

Stability and Regularity
of an Inverse Elliptic
Boundary Value Problem

David C. Dobson

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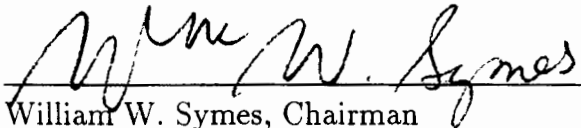
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David Clark Dobson

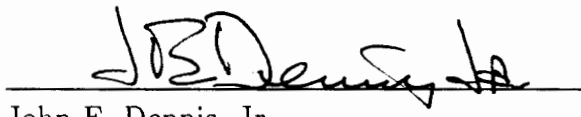
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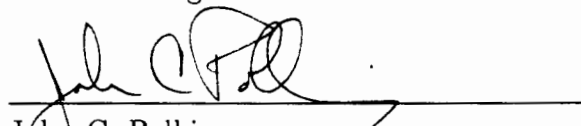
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Abstract

The *inverse conductivity problem* is that of recovering a spatially varying isotropic conductivity in the interior of some bounded region by means of steady-state measurements taken only at the boundary. In the underlying partial differential equation, the conductivity appears as a coefficient in an elliptic boundary value problem.

We first analyze the stability of the formal linearization of the inverse conductivity problem, establishing upper and lower bounds on the linearized map. Conditions are then established under which the forward map is regular, with computationally reasonable norms on the conductivity and the data. Certain smoothness assumptions on the conductivities are needed to prove regularity. A simple example is given to illustrate why the smoothness assumptions may be necessary. Finally, the inverse problem is formulated as a regularized minimization problem. The regularization penalizes rough conductivities, rendering the forward map regular and stabilizing the linearized inverse maps. The local convergence of a simple minimization scheme is established.

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For Tamara

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Chapter 1

Preliminaries

In many applications, the problem arises of determining some set of physical parameters by means of observing the behavior of a physical system. These kinds of problems are usually collectively referred to as *inverse problems*. It is typical that inverse problems exhibit some form of *ill-posedness*, making them notoriously difficult to solve computationally.

The topic of this thesis is a particular inverse problem, the problem of determining a spatially varying isotropic conductivity in the interior of a bounded region by means of steady-state measurements taken only at the boundary. We will refer to this problem as the *inverse conductivity problem*, although it goes by many other names (see [25] for a partial list). In terms of the underlying partial differential equation, the inverse conductivity problem is that of recovering a coefficient in an elliptic boundary value problem from knowledge of the Cauchy data.

The inverse conductivity problem has applications in medical imaging [3, 21], non-destructive testing, and geophysics [20, 39]. There has been a particularly rapid pace in the development of applications to medical imaging. In these applications,

information about the interior of the body may be obtained with relatively low-cost apparatus, without exposing the patient to high-frequency radiation.

In Section 1.1, the inverse conductivity problem is stated and preliminary ideas are discussed. A brief review of the recent literature concerning the well-posedness of this problem is given in Section 1.2. In Section 1.3, references describing some of the recently proposed methods for solving the inverse conductivity problem are cited.

This thesis will focus on aspects of the inverse conductivity problem relevant to the development of effective computational methods. One of the major difficulties faced in any procedure for solving the problem is *instability*. In Chapter 2 we will characterize the instability in a linearized version of the problem, with reasonable norms on the data and the solutions. An upper bound on the change in the data due to localized oscillatory perturbations in the conductivity is obtained in Section 2.3, providing a description of the resolution limit inherent in the problem. A similar lower bound is given in Section 2.4, providing a description of a (pessimistic) regularizing set.

The inverse conductivity problem is nonlinear. Most of the computational methods proposed to date rely on the assumption that some linearized problem is a good approximation to the nonlinear problem. Thus, the *regularity* of the forward map is assumed. In Chapter 3, we establish conditions under which the formal linearized problem can be proved to be a good local approximation to the nonlinear problem, with computationally reasonable norms on the data and the conductivities. The

sufficient conditions for regularity require a certain amount of smoothness of the conductivities, suggesting that the nonlinear problem is not well approximated by the linearized problem when the conductivities are rough. Under certain circumstances, this is indeed the case; a simple example is constructed which illustrates this phenomenon.

Stability and regularity are essential ingredients for the success of schemes such as Newton's method, which are based on the successive solution of linearized subproblems. Since neither property can be guaranteed when rough conductivities are allowed, it is reasonable to restrict the class of admissible conductivities to exclude offensive individuals. In Chapter 4, the inverse conductivity problem is posed as a least-squares problem, with a regularization term which penalizes oscillations in the conductivity. The regularization renders the problem both stable and regular. It is shown that a simple method—a globalized Gauss-Newton method, the so-called Levenberg-Marquardt method—converges locally, with a q -linear convergence rate. As far as we know, this is the first local convergence analysis for a minimization method applied to a least-squares formulation of the inverse conductivity problem.

1.1 Introduction to the problem

The problem may be stated as follows. Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$. Let γ be a bounded measurable function on Ω with $\gamma(x) \geq a > 0$ in $\bar{\Omega}$. The function γ will be called the *conductivity*.

The physics of the applications of interest are modelled by the Neumann problem

$$\begin{aligned}\nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in } \Omega \\ \gamma \frac{\partial u}{\partial \eta} &= f \quad \text{on } \partial \Omega\end{aligned}\tag{1.1}$$

where η denotes the unit outward normal on $\partial \Omega$. In the context of steady-state electrical conductivity, u represents the voltage potential and $\gamma \nabla u$ is the vector of current flow. The function f represents an applied current flux density. We will sometimes refer to f as a *current*.

For each $f \in H^{-1/2}(\partial \Omega)$ with $\int_{\partial \Omega} f = 0$, the problem (1.1) has a unique solution $u \in H^1(\Omega)$, modulo constant functions. If (1.1) is augmented, for example, with the condition that $\int_{\partial \Omega} u = 0$, then the solution is uniquely determined. Thus, ignoring constant functions in both the domain and the range spaces, the linear operator

$$\Lambda_\gamma : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial \Omega)$$

defined by

$$\Lambda_\gamma f = u|_{\partial \Omega},$$

where u satisfies (1.1), is well defined. The function $\Lambda_\gamma f$, which represents voltage potential on $\partial \Omega$, will sometimes be called a *measurement*.

We will consider the operator Λ_γ to be the data in the problem, for we can in principle apply any current f on the boundary and measure the corresponding voltage. The operator Λ_γ is called the *Neumann-to-Dirichlet map*. We choose to consider the map from Neumann data to Dirichlet data rather than its inverse because it is a

smoothing operator, hence easier to approximate numerically. Since fractional-order Sobolev norms are expensive to calculate numerically, we will regard Λ_γ as an operator from $L^2(\partial\Omega)$ into $L^2(\partial\Omega)$ and again consistently ignore constant functions.

We call $\gamma \mapsto \Lambda_\gamma$ the *forward map*. The inverse problem is: Given the data Λ_γ , find γ ; that is, invert the forward map.

Even though (1.1) is a linear partial differential equation, the dependence of the boundary values of the solutions on the conductivity coefficient γ is nonlinear. It is useful to *linearize* the problem using formal perturbational methods. Given a conductivity perturbation $\delta\gamma$ with $\delta\gamma|_{\partial\Omega} = 0$, one can expand the equation

$$\begin{aligned}\nabla \cdot (\gamma + \delta\gamma)\nabla(u + \delta u) &= 0 \\ (\gamma + \delta\gamma)\frac{\partial(u + \delta u)}{\partial\eta} &= f\end{aligned}$$

and discard nonlinear terms to see that formally, the linear perturbation δu in the solution u to (1.1) is the solution to

$$\begin{aligned}\nabla \cdot (\gamma\nabla\delta u) &= -\nabla \cdot (\delta\gamma\nabla u) \quad \text{in } \Omega \\ \gamma\frac{\partial\delta u}{\partial\eta} &= 0 \quad \text{on } \partial\Omega.\end{aligned}\tag{1.2}$$

The linearized equation (1.2) will be important in what follows. Chapter 2 will be concerned with the characterization of the set of “almost null” vectors of the linear operator $D\Lambda_\gamma$ defined by

$$(D\Lambda_\gamma f)(\delta\gamma) = \delta u|_{\partial\Omega}.$$

Chapter 3 will be concerned with establishing conditions under which the approximation

$$\Lambda_{\gamma+\delta\gamma}f \approx \Lambda_{\gamma}f + (D\Lambda_{\gamma}f)(\delta\gamma) \quad (1.3)$$

is valid. In Chapter 4, the convergence of a simple least-squares minimization scheme based on the approximation (1.3) will be established.

Several variations of the inverse conductivity problem, as we have stated it, have been studied. For example, one could allow the conductivity function to be anisotropic. In this case, the conductivity is not determined by the Dirichlet-to-Neumann map. However, much work has gone into understanding exactly what properties of the conductivity *can* be determined. This problem is closely related to the geometric problem of finding the Riemannian metric on a manifold with boundary from the Dirichlet-to-Neumann map of the Laplacian. See, for example, the work by J. Lee and G. Uhlmann [26], and J. Sylvester [34].

It is also interesting to study the problem of finding discontinuities or inclusions (for example, cracks or bubbles) in an otherwise relatively homogeneous background. This is in some sense a more computationally plausible problem, since there are “fewer parameters” to be determined. Some of the recent work in this area can be found in the papers by V. Isakov [17], A. Friedman and M. Vogelius [13], and F. Santosa and M. Vogelius [30]. Additional references can be found in these papers.

Another interesting variation on the inverse conductivity problem is to consider a discrete version of the problem: Determine the resistors in a network from measure-

ments taken at boundary nodes. J. Morrow and E. Curtis [10] have given an algorithm to iteratively determine all the resistors in a network by isolating resistors within the network with the boundary data. This approach has import for finite-dimensional approximations of the continuous problem with finite elements, for example.

1.2 Well-posedness

The classical questions of well-posedness for the inverse conductivity problem have been studied extensively over the past several years. A. P. Calderón was the first to investigate the question of uniqueness for the inverse conductivity problem. In [6], Calderón proved that the derivative of the forward map about a constant conductivity is injective. Unfortunately, the derivative operator is compact, so local uniqueness cannot be obtained by means of the Inverse Function Theorem. Kohn and Vogelius were the first to establish a uniqueness result for the fully nonlinear problem. Using estimates from highly oscillatory boundary data, Kohn and Vogelius proved in [23] that γ is determined to infinite order on the boundary. Thus real-analytic conductivities are uniquely determined. In [24], Kohn and Vogelius were able to extend this result to the case where γ is known to be piecewise analytic in Ω . Sylvester and Uhlmann [35] obtained a local uniqueness result by constructing asymptotic solutions to the Dirichlet problem which are accurate near constant conductivities. Again using asymptotic solutions, Sylvester and Uhlmann [36] proved that if $\gamma \in C^\infty(\bar{\Omega})$ and $n \geq 3$ then γ is uniquely determined. This result was extended, first by A. Nachman

[27] to $\gamma \in C^{1,1}(\bar{\Omega})$, then by Chanillo [7] to $\gamma \in W^{2,p}(\Omega)$, where $p > n/2$. Sun [33] has made some progress toward the uniqueness question for more general classes of conductivities in two dimensions, but this problem is as far as we know still essentially an open question.

Continuous dependence has also been studied. Using the representation of the Dirichlet-to-Neumann map as a pseudodifferential operator, Sylvester and Uhlmann proved in [37] that if γ is smooth, then γ and all its normal derivatives *evaluated at the boundary* depend continuously on the Dirichlet-to-Neumann map. Alessandrini [2] has proved that for $n \geq 3$, γ is very weakly continuously dependent on the Dirichlet-to-Neumann map under certain *a priori* smoothness assumptions on γ . In the same paper, Alessandrini shows that continuous dependence fails to hold with no smoothness assumptions on γ .

1.3 The reconstruction problem

The computational problem of finding an approximate solution to the inverse conductivity problem is generally referred to as the *reconstruction problem*. Many techniques for solving the reconstruction problem have been proposed and implemented.

Among the more interesting approaches, Santosa and Vogelius [29] have studied an algorithm for solving a linearized two-dimensional approximation to the inverse conductivity problem due to D.C. Barber and B.H. Brown. They have shown that Barber and Brown's backprojection operator may be viewed as part of an approximate inverse

of a generalized Radon transform. They have used the backprojection operator as a preconditioner as part of a more general iterative algorithm, with promising results. Kohn and Vogelius [25] have proposed a relaxed variational method, using ideas from homogenization theory and optimal design. This method was later implemented by Kohn and A. McKenney [22], with reasonably good results.

A widely used approach to solving inverse problems is to formulate a minimization problem whose solution also solves the inverse problem. The inverse problem can then be approached with optimization techniques. This approach has merit from a statistical (data fitting) point of view (see for example Tarantola [38]), and also from a computational point of view. Regularization and globalization strategies have been developed for dealing with unstable and highly nonlinear optimization problems. In addition, practical optimization methods are available which are well-tested and well-understood; see Dennis and Schnabel [12] for a description of nonlinear unconstrained optimization methods. Yorkey et al. [40] and Breckon and Pidcock [5] have obtained good results with output least-squares formulations of the inverse conductivity problem. In [40] an output least-squares approach was compared with several other methods, and was found to have somewhat better performance in terms of accuracy and efficiency. However, the severe instability of the problem was difficult to correct. Jiang [19] has formulated the inverse conductivity problem as a constrained minimization problem and applied an augmented Lagrangian method. Jiang has also

obtained a convergence proof for his method under certain assumptions; however, no convergence rate was obtained.

Chapter 2

Stability of the Inverse Map

Any method for solving the reconstruction problem faces difficulties in the presence of inexact data. The difficulties arise from the underlying instability of the inverse problem. In methods based on linearization of the forward map $\gamma \mapsto \Lambda_\gamma$, instability manifests itself as ill-conditioned or singular linear systems.

In this chapter, we analyze the stability of the inverse map. In Section 2.1, we demonstrate how instability can arise in the inverse map from highly oscillatory conductivities. In Section 2.2, we look at the related ideas of distinguishability, resolution, and stabilization. In Section 2.3, we obtain an upper bound on the linearized map in terms of an orthonormal wavelet basis. This upper bound limits the maximum resolution possible in the linearized problem. In Section 2.4, we describe a lower bound on the linearized map in terms of the Fourier transform, essentially given by Calderón [6], which leads to a pessimistic stabilization of the linearized inverse problem.

2.1 Discontinuity of the inverse map

In this section we wish to merely indicate that the inverse map is discontinuous in a practical sense, and give some motivation for why this is so. In the paper by

Alessandrini [2] (see also Isaacson [16]), an example is constructed of a sequence of conductivities $\{\gamma_k\}$ and a constant conductivity γ such that $\|\gamma_k - \gamma\|_{L^\infty(\Omega)} = C$ for all k , but $\|\Lambda_{\gamma_k} - \Lambda_\gamma\|_* \rightarrow 0$ as $k \rightarrow \infty$, where $\|\cdot\|_*$ denotes the operator norm. Thus, without *a priori* restrictions on the class of conductivities, the inverse map $\Lambda_\gamma \rightarrow \gamma$ is discontinuous as a map from $L[H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)] \rightarrow L^\infty(\Omega)$.

