad \text{for all } \xi \in \mathcal{M}_h,$$

then we need to show that $Q = 0$. Take $\xi = Q$, note that by (1.7),

$$\left(\tilde{\mathcal{P}}(a\nabla Q), \nabla Q\right)_T = \left(a^{-1}\tilde{\mathcal{P}}(a\nabla Q), a\nabla Q\right)_T = \left(a^{-1}\tilde{\mathcal{P}}(a\nabla Q), \tilde{\mathcal{P}}(a\nabla Q)\right)_T,$$
$$\left(\tilde{\mathcal{P}}(ab\mathcal{P}_{\mathcal{W}_h} Q), \nabla Q\right)_T = \left(a^{-1}\tilde{\mathcal{P}}(ab\mathcal{P}_{\mathcal{W}_h} Q), a\nabla Q\right)_T = \left(b\mathcal{P}_{\mathcal{W}_h} Q, \tilde{\mathcal{P}}(a\nabla Q)\right)_T,$$

and then apply (1.2) to see that $\|\tilde{\mathcal{P}}(a\nabla Q)\|_{(L^2(\Omega))^n} = 0$ and $(d\mathcal{P}_{\mathcal{W}_h} Q, \mathcal{P}_{\mathcal{W}_h} Q) = 0$. The former requires that the $\tilde{\mathcal{P}}$-projection of $a\nabla Q$ be zero on each $T \in \mathcal{T}_h$:

$$(a^{-1}a\nabla Q, v)_T = 0 \quad \text{for all } v \in \mathcal{V}_h(T).$$

We therefore require of the space $\mathcal{M}_h$ the first condition:

(C1) For $\xi \in \mathcal{M}_h$, if $(\nabla\xi, v)_T = 0$ for all $v \in \mathcal{V}_h(T)$ and all $T \in \mathcal{T}_h$, and if $(d\mathcal{P}_{\mathcal{W}_h}\xi, \mathcal{P}_{\mathcal{W}_h}\xi) = 0$, then $\xi = 0$.

We now consider the equivalence of the two schemes (1.5) and (1.8). It is convenient to take Q as given by (1.8) and let U, P, and λ be given by (1.9). We then show that (1.5) results.

By the definitions (1.9), definitions (1.6)–(1.7), and finally integration by parts, we see that for any $v \in \tilde{\mathcal{V}}_h$,

$$\begin{aligned}
(a^{-1}U, v) &- \sum_{T\in\mathcal{T}_h} \left[(P, \nabla\cdot v)_T - (\lambda, v\cdot\nu_T)_{\partial T\setminus\partial\Omega}\right] + (bP, v) \\
&= -\left(a^{-1}\tilde{\mathcal{P}}[a(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q - c)], v\right) \\
&\quad - \sum_{T\in\mathcal{T}_h} \left[(\mathcal{P}_{\mathcal{W}_h} Q, \nabla\cdot v)_T - (\mathcal{P}_{\mathcal{L}_h} Q, v\cdot\nu_T)_{\partial T\setminus\partial\Omega}\right] + (b\mathcal{P}_{\mathcal{W}_h} Q, v) \\
&= -\sum_{T\in\mathcal{T}_h} \left(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q - c, v\right)_T \\
&\quad - \sum_{T\in\mathcal{T}_h} \left[(Q, \nabla\cdot v)_T - (Q, v\cdot\nu_T)_{\partial T\setminus\partial\Omega}\right] + (b\mathcal{P}_{\mathcal{W}_h} Q, v) \\
&= \sum_{T\in\mathcal{T}_h} \left[-(\nabla Q - c, v)_T + (\nabla Q, v)_T\right] + (g, v\cdot\nu)_{\Gamma_1} \\
&= (c, v) + (g, v\cdot\nu)_{\Gamma_1};
\end{aligned} \tag{2.1}$$

this is (1.5a).

For (1.5b–c), we integrate the first term on the left side of (1.8) by parts to see that for any $\xi \in \mathcal{M}_h$,

$$\sum_{T\in\mathcal{T}_h} \left(\tilde{\mathcal{P}}[a(\nabla Q + b\mathcal{P}_{\mathcal{W}_h} Q - c)], \nabla\xi\right)_T = \sum_{T\in\mathcal{T}_h} \left[(\nabla\cdot U, \xi)_T - (U\cdot\nu_T, \xi)_{\partial T}\right]; \tag{2.2}$$

hence, introducing some projection operators, (1.8) becomes

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (\nabla\cdot U, \mathcal{P}_{\mathcal{W}_h}\xi)_T + (dP, \mathcal{P}_{\mathcal{W}_h}\xi) - \sum_{T\in\mathcal{T}_h} (U\cdot\nu_T, \mathcal{P}_{\mathcal{L}_h}\xi)_{\partial T\setminus\partial\Omega}\\ = (f, \mathcal{P}_{\mathcal{W}_h}\xi) \quad \text{for all } \xi\in\mathcal{M}_h,\end{aligned} \tag{2.3}$$

where $\mathcal{P}_{\mathcal{L}_h}\xi$ on ∂T is defined on the trace of ξ from within T. To separate out the effects on ∂T, we require the following condition on $\mathcal{M}_h$:

