

Layered Velocity Inversion:
A Model Problem from
Reflection Seismology

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Abstract

A simple model problem in exploration seismology requires that a depth-varying sound velocity distribution be estimated from reflected sound waves. For various physical reasons, these reflected signals or echoes have very small Fourier coefficients at both very high and very low frequencies. Nonetheless, both geophysical practice, based on heuristic considerations, and recent numerical evidence indicate that a spectrally complete estimate of the velocity distribution is often achievable. We prove a theorem to this effect, showing that “sufficiently rough” velocity distributions may be recovered from reflected waves under some restrictions, independently of the very low- or high-frequency content of the data. The main restriction is that the velocity depend only on a single (depth) variable; only in this case are sufficiently refined propagation-of-singularity results available. The proof is based on a novel variational principle, from which numerical algorithms have been derived. These algorithms have been implemented and used to estimate velocity distributions from both synthetic and field reflection seismograms.

1 Introduction.

A simple model of the physical setting for reflection seismology is constant-density *linear acoustics*, in which the sound velocity field $c(x)$ ($x \in \mathbf{R}^3$) is connected to the pressure field $u(x, t)$ *via* the wave equation:

$$\frac{1}{c^2(x)} \frac{\partial^2 u(x, t)}{\partial t^2} - \Delta u(x, t) = f(t)\delta(x)$$
$$u \equiv 0, \quad t < 0$$

The right-hand side represents an isotropic point dilatational energy source radiating with time varying (transient) intensity $f(t)$ (“the *source wavelet*”). The *seismogram* is a sampling of the pressure u at a number of “receiver” points. We adopt the idealization that these points form the continuum $\{x_3 =: z = 0\}$ (“the surface (of the earth)”) and that the measurement of u is also continuous in time for some time interval $0 \leq t \leq t_{\max}$. Regarding the source (i.e. $f(t)$) as known, the pressure field, hence the seismogram, becomes a function of the sound velocity:

$$s[c] := u|_{z=0}$$

In this simple model, the fundamental problem of reflection seismology is to estimate c from $s[c]$, i.e. to solve a functional equation of the form

$$s[c] = s_{\text{data}}$$

possibly in some least-error sense accommodating the possibility (virtual certainty!) of inconsistent data error.

This model is grossly inadequate for some practical purposes, as it ignores significant physics of seismic wave generation and propagation. Nonetheless, it forms the basis for most contemporary seismic data processing (see for instance Yilmaz (1987)), and it exhibits two fundamental features of the “real” problem:

- (i) $s[c]$ is nonlinear in c ;
- (ii) f should be chosen to suppress Fourier components in $s[c]$ at “low” and “high” frequencies.

Item (i) is simply the nonlinearity of solutions of linear equations as functions of their coefficients. Item (ii) is required by observations of the spectra of reflection seismograms: for various physical reasons, Fourier components at very low ($< 4\text{Hz}$) and very high ($> 80\text{ Hz}$) temporal frequencies are essentially missing from real reflection seismograms.

The suppression of high-frequency components simply means that $c \mapsto s[c]$ is a smoothing operator. Techniques for management of the resulting high-frequency instability are well-known — see e.g. Tihonov and Arsenin (1976), Miller (1970), Payne (1974).

In contrast, the instability resulting from the lack of low-frequency data has been little discussed in the mathematical literature on inverse problems, even though it is nearly ubiquitous in real world parameter estimation problems based on wave propagation. This *low-frequency lacuna* is a striking feature of reflection seismology, in particular, and the possible ambiguities resulting from spectral incompleteness of data have sparked considerable discussion within the geophysical research community.

The present paper is devoted to the proof of a uniqueness and continuous dependence result for a restricted version of the inverse problem described above, in which the estimates are independent of the behaviour of $\hat{f}(\omega)$ near $\omega = 0$. We shall show that c is well-determined by $s[c]$ when c is *sufficiently non-smooth*.

This rather strange sounding requirement is natural in view of the application to reflection seismology: rapid changes in the mechanical properties of rock are entirely responsible for the return of substantial echoes to the surface, hence for the information content of seismic reflection data. The times of arrival of these echoes — or rather, the signals which simulate them in the model described above — carry information about the slowly varying components of c , completely independently of the low-frequency behavior of f . On the other hand, the identifiability of these echoes depends on the wave nature of the seismic disturbance, in other words on the propagation of singularity (or regularity), according to geometric optics. This propagation property clearly requires some smoothness of the coefficients; thus the conditions necessary for the creation of strong reflected signals are in tension with those necessary for their propagation.

We show in this paper that this tension can be resolved at least in a special

case. The resolution requires certain estimates concerning propagation of regularity currently available only under the additional constraint:

The sound velocity c is a function only of the “depth” variable $x_3(= z)$.

That is, the result detailed here applies to layered fluid models only. The necessary technical results for this class of models were established in previous papers of the author (Symes 1981, 1983, 1986a, b). These are essentially energy estimates and are related to earlier work of Rauch and Taylor (1974) and Kreiss (1967) on mixed problems for linear hyperbolic systems in two independent variables. Other authors basing results about 1-D inverse problems for hyperbolic equations on the same ideas include Fawcett (1984), Suzuki (1988).

A few remarks concerning the prospects for weakening the layered medium assumption in these arguments may be found in the next section.

The paper is organized as follows: Section 2 gives a precise statement of the main results, a brief review of the literature, and discussion of related issues. Section 3 introduces the *plane-wave decomposition* and estimates for the plane-wave problems from the author’s previous work; this material forms the technical basis for the rest of the paper. As we are concerned mostly with the information independent of the low-frequency behaviour of f , we introduce in Section 4 the (temporary) assumption that f is a compactly supported measure, defining an elliptic convolution operator bounded on $L^2(\mathbf{R})$. Under this assumption, we prove an estimate of Gårding type for the derivative of the plane-wave seismogram map. In order to do more, it is necessary to consider the various plane-wave problems simultaneously. Each plane-wave *model* is parameterized by the vertical plane-wave velocity, viewed as a function of (its own) travel-time. These models are *a priori* independent. We derive a *coherency condition* in Section 5, equivalent to the existence of a (single) velocity profile $c(z)$ from which all of the plane-wave models are derived (i.e. “every (plane wave) experiment sees the same earth”). In Section 6 we show that a least-squares version of the inverse problem stated above, posed in terms of the plane-wave models and augmented by the coherency condition (as a so-called penalty term) has a positive-definite Hessian (second derivative) operator at a consistent (zero-residual) solution provided that the

corresponding velocity profile is sufficiently rough, as stated above. Our main result follows immediately *via* the implicit function theorem. So far we have maintained the elliptic assumption concerning f ; this is dropped in Section 7, for the usual price, paid for the solution of compact operator equations, of *a priori* constraints on the smoothness of c .

2 Statement of Main Result, Discussion.

In this section we give a precise statement of the major result of this paper, followed by a brief review of the literature and a conceptual overview of the problem described above.

Since the principal goal of the present work is the production of a “solution” of the inverse problem stated above with continuity properties independent of the *low-frequency* behavior of the source wavelet f , we make the temporary assumption that:

$f \in \mathcal{E}'(\mathbf{R})$ is a Borel measure, defining a bounded elliptic convolution operator on $H^1(\mathbf{R})$: i.e. for positive K_0, K_1, K^* , $\phi \in H^1(\mathbf{R})$

$$\begin{aligned} K_1 \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(\mathbf{R})} - K_0 \|\phi\|_{L^2(\mathbf{R})} \\ \leq \left\| f * \frac{\partial \phi}{\partial t} \right\|_{L^2(\mathbf{R})} \leq K^* \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(\mathbf{R})}. \end{aligned} \quad (2.1)$$

Also, $\text{supp } f \subset \{t \in \mathbf{R} : t \geq 0\}$.

Such a distribution necessarily has a finite *first moment*

$$m_f^1 = \sup_{\phi \in C^\infty} \frac{\langle f, t\phi \rangle}{\|\phi\|_{L^\infty}}$$

The “elliptic” assumption (2.1) will be weakened, to the extent possible, in Section 7. The simplest example of wavelets (i.e. kernels) f having the elliptic property (2.1) are obtained by subtracting from a slightly shifted Dirac delta function a smooth approximation.

The conventional, though perhaps ill-founded, choice of measure for the seismogram error is some weighted version of the L^2 -norm — see Tarantola (1987), Chapter 6. We shall adopt this choice also. The elliptic nature of f precludes square-integrability of $s[c]$, however, as can easily be seen, so we choose what amounts to a very singular “weight”: we define, for suitably small *slowness* p ,

$$S[c](t, p) = \int \int dx_1 dx_2 \frac{\partial}{\partial t} s[c](x_1, x_2, t + px_1)$$

That is, S is a version of the Radon-transform of s , which we shall further restrict to a rectangle $\{(t, p): 0 < t \leq T_1, P_1 \leq p \leq P_2\} =: R_1$. It is easily seen that, for smooth c , any $T_1 > 0$, and P_2 sufficiently small, S is square-integrable (Santosa and Symes (1988)). Thus the basic data set of this paper will be a member of $L^2(R_1)$.

Another piece of notation needed to state our main result is used in our method of measuring “roughness”: as mentioned in the introduction, a stable solution of the inverse problem can only be expected for sufficiently non-smooth coefficient c .

The “roughness” measure depends on an arbitrary Dirac kernel $h_1 \in C_0^\infty(\mathbf{R})$ satisfying

$$h_1 \geq 0, \quad h_1(0) > 0, \quad \int h_1 = 1$$

Set

$$h_\epsilon(z) = \frac{1}{\epsilon} h_1\left(\frac{z}{\epsilon}\right)$$

For $Z_0, \epsilon, \Delta > 0$, $c \in H_{\text{loc}}^1$, define

$$\begin{aligned} \bar{r}[c](Z_0, \Delta) &= \sup_{0 \leq z \leq Z_0} \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |c'|^2 \\ r_*[c](Z_0, \epsilon, \Delta) &= \inf_{0 \leq z \leq Z_0} \frac{\epsilon^2}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |h'_\epsilon * c'|^2 \\ r^*[c](Z_0, \epsilon, \Delta) &= \sup_{0 \leq z \leq Z_0} \frac{\epsilon^2}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |h'_\epsilon * c'|^2 \end{aligned}$$

These are local average measures of fluctuation. For example, r_* is a sizable fraction of \bar{r} , r^* when c has significant Fourier components at frequencies proportional to $\frac{1}{\epsilon}$, locally uniformly on the length scale Δ .

The geometry of the plane-wave problem which will occupy most of this paper is determined by the travel-time function

$$\tau(z, p) = 2 \int_0^z \sqrt{\frac{1}{c^2} - p^2}$$

which gives the time necessary for a point on a planar wavefront at slowness p , fixed horizontal coordinates, to travel to depth z and back to the surface.

The earlier results of the author (Symes 1986a,b) imply that S extends to a bounded, continuous map on the bounded set $\Sigma_c \subset H_{\text{loc}}^1(\mathbf{R})$ parameterized by positive numbers T_0, P_1, c_0, c_1, c^* , according to

$$\begin{aligned} \Sigma_c = \{ & c \in H_{\text{loc}}^1(\mathbf{R}) : c(z) = c_0, z < 0; \text{ for} \\ & Z_0 > 0 \text{ so that } T_0 = \tau(Z_0, P_1), \\ & \|\log c\|_{H^1[0, Z_0]} \leq c^*; \\ & c(z) = c_1 \text{ for } z \geq Z_0\} \end{aligned}$$

We have also shown, however, that this extension is *not* locally Lipschitz-continuous, and certainly not differentiable, in these metrics. S does become differentiable when the domain is metricized more strictly (H^3), but then the derivative fails to have a lower bound. Thus the implicit function theorem does not apply to the solution of least-squares problems for S . The computational consequences of this pathology are also striking (Santosa and Symes 1986).

A suitable family of “rough” subsets Σ'_c of Σ_c depends on positive parameters $M_1, M_2, \bar{\epsilon}$ and $\bar{\Delta}$ according to

$$\begin{aligned} \Sigma'_c = \{ & c \in \Sigma_c : \text{for } Z_0 > 0 \text{ such that} \\ & T_0 = \tau(Z_0, P_1) \text{ and some } 0 < \epsilon \leq \bar{\epsilon}, \quad 0 < \Delta \leq \bar{\Delta}, \\ & \text{the following inequalities hold:} \\ & M_1 \leq r_*(Z_0, \epsilon, \Delta), \\ & M_2 r_*(Z_0, \epsilon, \Delta) \geq \max(\bar{r}(Z_0, \Delta), r^*(Z_0, \epsilon, \Delta))\} \end{aligned}$$

We shall verify that Σ'_c is nonempty for suitable choices of parameters.

The main result of our paper is:

Theorem 1 *Suppose that $0, T_0 < T_1, 0 \leq P_1 < P_2, 0 \leq K_0, 0 < K_1 \leq K^*$ $0 < c_0, c_1, c^*$ are given. Then there exist constants $M_1, M_2, \bar{\epsilon}, \bar{\Delta}, \bar{m}$, and L^* depending on $T_0, T_1, P_1, P_2, K_0, K_1, K^*, c_0, c_1$, and c^* so that if $f \in \mathcal{E}'(\mathbf{R})$ satisfies (2.1) and*

$$m_f^1 \leq \bar{m}$$

then Σ'_c is nonempty and there exists an open neighborhood U of the set

$$\{S[c] \in L^2(R_1) : c \in \Sigma'_c\}$$

and a map

$$I : U \longrightarrow L^2_{\text{loc}}(\mathbf{R})$$

so that

- (i) for $c \in \Sigma'_c$, $IS[c] = c$;
- (ii) for $D_1, D_2 \in U$, $Z_1 > 0$

$$\begin{aligned} & \|I(D_1) - I(D_2)\|_{L^2[0, Z_1]} \\ & \leq L^* \|D_1 - D_2\|_{L^2(R_1)} \end{aligned}$$

Thus we obtain a continuous left-inverse for S , under various constraints. The requirement that c be constant for large z is simply a way of controlling c at depths below the zone influencing the seismogram. That the zone of constancy begins at Z_0 such that $T_0 = \tau(Z_0, P_1)$, rather than $T_1 = \tau(Z_0, P_1)$, is an unfortunate side-effect of the “width” of the source wavelet f : since $\text{supp } f$ is not a point, the depth interval in which the seismogram gives sure control over the velocity coefficient is strictly smaller than the depth interval needed to compute the seismogram. Given an arbitrary velocity profile with mean c_1 near $z = Z_0$, one can of course truncate it to a member of Σ_c , i.e. by setting the velocity constant ($= c_1$) for $z > Z_0$, as in the definition. The corresponding seismograms are then different only in the “gap” ($T_0 \leq t \leq T_1$). If the original profile obeys the uniform roughness conditions as in the definition of Σ'_c , then it follows from arguments similar to those in Sections 3 and 4 that the L^2 -norm of the difference of seismograms is $O(m_j^1)$. The theorem then gives the same qualitative estimate for the error due to application of I . We leave to the reader the formulation of a theorem embodying this extension of our results.