For computational reasons, in the remainder of the thesis we will be concerned with the behavior of the inverse map with the $L^2(\Omega)$ norm on the conductivities. In this section we will indicate that even with the $L^2(\Omega)$ norm on the conductivities, the inverse map is discontinuous in a practical sense. In particular, we will show that there is a sequence of conductivities $\{\gamma_k\}$ and a constant conductivity γ such that for $1 \leq p \leq \infty$, we have $\|\gamma_k - \gamma\|_{L^p(\Omega)} = C(p)$ for all k , and for each $f \in L^2(\partial\Omega)$ with $\int f = 0$,

$$\|(\Lambda_{\gamma_k} - \Lambda_\gamma)f\|_{L^2(\partial\Omega)} \rightarrow 0. \quad (2.1)$$

Thus the inverse map is discontinuous with the $L^2(\Omega)$ norm on the conductivities and the topology of strong operator convergence on the Neumann-to-Dirichlet maps.

It will be helpful to use some ideas from homogenization theory (see for example, Bensoussan, Lions, Papanicolaou [4]). Let

$$L_\gamma : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$$

be a uniformly elliptic operator of the form

$$L_\gamma u = \nabla \cdot (\gamma \nabla u).$$

A sequence of operators L_{γ_k} is said to be *G-convergent* to the operator L_γ (written $L_{\gamma_k} \xrightarrow{G} L_\gamma$) if for each $g \in H^{-1}(\Omega)$, we have

$$L_{\gamma_k}^{-1}g \rightharpoonup L_\gamma^{-1}g \quad \text{in } H_0^1(\Omega).$$

It is well-known that G-convergence does not correspond to any $L^p(\Omega)$ convergence of the conductivity coefficients. The class of isotropic (scalar) conductivities is not closed under G-convergence. It can be shown (see Spagnolo [32]) that if $L_{\gamma_k} \xrightarrow{G} L_\gamma$ then for the solutions u_k, u to the Neumann problems

$$\left\{ \begin{array}{l} L_{\gamma_k} u_k = 0 \\ \gamma_k \frac{\partial u_k}{\partial \eta} \Big|_{\partial \Omega} = f \\ \int_{\partial \Omega} u_k = 0 \end{array} \right. \quad \left\{ \begin{array}{l} L_\gamma u = 0 \\ \gamma \frac{\partial u}{\partial \eta} \Big|_{\partial \Omega} = f \\ \int_{\partial \Omega} u = 0 \end{array} \right. \quad (2.2)$$

we have $u_k \rightharpoonup u$ in $H^1(\Omega)$.

We will use the following simple lemma to show that G-convergence implies the strong operator convergence of the corresponding Neumann-to-Dirichlet maps.

Lemma 2.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. Then the trace imbedding $H^1(\Omega) \rightarrow L^2(\partial \Omega)$ is compact.

Proof The proof essentially follows from the Rellich-Kondrachov Theorem (see Adams [1]) except for some technicalities, which we merely outline how to circumvent. The regularity of $\partial \Omega$ implies the existence of a strong extension operator

$$E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n).$$

The idea is to choose a finite open cover $\{U_j\}_{j=1}^N$ of $\partial\Omega$ in \mathbb{R}^n , a partition of unity $\sum v_j = 1$ subordinate to $\{U_j\}$, and a set of smooth maps

$$\psi_j : B = \{y \in \mathbb{R}^n : |y| < 1\} \rightarrow U_j$$

such that $U_j \cap \partial\Omega = \psi_j(B_0)$, $B_0 = \{y \in B : y_n = 0\}$. Then given a bounded sequence $\{f_i\} \subset H^1(\Omega)$, the sequence $\{v_1 E f_i \circ \psi_1^{-1}\}$ is bounded in $H^1(B)$. By the Rellich-Kondrachov Theorem, there is a convergent subsequence $\{v_1 E f_{i_1} \circ \psi_1^{-1}|_{B_0}\}$ in $L^2(B_0)$. One can continue recursively, extracting convergent subsequences in this manner for $j = 2, \dots, N$ so that $\{v_j E f_{i_N} \circ \psi_j^{-1}\}$ converges in $L^2(B_0)$ for all j . After composition with ψ_j , we have $\{v_j E f_{i_N}\}$ converging in $L^2(\partial\Omega \cap U_j)$ for all j . Finally, summation of the partition of unity and application of the triangle inequality shows that $\{f_{i_N}|_{\partial\Omega}\}$ converges in $L^2(\partial\Omega)$. \square

Since weak $H^1(\Omega)$ convergence becomes strong $L^2(\partial\Omega)$ convergence under the compact trace imbedding map, the following proposition follows immediately from Lemma 2.1.

Proposition 2.1 If $L_{\gamma_k} \xrightarrow{G} L_\gamma$, then $u_k|_{\partial\Omega} \rightarrow u|_{\partial\Omega}$ strongly in $L^2(\partial\Omega)$.

Let $\Omega = [0, 1]^n$, the unit cell in \mathbb{R}^n . Let γ_0 be the piecewise constant function on Ω with values $a_1 > 0$ on a half-size cell centered in $[0, 1]^n$, and $a_2 > 0$ elsewhere, extended by periodicity to all of \mathbb{R}^n . One can show that the homogenized limit (the ‘‘effective conductivity’’) of γ_0 , when all variables are scaled down at the same rate, is a constant, isotropic conductivity. Let $\gamma_k(x) = \gamma_0(kx)$, where kx denotes

multiplication by k in each component, so that γ_k has structure on the scale $1/k$. Denote the homogenized limit of γ_k as $k \rightarrow \infty$ by γ . It follows that $L_{\gamma_k} \xrightarrow{G} L_\gamma$. It is easy to check that for $1 \leq p \leq \infty$, $\|\gamma_k - \gamma\|_{L^p(\Omega)} = C(p) > 0$ for all k .

Then, as established above, given any $f \in L^2(\partial\Omega)$ with $\int f = 0$ the boundary values $u_k|_{\partial\Omega}$ of the solutions to (2.2) converge strongly in $L^2(\partial\Omega)$, yielding (2.1).

This example illustrates one mechanism of instability in the inverse problem—highly oscillatory conductivities. We will show in Chapter 3 that highly oscillatory conductivities can also give rise to nonlinearity of the forward map. For the remainder of this chapter, the goal will be to attempt to characterize the sensitivity (or lack thereof) of the data in a linearized problem to oscillatory perturbations in the conductivity.

2.2 Distinguishability, resolution, and stabilization

Let ϵ be a positive number corresponding to the *noise level* in the data. By this we mean that each measurement $\Lambda_\gamma f$ is inaccurate in $L^2(\partial\Omega)$ by no more than the fixed amount ϵ . The problem of characterizing the information content of the data may be viewed as that of characterizing the inverse image under the forward map $\gamma \mapsto \Lambda_\gamma$ of an ϵ -error ball in the data space.

Roughly following D. Isaacson [16], we will say that a conductivity perturbation $\delta\gamma$ is *indistinguishable* relative to some fixed conductivity γ if for every $f \in L^2(\partial\Omega)$

with $\int f = 0$ and $\|f\| = 1$, we have

$$\|(\Lambda_{\gamma+\delta\gamma} - \Lambda_{\gamma})f\|_{L^2(\partial\Omega)} \leq \epsilon.$$

Thus, indistinguishable conductivity perturbations $\delta\gamma$ change the data by an amount too small to measure. The set of indistinguishable perturbations is the inverse image of an ϵ -error ball in the data space in the operator norm $\|\cdot\|_{L^2(\partial\Omega), L^2(\partial\Omega)}$.

In [16], D. Isaacson makes some explicit calculations in the case where Ω is the unit disc in R^2 , $\gamma \equiv 1$, and $\delta\gamma$ is a constant perturbation on a disc of radius $R < 1$. Within this two-parameter space of conductivities, a complete, explicit characterization of the indistinguishable perturbations is obtained.

In the case where $\delta\gamma$ is a constant perturbation on a disc not necessarily concentric with Ω , Seagar [31] uses conformal maps to obtain a similar description of indistinguishable perturbations. As one might expect, perturbations near the boundary of Ω are dramatically easier to detect than perturbations near the center.

Let us now consider the formal linearization of the problem. Let $D\Lambda_1$ be the linearized map described in Section 1.1, where the linearization is about the conductivity $\gamma \equiv 1$. Thus, $D\Lambda_1(\delta\gamma)f = \delta u|_{\partial\Omega}$, where δu solves

$$\begin{aligned} \Delta\delta u &= -\nabla \cdot \delta\gamma \nabla u \quad \text{in } \Omega \\ \nabla\delta u \cdot \eta &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and u is a harmonic function with Neumann data f . Associated with $D\Lambda_1$ is the bilinear form DQ_1 on $L^2(\partial\Omega) \times L^2(\partial\Omega)$ defined by

$$DQ_1(\delta\gamma)(f, g) = \int_{\partial\Omega} g(D\Lambda_1(\delta\gamma)f). \quad (2.3)$$

If v is a harmonic function with Neumann data g and we assume that $\delta\gamma|_{\partial\Omega} = 0$, integrating (2.3) by parts shows that

$$DQ_1(\delta\gamma)(f, g) = - \int_{\Omega} \delta\gamma \nabla u \cdot \nabla v. \quad (2.4)$$

Let us define a norm $\|\cdot\|_*$ on $DQ_1(\delta\gamma)(\cdot, \cdot)$ by

$$\|DQ_1(\delta\gamma)\|_* = \sup_{\|f\|_{L^2(\partial\Omega)}, \|g\|_{L^2(\partial\Omega)} \leq 1} |DQ_1(\delta\gamma)(f, g)|.$$

From (2.3), it is clear that $\|\cdot\|_*$ on $DQ_1(\delta\gamma)$ is equivalent to the operator norm $\|\cdot\|_{L^2(\partial\Omega), L^2(\partial\Omega)}$ on $D\Lambda_1$.

We would like to invert the map

$$D\Lambda_1 : L^2(\Omega) \rightarrow L[L^2(\partial\Omega), L^2(\partial\Omega)].$$

Unfortunately, as Calderón showed in [6], although this map is injective, the inverse is unbounded. Thus, any practical inversion scheme must stabilize or regularize the problem in some way. Assuming as before that ϵ -bounded errors are present in the measurements, by the linearity of $D\Lambda_1$ the solution to the linearized inverse problem, $\delta\gamma$, is well-determined up to perturbations μ in the set

$$A_\epsilon = \{\mu \in L^2(\Omega) : \|D\Lambda_1 \mu\|_{L^2(\partial\Omega), L^2(\partial\Omega)} \leq \epsilon\}.$$

Thus A_ϵ is the set of indistinguishable perturbations for the linearized problem. The set A_ϵ may be thought of as the “almost null” set of the operator $D\Lambda_1$.

An explicit characterization of A_ϵ would constitute an exact description of the information content of $D\Lambda_1$, as described at the beginning of this section. For the sake of exposition, imagine for a moment that we have obtained a singular value decomposition (SVD) for $D\Lambda_1$,

$$D\Lambda_1 = U\Sigma V^*,$$

where U and V are unitary, and Σ is “diagonal”. Then $\|D\Lambda_1\mu\| = \|\Sigma V^*\mu\|$, so that

$$A_\epsilon = \{\mu \in L^2(\Omega) : \|\Sigma V^*\mu\| \leq \epsilon\}$$

gives an explicit characterization of A_ϵ in terms of the singular values and right singular vectors of $D\Lambda_1$. This characterization has two important interpretations. First, it completely describes the set of indistinguishable perturbations in the linearized problem. Since for a given noise level ϵ all conductivities in the set $\delta\gamma + A_\epsilon$ are equally valid solutions, this gives an explicit description of the maximum resolution of the problem. Without *a priori* information, the “blurry” conductivity $\delta\gamma + A_\epsilon$ is the answer to the inversion problem. Second, the SVD provides an optimal stabilization, in the sense that the stability bound

$$\|D\Lambda_1(\delta\gamma)\| \geq c\|\delta\gamma\| \tag{2.5}$$

can be assured by restricting $\delta\gamma$ to the stabilizing set

$$\mathcal{S} = \{\delta\gamma : \|\Sigma V^*\delta\gamma\| \geq c\|\delta\gamma\|\}.$$

The stabilization is optimal because \mathcal{S} is the largest set for which (2.5) holds.

But this is only wishful thinking. Since the range space $L[L^2(\partial\Omega), L^2(\partial\Omega)]$ is not a Hilbert space, a SVD for the map DA_1 does not exist. This could be remedied by imposing Hilbert space structure on $L[L^2(\partial\Omega), L^2(\partial\Omega)]$, say, with the Hilbert-Schmidt norm, but this would nullify our seemingly reasonable assumptions about the nature of the errors in the problem. Also, in practice the SVD is expensive to compute, and probably cannot be written down explicitly.

However, it is still possible to obtain explicit descriptions of subsets and supersets of A_ϵ from which an upper bound on the maximum resolution for the problem and a sub-optimal stabilization set, respectively, can be obtained. In the following two sections, we will find operators B and C , diagonal with respect to certain bases, such that the bounds

$$\|DQ_1(\mu)\|_* \leq \|B\mu\| \tag{2.6}$$

and

$$\|DQ_1(\mu)\|_* \geq \|C\mu\| \tag{2.7}$$

hold. The operator B is diagonal in terms of a wavelet basis; the operator C is diagonal in terms of the Fourier transform. We would like to think of the basis sets as approximations (however crude) of the right singular vectors.

From the equivalence of the norms for DQ_1 and DA_1 ,

$$B_\epsilon = \{\mu \in L^2(\Omega) : \|B\mu\| \leq \epsilon\} \subset A_\epsilon \tag{2.8}$$

and

$$C_\epsilon = \{\mu \in L^2(\Omega) : \|C\mu\| \leq \epsilon\} \supset A_\epsilon.$$

The operator B is constructed in the next section. The construction of C , which was essentially given by Calderón in [6], is given in Section 2.4.

2.3 Upper bounds on the linearized map

The general approach we have in mind is to choose some orthonormal set $\{\psi_\nu\}_{\nu \in \mathcal{N}}$ in $L^2(\Omega)$, where \mathcal{N} is some index set, and to expand conductivity perturbations in terms of this basis, i.e.,

$$\mu(x) = \sum \alpha_\nu \psi_\nu(x)$$

where

$$\alpha_\nu = \int_\Omega \mu(x) \psi_\nu(x) dx.$$

If the “moment functions” $\nabla u \cdot \nabla v$ from expression (2.4) are also expressed in terms of this basis,

$$(\nabla u \cdot \nabla v)(x) = \sum \beta_\nu \psi_\nu(x),$$

where

$$\beta_\nu = \int_\Omega (\nabla u \cdot \nabla v)(x) \psi_\nu(x) dx,$$

then from (2.4),

$$|DQ_1(\mu)(f, g)| = \left| \sum \alpha_\nu \beta_\nu \right| \leq \sum |\alpha_\nu| |\beta_\nu|. \quad (2.9)$$

Thus an upper bound on $\|DQ_1(\mu)\|_*$ can be obtained by finding upper bounds $b(\nu) \geq |\beta_\nu|$ which hold for all $f, g \in L^2(\partial\Omega)$, or equivalently, for all “moment functions” $\nabla u \cdot \nabla v$.