(C2) For $\xi \in \mathcal{M}_h$, its projection $\mathcal{P}_{\mathcal{L}_h}\xi$ can be uniquely defined on each $e \in \mathcal{E}_h$, and for any $(w,\mu) \in \mathcal{W}_h \times \mathcal{L}_h$, there exist $\xi_1, \xi_2 \in \mathcal{M}_h$, such that

$$\text{(i)} \begin{cases} \mathcal{P}_{\mathcal{W}_h}\xi_1 = 0, \\ \mathcal{P}_{\mathcal{L}_h}\xi_1 = \mu, \end{cases} \qquad \text{and} \qquad \text{(ii)} \begin{cases} \mathcal{P}_{\mathcal{W}_h}\xi_2 = w, \\ \mathcal{P}_{\mathcal{L}_h}\xi_2 = 0. \end{cases}$$

The ξ_2 give us (1.5b) while the ξ_1 give us (1.5c).

We have shown the following theorem.

THEOREM 1. *For a given mixed finite element method (1.4) or (1.5) such that $\mathcal{W}_h = \nabla\cdot\mathcal{V}_h$, the projection finite element method (1.8) is well-defined and equivalent to it by the relations (1.9) if $\mathcal{M}_h$ satisfies (C1) and (C2).*

We remark that for a given mixed method, the choice of $\mathcal{M}_h$, if it exists, is not necessarily unique. Many examples are given in Sections 4–7 below.

We have the following converse to Theorem 1.

THEOREM 2. *If a given projection finite element method (1.8) with projection space $\mathcal{V}_h$ (and $\mathcal{W}_h$ and $\mathcal{L}_h$ defined from $\mathcal{V}_h$) satisfies (C1), (C2), and the property that for any $\xi \in \mathcal{M}_h$ such that $\mathcal{P}_{\mathcal{L}_h}\xi = 0$,*

$$\sup_{v\in\mathcal{V}_h\setminus\{0\}} \frac{\sum_{T\in\mathcal{T}_h}(\nabla\xi, v)_T}{\|v\|_{(L^2(\Omega))^n}} \geq \kappa_h \|\mathcal{P}_{\mathcal{W}_h}\xi\|_{L^2(\Omega)} \tag{2.4}$$

for some $\kappa_h > 0$, then $\mathcal{V}_h$ gives rise to an equivalent mixed method (1.4) or (1.5) in which $\mathcal{V}_h$ and $\mathcal{W}_h$ satisfy the inf-sup condition [2] for the constant κ_h: For any $w \in \mathcal{W}_h$,

$$\sup_{v\in\mathcal{V}_h\setminus\{0\}} \frac{(w, \nabla\cdot v)}{\|v\|_{(L^2(\Omega))^n}} \geq \kappa_h \|w\|_{L^2(\Omega)}.$$

Moreover, if (2.4) holds uniformly in h, i.e., $\kappa_h = \kappa$ is independent of h, then also the inf-sup condition holds uniformly in h.

Proof. For $w \in \mathcal{W}_h$, we can choose by (C2) $\xi \in \mathcal{M}_h$ such that $\mathcal{P}_{\mathcal{W}_h}\xi = -w$ and $\mathcal{P}_{\mathcal{L}_h}\xi = 0$. For this ξ, (2.4) is the inf-sup condition after an integration by parts. ∎

3. On the local construction of $\mathcal{M}_h$. It is not yet clear whether an appropriate $\mathcal{M}_h$ can be constructed for a given mixed method. In this section we consider the question of how to construct such an $\mathcal{M}_h$. We do not discuss problems associated with the outer boundary of the domain, but instead concentrate on the local spaces defined on some $T \in \mathcal{T}_h$ with edges or faces $e \in \mathcal{E}_h$.

We begin by noting that dimensional considerations for satisfying (C1)–(C2) easily show the following corollary of Theorem 1, wherein $\mathcal{M}_h(T) = \mathcal{M}_h|_T$ and $\mathcal{L}_h(e) = \mathcal{L}_h|_e$.

COROLLARY 1. *If a given mixed finite element method (1.4) or (1.5) (with $\mathcal{W}_h = \nabla \cdot \mathcal{V}_h$) is equivalent to the projection finite element method (1.8) by the relations (1.9), then, for each $T \in \mathcal{T}_h$ such that $\partial T \cap \partial \Omega = \emptyset$,*

$$\dim(\mathcal{W}_h(T)) + \sum_{e \subset \partial T} \dim(\mathcal{L}_h(e)) \le \dim(\mathcal{M}_h(T)) \le \dim(\mathcal{V}_h(T)) + 1.$$

This result can be used to bound the dimension of $\mathcal{M}_h(T)$; it may even show that $\mathcal{M}_h(T)$ cannot exist for some novel mixed methods.