A more subtle consequence of this gap is that the value of c in this “basement” region must be specified *a priori*, i.e. the condition that $c(z) = c_1$ for $z \geq Z_0$. It seems clear that this additional piece of data should have little influence on the values of c at shallower depths — and, to the extent that it does, should be determined by S as well. This may be a fruitful subject for

further work; some related ideas are discussed in Sacks and Santosa (1987). In any case the author does not see at present how to formulate a convenient theorem without such a restriction.

It is easy to see that the Lipschitz estimate (ii) cannot be strengthened much. See Symes (1986b) for instance. In particular it is not possible to replace L^2 by H^1 on the left-hand side.

Estimates of the sort presented in *Theorem 1* are only of qualitative importance. Numerical evidence (Symes 1988b,c,d) indicates that typical values of the Lipschitz constant L^* are very large. On the other hand, restriction to a submanifold of Σ'_c of small codimension diminishes L^* to a useful magnitude, while leaving enough freedom in the model that some information about c is still obtained from the data. Analysis of an approximation to S in Symes 1988a,b illustrates this feature. A full understanding of the need for this “residual regularization” is still lacking at this writing.

The significance of the present result lies in its reliance on the implicit function theorem: i.e. the stability follows directly from linearization stability, and any residual ill-conditioning can be improved by the straightforward addition of linear constraints. This is surprising — and, perhaps, of practical importance — since the implicit function theorem cannot be applied to S directly, as noted above, for elliptic f .

The left inverse I will be produced *via* the solution of an auxiliary least-squares problem, developed in Section 5. In Section 7 we remove the elliptic requirement on f , to a certain extent: for suitable $f \in C_0^\infty$, we obtain an approximate left inverse to replace I , which depends on a choice of regularization.

We conclude this section with a brief review of the literature and of the background of the present approach. Recent papers treating inverse problems for hyperbolic equations include Blagovaschenski (1974), Bamberger *et al.* (1977, 1979), Bube and Burridge (1983), Burridge (1980), Carroll and Santosa (1982), Santosa and Schwetlick (1982), Bruckstein *et al.* (1983), Ramm (1987), Fawcett (1984), Symes (1986a, b), and Suzuki (1988). (This list is by no means exhaustive.) Most earlier papers in the mathematical literature on inverse problems for partial differential equations concern obstacle scattering (Lax and Phillips (1968), Majda and Taylor (1976)) or quantum mechanical inverse scattering problems (Chadan and Sabatier (1977)). Prior

work on inverse problems in wave propagation is to be found mostly in the geophysical literature: notable examples include Goupillaud (1961), Ware and Aki (1969), and Gerver (1970). Note however that the seminal paper in this field — Gel’fand and Levitan (1951) — may be viewed as a discussion of an inverse problem in wave propagation: see Symes (1979).

While a few of the above references provide rigorous treatment of the central uniqueness, existence, and continuous dependence issues, none treat the “bandlimited” problem described in Section 1 in a satisfactory way: without exception these works either assume the low-frequency content problem away, or ignore it. The problem is well-known to geophysical researchers, and its severity is explicitly illustrated in Pao, Santosa and Symes (1984) and Gray and Symes (1985).

Note that even the known uniqueness theorems for several-dimensional inverse problems in wave propagation either require data in a frequency interval $[0, \omega)$ (Sacks and Symes (1985), Sun (1987), Ramm (1986)) or do not allow the reflection configuration (sources and receivers separated from the target region by a hyperplane (Nachman (1987), Rakesh and Symes (1988))). Thus the present paper is the only instance, to the author’s knowledge, in which any version of the reflection inverse problem has been shown to be well-posed in the presence of a low-frequency lacuna.

Nonetheless, evidence of two sorts indicates that velocities, at least, are quite well-determined by bandlimited data. First conventional seismic data-processing, as practiced by academic and industrial reflection seismologists, appears to produce such information. While based on numerous drastic approximations, the so-called “velocity analysis” procedures incorporate a great deal of data-driven insight; see for example Yilmaz (1987), Ch. 3. A side-effect of our work is to provide a partial but rigorous mathematical basis for these important techniques.

Second, recent numerical investigations of the nonlinear least-squares problem

$$\min_c \|s[c] - s_{\text{data}}\|_{L^2}^2 \quad (2.2)$$

have turned up more direct evidence that bandlimited seismograms determine velocity profiles: See Kolb *et al.* (1986), MacAulay (1985), Gauthier *et al.* (1986), Mora (1987). All of these papers also reveal that any reasonable setting of (2.2) results in a very difficult optimization problem. The reasons

for both the success and the difficulty of this so-called “least-squares inversion” are noted briefly in Section 3, and explained in great detail in Santosa and Symes (1986), to which we refer the reader for extensive discussion. In any case, the computational difficulty of (2.2) was the main motivation for the work reported here, which relies on a different, “relaxed” least-squares problem (Section 5).

In this paper, we give only a qualitative analysis of this “relaxed” problem, which we call the *coherency optimization problem*, leading to *Theorem 1*. A quantitative analysis of an approximation appears in Symes (1988a), and numerical experiments are reported in Symes (1988b,c,d) establishing the feasibility of the optimization, its relative insensitivity to noise and its favorable comparison to (2.2) regarding computational efficiency. In Symes (1988d) the technique is applied to field reflection seismograms with quite satisfactory results.

A very important remaining question concerns the extension of these results to other models, notably to non-layered velocities (i.e. c depending on all space variables). As shown in Symes (1988a,b), an approximation to the coherency optimization problem can be formulated for the general nonlayered fluid model. A full-blown extension of our results awaits better understanding of propagation-of-regularity for hyperbolic equations with nonsmooth coefficients, and implications for the relation between solutions and coefficients, analogous to the results for problems in two independent variables detailed in Section 3.

3 Preliminary Considerations: Reduction to Plane Waves, Properties of the One-dimensional Forward Map

We assume that seismograms are given on an open set Ω of the space-time boundary of cylinder form:

$$\Omega = \Omega' \times [0, t_{\max}]$$

with Ω' a neighborhood of the “source” point $x = 0$. We shall also assume that all velocity profiles $c : \mathbf{R} \rightarrow \mathbf{R}^+$ satisfy

$$0 < c_{\min} \leq c(z) \leq c_{\max}, \quad z \geq 0 \quad (3.1)$$

for *a priori* fixed c_{\min}, c_{\max} . Whenever convenient we will also think of (3.1) as an $L^\infty(\mathbf{R})$ -bound on $\log c$.

The principal technical device of this paper is the introduction of the Radon-transformed field

$$U(p, z, t) := \int_{\mathbf{R}^2} dx u(x, z, t + p \cdot x_1), \quad p \in \mathbf{R}.$$

Standard arguments show that U is well-defined for small p, t under the assumptions made so far. For an attempt to maximize the domain of definition of U , see Santosa and Symes (1988).

A straightforward calculation shows that for suitably small $p \geq 0$ so that $c_{\max}p < 1$,

$$\begin{aligned} \frac{1}{v^2(z, p)} \frac{\partial^2 U}{\partial t^2}(z, t) - \frac{\partial^2 U}{\partial z^2}(z, t) &= \delta(z) f(t) \\ U &\equiv 0, \quad t \ll 0 \end{aligned} \quad (3.2)$$

where the *vertical wave velocity* $v(z, p)$ (or $v[c]$, to emphasize the dependence on c) is defined by

$$v[c](z, p) = \frac{c(z)}{\sqrt{1 - c^2(z)p^2}}$$

Because of the *a priori* bounds (3.1), the support of u , hence of s , is contained in a cone

$$c_{\max} t \geq \sqrt{|x|^2 + z^2}$$

Therefore, for sufficiently small $p_{\max} (< \frac{1}{c_{\max}})$ there exists $\tau_{\max} \leq 0$ so that for

$$R = \{(t, p) : 0 \leq t \leq \tau_{\max}, |p| \leq p_{\max}\}$$

we have for $(t, p) \in R$

$$\begin{aligned} S[c](t, p) &:= \frac{\partial U}{\partial t}(p, 0, t) - f(t) = \\ &\int_{\mathbf{R}^2} dx (\mathbf{1}_{\Omega} \frac{\partial u}{\partial t}(x, z, t + px_1) - f(t + px_1) \delta(x)) \end{aligned} \quad (3.3)$$

i.e. the domains of integration of the Radon integrals intersect the support of u inside Ω . We assume tacitly in the sequel that all (t, p) domains satisfy this constraint.

From standard facts about the Radon transform (e.g. Helgason [1980], Section 1.4) it follows that, if $s[c]$ were square-integrable, we would have

$$\begin{aligned} \|S[c]\|^2 &:= \int \int_R dt dp p |S[c](t, p)|^2 \\ &\leq \|s[c]\|_{L^2(\Omega)}^2 \end{aligned} \quad (3.4)$$

(see alternatively Santosa and Symes (1986, Appendix B)). As noted in the introduction, $s[c]$ is not square-integrable, but it can be shown that a version of (3.4) holds in which the r.h.s. is replaced by the norm of a “pseudodifferential projection” of $s[c]$.

Now we recall some facts about the one-dimensional seismogram map $S[c](\cdot, p) =: S_0[v]$ (fixed p) for which references are Symes (1986a,b). It is convenient to include explicitly the source wavelet temporarily in the notation. That is, write $S_0[v, f]$ for the map defined by the solution of (3.2), (3.3) followed by restriction to fixed p . Recall that f is assumed to define an *elliptic convolution operator of order zero*. For the choice $f = \delta$, S_0 defines a bounded continuous map from

$$H^{1,+}(\mathbf{R}, v_0) = \{v \in H^1(\mathbf{R}) : v \equiv v_0 \text{ for } z < 0, \log v \in H_{\text{loc}}^1(\mathbf{R})\}$$

into $L^2[0, T]$ for any $T > 0$, but $S_0[\cdot, \delta]$ is not locally uniformly continuous (Symes 1986b). In order to recover the necessary degree of regularity for the arguments to follow, we introduce the “travel-time velocity” $\tilde{v}[c]$ defined by

$$\tilde{v} \circ \tau = v$$

where

$$\tau(z) = 2 \int_0^z \frac{1}{v}$$

is the (one-way) travel-time. More discussion of the map $c \mapsto \tilde{v}[c]$ appears in Section 5; see also Symes (1986a). A short calculation shows that \tilde{U} , defined by

$$\tilde{U}(\tau(z), t) = U(z, t)$$

satisfies

$$\frac{1}{\tilde{v}(x)} \frac{\partial^2 \tilde{U}}{\partial t^2}(x, t) - \frac{\partial}{\partial x} \frac{1}{\tilde{v}(x)} \frac{\partial \tilde{U}}{\partial x}(x, t) = \frac{1}{v_0} f(t) \delta(x)$$

Set

$$\tilde{S}_0[\tilde{v}, f](t) = \frac{\partial \tilde{U}}{\partial t}(0, t)$$

We recapitulate a number of properties of \tilde{S}_0 . With exceptions noted below, all of these may be found in Symes (1986a). Note that $\tilde{S}_0[\tilde{v}, f] \equiv S_0[v, f]$ if $\tilde{v} \circ \tau = v$. \tilde{S}_0 also defines a bounded map: $H_{\text{loc}}^{1,+}(\mathbf{R}, v_0) \rightarrow L_{\text{loc}}^2(\mathbf{R})$ for any $v_0 > 0$. Moreover, \tilde{S}_0 is actually of class C^2 , viewed as a map: $H^{1,+}([0, T/2], v_0) \rightarrow L^2[0, T]$. The derivative is given by the formal perturbation ($\delta\tilde{v} \in H_{\text{loc}}^1(\mathbf{R})$, $\delta\tilde{v} \equiv 0, z < 0$)

$$\begin{aligned} & \left(\frac{1}{\tilde{v}(x)} \right) \frac{\partial^2 \delta\tilde{U}}{\partial t^2}(x, t) - \frac{\partial}{\partial x} \left(\frac{1}{\tilde{v}(x)} \frac{\partial \delta\tilde{U}}{\partial x} \right)(x, t) \\ &= \frac{\partial}{\partial x} \left(\frac{\delta\tilde{v}}{\tilde{v}^2} \frac{\partial \tilde{U}}{\partial x}(x, t) \right), \quad x \geq 0 \\ & \delta\tilde{U} = 0, \quad t \ll 0 \end{aligned}$$

$$D\tilde{S}_0[\tilde{v}, f]\delta\tilde{v} = \frac{\partial \delta\tilde{U}}{\partial t} \Big|_{x=0},$$

so that

$$\begin{aligned} & \|\tilde{S}_0[\tilde{v} + \delta\tilde{v}, f] - \tilde{S}_0[\tilde{v}, f] - D\tilde{S}_0[\tilde{v}, f] \cdot \delta\tilde{v}\|_{L^2[0,T]} \\ & = o\left(\|\delta\tilde{v}\|_{H^1([0,T/2])}\right) \end{aligned}$$

For $f = \delta$, more is true: for constants $C_-, C_+ > 0$ depending on $\|\log \tilde{v}\|_{H^1[0,T/2]}$ and on T ,

$$\begin{aligned} C_- \left\| \frac{\partial \delta\tilde{v}}{\partial x} \right\|_{L^2[0,T/2]} & \leq \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0,T]} \\ & \leq C_+ \left\| \frac{\partial \delta\tilde{v}}{\partial x} \right\|_{L^2[0,T/2]} \end{aligned} \tag{3.5}$$

Also, for $f = H$ (the Heaviside function), there exists C_0 depending on $\|\log \tilde{v}\|_{H^1[0,T/2]}$ and on T so that

$$\|D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2[0,T]} \leq C_0 \|\delta\tilde{v}\|_{L^2[0,T/2]} \tag{3.6}$$

Note that for $\delta\tilde{v} \in H_{\text{loc}}^1$, $D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v} \in H_{\text{loc}}^1$ and

$$D\tilde{S}_0[\tilde{v}, \delta] = \frac{\partial}{\partial t} D\tilde{S}_0[\tilde{v}, H]$$

Also

$$\begin{aligned} \tilde{S}_0[\tilde{v}, f] & = f * \tilde{S}_0[\tilde{v}, \delta] \\ D\tilde{S}_0[\tilde{v}, f] & = f * D\tilde{S}_0[\tilde{v}, \delta] \end{aligned}$$

All of these results are based on simple local energy estimates; with the exception of (3.6) they are stated explicitly in Symes (1986a). The Heaviside estimate (3.6) is not given there, but the proof presents no novelties; it will be given in any case in the forthcoming monograph on layered inverse problems by Santosa and the author.