As mentioned in the previous section, we would like $\{\psi_\nu\}$ to approximate in some sense the right singular vectors of the map $D\Lambda_1$. We make the observation that high-frequency perturbations in the conductivity are harder to detect than low-frequency perturbations, and that localized perturbations are much easier to detect near the boundary than away from the boundary. These facts suggest that a description of the perturbations is needed which is localized in both space and frequency. For this reason, we will make use of a so-called *wavelet* or *multiscale* basis.

We will use tensor products of the compactly supported wavelet bases discovered by I. Daubechies [11]. Given any $m \in \mathbb{N}$, the Daubechies construction gives a family $\{\psi_{j,k}\}$, where $(j, k) \in Z \times \Gamma$ for some discrete index set Γ . The indices j and k refer to the scale and location, respectively, of the wavelet $\psi_{j,k}$. The family $\{\psi_{j,k}\}$ has the following properties:

1. $\psi_{j,k} \in C^m(\mathbb{R}^n)$ for all j, k .
2. $\{\psi_{j,k}\}$ is a complete orthonormal set in $L^2(\mathbb{R}^n)$.
3. $\int_{\mathbb{R}^n} x^\alpha \psi_{j,k} = 0$ for all j, k , where $|\alpha| \leq K$, and K increases linearly with m .

4. For each $k \in \Gamma$, there is a point $i = (i_1, \dots, i_n) \in Z^n$ such that $\text{supp } \psi_{j,k} \subset Q_{j,k} = \{x \in R^n : \max\{|x_1 - 2^{-j}i_1|, \dots, |x_n - 2^{-j}i_n|\} \leq C2^{-j}\}$, where C increases linearly with m .

Denote the pair $(j, k) = \nu$. Let

$$\mathcal{N} = \{\nu \in Z \times \Gamma : \text{supp } \psi_\nu = Q_\nu \subset \Omega\}.$$

Then $\{\psi_\nu\}_{\nu \in \mathcal{N}}$ is an orthonormal set in $L^2(\Omega)$, although it is not necessarily complete. Complete wavelet bases for $L^2(\Omega)$ have been constructed (see [18] and Section 4.2); however, the compact support of the Daubechies wavelets facilitates simple estimates in what follows.

Let $\nu \in \mathcal{N}$, then $Q_\nu \subset \Omega$. Let $x_0 \in Q_\nu$. Choose an integer r such that $2(r-1) \leq K$, where K is given by property 3. Let p_r and q_r denote the r th-order Taylor series approximations to u and v , respectively, at x_0 . Then $\nabla p_r \cdot \nabla q_r$ is a $2(r-1)$ degree polynomial. Using properties 3 and 4, and the fact that

$$\|\psi_\nu\|_1 \leq |Q_\nu|^{1/2} \|\psi_\nu\|_2 = |Q_\nu|^{1/2},$$

we have

$$\begin{aligned} |\beta_\nu| &= \left| \int_\Omega \psi_\nu(x) (\nabla u \cdot \nabla v)(x) dx \right| \\ &= \left| \int_{Q_\nu} \psi_\nu(x) [(\nabla u \cdot \nabla v)(x) - (\nabla p_r \cdot \nabla q_r)(x - x_0)] dx \right| \\ &\leq |Q_\nu|^{1/2} \|\nabla u \cdot \nabla v - \nabla p_r \cdot \nabla q_r\|_{L^\infty(Q_\nu)} \\ &\leq |Q_\nu|^{1/2} \cdot \max\{\|\nabla u - \nabla p_r\| \|\nabla v + \nabla q_r\|, \|\nabla u + \nabla p_r\| \|\nabla v - \nabla q_r\|\}, \end{aligned}$$

where all the norms in the last expression are $L^\infty(Q_\nu)$. By switching u and v if necessary, we may assume that the maximum is attained by the first argument.

The term $\|\nabla u - \nabla p_r\| \|\nabla v + \nabla q_r\|$ can be estimated by using knowledge of the Green's function (or in this case the so-called Neumann function) $N(x, y)$ for the differential equation at hand. Let $x \in Q_\nu$. Then

$$u(x) = \int_{\partial\Omega} N(x, y) f(y) dS_y.$$

Let $P_r(x - x_0)(y)$ be the r th-order Taylor series approximation to $N(x, y)$ in the x variables, about the point x_0 . Assuming $|x - y| \geq \text{dist}(Q_\nu, \partial\Omega) > 0$, $P_r(x - x_0)(y)$ is well-defined and bounded. It can be checked that

$$p_r(x - x_0) = \int_{\partial\Omega} P_r(x - x_0)(y) f(y) dS_y.$$

Hence, using the fact that $\|f\|_{L^1(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)} \leq C$, we have

$$\begin{aligned} \|\nabla u - \nabla p_r\|_{L^\infty(Q_\nu)} &= \sup_{x \in Q_\nu} \left| \int_{\partial\Omega} [\nabla_x N(x, y) - \nabla_x P_r(x - x_0)(y)] f(y) dS_y \right| \\ &\leq C \sup_{x \in Q_\nu} \sup_{y \in \partial\Omega} |\nabla_x [N(x, y) - P_r(x - x_0)(y)]|. \end{aligned} \quad (2.10)$$

Note that (2.10) is otherwise independent of f . The term $\|\nabla v + \nabla q_r\|$ may be estimated similarly. Since the size of Q_ν decreases like 2^{-j} , and $N(x, y)$ changes slowly for $|x - y|$ large, the terms (2.10) become small very quickly as j increases, and the decay is faster for ν such that $\text{dist}(Q_\nu, \partial\Omega)$ is larger. Denote the final bound

$$|\beta_\nu| \leq |Q_\nu|^{1/2} \|\nabla u - \nabla p_r\| \|\nabla v + \nabla q_r\| \leq b(\nu).$$

Define the operator B over \mathcal{N} by

$$B(\delta\gamma)(\nu) = b(\nu)\langle\delta\gamma, \psi_\nu\rangle_{L^2(\Omega)}.$$

Then from (2.9), we have established the following proposition.

Proposition 2.2 If $\delta\gamma \in \text{span}\{\psi_\nu\}_{\nu \in \mathcal{N}}$, then

$$\|DQ_1(\delta\gamma)\|_* \leq \|B(\delta\gamma)\|_{L^1(\mathcal{N})}.$$

We have made explicit calculations for $b(\nu)$ in the case where u and v are harmonic functions in the unit disc and the functions $\{\psi_\nu\}$ are tensor products of the Haar basis for $L^2(\mathbb{R})$. In this case, $N(x, y)$ is explicitly available, and $P_r(x - x_0)(y) = N(x_0, y)$. Thus, (2.10) is easy (but tedious) to calculate. Sparing the reader the details, a graph of $b(\nu)$ for several scales j is shown in Figure 1, illustrating the rapid loss of information away from the boundary.

As we mentioned in the last section, the utility of these estimates is that given a noise level ϵ , they provide an explicit description of a large class of indistinguishable conductivity perturbations, namely the set B_ϵ from (2.8). Thus we have established absolute limits on the resolution possible in the linearized problem.

2.4 Lower bounds on the linearized map

This bound is a simple application of the harmonic functions first used in this context by Calderón [6]. The gist of this bound is well-known. Calderón used the harmonic

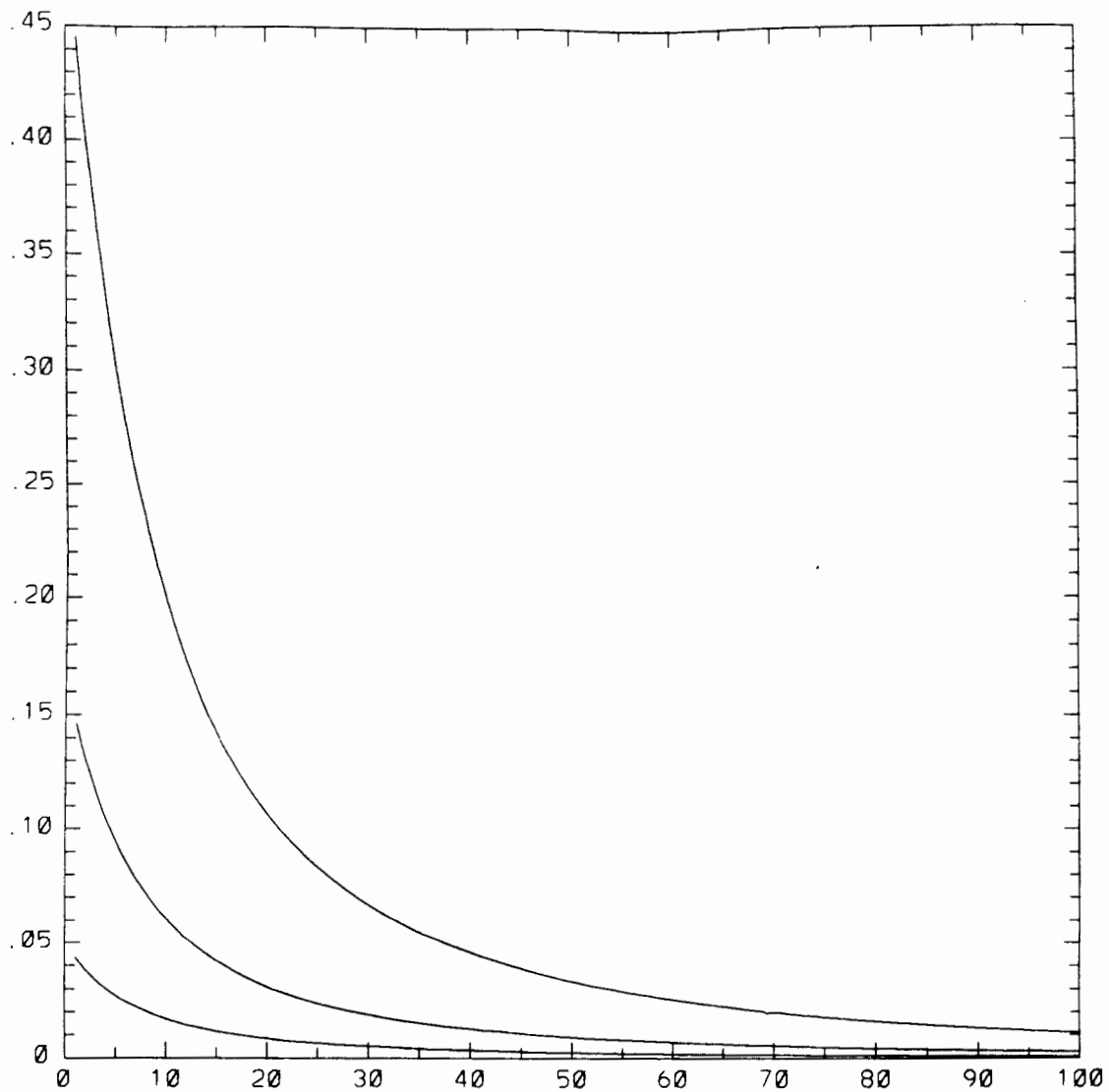


Figure 2.1 Magnitude of $b(\nu)$ versus distance from boundary for scales $j = 4, 5, 6$.

functions

$$u(x) = e^{(i\xi+\zeta)\cdot x}, v(x) = e^{(i\xi-\zeta)\cdot x},$$

where $\xi, \zeta \in R^n$, with $\xi \cdot \zeta = 0$ and $|\xi| = |\zeta|$, in the linearized map (2.4) to show that the linearized map is injective. In this case, the right-hand side of (2.4) becomes

$$|\xi|^2 \int_{\Omega} \delta\gamma(x) e^{i\xi\cdot x} dx = |\xi|^2 \widehat{\delta\gamma}(\xi), \quad (2.11)$$

where $\delta\gamma$ is extended to be zero outside Ω . Calderón also used this bound to obtain estimates for perturbations in the fully nonlinear problem. But these estimates are obtained assuming exact data, with no restriction on the magnitude of the boundary values. The functions u, v increase in magnitude exponentially with $|\xi|$ at the boundary, violating the practical “finite energy” condition that $\|f\| \leq 1$. Our lower bound follows immediately from scaling u and v appropriately and accounting for ϵ -errors in the measurements.

Let Ω be contained in a ball of radius R . Scale u and v by $ce^{-R|\xi|}$, where c is chosen so that $f_{\xi} = u|_{\partial\Omega}$ and $g_{\xi} = v|_{\partial\Omega}$ satisfy

$$\|f_{\xi}\|_{L^2(\partial\Omega)}, \|g_{\xi}\|_{L^2(\partial\Omega)} \leq 1.$$

Then (2.11) and (2.4) yield

$$|DQ_1(\delta\gamma)(f_{\xi}, g_{\xi})| = c|\xi|^2 e^{-2R|\xi|} |\widehat{\delta\gamma}(\xi)|. \quad (2.12)$$

Define the operator C by

$$C(\delta\gamma)(\xi) = c|\xi|^2 e^{-2R|\xi|} |\widehat{\delta\gamma}(\xi)|.$$

Taking the supremum of (2.12) over $\xi \in \mathbb{R}^n$, and using the definition of $\|\cdot\|_*$, we immediately obtain the following fact.

Proposition 2.3

$$\|C(\delta\gamma)\|_{L^\infty(\mathbb{R}^n)} \leq \|DQ_1(\delta\gamma)\|_*.$$

Thus, $\widehat{\delta\gamma}$ can at least be determined with exponentially increasing error as the frequency is increased.

As described in Section 2.2, stable inversion of the linearized problem requires restricting the set of possible solutions to a stabilizing set \mathcal{S} so that a bound such as

$$\|D\Lambda_1(\delta\gamma)\| \geq c\|\delta\gamma\| \tag{2.13}$$

holds for all $\delta\gamma \in \mathcal{S}$. By Proposition 2.3, one such set is

$$\mathcal{S}' = \{\delta\gamma \in L^2(\Omega) : \|C(\delta\gamma)\|_{L^\infty(\mathbb{R}^n)} \geq c\|\delta\gamma\|_{L^2(\Omega)}\}.$$

However, the condition $\delta\gamma \in \mathcal{S}'$ is probably an overly pessimistic restriction. The restriction $\delta\gamma \in \mathcal{S}'$ penalizes high frequencies regardless of their spatial localization, insisting that no sharp features be present anywhere. But experience with the problem and the estimates from the preceding section seem to indicate that localized details can be recovered if they are close enough to the boundary.

A somewhat more liberal stabilization scheme will be used in Chapter 4. The problem will be parametrized in terms of a wavelet basis. Although regularity considerations (which will be made precise in the next chapter) will force the suppression

of rough conductivities all over the domain, the wavelet parametrization allows for the specification of more stringent regularization away from the boundary. The stability estimates from Section 2.3 provide a heuristic for a prudent level of stabilization at a particular point in the domain.

Chapter 3

Regularity of the Forward Map

In this chapter, we will study the regularity properties of the forward map. We wish to determine conditions under which the linearized problem gives a sufficiently good local approximation to the nonlinear problem that it can be safely used as part of an iterative reconstruction method. It turns out that the strength of the regularity results we can prove is heavily dependent on the norm placed on the conductivities.