We now localize the condition (C1) as follows:

(C1′) For $\xi \in \mathcal{M}_h(T)$, if $(\nabla\xi, v)_T = 0$ for all $v \in \mathcal{V}_h(T)$, then ξ is constant on T.

LEMMA 1. *Suppose that $\mathcal{V}_h \times \mathcal{W}_h$ is a mixed finite element space such that $\mathcal{W}_h = \nabla \cdot \mathcal{V}_h$, $P_0(T) \subset \mathcal{W}_h(T)$ for each $T \in \mathcal{T}_h$, and $P_0(e) \subset \mathcal{L}_h(e)$ for each $e \in \mathcal{E}_h$. If $\mathcal{M}_h$ satisfies (C1′) for each $T \in \mathcal{T}_h$ and (C2), then $\mathcal{M}_h$ satisfies (C1).*

Proof. For some $\xi \in \mathcal{M}_h$, suppose that $(\nabla\xi, v)_T = 0$ for all $v \in \mathcal{V}_h(T)$ and $T \in \mathcal{T}_h$, and $(d\mathcal{P}_{\mathcal{W}_h}\xi, \mathcal{P}_{\mathcal{W}_h}\xi) = 0$. We conclude from (C1′) that ξ is constant on each T. Since (C2) requires a unique definition of $\mathcal{P}_{\mathcal{L}_h}\xi$, in fact ξ is a constant on all of Ω. Finally, either $\Gamma_1 \ne \emptyset$ or $d > 0$ implies that $\xi = 0$. □

The mixed method spaces that we consider have the property that there exists a projection operator $\Pi_h : \left(H^1(T)\right)^n \to \mathcal{V}_h(T)$ such that

$$\nabla \cdot (\Pi_h v) = \mathcal{P}_{\mathcal{W}_h}(\nabla \cdot v), \tag{3.1a}$$

$$(\Pi_h v) \cdot \nu = \mathcal{P}_{\mathcal{L}_h}(v \cdot \nu). \tag{3.1b}$$

We shall exploit this fact in the following way.

THEOREM 3. *Suppose that T is convex and that $\mathcal{V}_h(T) \times \mathcal{W}_h(T)$ is a mixed finite element space such that $\mathcal{W}_h(T) = \nabla \cdot \mathcal{V}_h(T)$, $P_0(T) \subset \mathcal{W}_h(T)$, $P_0(e) \subset \mathcal{L}_h(e)$ for each $e \subset \partial T$, and there exists an operator $\Pi_h : \left(H^1(T)\right)^n \to \mathcal{V}_h(T)$ satisfying (3.1). If $\mathcal{M}_h(T)$ is a space of functions such that*

$$\dim\{\mathcal{M}_h(T)\} = \dim\{\mathcal{W}_h(T)\} + \sum_{e \subset \partial T} \dim\{\mathcal{L}_h(e)\}$$

with (unisolvent) degrees of freedom described by

(DF1) *$(\xi, w)_T$ for all w in a basis of $\mathcal{W}_h(T)$,*

(DF2) *$(\xi, \lambda)_e$ for all λ in a basis of $\mathcal{L}_h(e)$, for each $e \subset \partial T$,*

then $\mathcal{M}_h(T)$ satisfies (C1′) and (C2).

Proof. The hypothesis (DF) gives (C2), so we need only show (C1′). Let $A_S(\psi) = (\psi, 1)_S/(1, 1)_T$ denote a type of average of a function $\psi(x)$ on $S \subset T$. For $\xi \in \mathcal{M}_h(T)$, if $\zeta = \xi - A_T(\xi)$ and

$$(\nabla\xi, v)_T = (\nabla\zeta, v)_T = -(\zeta, \nabla \cdot v)_T + \sum_{e \subset \partial T} (\zeta, v \cdot \nu)_e = 0 \tag{3.2}$$

for all $v \in \mathcal{V}_h(T)$, then we need to show that $\zeta = 0$.

Given any $w \in \mathcal{W}_h$, there is some $\tilde{v} \in \mathcal{V}_h$ such that $\nabla \cdot \tilde{v} = w$. Solve the problem

$$\Delta\psi = A_{\partial T}(\tilde{v}\cdot\nu) \quad \text{in } T,$$
$$\nabla\psi\cdot\nu = \tilde{v}\cdot\nu \quad \text{on } \partial T,$$

and set $v = \tilde{v} - \Pi_h\nabla\psi \in \mathcal{V}_h$. Then (3.1) implies that $v\cdot\nu = 0$ on ∂T and $\nabla\cdot v = w - A_{\partial T}(\tilde{v}\cdot\nu)$. As a consequence, (3.2) implies that $\mathcal{P}_{\mathcal{W}_h}\zeta = 0$.