4 Ellipticity for the One-dimensional Forward Map

We have assumed (until Section 7) that $f \in \mathcal{E}'(\mathbf{R})$ defines an elliptic convolution operator of order zero. Since $D\tilde{S}_0[\tilde{v}, \delta]$ for $\log \tilde{v} \in H_{\text{loc}}^1$ is invertible, ((3.6), (3.7)), it seems clear that $D\tilde{S}_0[\tilde{v}, f] = f * \tilde{S}_0[\tilde{v}, \delta]$ should be “elliptic” as well. The purpose of this section is to formulate and prove a precise result along these lines, keeping track of the dependence of various constants on the H^1 -norm of \tilde{v} , the time- and depth-intervals used, etc., etc. The result is a Gårding-type estimate for $D\tilde{S}_0$, which requires that $D\tilde{S}_0$ be given on $[0, T_1]$, $T_1 > T_0$, to estimate $\delta\tilde{v}$ on $[0, T_0]$. It is clear that a little “extra” data is required, since the support of f is not assumed to be a point. In fact, the principal constant intervening in the estimates is the first moment of f , which measures its “spread”, and is related in the estimates to the size of the necessary “margin” $T_1 - T_0$.

Select a cut-off function $\psi \in C^\infty(\mathbf{R})$ with $\psi(t) \equiv 0$ for $t > T_1$ and $\psi(t) \equiv 1$ for $t \leq T_0$, and define the smoothly cut-off version of \tilde{S}_0 :

$$\tilde{S}_\psi := \psi \tilde{S}_0$$

so also

$$D\tilde{S}_\psi := \psi D\tilde{S}_0$$

In the following, we will apply the estimates of the previous section on various t -intervals. Since the constants C_-, C_+ , etc. depend on the length of the interval, we will include the length explicitly in the notation, for the moment. Thus for estimates on the interval $[0, T]$, C_- becomes $C_-[T]$, etc. These constants depend on the H^1 -norm of $\log \tilde{v}$ on the appropriate intervals.

Recall that “elliptic”, applied to f , means the inequalities (2.1), which we recall here for convenience: for $\phi \in H^1(\mathbf{R})$,

$$\begin{aligned} K^* \left\| \frac{\partial \phi}{\partial x} \right\|_{L^2(\mathbf{R})} &\geq \left\| f * \frac{\partial \phi}{\partial x} \right\|_{L^2(\mathbf{R})} \\ &\geq K_1 \left\| \frac{\partial \phi}{\partial x} \right\|_{L^2(\mathbf{R})} - K_0 \|\phi\|_{L^2(\mathbf{R})} \end{aligned} \tag{4.1}$$

$\text{supp } f \subset \mathbf{R}^+$

From (3.5) and (3.6), for $\log \tilde{v} \in H_{\text{loc}}^1(\mathbf{R})$, $\delta \tilde{v} \in H_{\text{loc}}^1(\mathbf{R})$, $\tilde{v}(x) = v_0$ for $x < 0$:

$$\begin{aligned} \|D\tilde{S}_\psi[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2(\mathbf{R})} &\leq \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0, T_1]} \\ &\leq C_+[T_1] \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2[0, T_1/2]} \end{aligned} \quad (4.2)$$

while

$$\begin{aligned} \|D\tilde{S}_\psi[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2(\mathbf{R})} &\geq \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0, T_0]} \\ &\geq C_-[T_0] \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2[0, T_0/2]} \end{aligned} \quad (4.3)$$

Also

$$\|D\tilde{S}_\psi[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R}^+)} \leq C_0[T_1] \|\delta\tilde{v}\|_{L^2[0, T_1/2]} \quad (4.4)$$

Now

$$\begin{aligned} D\tilde{S}_\psi[\tilde{v}, f]\delta\tilde{v} &= \psi f * D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v} \\ &= f * D\tilde{S}_\psi[\tilde{v}, \delta]\delta\tilde{v} \\ &\quad + e \end{aligned}$$

where for e we have the standard commutator estimate

$$\|e\|_{L^2(\mathbf{R})} \leq \|\psi'\|_{L^\infty} m_f^1 \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0, T_1]}$$

Here m_f^1 denotes the measure norm of the distribution

$$\phi \longmapsto \langle |f|, t\phi \rangle$$

i.e. the first moment of f , as explained in Section 2.

Combine this estimate with (4.1)–(4.4) to get

$$\begin{aligned}
& \|D\tilde{S}_0[v, f]\delta\tilde{v}\|_{L^2[0, T_1]} \geq \|D\tilde{S}_\psi[\tilde{v}, f]\delta\tilde{v}\|_{L^2(\mathbf{R})} \\
& \geq \|f * \psi \frac{\partial}{\partial t} D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} - \|e\|_{L^2(\mathbf{R})} \\
& \geq \|f * \frac{\partial}{\partial t} \psi D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} \\
& \quad - \|f * \frac{\partial \psi}{\partial t} D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} - \|e\|_{L^2(\mathbf{R})} \\
& \geq K_1 \|\frac{\partial}{\partial t} \psi D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} - K_0 \|\psi D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} \\
& \quad - K^* \|\frac{\partial \psi}{\partial t} D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2(\mathbf{R})} - \|e\|_{L^2(\mathbf{R})} \\
& \geq K_1 \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0, T_0]} - K_0 \|D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2[0, T_1]} \\
& \quad - K^* \|D\tilde{S}_0[\tilde{v}, H]\delta\tilde{v}\|_{L^2[T_0, T_1]} \|\frac{\partial \psi}{\partial t}\|_{L^\infty(\mathbf{R})} \\
& \quad - m_f^1 \|\frac{\partial \psi}{\partial t}\|_{L^\infty(\mathbf{R})} \|D\tilde{S}_0[\tilde{v}, \delta]\delta\tilde{v}\|_{L^2[0, T_1]} \\
& \geq K_1 C_-[T_0] \|\frac{\partial \delta \tilde{v}}{\partial x}\|_{L^2[0, T_0/2]} \\
& \quad - C_0[T_1] \left(K_0 + K^* \|\frac{\partial \psi}{\partial t}\|_{L^\infty(\mathbf{R})} \right) \|\delta\tilde{v}\|_{L^2[0, T_1/2]} \\
& \quad - m_f^1 \|\frac{\partial \psi}{\partial t}\|_{L^\infty(\mathbf{R})} C_+[T_1] \|\frac{\partial \delta \tilde{v}}{\partial x}\|_{L^2[0, T_1/2]}
\end{aligned}$$

The upshot of all of this is the inequality

$$\begin{aligned}
& \|D\tilde{S}_0[\tilde{v}, f]\delta\tilde{v}\|_{L^2[0, T_1]} \geq K_1 C_-[T_0] \|\frac{\partial \delta \tilde{v}}{\partial x}\|_{L^2[0, T_0/2]} \\
& \quad - m_f^1 \|\frac{\partial \psi}{\partial x}\|_{L^\infty(\mathbf{R})} C_+[T_1] \|\frac{\partial \delta \tilde{v}}{\partial x}\|_{L^2[0, T_1/2]} \\
& \quad - C_0[T_1] \left(K_0 + K^* \|\frac{\partial \psi}{\partial x}\|_{L^\infty(\mathbf{R})} \right) \|\delta\tilde{v}\|_{L^2[0, T_1/2]}
\end{aligned}$$

Remark. As noted in Symes (1986a), in general the constants $C_+[T]$, $C_0[T]$ increase with T , while $C_-[T]$ decreases.

Remark. It is worth noting the relation of the various bounds involving f to its Fourier transform. Indeed, obviously

$$K^* \geq \|\hat{f}\|_{L^\infty(\mathbf{R})}$$

whereas K_0, K_1 are related to the detailed behaviour of the Fourier transform near $\omega = 0$. Suppose that, for some K_* , $\Omega > 0$,

$$|\hat{f}(\omega)| \geq K_* \quad \text{for } |\omega| > \Omega \quad (4.5)$$

(i.e. $[\Omega, \infty)$ constitutes the “passband” of \hat{f} , measured with tolerance K_*). Then for $\phi \in H^1(\mathbf{R})$, it is easy to see that

$$\left\| f * \frac{\partial \phi}{\partial t} \right\|_{L^2(\mathbf{R})} \geq K_* \left[\left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(\mathbf{R})} - \Omega \|\phi\|_{L^2(\mathbf{R})} \right]$$

which gives the relations

$$K_1 \geq K_*, \quad K_0 \leq \Omega K_*$$

Note also the effect of scaling: if f satisfies (4.6), then

$$f_\epsilon(t) := \frac{1}{\epsilon} f\left(\frac{t}{\epsilon}\right)$$

satisfies (4.6) with $\Omega = \Omega/\epsilon$, while $m_{f_\epsilon}^1 = \epsilon m_f^1$. Thus $K_1 = \mathcal{O}(1)$, $K_0 = \mathcal{O}(\frac{1}{\epsilon})$, and $m_{f_\epsilon}^1 = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$.

It is clear from the preceding discussion and the form of (4.5) that, for any prescribed $T_1 > T_0, \psi$ as above, and any “base” source wavelet f , a scaled version of f will have small enough first moment that

$$m_f^1 \left\| \frac{\partial \psi}{\partial x} \right\|_{L^\infty(\mathbf{R})} C_+[T_1] \leq \frac{1}{2} K_1 C_-[T_0]$$

with fixed K_1 (independent of scaling). Note that we can certainly choose ψ so that

$$\left\| \frac{\partial \psi}{\partial x} \right\|_{L^\infty(\mathbf{R})} \leq \frac{2}{T_1 - T_0}$$

These observations establish the non-vacuousness of

Theorem 2 *Choose $T_1 > T_0 > 0$, and $K^* > K_1 > 0$. Then for any $\log \bar{v} \in H_{\text{loc}}^1(\mathbf{R})$, there exists $\bar{m}, L_2, L_1, L_0 > 0$ depending on \bar{v} and on T_1, T_0, K^*, K_1 , and so that if f satisfies (4.1) with some $K_0 > 0$ and*

$$m_f^1 \leq \bar{m}$$

then for $\delta\tilde{v} \in H_{\text{loc}}^1(\mathbf{R})$,

$$\begin{aligned} & \|D\tilde{S}_0[\tilde{v}, f]\delta\tilde{v}\|_{L^2[0, T_1]} + m_f^1 L_2 \left\| \frac{\partial\delta v}{\partial x} \right\|_{L^2[T_0/2, T_1/2]} \\ & \geq L_1 \left\| \frac{\partial\delta v}{\partial x} \right\|_{L^2[0, T_0/2]} - L_0 \|\delta v\|_{L^2[0, T_1/2]} \end{aligned}$$

5 The Optimum Coherency Principle

While we have shown that the linearized one-dimensional forward map is elliptic under the circumstances which concern us, it is certainly not boundedly invertible — or rather, the hypotheses concerning f do not imply any uniform bound on the inverse. This circumstance is widely remarked in the literature; for a sampler, see Santosa and Symes (1986), where numerical examples are also given (see especially Chapters 6 and 7). Recall, however, the provenance of the one-dimensional problem: it governs the propagation of a plane wave, the surface data for which are identical to the Radon transform of the point source surface data at fixed slowness (or angle). The possibility remains that the collection of *all* (precritical) plane-wave data might constrain the velocity estimate more severely than does a single plane-wave component.

In this and the next section we confirm this possibility. Since we will consider the data in an interval of slownesses $P_1 \leq p \leq P_2$, we will work with a suite of travel-time velocity models $\{\tilde{v}(t, p) : 0 \leq t \leq T, P_1 \leq p \leq P_2\}$. Recall from Section 3 that these are derived from velocity profiles $c(z)$; accordingly we begin with the question: what condition must $\tilde{v}(t, p)$ satisfy in order that $\tilde{v} = \tilde{v}[c]$ for some c ? That is, we seek an operator whose kernel is identical to the range of $c \rightarrow \tilde{v}[c]$. We will call membership in the null space of the required operator (or in the range of $c \rightarrow \tilde{v}[c]$) the *coherency condition*, since coherence of the travel-time velocities \tilde{v} is then forced across various values of p : all are representations of the same mechanical model, in different coordinate systems.

Denote by $\zeta(t, p)$ the inverse of the two-way travel time function $\tau(z, p)$, i.e.

$$t = 2 \int_0^{\zeta(t, p)} \frac{dz}{v(z, p)}$$

Then clearly

$$\zeta(t, p) = \int_0^{t/2} d\tau \tilde{v}(\tau, p)$$

We will now regard ζ as being *defined* by this formula, hence as a functional of \tilde{v} . So, given \tilde{v} we can compute ζ , whether $\tilde{v} = \tilde{v}[c]$ for some c , or not.

