Polyhedral Results
for the Precedence-Constrained
Knapsack Problem

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Abstract

A problem characteristic common to a number of important integer programming problems is that of precedence constraints: a transitive collection of constraints of the form $x_j \leq x_i$ with $0 \leq x_i \leq 1, 0 \leq x_j \leq 1, x_i, x_j$ integer. Precedence constraints are of interest both because they arise frequently in integer programming applications and because the convex hull of feasible integer points is the same as the region obtained by relaxing the integrality restrictions. This paper investigates the polyhedral structure of the convex hull of feasible integer points when the precedence constraints are complicated by an additional constraint.
1 Introduction

A problem characteristic common to a number of important integer programming problems is that of precedence constraints: a transitive collection of constraints of the form \( x_j \leq x_i \) where \( 0 \leq x_i \leq 1 \), \( 0 \leq x_j \leq 1 \), \( x_i, x_j \) integer. Such constraints are used to model logical precedence conditions such as "retail outlet \( j \) cannot be stocked from warehouse \( i \) unless warehouse \( i \) is built," or logical forcing conditions such as "if the generator is on in period \( j \) it must also be on in period \( i \)." Beyond the extensive practical value of precedence constraints for modeling, precedence constraints are of theoretical and computational interest because by themselves they define a polyhedron with integer vertices. Integer programs defined completely by precedence constraints can therefore be solved by relaxing the integrality restrictions and solving the resulting linear program.

More formally, let \( P_{LP} \) be the polyhedron obtained by relaxing the integrality restrictions of an integer program and let \( P \) be the convex hull of feasible integer points for the same problem. It is always true that \( P \subseteq P_{LP} \), and the observation made above is that an integer program defined entirely by precedence constraints has the very special property \( P = P_{LP} \). The purpose of the present work is to explore the polyhedral structure of \( P \) when it is defined not only by a collection of precedence constraints but is complicated by an additional constraint. It is well-known that optimizing a linear function on \( P \) when it is defined by the constraints \( 0 \leq x_i \leq 1 \) together with an arbitrary additional constraint — the so-called knapsack problem — is already NP-complete. As the knapsack problem is nothing more than a special case of the precedence-constrained knapsack problem it follows that this latter problem is also NP-complete.

While the simple knapsack problem is known to be solvable in pseudopolynomial time using dynamic programming, there is no similar formulation for the precedence-constrained knapsack problem when the variables are binary. Polyhedral solution procedures may thus prove to be the most efficient methods for solving this problem. As there are a host of important problems that can be formulated as precedence-constrained knapsack problems — for example, capital budgeting
under precedence constraints — the present work has direct practical consequences. Further, it is envisioned that the present work can be effectively used in the solution of general integer programming problems in much the same way as polyhedral results for the simple knapsack problem were used by Crowder, Johnson, and Padberg in their Lanchester prize-winning paper [1983].

While motivation for studying the precedence-constrained knapsack problem was initially provided by the properties of precedence constraints, much of the present work can be interpreted as extending polyhedral results for the simple knapsack problem. In fact, an important aspect of the present work is to examine how results for the precedence-constrained knapsack problem compare and contrast with those for the simple knapsack problem. In many ways the results for the simple knapsack problem extend quite naturally to the more general case. Yet the additional structure of the precedence-constrained problem is sufficiently rich that results for the simple knapsack problem do not fully capture it.

The following section provides necessary background material and notation for the results presented in the remaining sections. It is assumed that the reader is familiar with basic concepts related to polyhedral theory. Section 3 investigates conditions under which many of the constraints in a natural integer programming formulation of the precedence-constrained knapsack problem are facets of $P$. In Section 4 it is demonstrated that the well-known lifting procedure for the simple knapsack problem can be generalized to the case of the precedence-constrained knapsack problem. Section 5 introduces two classes of facets not found in the integer programming formulation of the problem, and Section 6 explores conditions under which a class of homogeneous facets exists.