Now for $e \subset \partial T$, take any $\lambda \in \mathcal{L}_h(e)$ and then any $\tilde{v} \in \mathcal{V}_h$ such that $\tilde{v}\cdot\nu = \lambda$ on e. Solve the problem

$$\Delta\psi = \nabla\cdot\tilde{v} - A_T(\nabla\cdot\tilde{v}) + A_{\partial T\setminus e}(\tilde{v}\cdot\nu) \quad \text{in } T,$$
$$\nabla\psi\cdot\nu = \tilde{v}\cdot\nu \quad \text{on } \partial T\setminus e,$$
$$\nabla\psi\cdot\nu = 0 \quad \text{on } e,$$

and again set $v = \tilde{v} - \Pi_h\nabla\psi \in \mathcal{V}_h$. Then (3.1) and (3.2) imply that $\mathcal{P}_{\mathcal{L}_h}\zeta = 0$ on e.

By the unisolvence of the degrees of freedom, we conclude that $\zeta = 0$. □

4. General projection finite element spaces. We now establish a theorem which defines, for a given mixed method in a fairly general class, a space $\mathcal{M}_h$ which gives rise to an equivalent projection method. We assume that each $T \in \mathcal{T}_h$ is a simplex, rectangular parallelepiped, or a prism, although it is enough to assume that T is convex and has flat edges or faces. Let m denote the number of edges (if $n = 2$) or faces (if $n = 3$) in ∂T, and for each $i = 1, ..., m$, let $\ell_i(x)$ denote the affine function which is zero on $e_i \subset \partial T$ and (say) one on the opposite face, edge, or vertex. Define the bubble functions

$$B_0(T) = \prod_{j=1}^{m} \ell_j \quad \text{and} \quad B_i(T) = \prod_{\substack{j=1\\ j\neq i}}^{m} \ell_j \quad \text{for } i = 1, ..., m.$$

THEOREM 4. *Suppose that $\mathcal{V}_h(T) \times \mathcal{W}_h(T)$ is a mixed finite element space such that $\mathcal{W}_h(T) = \nabla\cdot\mathcal{V}_h(T)$, $P_0(T) \subset \mathcal{W}_h(T)$, $P_0(e) \subset \mathcal{L}_h(e)$, and there exists an operator $\Pi_h : \left(H^1(T)\right)^n \to \mathcal{V}_h(T)$ satisfying (3.1). Define*

$$\mathcal{M}_h(T) = \big(B_0(T)\mathcal{W}_h(T)\big) \oplus \left(\sum_{i=1}^{m} B_i(T)\mathcal{L}_h(e_i)\right),$$

where $\mathcal{L}_h(e_i)$ is extended to T as a constant in the direction perpendicular to e_i. Then $\dim\{\mathcal{M}_h(T)\} = \dim\{\mathcal{W}_h(T)\} + \sum_{i=1}^m \dim\{\mathcal{L}_h(e_i)\}$ *and the degrees of freedom defined by (DF) in Theorem 3 are unisolvent.*

Proof. The dimension of $\mathcal{M}_h(T)$ is easily verified since $B_0(T)\mathcal{W}_h(T)$ consists of functions which are zero on each edge or face, while the functions in each $B_i(T)\mathcal{L}_h(e_i)$ are zero on $e_j \neq e_i$ but nonzero on e_i (unless the function is identically zero).

Let $v \in \mathcal{M}_h(T)$ be chosen so that the (DF) are zero. Decompose v into $B_0(T)w + \sum_{i=1}^m B_i(T)\lambda_i$ for some $w \in \mathcal{W}_h(T)$ and $\lambda_i \in \mathcal{L}_h(e_i)$. Then (DF2) shows that

$$0 = (v, \lambda_i)_{e_i} = (B_i(T)\lambda_i, \lambda_i)_{e_i},$$

which implies that λ_i on e_i is zero, and thus also its extension to T. Finally, (DF1) shows that

$$0 = (v, w)_T = (B_0 w, w),$$

so $w = v = 0$. □

We remark that these spaces are conforming in the sense that when they are pieced together to form $\mathcal{M}_h$ on Ω, the resulting functions are continuous. This theorem applies to several known mixed methods.