Thus we can compute the quantity

$$\gamma[\tilde{v}] = \left[\frac{1}{(\tilde{v} \circ \zeta^{-1})^2} + p^2 \right]$$

(Here $\tilde{v} \circ \zeta^{-1}(z, p) = \tilde{v}(\zeta^{-1}(z, p), p)$). Referring to the definitions (Section 3), we see that if $\tilde{v} = \tilde{v}[c]$, then

$$\gamma[\tilde{v}] = \frac{1}{c^2}$$

and is thus independent of p : that is,

$$\tilde{v} = \tilde{v}[c] \Rightarrow \frac{\partial}{\partial p} \gamma[\tilde{v}] \equiv 0 \quad (5.1)$$

This last condition still involves the travel-time change of variables, so does not define a sufficiently regular function of \tilde{v} (recall the discussion in Section 3). Define instead

$$\begin{aligned} \tilde{Q}[\tilde{v}] &= -\frac{\tilde{v}^3}{2} \left[\frac{\partial}{\partial p} \gamma[\tilde{v}] \right] \circ \zeta \\ &= -\frac{\tilde{v}^3}{2} \left[\frac{\partial}{\partial p} (\tilde{v} \circ \zeta^{-1})^{-2} \right] \circ \zeta - p\tilde{v}^3 \\ &= \left\{ \frac{\partial \tilde{v}}{\partial p} + \left(\frac{\partial}{\partial p} \zeta^{-1} \right) \circ \zeta \frac{\partial \tilde{v}}{\partial t} \right\} - p\tilde{v}^3 \end{aligned} \quad (5.2)$$

A short chain-rule calculation gives

$$\left(\frac{\partial}{\partial p} \zeta^{-1} \right) \circ \zeta = -\frac{2}{\tilde{v}} \int_0^{t/2} \frac{\partial \tilde{v}}{\partial p}$$

which relation allows us to view $\tilde{Q}[\tilde{v}]$ as a functional of \tilde{v} . Clearly, from (5.1)

$$\tilde{v} = \tilde{v}[c] \Rightarrow \tilde{Q}[\tilde{v}] = 0$$

It will be important to define the coherency condition in such a way as to have the largest possible domain contained in $H_{\text{loc}}^{1,0}(\mathbf{R} \times [P_1, P_2])$ (which will be the natural domain for the forward map, defined below). An obvious choice is H^1 , but \tilde{Q} is not continuous in that topology. As it happens, we

can replace \tilde{Q} with another operator having the same kernel, but which is continuous (even C^∞) in the H^1 sense.

Suppose temporarily that $\tilde{v} = \tilde{v}[c]$ for some c . Then

$$\begin{aligned} \frac{1}{\tilde{v}} \tilde{Q}[\tilde{v}] &= \frac{1}{\tilde{v}} \frac{\partial \tilde{v}}{\partial p} - \frac{2}{\tilde{v}^2} \left(\int_0^{t/2} \frac{\partial \tilde{v}}{\partial p} \right) \frac{\partial \tilde{v}}{\partial t} - p \tilde{v}^2 \\ &= \frac{\partial}{\partial t} \left(\frac{2}{\tilde{v}} \int_0^{t/2} \frac{\partial \tilde{v}}{\partial p} - 2p \int_0^{t/2} \tilde{v}^2 \right) \\ &= 0 \end{aligned}$$

Thus

$$\tilde{v} = \tilde{v}[c] \Rightarrow \frac{2}{\tilde{v}} \int_0^{t/2} \frac{\partial \tilde{v}}{\partial p} = 2p \int_0^{t/2} \tilde{v}^2 \quad (5.3)$$

The map

$$Q[\tilde{v}] := \frac{\partial \tilde{v}}{\partial p} - 2p \left(\int_0^{t/2} \tilde{v}^2 \right) \frac{\partial \tilde{v}}{\partial t} - p \tilde{v}^3$$

therefore also satisfies

$$\tilde{v} = \tilde{v}[c] \Rightarrow Q[\tilde{v}] = 0 \quad (5.4)$$

On the other hand, for any $T > 0$, $R = [0, T] \times [P_1, P_2]$, Q obviously defines a C^2 -map

$$Q : H^1(R) \rightarrow L^2(R).$$

The following converse to (5.4) shows that $Q = 0$ is an adequate coherency condition:

Lemma 1 *Suppose that $\log \tilde{v} \in H^1(R)$ and $Q[\tilde{v}] = 0$. Then for some $Z > 0$ and some $c \in H^1[0, Z]$,*

$$\tilde{v} = \tilde{v}[c]|_R$$

Proof: Integrate in t :

$$\begin{aligned} 0 &= \int_0^{t/2} dt' Q[\tilde{v}](t', p) \\ &= \int_0^{t/2} dt' \left\{ \frac{\partial \tilde{v}}{\partial p}(t', p) - 2p \left[\int_0^{t'/2} dt'' \tilde{v}^2(t'', p) \right] \frac{\partial \tilde{v}}{\partial t}(t', p) - p \tilde{v}^3(t', p) \right\} \\ &= \int_0^{t/2} dt' \frac{\partial \tilde{v}}{\partial p}(t', p) - 2p \tilde{v}(t, p) \int_0^{t/2} dt' \tilde{v}^2(t', p) \end{aligned}$$

after integration by parts, so we once again recover the relation

$$\frac{1}{\tilde{v}} \int_0^{t/2} \frac{\partial \tilde{v}}{\partial p} = 2p \int_0^{t/2} \tilde{v}^2$$

It follows immediately that $\tilde{Q}[\tilde{v}] = 0$ as well, which is equivalent to

$$\frac{\partial}{\partial p} \gamma[\tilde{v}] = 0$$

Set

$$\begin{aligned} R_\zeta &= \{(z, p) : P_1 \leq p \leq P_2, 0 \leq z \leq \zeta(T, p)\} \\ v &= \tilde{v} \circ \zeta^{-1} \end{aligned}$$

It is easily checked that $\log v \in H^1(R_\zeta)$. On the other hand with

$$\begin{aligned} c(z) &:= \left(\frac{1}{v(z, p)^2} + p^2 \right)^{-1/2} \\ &= \gamma[\tilde{v}]^{-1/2}(z, p) \end{aligned}$$

we have $c \in H^1[0, Z]$, $Z = \sup_{P_1 \leq p \leq P_2} \zeta(T, p)$, and

$$v(z, p) = \frac{c(z)}{\sqrt{1 - c^2(z)p^2}}$$

$$\zeta^{-1}(z, p) = 2 \int_0^z \frac{1}{v}$$

whence $\zeta^{-1} = \tau$, and the conclusion follows.

q.e.d.

We now turn to the definition of the multi-plane-wave forward map. For simplicity, define seismograms on the *data rectangle*

$$R_1 = [0, T_1] \times [P_1, P_2]$$

Choose $c_0, c_1 > 0$ with $c_0 P_2, c_1 P_2 < 1$, $T_2 \geq T_0$, and $\ell^* > 0$ and set

$$\begin{aligned} \Sigma &= \{ \tilde{v} \in H_{\text{loc}}^1(\mathbf{R} \times [P_1, P_2]) : \\ \tilde{v}(x, p) &= \frac{c_0}{\sqrt{1 - c_0 p^2}}, \quad x < 0, \\ \tilde{v}(x, p) &= \frac{c_1}{\sqrt{1 - c_1^2 p^2}}, \\ &\quad x \geq \frac{1}{2} T_2 \\ &\| \log \tilde{v} \|_{H^1([0, \frac{1}{2} T_2] \times [P_1, P_2])} < \ell^* \} \end{aligned}$$

and its “tangent space”

$$\begin{aligned} \Sigma \cdot &= \{ \delta \tilde{v} \in H_{\text{loc}}^1(\mathbf{R} \times [P_1, P_2]) : \\ \delta v(x, p) &= 0, \quad x \leq 0 \text{ or } x \geq \frac{1}{2} T_2 \} \end{aligned}$$

Only a finite interval in t is needed for the arguments which follow. With $T_2 > T_1$ to be determined below, set

$$\tilde{R}_2 = [0, T_2/2] \times [P_1, P_2]$$

Identify elements of Σ and $\Sigma \cdot$ with their restrictions to \tilde{R}_2 , and topologize Σ and $\Sigma \cdot$ as subsets of $H^1(\tilde{R}_2)$.

With these conventions, define the forward map

$$\tilde{S} : \Sigma \longrightarrow L^2(R_1)$$

by

$$\tilde{S}[\tilde{v}](t, p) = \tilde{S}_0[\tilde{v}(\cdot, p), f](t)$$

with \tilde{S}_0 as in Section 4. From the results stated there and in Section 3, it follows that \tilde{S} is of class C^2 , with derivatives bounded in terms of c_0, T_1, f , and ℓ^* .

The derivative $D\tilde{S}[\tilde{v}]$ also obeys an “elliptic” estimate. To state this, set

$$\tilde{R}_0 = [0, T_0/2] \times [P_1, P_2], \quad \tilde{R}_1 = [0, T_1/2] \times [P_1, P_2].$$

Then from *Theorem 2* follows

Theorem 3 Given $P_2 > P_1 \geq 0$, $T_1 > T_0 > 0$, $K^* \geq K_1 > 0$, and $\ell^* > 0$, there exist $\bar{m}, L_0, L_1, L_2, L_1 > 0$, so that for $K_0 > 0$ and $f \in \mathcal{E}'(\mathbf{R})$ satisfying (4.1) and

$$m_f^1 \leq \bar{m}$$

so that for $\tilde{v} \in \Sigma$, $\delta\tilde{v} \in \Sigma$

$$\begin{aligned} & \|D\tilde{S}[\tilde{v}]\delta\tilde{v}\|_{L^2(\mathcal{R}_1)} + m_f^1 L_2 \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\mathcal{R}_1 \setminus \mathcal{R}_0)} \\ & \geq L_1 \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\mathcal{R}_0)} - L_0 \|\delta\tilde{v}\|_{L^2(\mathcal{R}_1)} \end{aligned}$$

Moreover, for each $p \in [P_1, P_2]$,

$$\begin{aligned} & \|D\tilde{S}[\tilde{v}]\delta\tilde{v}(\cdot, p)\|_{L^2[0, T_1]} + m_f^1 L_2 \left\| \frac{\partial\delta\tilde{v}}{\partial x}(\cdot, p) \right\|_{L^2[T_0, /2, T_1/2]} \\ & \geq L_1 \left\| \frac{\partial\delta\tilde{v}}{\partial x}(\cdot, p) \right\|_{L^2[0, T_0/2]} - L_0 \|\delta\tilde{v}(\cdot, p)\|_{L^2[0, t_1/2]} \end{aligned}$$

In view of *Lemma 1* and the obvious relation

$$\tilde{S}[\tilde{v}[c]] = S[c]$$

we can now state a version of the inverse problem closely related to the least-squares problem (2.2), as

$$\begin{aligned} & \text{minimize } \|\tilde{S}[\tilde{v}] - D\|_{L^2(\mathcal{R}_1)}^2 \text{ over } \tilde{v} \in \Sigma \\ & \text{subject to } Q[\tilde{v}] = 0 \end{aligned} \tag{5.5}$$

In fact, a solution to this problem clearly yields a solution to (2.2) on a suitable depth interval. On the other hand, this problem would appear to have the advantage of regularity: both the objective and constraint functions are of class C^2 . Moreover, it is possible to show that, under the circumstances described in *Theorem 4* below, the Hessian operator of the objective function is positive definite on the null space of the linearized constraints, at a consistent data set, i.e. when

$$D = S[c^*]$$

for suitable c^* .

Unfortunately these properties are insufficient to yield a stable-local-existence result. To motivate the next step in the development, we digress, with a brief review of Lagrangian theory for constrained optimization, and a simple but closely related example.

Suppose that X, Y are Hilbert spaces, $f : U \rightarrow \mathbf{R}, g : U \rightarrow Y$ smooth on an open set $U \subset X$. As is well-known (Luenberger, 1973), a local solution of the constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x \in U} & f(x) \\ \text{subject to} & g(x) = 0 \end{array}$$

is a critical point of the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle$$

We wish to apply the implicit function theorem, to assure that a solution is stable against (C^2 -) perturbations in f . The conditions necessary to apply the implicit function theorem to the critical-point problem

$$\text{grad}_{x, \lambda} \mathcal{L}(x, \lambda) = 0$$

are also sufficient to ensure the convergence of Newton's method (and, generally, of its computationally efficient quasi-Newton relatives). These amount to

$$\begin{aligned} & Dg(x)\delta x = 0 \\ \text{(i)} \quad & \implies \langle \delta x, [\text{Hess } f(x) + \langle \lambda, \text{Hess } g(x) \rangle], \delta x \rangle \\ & \geq \ell_1 \|\delta x\|^2 \end{aligned}$$

$$\text{(ii)} \quad \|Dg(x)^* \lambda\| \geq \ell_2 \|\lambda\|$$

for constants $\ell_1, \ell_2 > 0$. The so called "second-order sufficiency" condition (i) may be verified for the problem (5.3), as was mentioned above. The so-called "constraint qualification," however, fails, in the manner illustrated by the following simple example, which captures the main features of (5.5).

Let $X = \{u \in H^1(R) : \int \int_R u = 0\}$, $Y = L^2(R)$, $R = [0, 1]^2$ the unit square in \mathbf{R}^2 . Set for $u \in X$, $v \in Y$ given,

$$f(u) = \left\| \frac{\partial u}{\partial x_1} - v \right\|_{L^2(R)}^2$$

$$g(u) = \frac{\partial u}{\partial x_2}$$

and let us suppose that $v = v(x_1)$, so that the problem

$$\begin{array}{ll} \text{minimize}_{u \in X} & \text{grad } f(u) \\ \text{subject to} & g(u) = 0 \end{array}$$

has the unique, zero-residual solution

$$u(x_1, x_2) = \int_0^{x_1} dx v(x) + \text{const.}$$

The second-order sufficiency condition (i) is an immediate consequence of a form of Poincaré's inequality (Nečas, 1967). The adjoint $Dg^*(u)$ is given by

$$Dg^*(u) = -\mathcal{N} \frac{\partial}{\partial x_2}$$

where \mathcal{N} is the solution operator of the Neumann problem: $\mathcal{N}b = w$, where

$$\begin{aligned} (I - \Delta)w &= b & \text{in } R \\ \frac{\partial w}{\partial n} &= 0 & \text{on } \partial R \end{aligned}$$

Set $\lambda_k(x) = \cos k_1 \pi x_1 \cos k_2 \pi x_2$ for $k \in \mathbf{Z}^2$. Then

$$\|Dg^*(u)\lambda_k\|_{H^1(R)} = \frac{2k_2}{\sqrt{1 + k_1^2 + k_2^2}}$$

which can be made as small as one likes by taking k_1 large. Thus the constraint qualification (ii) fails.