With the $L^\infty(\Omega)$ norm, which is in some sense the natural norm for the problem, the forward map has the desired regularity properties. However, with the $L^2(\Omega)$ norm, which is far preferable from a computational standpoint, we need to impose some additional smoothness on the conductivities to get sufficiently strong regularity estimates. These results, along with some related estimates necessary to prove the convergence of an iterative method, are established in Section 3.1. In Section 3.2, we show that with stronger $L^p(\Omega)$ norms on the conductivities, or with a weaker notion of regularity, it is possible to weaken the smoothness assumptions. In Section 3.3 we construct a simple example in one dimension which “almost” violates the regularity estimates, thereby illustrating why the smoothness assumptions may be necessary.

Let us begin by fixing notation and describing the problem. Define the map

$$F : \mathcal{D} \subset L^2(\Omega) \rightarrow L^2(\partial\Omega)$$

by

$$F(\gamma) = \Lambda_\gamma f,$$

where f is fixed with $\|f\|_{L^2(\partial\Omega)} = 1$. Here

$$\mathcal{D} = \{\gamma \in L^\infty(\Omega) : \gamma(x) \geq a > 0, \text{supp}(\gamma - \tilde{\gamma}) \subset \Omega'\}, \quad (3.1)$$

where $\tilde{\gamma}$ is a fixed conductivity function and $\Omega' \subset\subset \Omega$ satisfies a uniform interior cone condition. Given a conductivity perturbation $\delta\gamma$ such that $\gamma + \delta\gamma \in \mathcal{D}$, let $\delta u \in H^1(\Omega)$ be the unique solution to the linearized problem

$$\begin{aligned} \nabla \cdot \gamma \nabla \delta u &= -\nabla \cdot \delta\gamma \nabla u \quad \text{in } \Omega \\ \gamma \frac{\partial \delta u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \delta u &= 0. \end{aligned} \quad (3.2)$$

Denote $\delta u|_{\partial\Omega} = DF(\gamma)(\delta\gamma)$.

Now let $p \in H^1(\Omega)$ solve

$$\begin{aligned} \nabla \cdot (\gamma + \delta\gamma) \nabla p &= -\nabla \cdot \delta\gamma \nabla \delta u \quad \text{in } \Omega \\ (\gamma + \delta\gamma) \frac{\partial p}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} p &= 0. \end{aligned} \quad (3.3)$$

By taking linear combinations, we see that

$$p|_{\partial\Omega} = F(\gamma + \delta\gamma) - F(\gamma) - DF(\gamma)(\delta\gamma).$$

Thus, the size of p indicates how well F is approximated by its formal linearization.

With only the condition that for some constant a ,

$$\gamma(x), (\gamma + \delta\gamma)(x) \geq a > 0 \text{ in } \Omega, \quad (3.4)$$

one can establish the bound

$$\|p\|_{L^2(\partial\Omega)} \leq C(\Omega, a) \|\delta\gamma\|_{L^\infty(\Omega)}^2. \quad (3.5)$$

This shows that on the set described by condition (3.4), F is locally uniformly well approximated by DF as a function on $L^\infty(\Omega) \rightarrow L^2(\partial\Omega)$. This is sufficient to show that F is Fréchet differentiable on the interior of the set (3.4). The inequality (3.5) was pointed out and proved to us by Michael Vogelius.

As a corollary to the $L^\infty(\Omega)$ bound (3.5), application of the Sobolev imbedding lemma immediately yields

$$\|p\|_{L^2(\partial\Omega)} \leq C(\Omega, a, s) \|\delta\gamma\|_{H_0^s(\Omega)}^2, \text{ where } s > n/2, \quad (3.6)$$

showing the regularity of F as a function on $H_0^s(\Omega) \rightarrow L^2(\partial\Omega)$. Thus we can obtain the regularity of F on a Hilbert space at the expense of strengthening the norm on the conductivities. However, practical considerations make it preferable to pose the problem over $L^2(\Omega)$, for at least two reasons. First, the stronger $H_0^s(\Omega)$ norm on the conductivities makes the inverse map less stable. Second, the $L^2(\Omega)$ inner product makes adjoint derivative calculations a simple matter of intergration by parts. Thus it will be worthwhile to find conditions under which the forward map is regular as a function on $L^2(\Omega)$.

3.1 Regularity theorems

To prove the regularity results, we will make use of several technical lemmas. The main idea in establishing most of these results is to use elliptic regularity to get bounds stronger than $L^2(\Omega)$ on the gradients of the solutions of the various differential equations. The first lemma gives conditions on the conductivities under which this can be done. In all of the following, C denotes a generic constant depending on Ω , with additional dependence as noted, whose value may vary from line to line.

Lemma 3.1 Given r with $2 < r \leq \infty$, let $\gamma \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > n(\frac{1}{2} - \frac{1}{r}) - 1$, and $\gamma(x) \geq a > 0$ in $\bar{\Omega}$. Let $f \in L^2(\partial\Omega)$ with $\int_{\partial\Omega} f = 0$. Then for each solution $u \in H^1(\Omega)$ of the problem

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 & \text{in } \Omega \\ \gamma \frac{\partial u}{\partial \eta} &= f & \text{on } \partial\Omega, \end{aligned} \tag{3.7}$$

we have

$$\|\nabla u\|_{L^r(\Omega')} \leq C \|\nabla u\|_{L^2(\Omega)}, \tag{3.8}$$

where $C = C(\Omega', a, k, r, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$.

Proof By standard elliptic regularity results (see Gilbarg and Trudinger [14]), for each solution u of (3.7),

$$\|u\|_{H^{k+2}(\Omega')} \leq C \|u\|_{H^1(\Omega)}, \tag{3.9}$$

where $C = C(\Omega', a, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$. We note that Ω' satisfies a uniform interior cone condition, so that we can apply the Sobolev imbedding lemma (see Adams [1]). In

the case $r = \infty$, we have $k + 2 > n/2 + 1$, so the imbedding lemma gives the existence of a continuous linear injection

$$H^{k+2}(\Omega') \rightarrow C_B^1(\Omega') = \{u \in C^1(\Omega') : \nabla u \in L^\infty(\Omega')\}.$$

In the case $r < \infty$, we have

$$r < \frac{2n}{n - 2(k + 1)},$$

so the Sobolev lemma gives the imbedding

$$H^{k+2}(\Omega') \rightarrow W^{1,r}(\Omega').$$

Hence, in both cases, we have

$$\|\nabla u\|_{L^r(\Omega')} \leq C(\Omega', k, r)\|u\|_{H^{k+2}(\Omega')} \quad (3.10)$$

for all $u \in H^{k+2}(\Omega')$. Let B be a ball containing Ω . Since $\partial\Omega$ is smooth, there is a strong 1-extension operator E for Ω . By the Poincaré inequality on convex domains,

$$\|Eu - Eu_B\|_{L^2(B)}^2 \leq C(B)\|\nabla Eu\|_{L^2(B)}^2, \quad \text{where } Eu_B = \frac{1}{|B|} \int_B Eu,$$

so that after some manipulation,

$$\|u\|_{H^1(\Omega)}^2 \leq C\|\nabla u\|_{L^2(\Omega)}^2 + |B|(Eu_B)^2. \quad (3.11)$$

Combining the bounds (3.9), (3.10), and (3.11), we have established that

$$\|\nabla u\|_{L^r(\Omega')} \leq C(\|\nabla u\|_{L^2(\Omega)}^2 + |B|(Eu_B)^2)^{1/2}. \quad (3.12)$$

If u is a particular solution of (3.7), then $u_0 = u - Eu_B$ is also a solution, with the property that $(Eu_0)_B = 0$. Hence (3.12) yields the desired inequality (3.8) for u_0 .

But since ∇u and C are independent of additive constants in u , (3.8) remains valid for all solutions u . \square

Lemma 3.2 Let $\gamma \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > n/2 - 1$, and $\gamma(x) \geq a > 0$ in $\bar{\Omega}$. Assume $\text{supp } \delta\gamma \subset \Omega'$. Then

$$\|\nabla \delta u\|_{L^2(\Omega)} \leq C \|\delta\gamma\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)}$$

where $C = C(\Omega', a, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$.

Proof Multiplying the linearized equation (3.2) by δu , integrating by parts, and using the fact that $\text{supp } \delta\gamma \subset \Omega'$, we have

$$\int_{\Omega} \gamma |\nabla \delta u|^2 = \left| \int_{\Omega'} \delta\gamma \nabla \delta u \cdot \nabla u \right|.$$

Hence after two applications of the Hölder inequality,

$$\|\nabla \delta u\|_{L^2(\Omega)}^2 \leq \frac{1}{a} \|\delta\gamma\|_{L^2(\Omega)} \|\nabla \delta u\|_{L^2(\Omega)} \|\nabla u\|_{L^\infty(\Omega')}.$$

The result now follows by Lemma 3.1 with $r = \infty$. \square

Lemma 3.3 Let $M \geq \gamma(x) \geq a > 0$. For each $u \in H^1(\Omega)$ satisfying (3.7), the following inequalities hold.

1. $\|\nabla u\|_{L^2(\Omega)} \leq \left(\frac{1}{a} \|f\|_{L^2(\partial\Omega)} \|\Lambda_\gamma f\|_{L^2(\partial\Omega)}\right)^{1/2}$
2. $\|\nabla u\|_{L^2(\Omega)} \leq C(a, M) \|f\|_{L^2(\partial\Omega)}$.

Proof The first inequality follows immediately from the Schwarz inequality applied to the energy identity

$$\int_{\Omega} \gamma |\nabla u|^2 = \int_{\partial\Omega} f \Lambda_{\gamma} f.$$

The second inequality is essentially proved in Sylvester and Uhlmann [37] (see proposition 3.2). One need only switch Dirichlet problems for Neumann problems and replace $H^{1/2}(\partial\Omega)$ estimates with the corresponding $H^{-1/2}(\partial\Omega)$ estimates. \square

Before proving the regularity of the forward map, we need to establish that the linearized map DF is bounded, since DF is supposed to approximate local changes in F uniformly well.

Proposition 3.1 If $\gamma \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > n/2 - 1$, and $\gamma(x) \geq a > 0$, then $DF(\gamma)(\cdot) : L^2(\Omega') \rightarrow L^2(\partial\Omega)$ is a bounded linear operator with

$$\|DF(\gamma)(\cdot)\|_{L^2(\Omega'), L^2(\partial\Omega)} \leq C_1, \quad (3.13)$$

where $C_1 = C(\Omega', a, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$.

Proof The linearity of DF is evident from its definition. Let $w \in H^1(\Omega)$ be a solution of

$$\begin{aligned} \nabla \cdot \gamma \nabla w &= 0 \quad \text{in } \Omega \\ \gamma \frac{\partial w}{\partial \eta} &= \delta u \quad \text{on } \partial\Omega. \end{aligned} \quad (3.14)$$

Let us hold γ fixed and use the notation $DF(\delta\gamma) = DF(\gamma)(\delta\gamma)$. Integrating by parts, applying equations (3.2) and (3.14), and using the fact that $\text{supp } \delta\gamma \subset \Omega'$, we obtain

$$\|DF(\delta\gamma)\|_{L^2(\partial\Omega)}^2 = \left| \int_{\partial\Omega} \delta u \left(\gamma \frac{\partial w}{\partial \eta} \right) \right| = \left| \int_{\Omega'} \delta\gamma \nabla w \cdot \nabla u \right|.$$

Applying the Hölder inequality twice,

$$\|DF(\delta\gamma)\|_{L^2(\partial\Omega)}^2 \leq \|\delta\gamma\|_{L^2(\Omega')} \|\nabla w\|_{L^\infty(\Omega')} \|\nabla u\|_{L^2(\Omega)}.$$

By Lemma 3.1, $\|\nabla w\|_{L^\infty(\Omega')} \leq C \|\nabla w\|_{L^2(\Omega)}$, where C is independent of $\delta\gamma$. Since γ is bounded in $L^\infty(\Omega)$ by its $C^{k,1}(\bar{\Omega})$ norm, application of Lemma 3.3 yields

$$\|DF(\delta\gamma)\|_{L^2(\partial\Omega)}^2 \leq C \|\delta\gamma\|_{L^2(\Omega')} \|\delta u\|_{L^2(\partial\Omega)} \|f\|_{L^2(\partial\Omega)}, \quad (3.15)$$

where $C = C(\Omega', a, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$. But $\delta u|_{\partial\Omega} = DF(\delta\gamma)$ and $\|f\| = 1$, so (3.15) is the desired result. \square

With all the preliminaries taken care of, we can now prove the main result of this section, which gives conditions under which the forward map is regular over $L^2(\Omega)$. These conditions restrict the class of admissible conductivities to a bounded set in $C^{k,1}(\bar{\Omega})$ where k depends on the dimension of the problem. The necessity of these restrictions will be investigated later.

Theorem 3.1 Let $\gamma, (\gamma + \delta\gamma) \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > \frac{n}{2} - 1$. Assume that

1. $\text{supp } \delta\gamma \subset \Omega'$,

2. $\gamma(x), (\gamma + \delta\gamma)(x) \geq a > 0$ in $\bar{\Omega}$, and
3. $\|\gamma\|_{C^{k,1}(\bar{\Omega})}, \|\gamma + \delta\gamma\|_{C^{k,1}(\bar{\Omega})} \leq M$.

Then

$$\|F(\gamma + \delta\gamma) - F(\gamma) - DF(\gamma)(\delta\gamma)\|_{L^2(\partial\Omega)} \leq C_2 \|\delta\gamma\|_{L^2(\Omega)}, \quad (3.16)$$

where $C_2 = C(\Omega', a, k, M)$.

Proof We need to estimate $\|p\|_{L^2(\partial\Omega)}^2$, where p is defined in (3.3). Let w solve

$$\begin{aligned} \nabla \cdot (\gamma + \delta\gamma) \nabla w &= 0 \text{ in } \Omega \\ \gamma \frac{\partial w}{\partial \eta} &= p \text{ on } \partial\Omega. \end{aligned}$$

Integration by parts and (3.3) yield

$$\begin{aligned} \|p\|_{L^2(\partial\Omega)}^2 &= \left| \int_{\partial\Omega} p(\gamma + \delta\gamma) \nabla w \cdot n \right| \\ &= \left| \int_{\Omega} w(\nabla \cdot \delta\gamma \nabla \delta u) \right| \\ &= \left| \int_{\Omega'} \delta\gamma \nabla \delta u \cdot \nabla w \right|, \end{aligned} \quad (3.17)$$

where we have used the fact that $\text{supp } \delta\gamma \subset \Omega'$. Applying the Schwarz inequality to (3.17) we obtain

$$\|p\|_{L^2(\partial\Omega)}^2 \leq C \|\delta\gamma\|_{L^2(\Omega)} \|\nabla \delta u\|_{L^2(\Omega)} \|\nabla w\|_{L^\infty(\Omega')}.$$

Using Lemma 3.1 and Lemma 3.3 to estimate $\|\nabla w\|_{L^\infty(\Omega')}$ and Lemma 3.2 to estimate $\|\nabla \delta u\|_{L^2(\Omega)}$, we obtain

$$\|p\|_{L^2(\partial\Omega)} \leq C \|\delta\gamma\|_{L^2(\Omega)}^2 \|\nabla u\|_{L^2(\Omega)}.$$

The result now follows from Lemma 3.3, the fact that $\|f\|_{L^2(\partial\Omega)} = 1$, and the definition of F . \square

In light of the relatively strong smoothness assumptions in Theorem 3.1, it is interesting that we can establish the Lipschitz continuity of F with no smoothness assumption on the conductivity perturbation $\delta\gamma$.