COROLLARY 2. *The mixed methods for second order elliptic problems using the spaces of Raviart and Thomas [10], Nedelec [9], Brezzi, Douglas, and Marini [5], Brezzi, Douglas, Durán, and Fortin [3], Brezzi, Douglas, Fortin, and Marini [4], and Chen and Douglas [8] on simplexes, rectangular parallelepipeds, and/or prisms in two and/or three dimensions can be formulated as equivalent projection finite element methods.*

We end the paper with three sections in which we construct some additional (and nonconforming) spaces that give rise to equivalent mixed methods. In each case, the mixed spaces satisfy the conditions of Theorem 3, so it remains only to define $\mathcal{M}_h(T)$ of the correct dimension and prove the unisolvence of (DF). Throughout, we let $P_k(T)$ denote the space of polynomials of total degree less than or equal to k defined on T, and we let $Q_{k,\ell,m}(T)$ denote the space of polynomials of degree less than or equal to k in x_1, ℓ in x_2, and m in x_3 (where m and x_3 are absent if $T \subset \mathbb{R}^2$).

5. Some additional spaces on triangles. In this section we consider finite element spaces for which T is a triangle. We make use of the barycentric coordinates $\hat{\ell}_i$, $i = 1, 2, 3$, defined to be the unique affine functions that take the value one at vertex i, and the value zero on the opposite edge. (Thus $\hat{\ell}_i = \ell_j$ in Section 4, where e_j is opposite vertex i.)

5.1. The Raviart-Thomas spaces on triangles. These spaces [10] are defined for each $k \geq 0$ by

$$\mathcal{V}_h^k(T) = \left(\mathcal{P}_k(T)\right)^2 \oplus \left((x_1, x_2)\mathcal{P}_k(T)\right),$$
$$\mathcal{W}_h^k(T) = P_k(T),$$
$$\mathcal{L}_h^k(e) = P_k(e).$$

We first recall what was previously known for the lowest order space. An $\mathcal{M}_h$ (of dimension 4) for this space is [1], [6]

$$\mathcal{M}_h(T) = P_1(T) \oplus \mathcal{B}_h(T),$$

where we define $\mathcal{B}_h(T)$ to be the span of either the P_3-bubble function,

$$B_3(x) = \hat{\ell}_1(x)\hat{\ell}_2(x)\hat{\ell}_3(x),$$

which vanishes on each edge, or the P_2-bubble function,

$$B_2(x) = 2 - 3\left(\hat{\ell}_1^2(x) + \hat{\ell}_2^2(x) + \hat{\ell}_3^2(x)\right),$$

which vanishes at the two quadratic Gauss points on each edge (recall that the Gauss points on [-1,1] are at $\pm 1/\sqrt{3}$).

For $\xi \in \mathcal{M}_h$, we can write $\xi = \xi_1 + \xi_2$ for $\xi_1 \in P_1(T)$ and $\xi_2 \in \mathcal{B}_h(T)$, and then the degrees of freedom for the element are normally given as the value of:

(i) $\displaystyle\int_T \xi(x)\,dx$;

(ii) ξ_1 at the midpoint of each edge $e \subset \partial T$.

(Note that (i) holds for ξ if $\mathcal{B}_h(T) = \mathrm{span}\{B_3\}$.) An equivalent set of degrees of freedom can be given by the value of (i) and

(ii′) $\int_e \xi(x)\,d\sigma(x)$ for each edge $e \subset \partial T$.

That (ii) and (ii′) are equivalent is easily seen since midpoint quadrature is exact for linear functions. These degrees of freedom are (DF1) and (DF2), and their unisolvence is known.

For the family as a whole, we define

$$\mathcal{M}_h^k(T) = \begin{cases} \{v \in \mathcal{P}_{k+3}(T) : v|e \in \mathcal{P}_{k+1}(e)\} & \text{if } k \text{ is even,} \\ \{v \in \mathcal{P}_{k+3}(T) : v|e \in \mathcal{P}_k(e) \oplus (\mathcal{P}_{k+2}(e) \setminus \mathcal{P}_k(e))\} & \text{if } k \text{ is odd.} \end{cases}$$

We first show that $\mathcal{M}_h^k(T)$ has the correct dimension. The dimension of $\mathcal{P}_{k+3}(T)$ is $\frac{1}{2}(k+5)(k+4)$, which is exactly six more than $\dim\{\mathcal{W}_h(T)\} + 3\dim\{\mathcal{L}_h(e)\} = \frac{1}{2}(k+8)(k+1)$. For simplicity, assume that k is even; the odd case is similar. For any $\xi \in \mathcal{P}_{k+3}(T)$, we can write that

$$\xi(x) = \sum_{0 \le i+j \le k+3} a_{i,j} \hat{\ell}_1^i(x) \hat{\ell}_2^j(x)$$

for some constants $a_{i,j}$. If now $\xi \in \mathcal{M}_h^k(T)$, then $\xi|_{e_1} \in P_{k+1}(e_1)$ implies that $a_{0,k+3} = a_{0,k+2} = 0$, and $\xi|_{e_2} \in P_{k+1}(e_2)$ implies that $a_{k+3,0} = a_{k+2,0} = 0$. On e_3, $\hat{\ell}_2 = 1 - \hat{\ell}_1$, so

$$\xi|_{e_3} = \sum_{0 \le i+j \le k+3} a_{i,j} \hat{\ell}_1^i (1 - \hat{\ell}_1)^j \in P_{k+1}(e_3)$$

implies that $\sum_{i+j=k+3} (-1)^j a_{i,j} = 0$ and $\sum_{i+j=k+2} j(-1)^j a_{i,j} = 0$. These six conditions are clearly independent, so $\mathcal{M}_h^k(T)$ has the correct dimension.