For similar but nonlinear problems such as (5.5), the nonzero singular values of Dg are proportional to the instantaneous radii of curvature of the

constraint set. The failure of the constraint qualification could conceivably be associated with the presence of arbitrarily small radii of curvature — indeed, this must be the case for uniformly nonlinear problems like (5.5). In essence, the constraint set for such a problem has a cusp at every point! No reasonable stability properties can be expected for the solutions of such a problem.

On the other hand, the Hessian operator of the “penalized” cost functional

$$f(u) + \sigma^2 \|g(u)\|_{L^2(R)}^2$$

is positive-definite for any $\sigma > 0$ — again, this is simply Poincaré’s inequality. This observation motivates the following construction.

For choices of time and slowness intervals $T_2 > T_1 > T_0 > 0$, $P_2 > P_1 \geq 0$, “data set” $D \in L^2(R_1)$ and “tuning” parameters $\sigma, \lambda \geq 0$, define for $\tilde{v} \in \Sigma$

$$J_{\sigma,\lambda}[\tilde{v}] = \frac{1}{2} \left\{ \left\| \tilde{S}[\tilde{v}] - D \right\|_{L^2(R_1)}^2 + \sigma^2 \|Q\tilde{v}\|_{L^2(\tilde{R}_2)}^2 + \lambda^2 \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}^2 \right\} \quad (5.6)$$

The first two summands in the definition of $J_{\sigma,\lambda}$ are motivated by analogy with the “Dirichlet” problem discussed above. The form of the “elliptic” estimate (*Theorem 3*) and the need, explained in the next section, to choose $T_2 > T_0$ motivate the last term. For example the elliptic estimate gives a bound on $\delta\tilde{v}$ only on the shorter interval $[0, T_0]$, so $\delta\tilde{v}$ must be bounded *a priori* for $t > T_0$.

We shall call the three terms on the r.h.s. of (5.6) “data,” “coherency,” and “extension” terms. Minimization of (5.6) is the “coherency optimization problem.”

6 Proof of the Main Theorem

We begin this section with the proof that the Hessian operator of $J_{\sigma,\lambda}$ as defined in Section 5 is *positive-definite* at a sufficiently rough, coherent \tilde{v} . The main idea is that the Hessian quadratic form consists of the “data” term dominating the high-frequencies in $\delta\tilde{v}$, and a “coherency” term essentially consisting of the product of the indefinite integral (in x) of $\delta\tilde{v}$ and a derivative of \tilde{v} . This latter term is thus the product of a smooth factor (depending on $\delta\tilde{v}$) and a rough factor (derivative of \tilde{v}). When the rough factor is rough enough, uniformly on the length scale of significant change in the smooth factor, then the product dominates the smooth factor. *Lemma 3* below makes this heuristic reasoning precise, and establishes a mathematical meaning for “sufficiently rough.” The “smooth factor” discussed here is the indefinite (t -) integral of $\delta\tilde{v}$; the estimate for it, together with the elliptic estimate from Section 5 and an interpolation argument, given a bound on $\delta\tilde{v}$ in terms of the “data,” “coherency,” and “extension” terms of the Hessian of $J_{\sigma,\lambda}$.

Recall from the preceding sections the geometry of the coherency optimization problem:

$$\begin{array}{ll} 0 < T_0 < T_1 < T_2 & \text{time limits} \\ 0 \leq P_1 < P_2 & \text{slowness limits} \\ \left. \begin{array}{l} R_\nu = [0, T_\nu] \times [P_1, P_2] \\ \tilde{R}_\nu = [0, T_\nu/2] \times [P_1, P_2] \end{array} \right\} \nu = 0, 1, 2 \\ \text{(data, model rectangles);} \end{array}$$

the function spaces involved in its setting:

$$\Sigma = \left\{ \tilde{v} \in H^1(\tilde{R}_2) : \tilde{v}(0, p) = \frac{c_0}{\sqrt{1 - c_0^2 p^2}}, \tilde{v}(T_2, p) = \frac{c_1}{\sqrt{1 - c_1^2 p^2}}, \right. \\ \left. \|\log \tilde{v}\|_{H^1(\tilde{R}_2)} < \ell^* \right\}$$

$$\dot{\Sigma} = \left\{ \delta\tilde{v} \in H^1(\tilde{R}_2) : \delta\tilde{v}(0, p) = 0 = \delta\tilde{v}(T_2, p) \right\}$$

and the maps:

$$\begin{aligned}\tilde{S} &: \Sigma \longrightarrow L^2(R_1) \\ D\tilde{S} &: \Sigma \times \Sigma \longrightarrow L^2(R_1) \\ Q &: \Sigma \longrightarrow L^2(\tilde{R}_2) \\ DQ &: \Sigma \times \Sigma \longrightarrow L^2(\tilde{R}_2)\end{aligned}$$

It will be essential in the arguments given below that T_2 be related appropriately to T_0 , as follows. For $\tilde{v} \in \Sigma$, the L^∞ -bound on \tilde{v} implies that

$$\begin{aligned}\zeta(T_0, P_2) &= \sup_{P_1 \leq p \leq P_2} \zeta(T_0, p) \\ &\leq \frac{1}{2} T_0 e^{\ell^*} := Z\end{aligned}$$

whence

$$\begin{aligned}\tau(Z, P_1) &= \sup_{P_1 \leq p \leq P_2} \tau(Z, p) \\ &\leq 2Z e^{\ell^*} = T_0 e^{2\ell^*}\end{aligned}$$

Set $T_2 = T_0 e^{2\ell^*}$. Then it follows that for any $p_1, p_2 \in [P_1, P_2]$,

$$\tau(\zeta(T_0, p_1), p_2) \leq T_2$$

Define

$$R_Z = [0, Z] \times [P_1, P_2]$$

Then for $\delta\tilde{v} \in H_{\text{loc}}^1(\mathbf{R})$, $\tilde{v} \in \Sigma$

$$\begin{aligned}\|\delta\tilde{v}\|_{H^1(\tilde{R}_0)} &\leq C \|\delta\tilde{v} \circ \tau\|_{H^1(R_Z)} \\ &\leq C' \|\delta\tilde{v}\|_{H^1(\tilde{R}_2)}\end{aligned}$$

and similarly for L^2 -norms.

Here we have introduced the habit, to which we shall uniformly adhere, of denoting by C, C', \dots constants which may be chosen uniformly over Σ . We shall make no attempt to identify optimum choices of such constants, so that the end result of our argument is only qualitative in nature.

Recall from Section 2 that the determination of c from band-limited data requires that c be “rough” in some sense. For convenience, we restate the basic “roughness” criterion, phrased in terms of an (arbitrary) Dirac kernel

$$h_\epsilon(z) = \frac{1}{\epsilon} h_1\left(\frac{z}{\epsilon}\right)$$

where h_1 satisfies the usual requirements:

$$h_1 \in C_0^\infty(\mathbf{R}), \quad h_1(0) > 0, \quad h_1 \geq 0, \\ \int h_1 = 1.$$

Then $\{\epsilon h'_\epsilon\}$ is bounded in L^1 , and defines a family of “low-cut” filters, i.e. convolution with $\epsilon h'_\epsilon$ suppresses Fourier components at frequencies $\leq 0(\frac{1}{\epsilon})$.

For $c \in H_{\text{loc}}^1(\mathbf{R})$, $\epsilon > 0$, $\Delta > 0$ define

$$r[c](z, \epsilon, \Delta) = \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |\epsilon h'_\epsilon * c|^2 \\ \bar{r}[c](z, \Delta) = \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |c|^2$$

and for any $Z_0 > 0$,

$$r_*[c](Z_0, \epsilon, \Delta) = \inf_{0 \leq z \leq Z_0} r[c](z, \epsilon, \Delta) \\ r^*[c](Z_0, \epsilon, \Delta) = \sup_{0 \leq z \leq Z_0} r[c](z, \epsilon, \Delta) \\ \bar{r}^*[c](Z_0, \Delta) = \sup_{0 \leq z \leq Z_0} \bar{r}[c](z, \Delta)$$

The main step in the proof of *Theorem 1* is embodied in:

Theorem 4 *There exist constants \bar{m} , M_1 , M_2 , ϵ_0 , and $\Delta_0 > 0$ depending on T_0, T_1, T_2, ℓ^* , K_0, K_1 , and K^* so that if*

- (i) $f \in \mathcal{E}'$ satisfies (4.1) and $m_f^1 \leq \bar{m}$;
- (ii) $\tilde{v} \in \sum$ is consistent with $D \in L^2(R_1)$, i.e.

$$S[\tilde{v}] = D$$

- (iii) $\tilde{v} \in \sum$ is coherent, $\tilde{v} = \tilde{v}[c]$ for $c \in H_{\text{loc}}^1(\mathbf{R})$;

(iv) for some $\epsilon, \Delta, Z_0 > 0$ with $\epsilon \leq \epsilon_0, \Delta \leq \Delta_0$, and

$$\tau(Z_0, P_1) = 2 \int_0^{Z_0} dz \sqrt{\frac{1}{c^2(z)} - P_1^2} \geq T_0$$

the following inequalities hold:

$$r_*[c](Z_0, \epsilon, \Delta) \geq M_1$$

$$M_2 r_*[c](Z_0, \epsilon, \Delta) \geq \max(r^*[c](Z_0, \epsilon, \Delta), \bar{r}[c](Z_0, \Delta))$$

Then there exists $\mu > 0$ depending on $T_0, T_1, T_2, \ell^*, K_0, K_1, \sigma$, and λ so that for $\delta\tilde{v} \in \Sigma$,

$$\langle \delta\tilde{v}, \text{Hess } J_{\sigma, \lambda} \delta\tilde{v} \rangle_{H^1(\bar{R}_2)} \geq \mu \|\delta\tilde{v}\|_{H^1(\bar{R}_2)}^2$$

Before giving the proof of *Theorem 4*, we digress to demonstrate the meaning of condition (iv) in the theorem, and show that the set of \tilde{v} satisfying it is non-empty.

Let $c_0, \delta c_1 \in C^\infty(\mathbf{R}) \cap L^\infty(\mathbf{R})$, so that

$$0 < r_* \leq \int_{z-1}^{z+1} |h'_1 * \delta c_1|^2 \leq r^*, \quad -\infty \leq z \leq \infty$$

and take $\bar{r} = \sup_{z \in \mathbf{R}} \int_{z-1}^{z+1} |\delta c_1|^2$. Clearly we can choose δc_1 so that $r^* \geq r_*$, \bar{r} achieve any prescribed values. On the other hand set

$$\delta c_\epsilon(z) = 2\epsilon \delta c_1\left(\frac{z}{\epsilon}\right)$$

$$c_\epsilon(z) = c_0(z) + \delta c_\epsilon(z)$$

for $0 \leq z \leq Z_0$, cut off to constants elsewhere. Then $\{\log c_\epsilon\}_{\epsilon \leq \epsilon_0}$ is bounded in $H^1[0, Z_0]$, for suitable ϵ_0 , but

$$r_* - \mathcal{O}(\epsilon) \leq \frac{1}{2\epsilon} \int_{z-\epsilon}^{z+\epsilon} |\epsilon h'_\epsilon * c'_\epsilon|^2 \leq r^* + \mathcal{O}(\epsilon)$$

$$\frac{1}{2\epsilon} \int_{z-\epsilon}^{z+\epsilon} |c'_\epsilon| = \bar{r} + \mathcal{O}(\epsilon)$$

Thus with the choice $\Delta = 2\epsilon$, the quantities

$$\begin{aligned} r_*[c_\epsilon](Z_0, \epsilon, 2\epsilon), \\ r^*[c_\epsilon](Z_0, \epsilon, 2\epsilon), \quad \bar{r}[c_\epsilon](Z_0, 2\epsilon) \end{aligned}$$

stand in more-or less fixed proportions throughout the set $\{c_\epsilon\}$. We conclude that, whatever the values of the constants $M_1, M_2, \bar{\epsilon}, \bar{\Delta}$ specified in *Theorem 4*, the set of travel-time velocities satisfying condition (iv) is non-empty — simply take $\tilde{v} = \tilde{v}[c_\epsilon]$ for sufficiently small ϵ , and a suitable choice of $c_0, \delta c_1$.

Also the meaning of condition (iv) is clear from this construction. As $\epsilon \rightarrow 0$, the perturbation δc_ϵ becomes smaller, but its derivative has uniformly bounded (above and below) mean-square over intervals of length ϵ , and this even after convolution with the oscillatory kernel $\epsilon h'_\epsilon$. Thus c_ϵ has significant oscillation everywhere on the scale ϵ , i.e., c_ϵ is “uniformly rough.”

In the estimates which follow, we will write for convenience

$$\begin{aligned} \delta Q &= DQ[\tilde{v}]\delta\tilde{v} \\ \delta\tilde{S} &= D\tilde{S}[\tilde{v}]\delta\tilde{v} \end{aligned}$$

Also, for any function u of (t, p) or (z, p) we shall denote by

$$\int u$$

the function

$$(t, p) \mapsto \int_0^t dt' u(t', p) \quad \text{or} \quad (z, p) \mapsto \int_0^z dz' u(z', p).$$

To begin the proof of *Theorem 4*, note that

$$\begin{aligned} &\langle \delta\tilde{v}, \text{Hess } J_{\sigma, \lambda}[\tilde{v}] \cdot \delta\tilde{v} \rangle \\ &= \|\delta\tilde{S}\|_{L^2(\tilde{R}_1)}^2 + \sigma^2 \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \\ &\quad + \lambda^2 \left\| \frac{\partial \delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)} \end{aligned} \tag{6.1}$$

because of the consistency and coherency assumptions.