2 Background and Notation

A partially ordered set $(V, \preceq)$ is a collection of elements $V$ together with a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. Using the obvious notation, $i \prec j$ will mean $i \preceq j$ and $i \neq j$. An element $j$ is said to cover $i$ if $i \prec j$ and there exists no element $k$ such that $i \prec k \prec j$.\[\]
Given a set $S \subseteq V$, we denote by $G(S)$ the set of elements $i \in S$ for which there are no $j \in S$ such that $j \prec i$. Likewise, we denote by $H(S)$ the set of elements $i \in S$ for which there are no $j \in S$ such that $i \prec j$.

A Hasse diagram is a directed graph $(V, E)$ with an edge $i-j \in E$ if and only if $j$ covers $i$. In order to emphasize this interpretation of the partial order the elements in $V$ generally will be referred to as vertices. In keeping with commonly accepted practice, when drawing a Hasse diagram edge direction will be implied by the relative vertical location of two vertices in the diagram — $j$ covers $i$ in such a diagram if there is an edge between $i$ and $j$ and $j$ is located above $i$ in the diagram. A lower ideal is a set $S \subseteq V$ such that if $j \in S$ and $i \preceq j$ then $i \in S$. Similarly, the lower ideal generated by a set $S$, denoted $\ell(S)$, is the set of $i$ such that $i \preceq j$ for some $j \in S$. For singleton sets $\{i\}$ we write $\ell(i)$ rather than $\ell(\{i\})$. The set of all lower ideals of a partially ordered set $(V, \preceq)$ will be denoted by $\mathcal{L}$ with $\mathcal{L}_S$ denoting the set of all lower ideals of the partially ordered set $(S, \preceq)$, $S \subseteq V$.

Given a set of elements $V$, let $x_i$ be a real-valued variable associated with element $i \in V$. For any set $S \subseteq V$, $\mathbb{R}^S$ will denote the $|S|$-dimensional space associated with the variables $x_i$, $i \in S$. The notation $x^S$ will be used for the incidence vector of $S$, namely, $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. The notation $x(S)$ will denote $\sum_{i \in S} x_i$, and as in the notation for the lower ideal of a set we write $x(i)$ rather than $x(\{i\})$ for singleton sets.

### 3 The Problem

Formally, the problem of interest can be stated as follows.

**The Precedence-Constrained Knapsack Problem (PK):**

Given a partially ordered set $(V, \preceq)$ and functions $w : V \to \mathbb{R}$, $a : V \to \mathbb{R}^+$, find an $S \in \mathcal{L}$ satisfying $a(S) \leq \alpha$ that maximizes $w(S)$. 

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Note that the coefficients \( a_i \) are restricted to be nonnegative. It will often prove useful to assume the elements of \( V \) or of some subset of \( V \) are indexed so that they satisfy the following property.

**Property 1** If \( i \leq j \) then \( i \leq j \).

Clearly there always exists such an indexing and in general many such indexings exist. Note that if \( a(\ell(i)) > \alpha \) for some \( i \in V \) then clearly \( i \) cannot be in any feasible solution to PK. Further, it is easy to determine if a vertex satisfies this inequality. We therefore assume henceforth that all \( i \in V \) satisfy \( a(\ell(i)) \leq \alpha \).

Associating a variable \( x_i \) with each \( i \in V \), a valid integer programming formulation of PK is the following.

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{\vert V \vert} w_i x_i \\
\text{s.t.} & \quad x_j \leq x_i \quad j \text{ covers } i \\
& \quad x_i \geq 0 \quad i \in H(V) \\
& \quad x_i \leq 1 \quad i \in G(V) \\
& \quad \sum_{i=1}^{\vert V \vert} a_i x_i \leq \alpha \\
& \quad x_i \text{ integer } \quad i \in V
\end{align*}
\]

Let \( P^{LP} \) denote the polyhedron defined by (1), (2), (3), and (4) and let \( P = \text{conv}(\{ x^S : S \in \mathcal{L}, a(S) \leq \alpha \}) \). Further, given a set \( S \subseteq V \), let \( P_S = \text{conv}(\{ x^{T\cap S} : T \in \mathcal{L}, a(T) \leq \alpha \}) \). Note that for \( S \in \mathcal{L} \), \( P_S = P \cap \mathbb{R}^S \).