Now we consider the unisolvence of (DF). Suppose that $\xi \in \mathcal{M}_h^k(T)$ has degrees of freedom (DF) equal to zero. The (DF2) imply that on each edge e, ξ is a Legendre polynomial of degree $k+1$ if k is even and $k+2$ if k is odd, i.e., of odd degree. Since the odd degree Legendre polynomials are odd functions, traversing ∂T, we see that ξ must vanish identically on the boundary. As a consequence, we write that $\xi = \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 w$ for some $w \in \mathcal{P}_k(T)$. Now (DF1) shows that $(\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 w, w)_T = 0$, which finally gives that $\xi = 0$.

We remark that if $k = 0$ we obtain the nonconforming method of Arnold and Brezzi.

5.2. The Brezzi-Douglas-Marini spaces on triangles. These spaces [5] can be defined for each $k \ge 1$ by

$$\mathcal{V}_h^k(T) = \big(P_k(T)\big)^2,$$
$$\mathcal{W}_h^k(T) = P_{k-1}(T),$$
$$\mathcal{L}_h^k(e) = P_k(e).$$

Let us define

$$\mathcal{M}_h^k(T) = \begin{cases} \{v \in \mathcal{P}_{k+2}(T) : v|e \in \mathcal{P}_{k+1}(e)\} & \text{if } k \text{ is even,} \\ \{v \in \mathcal{P}_{k+2}(T) : v|e \in \mathcal{P}_k(e) \oplus (\mathcal{P}_{k+2}(e) \setminus \mathcal{P}_k(e))\} & \text{if } k \text{ is odd.} \end{cases}$$

Since $\dim\{\mathcal{P}_{k+2}(T)\} = \frac{1}{2}(k+4)(k+3)$ is exactly three more than $\dim\{\mathcal{W}_h(T)\} + 3\dim\{\mathcal{L}_h(e)\} = \frac{1}{2}(k+6)(k+1)$, an argument as above shows that $\mathcal{M}_h^k(T)$ has the correct dimension. The unisolvence of (DF) is also shown as above.

6. Some additional spaces on rectangles. We now consider problems in which T is a rectangle; for simplicity assume that $T = [-1,1]^2$. We will make use of the Legendre polynomials $p_m(x_i)$ of degree m defined on the interval $[-1,1]$. We should also define the spaces

$$A_m^k(T) = \left\{ \sum_{i=0}^{k} [a_{i,1}p_{m+1}(x_1) + a_{i,2}p_{m+2}(x_1)]p_i(x_2) : a_{i,j} \in \mathbb{R} \right\},$$
$$B_m^k(T) = \left\{ \sum_{i=0}^{k} p_i(x_1)[b_{i,1}p_{m+1}(x_2) + b_{i,2}p_{m+2}(x_2)] : b_{i,j} \in \mathbb{R} \right\}.$$

6.1. The Raviart-Thomas spaces on rectangles. These spaces [10] are defined for each $k \geq 0$ by

$$\begin{aligned} \mathcal{V}_h^k(T) &= Q_{k+1,k}(T) \times Q_{k,k+1}(T), \\ \mathcal{W}_h^k(T) &= Q_{k,k}(T), \\ \mathcal{L}_h^k(e) &= P_k(e). \end{aligned}$$

We define

$$\mathcal{M}_h^k(T) = Q_{k+2,k}(T) \oplus Q_{k,k+2}(T) = Q_{k,k}(T) \oplus A_k^k(T) \oplus B_k^k(T).$$

We remark that if $k = 0$, then

$$\mathcal{M}_h^0(T) = \{a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_2^2 : a_i \in \mathbb{R}\} = \mathcal{M}_h^{0,*}(T) \oplus \operatorname{span}\{B_2\},$$

where

$$\mathcal{M}_h^{0,*}(T) = \{a_1 + a_2x_1 + a_3x_2 + a_4(x_1^2 - x_2^2) : a_i \in \mathbb{R}\},$$

and now the P_2-bubble function $B_2(x)$ on $T = [-1,1]^2$ is

$$B_2(x) = 4 - 3(x_1^2 + x_2^2),$$

which vanishes at the two quadratic Gauss points on each edge. Also, $\nabla\mathcal{M}_h^0(T) \subset \mathcal{V}_h^0(T)$.