The second term in (6.1) is most interesting. It follows immediately from the definition of Q that

$$\begin{aligned}
\delta Q &= \frac{\partial \delta \tilde{v}}{\partial p} - 2p \left(\int \tilde{v}^2 \right) \frac{\partial \delta \tilde{v}}{\partial x} - 4p \left(\int \tilde{v} \delta \tilde{v} \right) \frac{\partial \tilde{v}}{\partial x} - 3p \tilde{v}^2 \delta \tilde{v} \\
&= \left(\frac{\partial}{\partial p} (\delta \tilde{v} \circ \tau) \right) \circ \zeta - 4p \left(\int \tilde{v} \delta \tilde{v} \right) \frac{\partial \tilde{v}}{\partial x} - 3p \tilde{v}^2 \delta \tilde{v} \\
&= \left\{ \frac{\partial \delta v}{\partial p} - \frac{p}{2} \left(\int dz \delta v \right) \frac{\partial v^2}{\partial z} - 3pv^2 \delta v \right\} \circ \zeta
\end{aligned}$$

where we have written for convenience $\delta v = \delta \tilde{v} \circ \tau$. Note that δv is *not* the perturbation in v resulting from a perturbation in c .

Choose a test kernel $g \in C_0^\infty(\mathbf{R}_z)$ with $\|g\|_{L^1(\mathbf{R})} = 1$ and $\text{supp } g \subset [0, \infty)$. Then

$$\begin{aligned}
g * (\delta Q \circ \tau) &= \frac{\partial}{\partial p} g * \delta v - \frac{1}{2} p \left(\int dz \delta v \right) g * \frac{\partial v^2}{\partial z} + E_1 \\
&\quad - 3pg * (v^2 \delta v)
\end{aligned}$$

The error term E_1 is the commutator of a multiplication operator and convolution with g :

$$E_1 = p \left\{ \left(\int dz \delta v \right) g * \frac{\partial v^2}{\partial z} - g * \left(\left(\int dz \delta v \right) \frac{\partial v^2}{\partial z} \right) \right\}$$

for which a standard estimate gives

$$\|E_1(\cdot, p)\|_{L^2[0, T_2]} \leq C m_g^1 \|\delta v(\cdot, p)\|_{L^\infty(\mathbf{R})}$$

(We have used the notation

$$m_g^k = \int_{\mathbf{R}} dz z^k |g(z)|$$

for the moments of $|g|$ as explained before.)

Set

$$K_g = -\frac{1}{2} p \left(\int dz \delta v \right) g * \frac{\partial v^2}{\partial z}$$

The next goal is an L^2 estimate for K_g on the domain $R_Z = [0, Z] \times [P_1, P_2]$. Recall that Z is chosen so that

$$T_0 \leq \tau(Z, P_2) < \tau(Z, P_1) \leq T_2$$

First note that for its indefinite p -integral,

$$\int_{p_1}^{p_2} dp K_g = \int_{p_1}^{p_2} g * (\delta Q \circ \tau) - g * \delta v \Big|_{p=p_1}^{p_2} - \int_{p_1}^{p_2} dp E_1$$

Since composition with τ and ζ , the indefinite p -integral, and z -convolution with g are all bounded operators on $L^2_{\text{loc}}(\mathbf{R} \times [p_1, p_2])$,

$$\begin{aligned} \left\| \int_{p_1}^{p_2} dp K_g \right\|_{L^2[0, Z]} &\leq C \left\{ \|g * \delta v(\cdot, p_1)\|_{L^2[0, Z]} + \|g * \delta v(\cdot, p_2)\|_{L^2[0, Z]} \right. \\ &\quad \left. + \|\delta Q\|_{L^2(\tilde{R}_2)} + m_g^1 \left\| \frac{\partial \delta v}{\partial z} \right\|_{L^2(R_Z)} \right\} \end{aligned}$$

Now assume that

$$\int g = 0$$

and choose $\psi \in C_0^\infty(\mathbf{R})$ with $\psi \equiv 1$ on $[0, Z]$. For $p \in [P_1, P_2]$, denote by $\delta \tilde{v}$ the extension δv by a constant for $z > Z$. Then (since $\text{supp } g \subset \mathbf{R}^+$)

$$\begin{aligned} \|g * \delta v(\cdot, p)\|_{L^2[0, Z]}^2 &\leq \|g * \psi \delta \tilde{v}(\cdot, p)\|_{L^2[\mathbf{R}]}^2 \\ &= \frac{1}{\sqrt{2\pi}} \int dk |\hat{g}(k)|^2 |\widehat{\psi \delta \tilde{v}}(k)|^2 \\ &= \frac{1}{\sqrt{2\pi}} \int dk \frac{|\hat{g}(k)|^2}{|k|^2} |ik \widehat{\psi \delta \tilde{v}}(k)|^2 \\ &\leq \left(\sup_k \frac{|\hat{g}(k)|}{|k|} \right)^2 \left\| \frac{\partial}{\partial z} (\psi \delta \tilde{v}(\cdot, p)) \right\|_{L^2(\mathbf{R})}^2 \\ &\leq C \left(\sup_k \frac{|\hat{g}(k)|}{|k|} \right)^2 \left\| \frac{\partial \delta v}{\partial z}(\cdot, p) \right\|_{L^2([0, Z])}^2, \end{aligned}$$

since $\delta \tilde{v}$ is constant for $z > Z$ and $\delta v(0, p) \equiv 0$.

Define

$$\epsilon_g := \sup_k \frac{|\hat{g}(k)|}{|k|}$$

From the bound

$$\left\| \frac{\partial \delta v}{\partial z}(\cdot, p) \right\|_{L^2[0, Z]} \leq C \left\| \frac{\partial \delta \tilde{v}}{\partial x}(\cdot, p) \right\|_{L^2[0, T_2]}$$

valid for $\delta \tilde{v} \in \Sigma$, we get for any $P_1 \leq p_1 \leq p_2 \leq P_2$,

$$\begin{aligned} \left\| \int_{p_1}^{p_2} dp K_g \right\|_{L^2[0, Z]}^2 &\leq C \left\{ \epsilon_g \left[\left\| \frac{\partial \delta \tilde{v}}{\partial x}(\cdot, p_1) \right\|_{L^2[0, T_2/2]}^2 \right. \right. \\ &\quad \left. \left. + \left\| \frac{\partial \delta \tilde{v}}{\partial x}(\cdot, p_2) \right\|_{L^2[0, T_2/2]}^2 \right] + \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \right. \\ &\quad \left. + m_g^1 \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2)}^2 \right\} \end{aligned} \quad (6.2)$$

Next we estimate the p -derivative of K_g :

$$\begin{aligned} \frac{\partial K_g}{\partial p} &= -\frac{1}{2} \left[\left(\int dz \delta v \right) g * \frac{\partial v^2}{\partial z} \right. \\ &\quad \left. + p \left(\int dz \frac{\partial \delta v}{\partial p} \right) g * \frac{\partial v^2}{\partial z} + p \left(\int dz \delta v \right) g * \frac{\partial^2 v^2}{\partial p \partial z} \right] \end{aligned}$$

The first term is clearly dominated in $L^2[0, Z]$ for each p by a (Σ -dependent) multiple of

$$\|\delta \tilde{v}(\cdot, p)\|_{L^2[0, \frac{1}{2}T_2]}$$

From the definition of δQ ,

$$\int dz \frac{\partial \delta v}{\partial p} = \int dz \left\{ \delta Q \circ \tau - \frac{1}{2} p \left(\int dz \delta v \right) \frac{\partial v^2}{\partial z} - 3p \int v^2 \delta v \right\}$$

which is bounded by a (Σ -dependent) multiple of

$$\|\delta Q(\cdot, p)\|_{L^2[0, T_2]} + \|\delta \tilde{v}(\cdot, p)\|_{L^2[0, \frac{1}{2}T_2]}$$

Finally we note that

$$v = \frac{c}{\sqrt{1 - c^2 p^2}}$$

for some c with

$$\|c\|_{H^1[0, Z]} \leq C$$

so that for any $p \in [P_1, P_2]$,

$$\left\| \frac{\partial^2 v^2}{\partial z \partial p}(\cdot, p) \right\|_{L^2[0, Z]} \leq C$$

The upshot is the estimate

$$\begin{aligned} \left\| \frac{\partial K_g}{\partial p}(\cdot, p) \right\|_{L^2[0, Z]} &\leq C \left\{ \|\delta Q(\cdot, p)\|_{L^2[0, \frac{1}{2}T_2]} \right. \\ &\quad \left. + \|\delta \tilde{v}(\cdot, p)\|_{L^2[0, \frac{1}{2}T_2]} \right\} \end{aligned} \quad (6.3)$$

Integrating in p , we also get

$$\left\| \frac{\partial K_g}{\partial p} \right\|_{L^2(\mathbb{R}_Z)} \leq C(\|\delta Q\|_{L^2(\tilde{\mathbb{R}}_2)} + \|\delta \tilde{v}\|_{L^2(\tilde{\mathbb{R}}_2)}) \quad (6.4)$$

Now we combine (6.2) and (6.4) to estimate K_g via a simple interpolation inequality, which is a special case of Gilbarg and Trudinger (1983) *Theorem* 7.28:

For $u \in H^2[a, b]$, $\alpha > 0$:

$$\|u'\|_{L^2[a, b]}^2 \leq \alpha \|u''\|_{L^2[a, b]}^2 + \frac{C}{\alpha} \|u\|_{L^2[a, b]}^2 \quad (6.5)$$

where $C = C(|b - a|)$.

Apply (6.5) to $u(p) = \int_{p_0}^p dp' K_g(z, p')$ for arbitrary $p_0 \in [P_1, P_2]$, integrate the result in z , and use (6.2) and (6.4) to obtain

$$\begin{aligned} \|K_g\|_{L^2(\mathbf{R}_z)}^2 &\leq \frac{C}{\alpha} \left\{ \epsilon_g^2 \left([P_2 - P_1] \left\| \frac{\partial \delta \tilde{v}}{\partial x}(\cdot, p_0) \right\|_{L^2[0, \frac{1}{2}T_2]}^2 + \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{\mathbf{R}}_2)}^2 \right) \right. \\ &\quad \left. + (m_g^1)^2 \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{\mathbf{R}}_2)}^2 + \|\delta Q\|_{L^2(\tilde{\mathbf{R}}_2)}^2 \right\} + \alpha C' \left\{ \|\delta Q\|_{L^2(\tilde{\mathbf{R}}_2)}^2 + \|\delta \tilde{v}\|_{L^2(\tilde{\mathbf{R}}_2)}^2 \right\} \end{aligned}$$

With the choice

$$\alpha^* = \max(\epsilon_g, m_g^1),$$

$$\alpha_* = \min(\epsilon_g, m_g^1)$$

this becomes after integration in p_0 from P_1 to P_2

$$\begin{aligned} \|K_g\|_{L^2(\mathbf{R}_z)}^2 &\leq C \frac{(\alpha^*)^2}{\alpha_*} \left\{ \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{\mathbf{R}}_2)}^2 + \|\delta \tilde{v}\|_{L^2(\tilde{\mathbf{R}}_2)}^2 \right\} \\ &\quad + C'(\alpha^* + \frac{1}{\alpha_*}) \|\delta Q\|_{L^2(\tilde{\mathbf{R}}_2)}^2 \end{aligned} \tag{6.6}$$

The next step is to show that K_g actually dominates the indefinite integral of $\delta \tilde{v}$. It is at this point that some constraints on \tilde{v} (hence on c), other than coherency and membership in Σ , become necessary. Recall that

$$K_g = -\frac{1}{2} p \left(\int dz \delta v \right) g * \frac{\partial v^2}{\partial z}$$

Thus K_g is the product of a relatively smooth factor (the indefinite integral) and a relatively rough factor (the derivative). Clearly, some estimate of the smooth factor must be possible, provided that the rough factor is sufficiently *uniformly* rough. The following simple lemma gives a crude criterion of this type:

Lemma 2 *Suppose that $u, \Phi \in C^\infty(\mathbf{R})$. Set for $\Delta > 0$, $a < b$*

$$\begin{aligned} r(x, \Delta) &= \frac{1}{\Delta} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |u|^2 \\ r_*(\Delta) &= \inf_{x \in [a, b]} r(x, \Delta), \quad r^*(\Delta) = \sup_{x \in [a, b]} r(x, \Delta) \end{aligned}$$

Then for any $\Delta > 0$, (L^2 -norms):

$$\|\Phi u\|_{L^2[a, b]}^2 \geq \frac{r_*(\Delta)}{2} \|\Phi\|_{L^2[a, b]}^2 - \frac{16}{9} (r_*(\Delta) + r^*(\Delta)) \Delta^2 \|\Phi'\|_{L^2[a, b]}^2$$

Proof: Set

$$\bar{\Phi}_\Delta(x) = \frac{1}{\Delta} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} \Phi$$

Then the Cauchy-Schwarz inequality gives

$$|\Phi(y) - \bar{\Phi}_\Delta(x)| \leq \frac{4}{3} \Delta^{\frac{1}{2}} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi'|^2$$

for $x - \frac{\Delta}{2} \leq y \leq x + \frac{\Delta}{2}$. Thus

$$\begin{aligned} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi u|^2 &\geq \left[|\bar{\Phi}_\Delta(x)|^2 - \frac{16}{9} \Delta \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi'|^2 \right] \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |u|^2 \\ &\geq \Delta |\bar{\Phi}_\Delta(x)|^2 r_*(\Delta) - \frac{16}{9} \Delta^2 r^*(\Delta) \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi'|^2 \end{aligned}$$

Similarly

$$\Delta |\bar{\Phi}_\Delta(x)|^2 \geq \frac{1}{2} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi|^2 - \frac{16}{9} \Delta^2 \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi'|^2$$

Thus

$$\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi u|^2 \geq \frac{r_*(\Delta)}{2} \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi|^2 - \frac{16}{9} (r_*(\Delta) + r^*(\Delta)) \Delta^2 \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} |\Phi'|^2$$

Now sum both sides over $x = (k + \frac{1}{2})\Delta$, $k = 0, \dots, [\frac{b-a}{\Delta}]$ to obtain the result. q.e.d.