Proposition 3.2 Let $\gamma \in C^{k,1}(\bar{\Omega})$, where k is a given integer with $k > n/2 - 1$, and $\gamma(x) \geq a > 0$. For each $\delta\gamma$ satisfying

1. $\text{supp } \delta\gamma \subset \Omega'$,
2. $\gamma + \delta\gamma \geq a > 0$ in Ω
3. $\|\gamma + \delta\gamma\|_{L^\infty(\Omega)} \leq M$,

we have

$$\|F(\gamma + \delta\gamma) - F(\gamma)\|_{L^2(\partial\Omega)} \leq C\|\delta\gamma\|_{L^2(\Omega')}, \quad (3.18)$$

where $C = C(\Omega', a, M, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$.

Proof We have $F(\gamma + \delta\gamma) - F(\gamma) = v|_{\partial\Omega}$, where v solves

$$\begin{aligned} \nabla \cdot (\gamma + \delta\gamma)\nabla v &= -\nabla \cdot \delta\gamma\nabla u \quad \text{in } \Omega \\ (\gamma + \delta\gamma)\frac{\partial v}{\partial\eta} &= 0 \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} v &= 0. \end{aligned} \quad (3.19)$$

Let $w \in H^1(\Omega)$ satisfy

$$\begin{aligned} \nabla \cdot (\gamma + \delta\gamma)\nabla w &= 0 \quad \text{in } \Omega \\ (\gamma + \delta\gamma)\frac{\partial w}{\partial \eta} &= v \quad \text{on } \partial\Omega. \end{aligned} \tag{3.20}$$

Integrating by parts several times and using the identities (3.19) and (3.20) along with the fact that $\text{supp } \delta\gamma \subset \Omega'$ yields

$$\|v\|_{L^2(\partial\Omega)}^2 = \left| \int_{\partial\Omega} v(\gamma \frac{\partial w}{\partial \eta}) \right| = \left| \int_{\Omega'} \delta\gamma \nabla w \cdot \nabla u \right|.$$

Two applications of Hölder's inequality gives

$$\|v\|_{L^2(\partial\Omega)}^2 \leq \|\delta\gamma\|_{L^2(\Omega')} \|\nabla w\|_{L^2(\Omega')} \|\nabla u\|_{L^\infty(\Omega')}.$$

Using Lemma 3.1 and Lemma 3.3 to estimate $\|\nabla u\|$, and noting that $\|f\| = 1$, we obtain

$$\|v\|_{L^2(\partial\Omega)}^2 \leq C \|\delta\gamma\|_{L^2(\Omega')} \|\nabla w\|_{L^2(\Omega')}, \tag{3.21}$$

where $C = C(\Omega', a, k, \|\gamma\|_{C^{*,1}(\bar{\Omega})})$. Finally, since $\|\gamma + \delta\gamma\|_{L^\infty(\Omega)} \leq M$, we can estimate $\|\nabla w\|_{L^2(\Omega')}$ with Lemma 3.3, so that the result follows from (3.21). \square

In the following chapter, we will formulate a minimization problem over $L^2(\Omega')$, and prove the convergence of a minimization method applied to the problem. To do so will require a Lipschitz estimate on the adjoint derivatives. Let us first calculate the $L^2(\partial\Omega) \rightarrow L^2(\Omega)$ adjoint.

Given $DF(\gamma)(\delta\gamma) = \delta u|_{\partial\Omega}$, we define the formal adjoint $DF(\gamma)^*$ by the adjoint equation

$$\int_{\partial\Omega} DF(\gamma)(\delta\gamma)\phi = \int_{\Omega'} \delta\gamma DF(\gamma)^*(\phi), \quad (3.22)$$

where $\phi \in L^2(\partial\Omega)$ with $\int \phi = 0$. If $u^* \in H^1(\Omega)$ satisfies

$$\begin{aligned} \nabla \cdot (\gamma \nabla u^*) &= 0 \quad \text{in } \Omega \\ \gamma \frac{\partial u^*}{\partial \eta} &= \phi \quad \text{on } \partial\Omega \end{aligned}$$

then several integrations by parts along with the identity (3.2) reveal that

$$\int_{\partial\Omega} (\delta u)\phi = - \int_{\Omega'} \delta\gamma (\nabla u \cdot \nabla u^*).$$

Hence,

$$DF(\gamma)^*(\phi) = -\nabla u \cdot \nabla u^*. \quad (3.23)$$

Theorem 3.2 Under all the assumptions of Theorem 3.1 and the additional assumption that $\delta\gamma \in C^{k,1}(\bar{\Omega})$, for any $\phi \in L^2(\partial\Omega)$ with $\int \phi = 0$ we have

$$\|[DF(\gamma + \delta\gamma)^* - DF(\gamma)^*]\phi\|_{L^2(\Omega')} \leq C_3(\|\phi\|_{L^2(\partial\Omega)})\|\delta\gamma\|_{L^2(\Omega')}. \quad (3.24)$$

The constant C_3 has the form $C_3 = (C + K\|\phi\|_{L^2(\partial\Omega)})$ where

$$C = C(\Omega', a, k, \|\delta\gamma\|_{C^{k,1}(\bar{\Omega})}, \|\gamma\|_{C^{k,1}(\bar{\Omega})})$$

and

$$K = K(\Omega', a, k, \|\gamma\|_{C^{k,1}(\bar{\Omega})}).$$

Proof Let u, u^* be as above so that $DF(\gamma)^*(\phi) = -\nabla u \cdot \nabla u^*$. Let v, v^* satisfy the perturbational equations

$$\begin{aligned}\nabla \cdot (\gamma + \delta\gamma)\nabla v &= -\nabla \cdot \delta\gamma\nabla u \\ \gamma \frac{\partial v}{\partial \eta}|_{\partial\Omega} &= 0\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}\nabla \cdot (\gamma + \delta\gamma)\nabla v^* &= -\nabla \cdot \delta\gamma\nabla u^* \\ \gamma \frac{\partial v^*}{\partial \eta}|_{\partial\Omega} &= 0,\end{aligned}$$

so that

$$\begin{aligned}-[DF(\gamma + \delta\gamma)^* - DF(\gamma)^*]\phi &= (\nabla u^* + \nabla v^*) \cdot (\nabla u + \nabla v) - (\nabla u \cdot \nabla u^*) \\ &= (\nabla v \cdot \nabla u^*) + (\nabla u \cdot \nabla v^*) + (\nabla v \cdot \nabla v^*).\end{aligned}\tag{3.26}$$

Taking norms in (3.26) and applying the triangle inequality and Hölder's inequality to the right hand side,

$$\|[DF(\gamma + \delta\gamma)^* - DF(\gamma)^*]\phi\|_{L^2(\Omega')} \leq \|\nabla u^*\|_\infty \|\nabla v\|_2 + \|\nabla u\|_\infty \|\nabla v^*\|_2 + \|\nabla v\|_\infty \|\nabla v^*\|_2,$$

where all norms are over Ω' . Since $\|f\| = 1$, we have by Lemma 3.1 and Lemma 3.3 that $\|\nabla u\|_{L^\infty(\Omega')} \leq C$. By Lemma 3.2 and Lemma 3.3,

$$\|\nabla v\|_{L^2(\Omega')} \leq C\|\delta\gamma\|_{L^2(\Omega')}.\tag{3.27}$$

Similarly, $\|\nabla u^*\|_\infty \leq K\|\phi\|_{L^2(\partial\Omega)}$ and $\|\nabla v^*\|_2 \leq C$ so that

$$\|[DF(\gamma + \delta\gamma)^* - DF(\gamma)^*]\phi\|_{L^2(\Omega')} \leq \|\delta\gamma\|_{L^2(\Omega')} (C + \|\nabla v\|_{L^\infty(\Omega')}).\tag{3.28}$$

Inequalities (3.9) and (3.11) show that

$$\|u\|_{H^{k+2}(\Omega')} \leq C(\|\nabla u\|_{L^2(\Omega)}^2 + |B|(Eu_B)^2)^{1/2};$$

hence Lemma 3.3 yields

$$\|Du\|_{H^{k+1}(\Omega')} \leq C\|\nabla u\|_{L^2(\Omega)} \leq C. \quad (3.29)$$

The fact that $\|\delta\gamma\|_{C^{k,1}(\Omega)}$, $k \geq 1$, is bounded, combined with (3.29) shows that for some constant N ,

$$\|\nabla \cdot \delta\gamma \nabla u\|_{L^2(\Omega)} \leq N.$$

Since the right hand side of (3.25) is bounded in L^2 , elliptic regularity applies to give

$$\|v\|_{H^{k+2}(\Omega')} \leq C(\|v\|_{H^1(\Omega)} + N).$$

Then an argument similar to the proof of Lemma 3.1 shows that

$$\|\nabla v\|_{L^\infty(\Omega')} \leq C(\|\nabla v\|_{L^2(\Omega)} + N). \quad (3.30)$$

Substitution of (3.30) and (3.27) into (3.28) leaves us with the desired inequality. \square

3.2 Weaker regularity results

As mentioned at the beginning of this chapter, the smoothness assumptions in the regularity estimates of Theorem 3.1 can be reduced in two interesting cases.

In the first case, we consider strengthening the norm on the conductivities from $L^2(\Omega)$ to $L^q(\Omega)$, where $q > 2$. In the limiting case $q = \infty$, the bound (3.5) shows

that no smoothness is required whatsoever. It is natural to ask what happens for intermediate values of q . Theorem 3.1 can be easily modified as follows.

Theorem 3.3 Given q with $2 < q < \infty$, let $\gamma, (\gamma + \delta\gamma) \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > n(\frac{1}{2} - \frac{1}{r}) - 1$, and r satisfies $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Assume

1. $\text{supp } \delta\gamma \subset \Omega'$,
2. $\gamma(x), (\gamma + \delta\gamma)(x) \geq a > 0$ in $\bar{\Omega}$, and
3. $\|\gamma\|_{C^{k,1}(\bar{\Omega})}, \|\gamma + \delta\gamma\|_{C^{k,1}(\bar{\Omega})} \leq M$.

Then

$$\|F(\gamma + \delta\gamma) - F(\gamma) - DF(\gamma)(\delta\gamma)\|_{L^2(\partial\Omega)} \leq C\|\delta\gamma\|_{L^2(\Omega)}^2, \quad (3.31)$$

where $C = C(\Omega', a, k, q, M)$.

Note that in the particular case $n = 2$, the smoothness assumption on the coefficient is reduced from $C^{1,1}(\bar{\Omega})$ to $C^{0,1}(\bar{\Omega})$ (uniformly Lipschitz continuous) by merely strengthening the norm on $\delta\gamma$ from $L^2(\Omega)$ to $L^q(\Omega)$ for any $q > 2$. This “jump” is probably due to the fact that we have not considered coefficients with fractional smoothness properties, and suggests that the L^2 result in Theorem 3.1 may not be optimal.

Proof We begin as in the proof of Theorem 3.1. Apply Hölder’s inequality twice to (3.17) so that

$$\|p\|^2 L^2(\partial\Omega) \leq C\|\delta\gamma\|_{L^q(\Omega')}\|\nabla\delta u\|_{L^2(\Omega')}\|\nabla w\|_{L^r(\Omega')} \quad (3.32)$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. The proof now follows exactly as in Theorem 3.1, with a simple modification to Lemma 3.2 to get

$$\|\nabla \delta u\|_{L^2(\Omega)} \leq C \|\delta \gamma\|_{L^q(\Omega)} \|\nabla u\|_{L^2(\Omega)}.$$

□

The second case of special interest occurs when we relax the bound (3.16) to

$$\|F(\gamma + \delta \gamma) - F(\gamma) - DF(\gamma)(\delta \gamma)\|_{L^2(\partial \Omega)} = o(\|\delta \gamma\|_{L^2(\Omega)}),$$

which corresponds to the usual definition of Fréchet differentiability. In this case, for $n > 2$, we can relax the smoothness assumption on the perturbation to $C^{0,1}(\bar{\Omega})$.

Theorem 3.4 Let $\gamma \in C^{k,1}(\bar{\Omega})$, where k is an integer with $k > \frac{n}{2} - 1$,

let $\gamma + \delta \gamma \in C^{0,1}(\bar{\Omega})$. Assume that $n \geq 3$, and that

1. $\text{supp } \delta \gamma \subset \Omega'$,
2. $\gamma(x), (\gamma + \delta \gamma)(x) \geq a > 0$ in $\bar{\Omega}$, and
3. $\|\gamma\|_{C^{k,1}(\bar{\Omega})}, \|\gamma + \delta \gamma\|_{C^{0,1}(\bar{\Omega})} \leq M$.

Then there is an $s > 0$ such that

$$\|F(\gamma + \delta \gamma) - F(\gamma) - DF(\gamma)(\delta \gamma)\|_{L^2(\partial \Omega)} \leq C \|\delta \gamma\|_{L^2(\Omega)}^{1+s},$$

where $C = C(\Omega', a, k, M)$.

Proof The proof follows from estimate (3.32) with $2 < r < 1/(\frac{1}{2} - \frac{1}{n})$. In this case, Lemma 3.1 holds with $k = 0$. Estimating $\|\nabla\delta u\|$ and $\|\nabla w\|$ in the usual way, we obtain

$$\|p\|_{L^2(\partial\Omega)} \leq C\|\delta\gamma\|_{L^q(\Omega)}\|\delta\gamma\|_{L^2(\Omega)}.$$

Since $\delta\gamma$ is bounded in $L^\infty(\Omega)$ by its $C^{0,1}(\bar{\Omega})$ norm, the interpolation inequality

$$\|\delta\gamma\|_q \leq \|\delta\gamma\|_2^s \|\delta\gamma\|_\infty^{1-s},$$

where s is defined by $\frac{1}{q} = \frac{s}{2} + \frac{1-s}{r}$, yields the conclusion. \square

3.3 An example of nonregularity

In this section, we will construct a simple one-dimensional example illustrating one way in which uncontrolled nonlinearity in the forward map might arise. Although the example falls short of establishing the necessity of the smoothness assumptions in the regularity theorems of Section 3.1, it does show that the forward map does not have all the regularity properties we might hope.

Let $\Omega = (0, 1) \subset \mathbb{R}$. We linearize about $\gamma \equiv 1$, with Neumann data $f = (-1, 1)$, so that $u(x) = x$ solves

$$\begin{aligned} \frac{d^2u}{dx^2} &= 0 \quad \text{in } \Omega \\ u'(0) &= u'(1) = 1 \\ u(0) &= 0 \end{aligned}$$

and $F(1) = u(1) = 1$.