We need to show that the degrees of freedom (DF) are independent. It is trivial to verify that $\dim\{\mathcal{M}_h(T)\} = \dim\{\mathcal{W}_h(T)\} + 4\dim\{\mathcal{L}_h(e)\}$. Assume that the (DF) are zero for some $\xi \in \mathcal{M}_h^k(T) = \xi_1 + \xi_2 + \xi_3$, where $\xi_1 \in Q_{k,k}(T)$, $\xi_2 \in A_k^k(T)$, and $\xi_3 \in B_k^k(T)$. By the orthogonality of the Legendre polynomials, (DF1) is zero for $A_k^k(T)$ and $B_k^k(T)$, so (DF1) implies that $\xi_1 = 0$. On the two sides where $x_1 = \pm 1$, (DF2) for $B_k^k(T)$ is zero, but for $A_k^k(T)$ we have

$$\sum_{i=0}^{k} \int_{-1}^{1} [a_{i,1}p_{k+1}(\pm 1) + a_{i,2}p_{k+2}(\pm 1)]p_i(x_2)\lambda(x_2)\,dx_2 = 0 \quad \text{for all } \lambda \in P_k([-1,1]),$$

and so $a_{i,1}p_{k+1}(\pm 1) + a_{i,2}p_{k+2}(\pm 1) = 0$ for each i. Since the Legendre polynomials are alternately even and odd, we conclude that $a_{i,1} = a_{i,2} = 0$ for each i, i.e., $\xi_2 = 0$. Similarly on the sides where $x_2 = \pm 1$ we conclude that $\xi_3 = 0$, and so $\xi = 0$ and we have our unisolvence.

6.2. The Brezzi-Douglas-Marini spaces on rectangles. These spaces [5] are defined for each $k \geq 1$ as

$$\mathcal{V}_h^k(T) = \left(P_k(T)\right)^2 \oplus \operatorname{span}\left\{\operatorname{curl} x_1^{k+1}x_2, \operatorname{curl} x_1 x_2^{k+1}\right\},$$
$$\mathcal{W}_h^k(T) = P_{k-1}(T),$$
$$\mathcal{L}_h^k(e) = P_k(e),$$

where $\operatorname{curl} w = (-\partial w/\partial x_2, \partial w/\partial x_1)$. We define $\mathcal{M}_h^k(T) = P_{k-1}(T) \oplus A_{k-1}^k(T) \oplus B_{k-1}^k(T)$, and argue as above to verify the unisolvence of (DF).

6.3. The Brezzi-Douglas-Fortin-Marini spaces on rectangles. Also called reduced Brezzi-Douglas-Marini spaces [4], they can be defined for each $k \geq 0$ as

$$\mathcal{V}_h^k(T) = \left(P_{k+1}(T) \setminus \{x_2^{k+1}\}\right) \times \left(P_{k+1}(T) \setminus \{x_1^{k+1}\}\right),$$
$$\mathcal{W}_h^k(T) = P_k(T),$$
$$\mathcal{L}_h^k(e) = P_k(e).$$

Now we define $\mathcal{M}_h^k(T) = P_k(T) \oplus A_k^k(T) \oplus B_k^k(T)$. Again, we argue as in Subsection 6.1 to verify the unisolvence of (DF).

7. Some additional spaces on rectangular parallelepipeds. We now consider problems in which T is a rectangular parallelepiped in three dimensions. For simplicity, assume that $T = [-1,1]^3$. We will again make use of the Legendre polynomials $p_m(x_i)$ of degree m defined on the interval $[-1,1]$.

7.1. The Raviart-Thomas-Nedelec spaces on rectangular parallelepipeds. These spaces are the three dimensional analogues of the Raviart-Thomas spaces on rectangles, and they are defined [9], [10] for each $k \geq 0$ by

$$\mathcal{V}_h^k(T) = Q_{k+1,k,k}(T) \times Q_{k,k+1,k}(T) \times Q_{k,k,k+1}(T),$$
$$\mathcal{W}_h^k(T) = Q_{k,k,k}(T),$$
$$\mathcal{L}_h^k(e) = Q_{k,k}(e).$$

We define

$$\begin{aligned}\mathcal{M}_h^k(T) &= Q_{k+2,k,k}(T) \oplus Q_{k,k+2,k}(T) \oplus Q_{k,k,k+2}(T)\\ &= Q_{k,k,k}(T) \oplus A^k(T) \oplus B^k(T) \oplus C^k(T),\end{aligned}$$

where

$$A^k(T) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}[a_{i,j,1}p_{k+1}(x_1) + a_{i,j,2}p_{k+2}(x_1)]p_i(x_2)p_j(x_3) : a_{i,j,\ell} \in \mathbb{R} \right\},$$

$$B^k(T) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}p_i(x_1)[b_{i,j,1}p_{k+1}(x_2) + b_{i,j,2}p_{k+2}(x_2)]p_j(x_3) : b_{i,j,\ell} \in \mathbb{R} \right\},$$

$$C^k(T) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}p_i(x_1)p_j(x_2)[c_{i,j,1}p_{k+1}(x_3) + c_{i,j,2}p_{k+2}(x_3)] : c_{i,j,\ell} \in \mathbb{R} \right\}.$$