We shall apply *Lemma 2* to K_g , with the identifications

$$\begin{aligned} \Phi &\sim -\frac{1}{2} p \int \delta v(\cdot, p) \\ u &\sim g * \frac{\partial v^2}{\partial z}(\cdot, p) \end{aligned}$$

Note that

$$\frac{\partial v^2}{\partial z} = (1 - c^2 p^2)^{-2} c \frac{\partial c}{\partial z}$$

so that

$$g * \frac{\partial v^2}{\partial z} = (1 - c^2 p^2)^{-2} c g * \frac{\partial c}{\partial z} + E_2$$

We will assume that the length scale Δ is chosen so that $\text{supp } g \subset [\frac{\Delta}{2}, \frac{\Delta}{2}]$. Then a slight refinement of the standard error estimate gives

$$\frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |E_2| \leq C \Delta^2 r_\delta^*(\Delta) a_g(\Delta)$$

where we have written, for any compactly supported measure h , and $Z_0 > 0$ to be determined:

$$\begin{aligned} r_h^*(\Delta) &= \sup_{z \in [0, Z_0]} \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} \left| h * \frac{dc}{dz} \right|^2 \\ r_{h,*}(\Delta) &= \inf_{z \in [0, Z_0]} \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} \left| h * \frac{dc}{dz} \right|^2 \\ a_h(\Delta) &= \sup_{z \in [0, Z_0]} \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} \left(|h| * \left| \frac{dc}{dz} \right| \right)^2 \end{aligned}$$

For various Σ -dependent constants C_1 - C_4 , any $z \in [0, Z_0]$,

$$\begin{aligned} C_1 r_{g,*}(\Delta) - C_2 \Delta^2 r_\delta^*(\Delta) a_g(\Delta) \\ \leq \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} \left| g * \frac{\partial v^2}{\partial z} \right|^2 \\ \leq C_3 r_g^*(\Delta) + C_4 \Delta^2 r_g^*(\Delta) a_g(\Delta) \end{aligned}$$

Accordingly *Lemma 2* implies, so long as $Z_0 \leq Z$, for $p_0 \in [P_1, P_2]$

$$\begin{aligned} \|K_g(\cdot, p)\|_{L^2[0, Z]}^2 &\geq C p^2 r_{g,*}(\Delta) \left\| \int dz \delta v(\cdot, p) \right\|_{L^2[0, Z_0]}^2 \\ &\quad - C' (r_{g,*}(\Delta) + r_g^*(\Delta) + r_g^*(\Delta) a_g(\Delta)) \Delta^2 \|\delta v(\cdot, p)\|_{L^2[0, Z_0]}^2 \end{aligned} \quad (6.7)$$

These estimates have no force unless the quantities $r_{g,*}(\Delta)$ and

$$r_t^*(\Delta) = (r_{g,*}(\Delta) + r_g^*(\Delta) + r_g^*(\Delta) a_g(\Delta))$$

are comparable. These quantities depend on the parameters Δ and Z_0 and on the kernel g (all of which are still to be chosen) — and, of course, on the velocity profile c .

To further manipulate the norms of $\delta v, \delta \tilde{v}$, we require an estimate on the p -derivative: after some algebra,

$$\begin{aligned} & \frac{d}{dp} \left\| \int dz \delta v(\cdot, p) \right\|_{L^2[0, Z_0]}^2 \\ &= 2 \left\langle \int dz \delta v(\cdot, p), \int dz \left(\delta C \circ \tau + \frac{1}{2} p \left(\int dz \delta v \right) \frac{\partial v^2}{\partial z} + 3pv^2 \delta v \right) \right\rangle_{L^2[0, Z_0]} \\ &\leq C \left\| \int dz \delta v(\cdot, p) \right\|_{L^2[0, Z_0]}^2 + C' \|\delta C \circ \tau(\cdot, p)\|_{L^2[0, Z_0]}^2 \end{aligned}$$

Here we have employed an estimate on norms of products:

Lemma 3 For $f \in H^1[a, b]$, $g \in L^2[a, b]$, define

$$\int_a^z fg = h(z), \quad a \leq z \leq b$$

Then

$$\|h\|_0 \leq C \|f\|_1 \left\| \int_a g \right\|_0$$

for a constant C depending on a, b .

Proof: Since

$$h(z) = f(z) \int_a^z g - \int_a^z dz' f'(z') \int_a^{z'} g$$

we have

$$\begin{aligned} \|h\|_0 &\leq \|f\|_{L^\infty} \left\| \int g \right\|_0 + \left(\int_a^b dz \left[\int_a^z dz' f'(z') \int_0^{z'} g \right]^2 \right)^{1/2} \\ &\leq C \|f\|_1 \left\| \int g \right\|_0 + \int_a^b dz \left[\int_a^z dz' (f'(z'))^2 \right] \left[\int_a^z dz' \left[\int_0^{z'} g \right]^2 \right]^{1/2} \end{aligned}$$

whence the required inequality follows.

q.e.d.

Thus Gronwall's inequality gives

$$\begin{aligned} & \left| \left\| \int dz \delta v(\cdot, p) \right\|_{L^2[0, Z_0]}^2 - \left\| \int dz \delta v(\cdot, p_0) \right\|_{L^2[0, Z_0]}^2 \right| \\ &\leq C \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \end{aligned}$$

for any $p_0 \in [P_1, P_2]$. Thus

$$\begin{aligned} & \int_{P_1}^{P_2} dp p^2 \left\| \int dz \delta v(\cdot, p) \right\|_{L^2[0, Z_0]}^2 \\ & \geq \left(\left\| \int dz \delta v(\cdot, p_0) \right\|_{L^2[0, Z_0]}^2 - C \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \right) \frac{P_2^3 - P_1^3}{3} \end{aligned}$$

so that (6.7) implies, after integrating the preceding inequality in p_0 from P_1 to P_2 ,

$$\begin{aligned} & C \|K_g\|_{L^2(R_Z)}^2 \\ & \geq C' r_{g,*}(\Delta) \left\| \int dz \delta v \right\|_{L^2([0, Z_0] \times [P_1, P_2])}^2 \\ & - C'' \left\{ \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \right. \\ & \quad \left. + r_t^*(\Delta) \Delta^2 \|\delta v\|_{L^2([0, Z_0] \times [p_1, p_2])}^2 \right\} \end{aligned}$$

Now concatenate the above inequality with (6.6) to get

$$\begin{aligned} & r_{g,*}(\Delta) \left\| \int \delta v \right\|_{L^2([0, Z_0] \times [P_1, P_2])}^2 \\ & - C'' \left\{ \|\delta Q\|_{L^2(\tilde{R}_2)}^2 + r_t^*(\Delta) \Delta^2 \|\delta v\|_{L^2([0, Z_0] \times [P_1, P_2])}^2 \right\} \\ & \leq C \frac{(\alpha^*)^2}{\alpha_*} \|\delta \tilde{v}\|_{H^1(\tilde{R}_2)} + C' \left(\alpha^* + \frac{1}{\alpha_*} \right) \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \end{aligned}$$

Next recall that the L^2 -norms of δv and $\delta \tilde{v}$ are related by

$$\|\delta v\|_{L^2(R_Z)} \leq C \|\delta \tilde{v}\|_{L^2(\tilde{R}_2)}$$

as noted at the beginning of the section, which allows us to simplify the above inequality to:

$$\begin{aligned} & r_{g,*}(\Delta) \left\| \int dz \delta v \right\|_{L^2([0, Z_0] \times [P_1, P_2])}^2 \\ & \leq C \left(\frac{(\alpha^*)^2}{\alpha_*} + r_t^*(\Delta) \Delta^2 \right) \|\delta \tilde{v}\|_{H^1(\tilde{R}_2)}^2 \\ & \quad + C' \left(1 + \alpha^* + \frac{1}{\alpha_*} \right) \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \end{aligned} \tag{6.8}$$

It is convenient at this point to put the left-hand side in terms of $\delta\tilde{v}$ also, by means of *Lemma 3*. Since

$$\int_0^z dz' \delta v(z', p) = \int_0^{\tau(z, p)} dx \tilde{v}(x, p) \delta\tilde{v}(x, p)$$

we can apply *Lemma 3* with $f = (\tilde{v}(\cdot, p))^{-1}$, $g = \tilde{v}(\cdot, p) \delta\tilde{v}(\cdot, p)$, to get

$$\left\| \int \delta\tilde{v}(\cdot, p_0) \right\|_{L^2[0, \tau(Z_0, p_0)]} \leq C \left\| \int \delta v(\cdot, p_0) \right\|_{L^2[0, Z_0]}$$

whence (6.8) implies, after integration in p_0 from P_1 to P_2 ,

$$\begin{aligned} & r_{g, *}(\Delta) \left\| \int \delta\tilde{v} \right\|_{L^2(A_3)}^2 \\ & \leq C \left(\frac{(\alpha^*)^2}{\alpha_*} + r_i^*(\Delta) \Delta^2 \right) \|\delta\tilde{v}\|_{H^1(\tilde{R}_2)}^2 \\ & + C'(1 + \alpha^* + \frac{1}{\alpha_*}) \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \end{aligned} \quad (6.9)$$

Here A_3 is the region

$$\{(x, p) : P_1 \leq p \leq P_2, \quad 0 \leq x \leq \tau(Z_0, p)\}$$

At this point, we determine Z_0 : set

$$Z_0 = \zeta \left(\frac{1}{2} T_0, P_1 \right) = \inf_{p \in P_1, P_2} \zeta \left(\frac{1}{2} T_0, p \right)$$

Thus

$$A_3 \subset \tilde{R}_0$$

The interpolation inequality (6.5) and (6.9) now yield

$$\begin{aligned} \|\delta\tilde{v}\|_{L^2(A_3)}^2 & \leq \left[\beta + \frac{C}{\beta r_{g, *}(\Delta)} \left(\frac{(\alpha^*)^2}{\alpha_*} + r_i^*(\Delta) \Delta^2 \right) \right] \|\delta\tilde{v}\|_{H^1(\tilde{R}_2)}^2 \\ & + \frac{C'(1 + \alpha^* + (1/\alpha_*))}{\beta r_{g, *}(\Delta)} \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \end{aligned} \quad (6.10)$$

It follows immediately from the Dirichlet condition at $x = T_2$ in the definition of $\tilde{\Sigma}$ that

$$\|\delta\tilde{v}\|_{H^1(\tilde{R}_2)}^2 \leq \|\delta\tilde{v}\|_{H^1(\tilde{R}_0)}^2 + C \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}^2$$

It is only slightly more difficult to estimate

$$\|\delta\tilde{v}\|_{L^2(\tilde{R}_1 \setminus A_3)}^2 \leq C \left\{ \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}^2 + \|\delta Q\|_{L^2(\tilde{R}_2)}^2 \right\} \quad (6.11)$$

In fact, from the inequality

$$\left| \frac{\partial\delta v}{\partial p} \right|^2 + \left| \frac{\partial\delta v}{\partial x} \right|^2 \leq C \left(\left| \frac{\partial\delta v}{\partial x} \right|^2 + |\delta Q|^2 \right)$$

which holds at every point in \tilde{R}_2 , and the already-used inequality

$$\|\delta\tilde{v}\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)} \leq \sqrt{T_2 - T_0} \left\| \frac{\partial\delta v}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}$$

(Dirichlet condition at $x = T_2$!) follows the estimate

$$\|\delta\tilde{v}\|_{H^1(\tilde{R}_2 \setminus \tilde{R}_0)} \leq C \left(\left\| \frac{\partial\delta v}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)} + \|\delta Q\|_{L^2(\tilde{R}_2)} \right)$$

whence from the trace theorem

$$\|\delta\tilde{v}(\cdot, P_1)\|_{L^2[T_0, T_2]} \leq C \left\{ \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)} + \|\delta Q\|_{L^2(\tilde{R}_2)} \right\}$$

Any point in $\tilde{R}_1 \setminus A_3$ lies on a curve of the family

$$p \mapsto \tau(z, p)$$

which joins it to the segment $[T_0, T_2] \times \{P_1\}$, and δQ is a first order partial differential operator (acting on $\delta\tilde{v}$) whose principal part is a tangent vector

field to this family of curves. Now (6.11) follows from the standard energy estimate for hyperbolic systems.

Combining (6.11) with (6.10) we get

$$\begin{aligned} \|\delta\tilde{v}\|_{L^2(\tilde{R}_1)}^2 &\leq \left[\beta + \frac{C}{\beta r_{g,*}(\Delta)} \left(\frac{(\alpha^*)^2}{\alpha_*} + r_t^*(\Delta)\Delta^2 \right) \right] \|\delta\tilde{v}\|_{H^1(\tilde{R}_0)}^2 \\ &\quad + C' \left[\|\delta Q\|_{L^2(\tilde{R}_2)}^2 + \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}^2 \right] \end{aligned}$$

where C' depends on $\beta, \alpha_*, \alpha^*$, and $r_{g,*}$ as well. Now recall the conclusion of *Theorem 3* which we rewrite in a suggestive way:

$$\begin{aligned} &\|\delta\tilde{S}\|_{L^2(\tilde{R}_1)}^2 + (m_g^1)^2 L_2^2 \left\| \frac{\partial\delta\tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_1 \setminus \tilde{R}_0)}^2 \\ &\geq L_1^2 \|\delta\tilde{v}\|_{H^1(\tilde{R}_0)}^2 - (L_0^2 + L_1^2) \|\delta\tilde{v}\|_{L^2(\tilde{R}_1)}^2 \end{aligned}$$

Evidently *Theorem 4* has been proved if we can make the various choices deferred above so that

$$(L_0^2 + L_1^2) \left[\beta + \frac{C}{\beta r_{g,*}(\Delta)} \left(\frac{(\alpha^*)^2}{\alpha_*} + r_t^*(\Delta)\Delta^2 \right) \right] < L_1^2 \quad (6.12)$$

To understand the l.h.s. of (6.12) we first examine $\alpha = \max(\epsilon_g, m_g^1)$. We relieve the reader's suspense by identifying the test kernel g with the kernel $\epsilon h'_\epsilon$ appearing in the statement of the theorem, so that the issue becomes one of choosing $\epsilon > 0$. Note that with this identification

$$m_g^1 = \int dz |z \epsilon h'_\epsilon(z)| = \mathcal{O}(\epsilon)$$

and similarly

$$\epsilon_g = \mathcal{O}(\epsilon)$$

On the other hand, from the definitions,

$$\begin{aligned} r_g^*(\Delta) &= r^*(Z_0, \Delta, \epsilon) \\ r_{g,*}(\Delta) &= r_*(Z_0, \Delta, \epsilon) \\ r_g^*(\Delta) &= \bar{r}(Z_0, \Delta) \end{aligned}$$

while Young's inequality gives

$$a_g(\Delta) \leq C\bar{r}(Z_0, \Delta)$$

provided that $\epsilon \leq \Delta$, say, so that the support of h_ϵ is contained within a fixed number of Δ -intervals. In fact, we shall adopt the convention that $\epsilon \leq C\Delta^2$ and assume that Δ is sufficiently small that $\epsilon \leq \Delta$ as well. Then

$$r_{t,*}(\Delta) \leq C(r_*(Z_0, \epsilon, \Delta) + r^*(Z_0, \epsilon, \Delta) + \bar{r}(Z_0, \Delta)^2)$$

where the constant now depends on the test wavelet h_1 as well as on Σ — this will be the default dependence for constants “ C ” for the rest of the argument.