Let i be an integer, $i \geq 2$. Define $\delta\gamma_i$ on Ω as follows

$$\delta\gamma_i(x - \frac{1}{4}) = \begin{cases} 2ix & \text{if } x \in [0, \frac{1}{4i}) \\ \frac{1}{2} - 2ix & \text{if } x \in [\frac{1}{4i}, \frac{3}{4i}) \\ 2ix - \frac{1}{2} & \text{if } x \in [\frac{3}{4i}, \frac{1}{i}) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\delta\gamma_i$ has the following properties:

1. $\delta\gamma_i \in C^{0,1}(\bar{\Omega})$ for all i .
2. For $i \geq 2$, $\text{supp } \delta\gamma_i \subset \Omega' = [\frac{1}{4}, \frac{3}{4}]$.
3. $1/2 \leq 1 + \delta\gamma_i \leq 3/2$ in Ω , for all i .

Also, it is simple to verify that

$$\|\delta\gamma_i\|_{L^p(\Omega)} = C(p)i^{-1/p}, \quad \text{for } 1 \leq p < \infty. \quad (3.33)$$

We will show that for any $p < 2$ we have

$$\frac{|F(1 + \delta\gamma_i) - F(1) - DF(1)(\delta\gamma_i)|}{\|\delta\gamma_i\|_{L^p(\Omega)}^2} \rightarrow \infty \quad (3.34)$$

as $i \rightarrow \infty$. Since only the *uniform* Lipschitz continuity of the conductivity perturbations is violated, we will have “almost” shown that the smoothness assumptions in Theorem 3.1 are necessary in the case $n = 1$.

Inserting $\delta\gamma_i$ into the linearized equation,

$$\begin{aligned}\frac{d^2\delta u}{dx^2} &= -\frac{d\delta\gamma_i}{dx} \text{ in } \Omega \\ \delta u'(0) &= \delta u'(1) = 0 \\ \delta u(0) &= 0,\end{aligned}$$

we see that $\delta u \in H^1(\Omega)$ can be found by integration, and $DF(1)(\delta\gamma_i) = \delta u(1) = 0$.

Note that DF simply returns the *average* of the perturbation $\delta\gamma_i$ over the interval.

Explicitly solving

$$\frac{d}{dx}\left((1 + \delta\gamma_i)\frac{du_i}{dx}\right) = 0$$

along with the appropriate boundary conditions, we get

$$F(1 + \delta\gamma_i) = u_i(1) = \int_0^1 \frac{dx}{1 + \delta\gamma_i(x)} = 1 + \frac{1}{i}(\log(\frac{1 + \frac{1}{2i}}{1 - \frac{1}{2i}}) - 1). \quad (3.35)$$

Since the log term tends to zero as $i \rightarrow \infty$, there is a constant M such that for all i sufficiently large,

$$|F(1 + \delta\gamma_i) - F(1) - DF(1)(\delta\gamma_i)| \geq \frac{M}{i}.$$

Hence, by (3.33), for every $K > 0$, $p < 2$, there is an N large enough that

$$|F(1 + \delta\gamma_i) - F(1) - DF(1)(\delta\gamma_i)| \geq \frac{M}{i} \geq K \frac{C(p)}{i^{2/p}} = K \|\delta\gamma_i\|_{L^p(\Omega)}^2$$

for all $i \geq N$, proving (3.34).

Note that the same argument shows that

$$\frac{|F(1 + \delta\gamma_i) - F(1) - DF(1)(\delta\gamma_i)|}{\|\delta\gamma_i\|_{L^p(\Omega)}^p} \geq K > 0$$

for all i sufficiently large, so that in particular, selecting $p = 1$, the map F on $L^1(\Omega)$ is not Fréchet differentiable.

We suspect that a similar construction can be carried out, for example, on the unit disc in \mathbb{R}^2 . However, the calculations are more difficult, as explicit solutions are hard to obtain. Since the forward map is Lipschitz continuous by Proposition 3.2, one cannot construct a counterexample to the regularity theorems by simply violating the Lipschitz bound. Any counterexample to regularity must take the derivative into account.

This example can be trivially extended to the unit square by choosing solutions independent of one variable. However, this kind of extension requires $\delta\gamma$ to be also independent of one variable, violating the assumption that $\text{supp } \delta\gamma \subset \Omega' \subset\subset \Omega$.

Perhaps it is also worth noting that the loss of regularity exhibited in the example may be interpreted in the language of homogenization theory. Indeed, it is easy to calculate that for each $\delta\gamma_i$ above, $F(1 + \delta\gamma_i)$ is precisely the same as $\lim_{\epsilon \rightarrow 0} F(1 + \delta\gamma_{i,\epsilon})$, the homogenized value obtained as the limit of coefficients with periodic structure $\delta\gamma_i$ on the scale ϵ . As was noted, $DF(\delta\gamma_i)$ returns the average of $\delta\gamma_i$ over the interval. But it is well-known (see for example [4]) that in the one-dimensional case, the homogenized coefficient is obtained as the *harmonic* average of $\delta\gamma_i$, as reflected in equation (3.35). So the disparity between $F(1 + \delta\gamma_i)$ and $F(1) + DF(1)(\delta\gamma_i)$ arises from the nonlinearity in the homogenization formula. Then the relative sensitivity of

the $L^p(\Omega)$ norms to variations in $\text{supp } \delta\gamma_i$ determines the degree of regularity of F over $L^p(\Omega)$. In the limiting case $p = \infty$, F is regular, as we saw in (3.5).

Chapter 4

Convergence of a Minimization Scheme

Stability and regularity are essential ingredients for the success of schemes such as Newton's method, which are based on the successive solution of linearized subproblems. In this chapter, we will use the regularity results from the previous chapter to prove the local convergence of a minimization algorithm on an appropriately formulated problem. In Section 4.1, we motivate the formulation of a computationally feasible minimization problem with a regularization which controls both the stability and the regularity of the problem. In Section 4.2, an unbounded regularization operator is constructed in terms of a wavelet basis. It is shown that the operator controls the $C^{k,1}(\bar{\Omega})$ norm of the conductivity coefficients so that the regularity results of Chapter 3 apply. In Section 4.3, a globalized version of the Gauss-Newton method, usually called the Levenberg-Marquardt method, is described. This scheme is well-known and widely used; we have chosen it for its simplicity. Finally, in Section 4.4, we show that under simple assumptions which can be easily monitored during computation, the Levenberg-Marquardt method converges locally and q-linearly for the minimization problem described in Section 4.1.

4.1 A minimization problem

In this section we wish to motivate the formulation of a practical minimization problem whose solution is an approximate solution of the reconstruction problem.

Perhaps the most natural way to topologize the solution and data spaces are with the $L^\infty(\Omega)$ norm and the operator norm, respectively. The $L^\infty(\Omega)$ norm arises naturally from the weak formulation of problem (1.1), and as was pointed out in the previous chapter, regularity estimates are easy to obtain with the L^∞ norm on the conductivities. Let $\tilde{\gamma}$ be the conductivity to be reconstructed. Given the data operator $\Lambda_{\tilde{\gamma}}$, one may then be tempted to pose the minimization problem

$$\min_{\gamma \in \mathcal{D}CL^\infty(\Omega)} \|\Lambda_\gamma - \Lambda_{\tilde{\gamma}}\|_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)},$$

where \mathcal{D} is some admissible set of conductivities. However, from a computational standpoint, this problem is difficult to solve. There are two major difficulties. First, the solution space of conductivities is not a Hilbert space, so it is difficult to define gradients as part of an iterative scheme. Second, the operator norm is not necessarily differentiable, again raising difficulties for methods based on local linearizations of the problem. M. Overton [28] has recently developed methods which could be applied to the problem of minimizing the operator norm of $\Lambda_\gamma - \Lambda_{\tilde{\gamma}}$, but such eigenvalue optimization methods are necessarily more complicated than standard Newton or quasi-Newton schemes.

One way to remedy the practical difficulties with the problem above is to place the $L^2(\Omega)$ norm on the solution space, and the Hilbert-Schmidt norm on the data space. Thus, we pose the problem

$$\min_{\gamma \in \mathcal{D} \subset L^2(\Omega)} \|\Lambda_\gamma - \Lambda_{\tilde{\gamma}}\|_{HS}^2, \quad (4.1)$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm over $L^2(\partial\Omega)$. Since the Neumann problem (1.1) is not well-defined for arbitrary $\gamma \in L^2(\Omega)$, the class of admissible conductivities \mathcal{D} must be restricted to be bounded and positive. Also, since the primary concern is to find γ on the interior of Ω , it is reasonable to assume that $\tilde{\gamma}$ is known on a neighborhood of the boundary. (For example, in medical applications, the conductivity of the skin is approximately known *a priori*.) Thus we take \mathcal{D} to be the set

$$\mathcal{D} = \{\gamma \in L^\infty(\Omega) : \gamma(x) \geq a > 0, \text{supp}(\gamma - \tilde{\gamma}) \subset \Omega'\},$$

exactly as defined in (3.1).

Gisser, Isaacson, and Newell [15] have demonstrated with an example that if γ and $\tilde{\gamma}$ agree on a neighborhood of the boundary, then the eigenvalues of the difference operator $\Lambda_\gamma - \Lambda_{\tilde{\gamma}}$ decay exponentially. Thus, in this case, the difference operator is a Hilbert-Schmidt operator.

In fact, in the case $n = 2$, the Neumann-to-Dirichlet maps Λ_γ themselves are Hilbert-Schmidt operators, as we shall now prove. The proof is a simple consequence of the fact that for smooth γ , Λ_γ is a pseudodifferential operator of order -1 . We adopt the notation used by Chazarain and Piriou [9] and Sylvester and Uhlmann [37],

denoting the class of standard pseudodifferential operators of order m on a manifold X by $\mathcal{L}^m(X)$. The corresponding symbol class will be denoted $S^m(X \times \mathbb{R}^n)$.

Lemma 4.1 Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$ with smooth boundary $\partial\Omega$. If $\gamma(x) \geq a > 0$ in $\bar{\Omega}$ and $\gamma \in C^\infty(\bar{\Omega})$, then the Neumann-to-Dirichlet map Λ_γ is a standard pseudodifferential operator belonging to the class $\mathcal{L}^{-1}(\partial\Omega)$.

This fact is well-known and is outlined in [9]. In [37], the asymptotic expansion of the symbol of the Dirichlet-to-Neumann map is calculated.

Theorem 4.1 Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with smooth boundary $\partial\Omega$. Then for $\gamma \in C^\infty(\bar{\Omega})$, $\gamma(x) \geq a > 0$, the Neumann-to-Dirichlet map Λ_γ is a Hilbert-Schmidt operator ($\Lambda_\gamma \in HS$) on the space $L^2(\partial\Omega)$.

Proof The boundary $\partial\Omega$ is a simple closed curve, diffeomorphic to $S^1 = \mathbb{R}/2\pi Z$. Without loss of generality, assume that $\partial\Omega$ is parametrized by arclength x , and that $\partial\Omega$ has length 2π . Then the set

$$\{e^{ixk}\}, \text{ where } k \in Z$$

is complete and orthonormal in $L^2(\partial\Omega)$. Since γ does not change throughout the proof, we make the abbreviation $\Lambda = \Lambda_\gamma$. We will show that

$$\sum_k \|\Lambda e^{ixk}\|_{L^2(\partial\Omega)}^2 < \infty.$$

Let $\{U_j, \psi_j\}_{j=1}^N$, be a smooth coordinate chart on $\partial\Omega$, where $U_j \subset \partial\Omega$ and $\psi_j : U_j \rightarrow \tilde{U}_j \subset \mathbb{R}$. It is easy to construct a coordinate chart for S^1 such that $e^{ixk} = e^{i\tilde{x}k}$, for all $x \in U_j$, where $\tilde{x} = \psi_j(x) \in \tilde{U}_j$. For convenience, we assume that $\{U_j, \psi_j\}$ has this property. Let $\sum \phi_j = 1$ be a smooth partition of unity subordinate to $\{U_j\}$. Let $\mu_j \in C_0^\infty(U_j)$ with μ_j identical to 1 on $\text{supp } \phi_j$.

By Lemma 4.1 and the definition of a pseudodifferential operator on a manifold, the operator

$$\tilde{\Lambda}_j = \psi_j \circ \mu_j \Lambda \circ \psi_j^{-1} : C_0^\infty(\tilde{U}_j) \rightarrow C^\infty(\tilde{U}_j)$$

belongs to the class $\mathcal{L}^{-1}(\tilde{U}_j)$. The fact that $\tilde{\Lambda}_j \in \mathcal{L}^{-1}(\tilde{U}_j)$ is equivalent to the statement (see for example [9] Proposition 2.2, Chapter 4)

$$e^{-i\tilde{x}\xi} \tilde{\Lambda}_j(ae^{i\tilde{x}\xi}) \in S^{-1}(\tilde{U}_j \times \mathbb{R})$$

for all $a \in C_0^\infty(\tilde{U}_j)$. Hence, since $\text{supp } (\psi \circ \mu_j)$ is compact in \tilde{U}_j , by the definition of the symbol class $S^{-1}(\tilde{U}_j \times \mathbb{R})$, there is a constant C_j such that the estimate

$$|\tilde{\Lambda}_j(ae^{i\tilde{x}k})| = |e^{-i\tilde{x}k} \tilde{\Lambda}_j(ae^{i\tilde{x}k})| \leq C_j(1 + |k|)^{-1}$$

holds over \tilde{U}_j , for all $k \in Z$. Using the fact that $e^{ixk} = e^{i\tilde{x}k}$ in U_j , we can change coordinates, yielding

$$|\mu_j \Lambda \phi_j e^{ixk}| \leq C_j(1 + |k|)^{-1}. \quad (4.2)$$

Since the operator $(1 - \mu_j)\Lambda\phi_j$ has null essential support, we have $\Lambda\phi_j = \mu_j\Lambda\phi_j$ modulo $\mathcal{L}^{-\infty}(\partial\Omega)$. Thus, using the compactness of $\partial\Omega$, the estimate (4.2) holds (for some C_j) on all of $\partial\Omega$, with $\mu_j\Lambda$ replaced by Λ .

Finally, summation of the partition of unity yields

$$\sum_k \|\Lambda e^{ikx}\|_{L^2(\partial\Omega)}^2 \leq \sum_k \sum_j \|\Lambda \phi_j e^{ixk}\|_{L^2(\partial\Omega)}^2 \leq C \sum_k (1 + |k|)^{-2} < \infty, \quad (4.3)$$

proving that Λ is a Hilbert-Schmidt operator. \square

Theorem 4.1 can be easily generalized to dimension $n \geq 3$ in the case where $\partial\Omega$ is diffeomorphic to the torus T^{n-1} , so that the complex exponentials still give a basis for $L^2(\partial\Omega)$. In this case it can only be shown that the difference operator $(\Lambda_\gamma - \Lambda_{\tilde{\gamma}})$ is Hilbert-Schmidt, where γ and $\tilde{\gamma}$ agree to a given order on $\partial\Omega$. The additional hypothesis makes $(\Lambda_\gamma - \Lambda_{\tilde{\gamma}}) \in \mathcal{L}^{-s}(\partial\Omega)$ for some $s > 1$, so that the sum

$$\sum_{k \in \mathbb{Z}^{n-1}} (1 + |k|)^{-2s}$$

corresponding to (4.3) is finite even when k has more than one index. In fact, under the assumption that γ agrees with $\tilde{\gamma}$ to infinite order on the boundary (which certainly holds for $\gamma \in \mathcal{D}$), the difference operator is in $\mathcal{L}^{-\infty}(\partial\Omega)$. The only difficulty in extending the proof to more arbitrary domains is the technical problem of writing down a (globally) orthonormal basis for $L^2(\partial\Omega)$.