If $k=0$, then $\mathcal{M}_h^0(T) = \mathcal{M}_h^{0,*}(T) \oplus \operatorname{span}\{B_2\}$, where

$$\mathcal{M}_h^{0,*}(T) = \{A_1 + A_2x_1 + A_3x_2 + A_4x_3 + A_5(x_1^2 - x_2^2) + A_6(x_1^2 - x_3^2) : A_i \in \mathbb{R}\},$$

and now the P_2-bubble function $B_2(x)$ on T is

$$B_2(x) = 5 - 3(x_1^2 + x_2^2 + x_3^2),$$

which vanishes at the four tensor product quadratic Gauss points on each face.

To see the independence of the degrees of freedom (DF), note that $\dim\{\mathcal{M}_h^k(T)\} = \dim\{\mathcal{W}_h^k(T)\} + 6\dim\{\mathcal{L}_h^k(e)\}$ and argue as in Subsection 6.1 for the case of Raviart-Thomas spaces on rectangles.

7.2. The Brezzi-Douglas-Durán-Fortin spaces on rectangular parallelepipeds. These spaces [3] are the three dimensional analogues of the Brezzi-Douglas-Marini spaces on rectangles. They are defined for $k \geq 1$ by

$$\begin{aligned}\mathcal{V}_h^k(T) = \big(P_k(T)\big)^3 \oplus \operatorname{span}\big\{&\operatorname{curl}(0,0,x_1^{k+1}x_2), \operatorname{curl}(0,x_1x_3^{k+1},0), \operatorname{curl}(x_2^{k+1}x_3,0,0),\\ &\operatorname{curl}(0,0,x_1x_2^{i+1}x_3^{k-i}), \operatorname{curl}(0,x_1^{i+1}x_2^{k-i}x_3,0), \operatorname{curl}(x_1^{k-i}x_2x_3^{i+1},0,0)\big\},\end{aligned}$$

$$\mathcal{W}_h^k(T) = P_{k-1}(T),$$

$$\mathcal{L}_h^k(e) = P_k(e).$$

We define

$$\mathcal{M}_h^k(T) = P_{k-1}(T) \oplus A_{k-1}^k(T) \oplus B_{k-1}^k(T) \oplus C_{k-1}^k(T),$$

where

$$A_m^k(T) = \left\{ \sum_{0\leq i+j\leq k} [a_{i,j,1}p_{m+1}(x_1) + a_{i,j,2}p_{m+2}(x_1)]p_i(x_2)p_j(x_3) : a_{i,j,\ell} \in \mathbb{R} \right\},$$

$$B_m^k(T) = \left\{ \sum_{0\leq i+j\leq k} p_i(x_1)[b_{i,j,1}p_{m+1}(x_2) + b_{i,j,2}p_{m+2}(x_2)]p_j(x_3) : b_{i,j,\ell} \in \mathbb{R} \right\},$$

$$C_m^k(T) = \left\{ \sum_{0\leq i+j\leq k} p_i(x_1)p_j(x_2)[c_{i,j,1}p_{m+1}(x_3) + c_{i,j,2}p_{m+2}(x_3)] : c_{i,j,\ell} \in \mathbb{R} \right\}.$$

We can argue as above to verify the unisolvence of (DF).

7.3. The Brezzi-Douglas-Fortin-Marini spaces on rectangular parallelepipeds. These spaces [4] are also called reduced Brezzi-Douglas-Durán-Fortin spaces, and they can be defined for each $k \geq 0$ as

$$\begin{aligned}\mathcal{V}_h^k(T) = \left(P_{k+1}(T) \setminus \left\{\sum_{i=0}^{k+1} x_2^{k+1-i}x_3^i\right\}\right) \times \left(P_{k+1}(T) \setminus \left\{\sum_{i=0}^{k+1} x_1^{k+1-i}x_3^i\right\}\right) \times\\ \times \left(P_{k+1}(T) \setminus \left\{\sum_{i=0}^{k+1} x_1^{k+1-i}x_2^i\right\}\right),\end{aligned}$$

$$\mathcal{W}_h^k(T) = P_k(T),$$

$$\mathcal{L}_h^k(e) = P_k(e).$$

We define $\mathcal{M}_h^k(T) = P_k(T) \oplus A_k^k(T) \oplus B_k^k(T) \oplus C_k^k(T)$, where $A_k^k(T)$, $B_k^k(T)$, and $C_k^k(T)$ are defined in the previous subsection, and again we argue as in Subsection 7.1 to verify the unisolvence of (DF).

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