Note the uniform (over $c \in H_{\text{loc}}^1$ corresponding to $\tilde{v} \in \Sigma$) constraints, for all $\epsilon, \Delta > 0$:

$$\begin{aligned} r_*(Z_0, \epsilon, \Delta) &\leq r^*(Z_0, \epsilon, \Delta) \leq C \\ \bar{r}(Z_0, \Delta) &\leq C \\ r^*(Z_0, \epsilon, \Delta) &\leq C\bar{r}^*(Z_0, \Delta) \end{aligned}$$

Choose a lower bound M_1 and a relative lower bound M_2 , in the statement of *Theorem 4*, subject to these constraints, and assume that c satisfies condition (iv) of the theorem (recall that we have already met the constraint on Z_0 , viz.

$$T_0 = \tau(Z_0, P_1)).$$

That is,

$$\begin{aligned} M_1 &\leq r_*(Z_0, \epsilon, \Delta) \\ \bar{r}(Z_0, \Delta) &\leq M_2 r_*(Z_0, \epsilon, \Delta) \\ r^*(Z_0, \epsilon, \Delta) &\leq M_2 r_*(Z_0, \epsilon, \Delta) \end{aligned} \tag{6.13}$$

Then the ratio

$$\frac{C}{r_{g,*}(\Delta)} \left(\frac{(\alpha^*)^2}{\alpha_*} + r_t^*(\Delta)\Delta^2 \right)$$

is $\mathcal{O}(\epsilon + \Delta^2) = \mathcal{O}(\Delta^2)$ for coherent $\tilde{v} \in \Sigma$ for which the corresponding velocity profile c satisfies (6.13). Then we can choose $\beta = \mathcal{O}(\Delta)$ so that the l.h.s. of

(6.12) becomes $\mathcal{O}(\Delta)$ as well. In particular, for sufficiently small Δ , other choices as above, the l.h.s. of (6.12) is

$$\leq C(L_0^2 + L_1^2)\Delta < \frac{1}{2}L_1^2$$

whence finally

$$\frac{1}{2}L_1^2\|\delta\tilde{v}\|_{H^1(\tilde{R}_0)}^2 \leq \|\delta\tilde{S}\|_{L^2(R_1)}^2 + C\|\delta Q\|_{L^2(\tilde{R}_2)}^2 + C'\left\|\frac{\partial\delta\tilde{v}}{\partial x}\right\|_{L^2(\tilde{R}_2\setminus\tilde{R}_0)} \quad (6.14)$$

The r.h.s. of (6.14) is bounded by a multiple of the Hessian quadratic form, the factor now depending on the penalty constants σ and λ as well. This completes the proof of *Theorem 4*. q.e.d.

The proof of *Theorem 1* is now immediate. Given appropriate $T_0, T_1, T_2, K_0, K_1, K^*, P_1, P_2, c_0, c_1$, and c^* , let Σ_c denote the collection of $c \in H_{\text{loc}}^1(\mathbf{R})$ satisfying

$$(i) \quad c(z) = c_0, \quad z < 0$$

$$(ii) \quad \|\log c\|_{L^\infty(\mathbf{R})} \leq c^* \quad (\leq -\log P_2)$$

$$(iii) \quad c(z) = c_1 \quad \text{if} \quad 2\int_0^z \sqrt{\frac{1}{c^2} - P_1^2} \geq T_0.$$

For some $\ell^* = \ell(c^*)$,

$$\|\log c\|_{H^1[0, Z]} \leq c^* \longrightarrow \|\log \tilde{v}[c](\cdot, p)\|_{H^1[0, \frac{1}{2}T_0]} \leq \ell^*$$

Then $c \in \Sigma_c \Rightarrow \tilde{v}[c] \in \Sigma$, as defined before the statement of *Theorem 4*.

Let \bar{m} be as in the statement of *Theorem 3* and assume that $f \in \mathcal{E}'(\mathbf{R})$ is chosen to satisfy

$$K^* \left\| \frac{\partial\phi}{\partial x} \right\|_{L^2(\mathbf{R})} \geq \left\| f * \frac{\partial\phi}{\partial x} \right\|_{L^2(\mathbf{R})} \geq K_1 \left\| \frac{\partial\phi}{\partial x} \right\|_{L^2(\mathbf{R})} - K_0 \|\phi\|_{L^2(\mathbf{R})}$$

$$\text{for } \phi \in H^1(\mathbf{R})$$

$$m_f^1 \leq \bar{m}$$

Choose a test wavelet h_1 as described above. Then choose $M_1, M_2, \bar{\Delta}$, and $\bar{\epsilon}$ as in the statement of *Theorem 4* and define

$$\Sigma'_c = \{c \in \Sigma_c : \text{for } \Delta, \epsilon \text{ with} \\ 0 < \Delta \leq \bar{\Delta}, \quad 0 < \epsilon \leq \bar{\epsilon}, \text{ and } Z_0 \text{ satisfying}$$

$$2 \int_0^{Z_0} \sqrt{\frac{1}{c_2} - p_1^2} = T_0,$$

the following inequalities hold:

$$M_1 \leq r_*(Z_0, \epsilon, \Delta)$$

$$M_2 r_*(Z_0, \epsilon, \Delta) \geq \max(\bar{r}(Z_0, \Delta), r^*(Z_0, \epsilon, \Delta))\}$$

Then c is constant for $z > Z_0$, so $\tilde{v}[c]$ is constant (for each $p \in [P_1, P_2]$) for $t \geq \tau(Z_0, p)$, whence *a fortiori* for $t \geq T_0$.

The remarks after the statement of *Theorem 4* show that, for arbitrary (but consistent) choices of the various parameters, the set Σ'_c is nonempty.

Finally, assume that in the definition of $J_{\sigma, \lambda}$,

$$D = S[c] = \tilde{S}[\tilde{v}[c]]$$

for $c \in \Sigma'_c$. Then

$$J_{\sigma, \lambda}[\tilde{v}[c]] = 0$$

while *Theorem 4* shows that the Hessian of $J_{\sigma, \lambda}$ at $\tilde{v}[c]$ is positive-definite.

Therefore the implicit function theorem implies the existence of

1. an open neighborhood U in $L^2(R_1)$ of the set

$$\{S[c] : c \in \Sigma'_c\}$$

2. an open neighborhood V in Σ of the set

$$\{\tilde{v}[c] : c \in \Sigma'_c\}$$

so that for each $D \in U$, the problem

$$\text{minimize}_{\tilde{v} \in \Sigma} J_{\sigma, \lambda}[\tilde{v}]$$

has a unique solution $\tilde{v} = \tilde{I}[D] \in V$, which is moreover a Lipschitz continuous function of the data D .

Define the averaging operator

$$A : \Sigma \longrightarrow L^2_{\text{loc}}(\mathbf{R})$$

by

$$A\tilde{v}(z) = \frac{1}{P_2 - P_1} \int_{P_1}^{P_2} dp \left[\left(\frac{1}{\tilde{v}(\tau(z, p), p)} \right)^2 + p^2 \right]^{-1/2}$$

Remark. A performs a version of the operation “normal moveout correction, stack” from the reflection seismic data processing stream: see e.g. Yilmaz (1987), Section 1.4.

Then for $c \in \Sigma_c$,

$$A\tilde{v}[c] = c$$

Also, A is Lipschitz continuous in the topologies indicated in its definition.

Set

$$I = A \circ \tilde{I} : U \rightarrow L^2_{\text{loc}}$$

Then I has all of the properties indicated in the statement of *Theorem 1*. In particular, for $D_1, D_2 \in U$,

$$\|I(D_1) - I(D_2)\|_{L^2[0, Z]} \leq L^* \|D_1 - D_2\|_{L^2(R_1)}$$

for suitable L^* , depending on the various parameters defining Σ'_c and on f . This completes the proof of *Theorem 1*. q.e.d.

7 Non-Elliptic Sources.

In this section we give a very brief sketch of the state of affairs when f is smooth. The necessary regularization arguments have become quite commonplace, so we shall concentrate on the steps necessary to modify the proof of *Theorem 4*.

Thus, suppose that $f \in C_0^\infty(\mathbf{R})$: then the best “near-elliptic” estimate might have the form

$$\left\| f * \frac{\partial \phi}{\partial x} \right\|_{L^2(\mathbf{R})} \geq K_1 \left\| \frac{\partial \phi}{\partial x} \right\|_{L^2(\mathbf{R})} - K_0 \|\phi\|_{L^2(\mathbf{R})} - K_2 \left\| \frac{\partial^2 \phi}{\partial x^2} \right\|_{L^2(\mathbf{R})} \quad (7.1)$$

for $\phi \in H^2(\mathbf{R})$. The size of K_2 measures the “passband” of f : i.e. if $|\hat{f}(\omega)|$ is uniformly large in an interval $\Omega_\ell \leq |\omega| \leq \Omega_h$, then $K_2 = \mathcal{O}(1/\Omega_h)$.

The analogue of *Theorem 3* is

Theorem 5 *Given $P_2 > P_1 \geq 0$, $T_1 > T_0 > 0$, $K^* \geq K_1 > 0$, $K_0, K_2 \geq 0$, and $\ell^* > 0$, there exist $\bar{m}, L_0, L_1, L_2, L_3, > 0$, so that if $f \in C_0^\infty(\mathbf{R})$ satisfies (7.1) and $m_f^1 \leq \bar{m}$, then for $\tilde{v} \in \sum \cap H_{loc}^2(\mathbf{R})$, $\delta \tilde{v} \in \sum \cap H_{loc}^2(\mathbf{R})$*

$$\begin{aligned} \|D\tilde{S}[\tilde{v}]\delta\tilde{v}\|_{L^2(\tilde{R}_1)} &\geq L_1 \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_0)} \\ &\quad - L_0 \|\delta \tilde{v}\|_{L^2(\tilde{R}_1)} - m_f^1 L_3 \left\| \frac{\partial \delta \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_1 \setminus \tilde{R}_0)} \\ &\quad - L_2 \left\| \frac{\partial^2 \delta \tilde{v}}{\partial x^2} \right\|_{L^2(\tilde{R}_1)} \end{aligned}$$

The principal new ingredient in the proof is the higher-order estimate for the plane-wave problem

$$\|DS_0[\tilde{v}, \delta']\delta\tilde{v}\|_{L^2[0, T]} \leq C_2 \|\delta\tilde{v}\|_{H^2[0, 1/2T]}$$

for $\log \tilde{v} \in H_{loc}^2(\mathbf{R})$, $\delta \tilde{v} \in H_{loc}^2(\mathbf{R})$. See for instance Suzuki (1988) for similar estimates.

Most of the proof of *Theorem 4* goes through as before, except that now the smoothness constraint implicit in *Theorem 5* conflicts with the roughness conditions. For example, for a coherent $\tilde{v} \in \Sigma \cap H_{\text{loc}}^2(\mathbf{R})$, its corresponding $c \in H_{\text{loc}}^2(\mathbf{R})$ satisfies

$$\begin{aligned} \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |\epsilon h'_\epsilon * c'|^2 &= \frac{1}{\Delta} \int_{z-\frac{\Delta}{2}}^{z+\frac{\Delta}{2}} |\epsilon h_\epsilon * c''|^2 \\ &\leq C \frac{\epsilon^2}{\Delta} \|c''\|_{L^2[z-2\Delta, z+2\Delta]}^2 \end{aligned}$$

For the constraint $\epsilon = 0(\Delta)$, which we were bound to impose this gives

$$\leq C \Delta \|c''\|_{L^2[0, Z_0]}$$

So over any bounded set in $H^2(\tilde{R}_2)$, r_* is $\mathcal{O}(\Delta)$ over coherent travel-time velocities. Thus estimates of the sort developed in Section 6 can only succeed if

- (i) f is sufficiently “broadband” that (7.1) holds with small K_2 relative to Σ, K_0, K_1, K^* ;
- (ii) target velocities exist in the intersection of Σ'_c and a sufficiently large ball in $H^2(\tilde{R}_2)$, for which the regularized cost functional

$$\begin{aligned} J_{\sigma, \lambda, \rho}[\tilde{v}] &:= \frac{1}{2} \left\{ \|\tilde{S}[\tilde{v}] - D\|_{L^2(R_1)}^2 \right. \\ &+ \sigma^2 \|C[\tilde{v}]\|_{L^2(\tilde{R}_1)}^2 + \lambda^2 \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2 \setminus \tilde{R}_0)}^2 \\ &\left. + \rho^2 \left\| \frac{\partial \tilde{v}}{\partial x} \right\|_{L^2(\tilde{R}_2)}^2 \right\} \end{aligned}$$

takes a sufficiently small value, with $\rho = \mathcal{O}(K_2)$.

Then the Hessian of $J_{\sigma, \lambda, \rho}$ will be positive-definite at target velocities as described in (ii), while the *value* of $J_{\sigma, \lambda, \rho}$ will be small enough to conclude the existence of a nearby local minimizer. Since it will no longer be possible to have $J = 0$, only an approximation will be obtained even for data corresponding exactly to $c \in \Sigma'_c \cap H_{\text{loc}}^2(\mathbf{R})$.

The reader is referred to Symes (1986b) for details of a similar construction.

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