Of course, since we want a computational method, we must use a truncated approximation to the HS norm in (4.1). Since

$$\|\Lambda_\gamma - \Lambda_{\tilde{\gamma}}\|_{HS}^2 = \sum_\alpha \|(\Lambda_\gamma - \Lambda_{\tilde{\gamma}})f_\alpha\|_{L^2(\partial\Omega)}^2 \quad (4.4)$$

for every complete orthonormal set $\{f_\alpha\} \subset L^2(\partial\Omega)$, including the orthonormal set of eigenfunctions associated with the difference operator, a truncated approximation

yields either the conventional least-squares problem with fixed boundary data, or some variation of an “adaptive” formulation as studied by Gisser, Isaacson, and Newell [15], or Breckon and Pidcock [5]. Furthermore, in light of the rapid decay of the eigenvalues, a small number of boundary measurements should give a very good approximation to the Hilbert-Schmidt norm. As pointed out by D. Isaacson, if we assume that a fixed bounded noise level is present in the measurements, the signal-to-noise ratio deteriorates rapidly. Hence it is pointless to use boundary data with magnitude below the noise level when approximating (4.4).

By placing the $L^2(\Omega)$ norm on the solution space, we have created a new set of difficulties. In particular, as we saw in Chapter 3, we cannot guarantee that the forward map has sufficient regularity without imposing some additional smoothness restrictions on the solutions. Also, as was demonstrated in Chapter 2, the associated linear problems are not stable without some kind of regularization. However, by imposing smoothness constraints, we can stabilize the linearized problems and insure regularity of the forward map at the same time.

Let us first change variables to eliminate the requirement $\text{supp}(\gamma - \tilde{\gamma}) \subset \Omega'$. Since $\tilde{\gamma}$ is known on a neighborhood of the boundary, assume that we have obtained an extension $\gamma' \in C^{k,1}(\bar{\Omega})$ to $\tilde{\gamma}$ over Ω , i.e. $(\gamma' - \tilde{\gamma})|_{\Omega - \Omega'} = 0$, where $\Omega' \subset\subset \Omega$. For technical reasons which will become clear in the next section, we choose Ω' with a smooth boundary. Let $\rho \in L^2(\Omega')$, and make the change of variables $\gamma \mapsto \rho + \gamma'$ where the variation ρ is extended to Ω by zero.

For simplicity, let us approximate the Hilbert-Schmidt norm (4.4) with a single boundary measurement. We stress that this is only to make the notation simpler; all of the following holds with any finite sum approximating (4.4). As in Chapter 3, set $F(\gamma) = \Lambda_\gamma f$ where $f \in L^2(\partial\Omega)$ is fixed, with $\int_{\partial\Omega} f = 0$, and $\|f\| = 1$. Now take a measurement $g = F(\tilde{\gamma})$. Consider the Tikhonov regularized problem

$$\min_{\rho \in L^2(\Omega')} J_\beta(\rho) = \frac{1}{2} \|F(\rho + \gamma') - g\|_{L^2(\partial\Omega)}^2 + \frac{\beta}{2} \|B\rho\|_{L^2(\Omega')}^2 \quad (4.5)$$

where $\beta \in (0, \infty)$, and B is a possibly unbounded operator, defined on some subset of $L^2(\Omega')$.

The addition of the regularization term $\|B\rho\|^2$ constrains the admissible class of conductivities to the domain of B . By choosing B such that $\|\rho\|_{C^{k,1}(\bar{\Omega})}$ is controlled by $\|B\rho\|_{L^2(\Omega')}$, where $k > n/2 - 1$, we can effectively bound the conductivities $\gamma' + \rho$ in $C^{k,1}(\bar{\Omega})$, so that the regularity theorems of Chapter 3 apply. The construction of B is the topic of the next section.

4.2 Construction of a regularization operator

We will construct B in a somewhat roundabout fashion, for reasons to be explained shortly.

In [18], S. Jaffard and Y. Meyer construct orthonormal wavelet bases for $L^2(\Omega)$ (as opposed to the usual constructions for $L^2(\mathbb{R}^n)$). The wavelets have all the usual smoothness, localization, and vanishing moments properties, and they are shown to

naturally characterize functions in the Hölder spaces $C_0^r(\Omega)$ and the Sobolev spaces $H_0^s(\Omega)$. We refer the reader to [18] for details.

Let $\{\psi_\lambda\}_{\lambda \in \Lambda}$ be a wavelet basis of order m for $L^2(\Omega')$, as constructed by Jaffard and Meyer. This orthonormal set in $L^2(\Omega')$ has exponential localization properties in space, and each ψ_λ is in the class $C^{2m-2}(\Omega')$. We stick with the indexing notation in [18]. The index set is $\Lambda = \bigcup_{j=0}^{\infty} R_j$, where each R_j is a discrete set in Ω' which indexes location, and j denotes the scale.

Given a function $\rho \in L^2(\Omega')$, let α_λ denote the wavelet coefficients of ρ :

$$\alpha_\lambda = \int_{\Omega'} \rho(x) \psi_\lambda(x) dx.$$

Let $\{b_\lambda\}$ be a set of real numbers with $b_\lambda \geq 1$. With the set $\{b_\lambda\}$, we associate the operator B by defining

$$(B\rho)(x) = \sum_{\lambda \in \Lambda} b_\lambda \alpha_\lambda \psi_\lambda(x)$$

for all ρ for which the sum converges.

Lemma 4.2 Let $k \in \mathbb{N}$ be given. Assume that the wavelet basis $\{\psi_\lambda\}$ used in the construction of B is of order $m > k + n/4 + 3/2$, and that the coefficients $\{b_\lambda\}$ satisfy

$$b_\lambda \geq 2^{js}, \quad \lambda \in R_j, \quad j \in \mathbb{N}, \quad (4.6)$$

where $k < s - n/2 - 1 < 2k$. Then there is a constant $C = C(\Omega', k, m, s)$ such that if $\|B\rho\|_{L^2(\Omega')} < \infty$, then

$$\|\rho\|_{C^{k,1}(\Omega)} \leq C \|B\rho\|_{L^2(\Omega')},$$

when ρ is extended to $\bar{\Omega}$ by zero.

Proof Since $0 < s < 2m - 2$, the proof of Theorem 2 in [18] shows that

$$\|\rho\|_{H_0^s(\Omega')}^2 \leq C \sum_{j=0}^{\infty} \sum_{\lambda \in R_j} 4^{js} \alpha_\lambda^2 \leq \|B\rho\|_{L^2(\Omega')}^2. \quad (4.7)$$

Since $s > k + n/2 + 1$, we have the imbeddings

$$H_0^s(\Omega') \rightarrow C^{k+1}(\bar{\Omega}') \rightarrow C^{k,1}(\bar{\Omega}'). \quad (4.8)$$

The first imbedding follows from the Sobolev Imbedding Theorem, and the second comes from the fact that Ω' is bounded with smooth boundary (see Adams [1]).

Hence, combining (4.7) and (4.8),

$$\|\rho\|_{C^{k,1}(\bar{\Omega}')} \leq C \|B\rho\|_{L^2(\Omega')}.$$

Since $\rho \in H_0^s(\Omega')$, ρ and all derivatives are zero on $\partial\Omega'$, so that the extension by zero to Ω does not increase the $C^{k,1}(\bar{\Omega})$ norm. \square

The reason for this construction of B is that in practice there is evidence that the conditioning (stability) of the problem can be improved by reparametrizing the conductivities in terms of a wavelet basis. Chavent [8] has demonstrated this property for a similar coefficient determination problem. For the inverse conductivity problem one could use the freedom allowed by (4.6) to choose the weights b_λ larger for ψ_λ which are localized away from the boundary. This scheme forces ρ to be smoother away from the boundary, reflecting the limited maximum resolution described in Section 2.3.

Having constructed B , we now establish that the functional J_β defined in (4.5) has a minimizer whose $C^{k,1}(\bar{\Omega})$ norm does not blow up as β is increased.

Lemma 4.3 For all $\beta > 0$, the functional J_β has a minimizer ρ_β in the set

$$A = \{\rho \in L^2(\Omega') : \rho + \gamma' \geq a > 0\},$$

and $\rho_\beta \in C^{k,1}(\bar{\Omega})$. Furthermore, there is a constant M_1 such that for all β sufficiently large, $\|\rho_\beta\|_{C^{k,1}(\bar{\Omega})} \leq M_1$.

Proof The set $\{J_\beta(\rho) : \rho \in A\}$ is bounded below by zero, so it has an infimum which we denote l_β . Thus there is a sequence $\{\rho_i\} \subset A$ such that $J_\beta(\rho_i) \rightarrow l_\beta$. Given some number $M > l_\beta$, the proof of Lemma 4.2 shows that for all i sufficiently large,

$$\|\rho_i\|_{H_0^s(\Omega')}^2 \leq C \frac{\beta}{2} \|B\rho_i\|_{L^2(\Omega')}^2 \leq C J_\beta(\rho_i) \leq M. \quad (4.9)$$

Thus the sequence $\{\rho_i\}$ is contained in the set

$$A \cap \{\rho : \|\rho\|_{H_0^s(\Omega)} \leq M\}$$

which is weakly compact in $H_0^s(\Omega')$. So there is subsequence (which we still denote $\{\rho_i\}$) and an element $\rho_\beta \in H_0^s(\Omega')$ such that

$$\rho_i \rightharpoonup \rho_\beta, \text{ in } H_0^s(\Omega'),$$

with $\|\rho_\beta\|_{H_0^s(\Omega')} \leq M$, and $\rho \in A$. By the Rellich-Kondrachov Theorem, the imbedding $H_0^s(\Omega') \rightarrow L^2(\Omega')$ is compact, so $\rho_i \rightarrow \rho_\beta$ strongly in $L^2(\Omega')$. We note that the

hypotheses of Proposition 3.2 are satisfied, so J_β is continuous as a functional over $L^2(\Omega')$ in the sense that $\rho_i \rightarrow \rho_\beta$ implies $J_\beta(\rho_i) \rightarrow J_\beta(\rho_\beta) = l_\beta$. Thus we have established that ρ_β minimizes J_β . The fact that $\rho_\beta \in C^{k,1}(\bar{\Omega})$ follows from the inequality (4.9) and the imbedding (4.8).

To prove the second assertion, let us assume that the negation is true. Then there is a sequence of real numbers $\{\beta_i\}$ with $\beta_i \rightarrow \infty$, such that $\|B\rho_{\beta_i}\|_{L^2(\Omega')} \rightarrow \infty$. Then necessarily

$$J_{\beta_i}(\rho_{\beta_i}) \rightarrow \infty. \quad (4.10)$$

Let $\rho_0 \equiv 0$. Since $B\rho_0 = 0$, we have $J_{\beta_i}(\rho_0) = C$ for all i . From (4.10), there exists an index j such that $J_{\beta_j}(\rho_0) < J_{\beta_j}(\rho_{\beta_j})$, contradicting the fact that ρ_{β_j} minimizes J_{β_j} .

□

4.3 The Gauss-Newton method

In this section, we will present the Gauss-Newton method for solving problem (4.5). Each algorithm for solving a particular problem has advantages and disadvantages. Various methods for solving nonlinear least-squares problems, detailed discussion of the advantages and disadvantages of each method, and convergence analyses are given in Dennis and Schnabel, [12], Chapter 10. We have chosen a globalized version of the Gauss-Newton method, the Levenberg-Marquardt method.

Let us begin with a completely formal development of the Gauss-Newton method.

We differentiate $J_\beta(\rho)$ in the direction $\delta\rho$ to get

$$DJ_\beta(\rho)(\delta\rho) = \langle DF(\rho + \gamma')^*(F(\rho + \gamma') - g) + \beta B^* B \rho, \delta\rho \rangle_{L^2(\Omega)},$$

where $DF(\gamma)^*$ is defined in (3.22). Thus the “gradient” of J_β at ρ is the $L^2(\Omega)$ function

$$G(\rho) = DF(\rho + \gamma')^*(F(\rho + \gamma') - g) + \beta B^* B \rho.$$

We denote the bilinear second derivative operator for F at γ by $D^2F(\gamma)(\cdot, \cdot)$.

Formally,

$$D^2F(\gamma)(\delta\gamma, \delta\gamma) = \delta^2u|_{\partial\Omega},$$

where δ^2u solves the perturbational equation

$$\begin{aligned} \nabla \cdot \gamma \nabla \delta^2u &= -\nabla \cdot \delta\gamma \nabla \delta u \quad \text{in } \Omega \\ \gamma \frac{\partial \delta^2u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Taking the second derivative of J_β in the direction $\delta\rho$ and applying the chain rule for differentiation,

$$\begin{aligned} D^2J_\beta(\rho)(\delta\rho, \delta\rho) &= \langle D[DF(\rho + \gamma')^*(F(\rho + \gamma') - g) + \beta B^* B \rho](\delta\rho), \delta\rho \rangle_{L^2(\Omega')} \\ &= \langle DF(\rho + \gamma')^* DF(\rho + \gamma')(\delta\rho) \\ &\quad + D^2F(\rho + \gamma')^*(F(\rho + \gamma') - g, \delta\rho) + \beta B^* B \delta\rho, \delta\rho \rangle_{L^2(\Omega')}, \end{aligned}$$

where $D^2F(\gamma)^*$ is the bilinear form on $L^2(\partial\Omega) \times L^2(\Omega')$ which satisfies

$$\langle \phi, D^2F(\gamma)(\delta\rho, \delta\rho) \rangle_{L^2(\partial\Omega)} = \langle D^2F(\gamma)^*(\phi, \delta\rho), \delta\rho \rangle_{L^2(\Omega)}$$

for all $\phi \in L^2(\partial\Omega)$ with $\int \phi = 0$. We refer to the linear operator defined by

$$H_F(\rho)(\delta\rho) = \frac{1}{2}(DF(\rho+\gamma')^*DF(\rho+\gamma')(\delta\rho) + D^2F(\rho+\gamma')^*(F(\rho+\gamma') - g, \delta\rho) + \beta B^*B\delta\rho)$$

as the *Hessian*. The Hessian supplies the second derivative information for the functional J_β . Writing J_β in a truncated Taylor series expansion about ρ gives a quadratic model of J_β near ρ :

$$J_\beta(\rho + \delta\rho) \approx J_\beta(\rho) + DJ_\beta(\rho)(\delta\rho) + \frac{1}{2}D^2J_\beta(\rho)(\delta\rho, \delta\rho).$$

Newton's method is an iterative procedure which at each step selects the $\delta\rho$ which minimizes the quadratic model. This is equivalent to solving the corresponding linear equation

$$H_F(\rho)(\delta\rho) = -G(\rho). \quad (4.11)$$

In the hope that either $F(\rho + \gamma') - g$ is small, or the problem is nearly linear so that $DF^2(\gamma)^*$ itself is small, we discard the second order derivative term in H_F .

Denote

$$H(\rho)(\delta\rho) = DF(\rho + \gamma')^*DF(\rho + \gamma')(\delta\rho) + \beta B^*B\delta\rho. \quad (4.12)$$

We will refer to H as the *Gauss-Newton Hessian* operator. Given an initial estimate ρ_0 of the solution to (4.5), and assuming that $H(\rho)(\cdot)$ is invertible for each k , the Gauss-Newton method is the iteration

$$\rho_{k+1} = \rho_k - H(\rho_k)^{-1}G(\rho_k), \quad (4.13)$$

which is continued until convergence.

As an aside, we remark that formally, at least, it is easy to calculate all the derivatives above with well-known “adjoint state” techniques. In fact, assuming that the linear system (4.11) is solved by some iterative technique such as the conjugate gradient method or the conjugate residual method, this can be a useful computational tool. The simplicity of the $L^2(\partial\Omega) \rightarrow L^2(\Omega)$ adjoint is a good reason for using the $L^2(\Omega)$ norm on the conductivities. We have already computed

$$DF(\gamma)^*(\phi) = -\nabla u \cdot \nabla u^*$$

in (3.23). We can make a similar integration by parts computation to find the adjoint second derivative

$$D^2F(\gamma)^*(\phi, \delta\rho) = -\nabla \delta u \cdot \nabla u^*,$$

where u^* is exactly the same function as above. Since computation of the Gauss-Newton Hessian (4.12) by this technique requires solving partial differential equations to find u , u^* , and δu , we see that all the components in the second derivative term in the full Newton Hessian H_F have already been calculated. Thus H_F is essentially no more expensive to compute than H .

There are a number of potential problems with the Gauss-Newton method as we have described it. Of course, it is necessary to establish conditions under which the iteration (4.13) is well-defined and convergence can be guaranteed. These problems will be addressed in the next section. It turns out that a “globalization strategy” is necessary to establish even the local convergence of the Gauss-Newton method on

problem (4.5). We describe a globalization strategy which gives rise to a method commonly known as the *Levenberg-Marquardt* method.

The Levenberg-Marquardt method may be viewed as a trust-region method, where at each iteration, the linear subproblem is constrained so that the step $\delta\rho$ must lie inside a ball of a certain radius. The radius is chosen at each step on the basis of how much the quadratic model is “trusted” as an accurate description of the behavior of J_β . This is equivalent to replacing the iteration (4.13) with

$$\rho_{k+1} = \rho_k - (H(\rho_k) + \mu_k I)^{-1} G(\rho_k), \quad (4.14)$$

where $\mu_k = 0$ if the Gauss-Newton step lies inside the trust-region, and $\mu_k > 0$ otherwise. (Larger μ_k corresponds to a smaller trust-region radius.) If the trust-region radius is chosen properly at each step, one can show under minimal assumptions that the resulting algorithm is globally convergent; that is, it will converge to a local minimizer from any starting point. An adequate description of trust-region globalization strategies is far beyond the scope of this outline. See Dennis and Schnabel [12] for a complete discussion. The key fact we will use is that if J_β is differentiable in any reasonable sense, the trust-region can be chosen so that $J_\beta(\rho_{k+1}) \leq J_\beta(\rho_k)$ at each step k . In fact, most trust-region strategies insist on much stronger decrease conditions.

4.4 Convergence

There are a few problems standing in the way of a simple proof of the local convergence of the Levenberg-Marquardt method for problem (4.5).

The most serious problem is this: The constants in the regularity bounds in Theorem 3.1, Theorem 3.2, and the bound on DF in Proposition 3.1, which are crucial in establishing convergence, all depend upon the $C^{k,1}(\bar{\Omega})$ norms of the conductivities $\rho + \gamma'$, as well as the bound $\rho + \gamma' \geq a > 0$. To guarantee convergence, we must be able to maintain absolute bounds on $\|\rho_i + \gamma'\|_{C^{k,1}(\bar{\Omega})}$ and $\inf(\rho_i + \gamma')$ which hold for all iterates ρ_i , lest the regularity constants blow up. The following two assumptions allow for sufficient control.

Assumption 1 There is a constant $a > 0$ such that the sequence of iterates $\{\rho_i\}$ satisfies $\rho_i + \gamma' \geq a > 0$ in Ω .

Limited computational experience has indicated that this is a reasonable assumption. In fact, it is possible to remove the constraint by making the change of variables $\gamma \rightarrow \log(\gamma)$. In any case it is easy to test whether or not Assumption 1 has been violated during the course of the computation.

Assumption 2 The globalization strategy generates a sequence $\{\mu_i\}$ satisfying $\mu_i \leq b$ for some fixed b , which results in a non-increasing sequence of functional values $J_\beta(\rho_{i+1}) \leq J_\beta(\rho_i)$.

As noted in the last section, most trust-region strategies insist on much stronger decrease conditions than required by Assumption 2. As with the first assumption, it is easy to check whether Assumption 2 has been violated during the course of the computation.

The other problem, as discussed in [12], is that a crucial constant arises in the proof which is a combined relative measure of the nonlinearity and the residual size of the problem. This constant must be strictly less than one to guarantee local convergence. In the case of problem (4.5), the constant is essentially the quotient of the regularity bound $C_3 = (C + K\|F(\rho_\beta + \gamma') - g\|)$ from Theorem 3.2 and the smallest eigenvalue of the Hessian $H(\rho)$, which in this case is β . Increasing β tightens control on the $C^{k,1}(\bar{\Omega})$ norm of the coefficients, and hence should make C and K smaller. But at the same time the residual $\|F(\rho_\beta + \gamma') - g\|$ may increase. Fortunately, since the size of ρ_β is bounded by Lemma 4.3, and F is a continuous function, the size of $\|F(\rho_\beta + \gamma') - g\|$ is easy to bound.

Lemma 4.4 Given $N > 0$, there is a $\beta > 0$ such that for all $\rho \in L^2(\Omega')$ with $\|\rho\|_{C^{k,1}(\bar{\Omega})} \leq N$, and for some local minimizer ρ_β of J_β , we have $\beta > C_3 + b$, where

$$C_3 = C(\Omega', a, k, N, \|\rho_\beta + \gamma'\|_{C^{k,1}(\bar{\Omega})}) \\ + K(\Omega', a, k, \|\rho_\beta + \gamma'\|_{C^{k,1}(\bar{\Omega})})\|F(\rho_\beta + \gamma') - g\|_{L^2(\partial\Omega)}$$

is the constant appearing in Theorem 3.2 with $\gamma = \rho_\beta + \gamma'$ and with $\phi = F(\rho_\beta + \gamma') - g$. Here b is the constant from Assumption 2.

Proof From Lemma 4.3, we have $\|\rho_\beta\|_{C^{k,1}(\bar{\Omega})} \leq M_1$ for all β sufficiently large. By the triangle inequality, for all β sufficiently large,

$$\|\rho_\beta + \gamma'\|_{C^{k,1}(\bar{\Omega})} \leq M_2 \tag{4.15}$$

for some constant M_2 . Since $\rho_\beta + \gamma' \geq a > 0$ for all β , Proposition 3.2 yields

$$\|F(\rho_\beta + \gamma') - g\|_{L^2(\partial\Omega)} \leq C\|\rho_\beta + \gamma' - \tilde{\gamma}\|_{L^2(\Omega')}.$$

Thus by (4.15) and the triangle inequality, there is a constant M_3 such that

$$\|F(\rho_\beta + \gamma') - g\|_{L^2(\partial\Omega)} \leq M_3$$

for all β sufficiently large. Hence

$$C_3 = C(\Omega', a, k, N, M_2) + K(\Omega', a, k, M_2)M_3$$

is independent of β for β sufficiently large, so β can be chosen such that $\beta > C_3 + b$.

□

It is now relatively straightforward to prove the local convergence of the iterates in $L^2(\Omega')$ in the following sense.

Theorem 4.2 Under Assumptions 1,2, for every $M > 0$, there exists numbers $\epsilon, \beta, N > 0$, such that for every ρ_0 with

1. $\|\rho_0 - \rho_\beta\|_{L^2(\partial\Omega)} < \epsilon$,
2. $\|B\rho_0\|_{L^2(\Omega')} \leq M$,
3. $\rho_0 + \gamma' \geq a > 0$,

the sequence $\{\rho_i\}$ generated by the iteration (4.14) is well-defined, satisfies

$\|\rho_i + \gamma'\|_{C^{k,1}(\bar{\Omega})} \leq N$, and converges in $L^2(\Omega')$ to ρ_β at a q-linear rate. If

$F(\rho_\beta + \gamma') - g = 0$, and $\mu_k = O(\|DF(\rho_i + \gamma')^*(F(\rho_i + \gamma') - g)\|_{L^2(\Omega')})$,

then the convergence rate is q-quadratic.

Thus, for all starting points with bounded $C^{k,1}(\bar{\Omega})$ norm, the problem can be regularized enough to guarantee local convergence. We remark that in general, a smaller bound M on the $C^{k,1}(\bar{\Omega})$ norm of the initial iterate ρ_0 results in larger ϵ and smaller β and N , but only if $M > M_1$, where M_1 is given by Lemma 4.3.

Proof Remembering that $\rho + \gamma' \geq a > 0$, let

$$S = \sup_{\|B\rho\|_{L^2(\Omega')} \leq M} \|F(\rho + \gamma') - g\|_{L^2(\partial\Omega)}.$$

From Lemma 4.2 and the proof of Lemma 4.4, it is easy to see that $S < \infty$. Now choose β as given by Lemma 4.4, with $N = C(M^2 + S^2)^{1/2}$, where C is the constant given by Lemma 4.2. From Assumption 2, for $i \in \mathbb{N}$ we have

$$\frac{\beta}{2} \|B\rho_i\|^2 \leq J_\beta(\rho_i) \leq J_\beta(\rho_0) = \frac{1}{2} \|F(\rho_0 + \gamma') - g\|^2 + \frac{\beta}{2} \|B\rho_0\|^2,$$

so that Lemma 4.2 yields

$$\|\rho_i\|_{C^{k,1}(\bar{\Omega})} \leq C \|B\rho_i\|_{L^2(\Omega')} \leq C(\|B\rho_0\|^2 + \frac{1}{\beta} \|F(\rho_0 + \gamma') - g\|^2)^{1/2} \leq N, \quad (4.16)$$

which holds for all $i \in \mathbb{N}$.

We can now follow the structure of the proof of the local convergence of the Gauss-Newton method given in [12]. For every $\beta > 0$, since $DF(\gamma)^*DF(\gamma)$ is a nonnegative operator, and βB^*B satisfies $\beta \|B^*B\delta\rho\|_{L^2(\Omega')} \geq \|\delta\rho\|_{L^2(\Omega')}$ for all $\delta\rho \in \mathcal{D}(B^*B)$, the operator $H(\rho)^{-1}(\cdot)$ exists and

$$\|H(\rho)^{-1}\| \leq \frac{1}{\beta}. \quad (4.17)$$

Let us denote $H(\rho_k)(\cdot) = H_k(\cdot)$, $F(\rho_k + \gamma') = F_k$, and similarly for DF , DF^* , and G , even when $k = \beta$. We proceed by induction. We have

$$\begin{aligned}
\rho_\beta - \rho_1 &= \rho_\beta - \rho_0 - H_0^{-1}G_0 \\
&= -H_0^{-1}[G_0 + H_0(\rho_\beta - \rho_0)] \\
&= -H_0^{-1}[DF_0^*(F_0 - g) + \beta B^*B\rho_0 + DF_0^*DF_0(\rho_\beta - \rho_0) \\
&\quad + \beta B^*B(\rho_\beta - \rho_0) + \mu_0(\rho_\beta - \rho_0)] \\
&= -H_0^{-1}[DF_0^*(F_\beta - g) - DF_0^*(F_\beta - F_0 - DF_0(\rho_\beta - \rho_0)) \\
&\quad + \beta B^*B\rho_\beta + \mu_0(\rho_\beta - \rho_0)].
\end{aligned}$$

From Assumption 1 and (4.16) we see that all the hypotheses of Theorem 3.1 and Proposition 3.1 are satisfied. Let us denote the constant in Theorem 3.1 by C_2 , and the constant in Proposition 3.1 by C_1 . Application of (4.17), (3.16), and (3.13) yield

$$\begin{aligned}
\|\rho_\beta - \rho_1\| &\leq \frac{1}{\beta} \|DF_0^*(F_\beta - g) + \beta B^*B\rho_\beta \\
&\quad + DF_0^*(F_\beta - F_0 - DF_0(\rho_\beta - \rho_0)) + \mu_0(\rho_\beta - \rho_0)\| \\
&\leq \frac{1}{\beta} [\|DF_0^*(F_\beta - g) + \beta B^*B\rho_\beta\| \\
&\quad + \|DF_0^*\|_{L^2(\partial\Omega), L^2(\Omega')} \|F_\beta - F_0 - DF_0(\rho_\beta - \rho_0)\| + \mu_0 \|\rho_\beta - \rho_0\|] \\
&\leq \frac{1}{\beta} [\|DF_0^*(F_\beta - g) + \beta B^*B\rho_\beta\| + C_1 C_2 \|\rho_\beta - \rho_0\|^2 + \mu_0 \|\rho_\beta - \rho_0\|],
\end{aligned}$$

where $\|\cdot\| = \|\cdot\|_{L^2(\Omega')}$. It is easy to check that over $\mathcal{D}(B^*B)$, J_β is Gateaux differentiable, and $DJ_\beta(\rho_\beta)(\delta\rho) = \langle G(\rho_\beta), \delta\rho \rangle_{L^2(\Omega')}$. From the proof of Lemma 4.2, we see that $H_0^s(\Omega) \subset \mathcal{D}(B^*B)$, so $\mathcal{D}(B^*B)$ is a dense, convex subset of $L^2(\Omega)$. Since

ρ_β is a local minimizer of J_β , the variational inequality $DJ_\beta(\rho_\beta)(\delta\rho) = 0$ implies that

$$G(\rho_\beta) = DF_\beta^*(F_\beta - g) + \beta B^* B \rho_\beta = 0.$$

Hence we have

$$DF_0^*(F_\beta - g) + \beta B^* B \rho_\beta = [DF_0^* - DF_\beta^*](F_\beta - g). \quad (4.18)$$

All the hypotheses of Theorem 3.2 are also satisfied, so we can estimate (4.18) with (3.24) to obtain

$$\|\rho_\beta - \rho_1\| \leq \frac{1}{\beta} [(C_3 + \mu_0)\|\rho_\beta - \rho_0\| + C_1 C_2 \|\rho_\beta - \rho_0\|^2] \quad (4.19)$$

$$\leq \frac{1}{\beta} (C_3 + b + C_1 C_2 \epsilon) \|\rho_\beta - \rho_0\|. \quad (4.20)$$

By Lemma 4.4, $\beta > C_3 + b$, so let $\delta = \frac{C_3 + b}{\beta} < 1$. Then we can choose ϵ with $0 < \epsilon < (1 - \delta)/C_1 C_2$ so that the constant in (4.20) is strictly less than one, and the q-linear convergence follows. To see that the convergence rate is q-quadratic when $F(\rho_\beta + \gamma') - g = 0$, and $\mu_k = O(\|DF(\rho_i + \gamma')^*(F(\rho_i + \gamma') - g)\|_{L^2(\Omega')})$, we note that in this case (4.18) is zero, so we can choose $C_3 = 0$. The q-quadratic convergence rate then follows from (4.19). \square

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