As it is trivial to construct objective functions for which an integral optimal solution does not exist, it follows that \( P \neq P^{LP} \). However, while this implies that (1), (2), (3), and (4) do not constitute all of the facets of \( P \), most are, in fact, facets. We first make note of the dimension of \( P \) before considering conditions under which (1), (2), and (3) are facets of \( P \).
Proposition 1  \( P \) has dimension \( |V| \).

Proof: Assume \( V \) is indexed so that Property 1 is satisfied. Let \( M \) be a matrix with rows indexed \( 1, \ldots, |V| \) and let column \( i \) be the vector \( \mathbf{z}^{(i)} \). By Property 1 the matrix \( M \) is upper triangular with nonzeros on the diagonal. It follows that the columns of \( M \) together with the \( 0 \) vector form a set of \( |V| + 1 \) affinely independent points contained in \( P \).  

Proposition 2 The constraints (1) and (2) are facets of \( P \).

Proof: Consider any two \( i, j \in V \) such that \( j \) covers \( i \) and let \( J = \{k \in V : j \preceq k\} \). Assume that \( V \) is indexed so that Property 1 is satisfied and that in addition the indexing satisfies \( j = i + 1 \) and \( J = \{i+1, i+2, \ldots, |V|\} \). The existence of such an indexing is easily verified from the properties of partial orders. As in Proposition 1, let \( M \) be a matrix with rows indexed \( 1, \ldots, |V| \) and let column \( k \) be the vector \( \mathbf{z}^{(k)} \) so that the columns of \( M \) together with the \( 0 \) vector form a set of \( |V| + 1 \) affinely independent points. By Property 1 the columns \( k < i \) satisfy \( \mathbf{z}_i = \mathbf{z}_j = 0 \). By the definition of \( J \) and the fact that \( i \preceq j \), the columns \( k > i \) satisfy \( \mathbf{z}_i = \mathbf{z}_j = 1 \). It follows that the constraint \( \mathbf{z}_j \leq \mathbf{z}_i \) is a facet of \( P \).

Similarly, consider any \( i \in H(V) \) and assume \( V \) is indexed so that Property 1 is satisfied and that in addition \( i = |V| \). Letting \( M \) be the matrix described above, it is clear that the first \( |V| - 1 \) columns of \( M \) together with the \( 0 \) vector form a set of \( |V| \) affinely independent points satisfying \( \mathbf{z}_i = 0 \).

Proposition 3 A constraint of the form (3) is a facet of \( P \) if and only if

\[
a(\ell(j) \cup \{i\}) \leq \alpha
\]

for all \( j \in V \).

Proof: Assume \( V \) is indexed so that Property 1 is satisfied and that in addition \( i = 1 \). Clearly such an indexing exists since \( i \in G(V) \). Let \( M \) be a matrix with rows indexed \( 1, \ldots, |V| \) and let
column $j$ be the vector $x^{(j) \cup \{i\}}$. By assumption each vector $x^{(j) \cup \{i\}} \in P$ and clearly $x_1^{(j) \cup \{i\}} = 1$. By Property 1, $M$ is upper triangular with nonzeros on the diagonal. Affine independence of the columns of $M$ follows, completing half of the proof.

To complete the other half of the proof, assume there exists some $j \in V$ such that $a((j) \cup \{i\}) > \alpha$. This implies that any point $x^S \in P$ satisfying $x_i^S = 1$ must also satisfy $x_j^S = 0$; that is, $x^S$ resides in the $(|V| - 2)$-dimensional affine space $R^V \cap \{x \in R^V : x_i = 1\} \cap \{x \in R^V : x_j = 0\}$. As there can be at most $|V| - 1$ such points, $x_i \leq 1$ cannot be a facet of $P$. 

4 Liftings

Unlike the valid inequalities discussed in the previous section the valid inequalities discussed in the remaining sections are not generally facets of $P$ but are instead facets of a lower-dimensional polyhedron $P_S$ with $S \in \mathcal{L}$. Similar results arise in the study of the polyhedral structure of many problems. For the simple knapsack problem Padberg [1975] showed how facets of knapsack problems defined on subsets of $V$ could be algorithmically lifted to facets of the full knapsack problem on $V$. Padberg’s result was an instance of a more general result proved by Nemhauser and Trotter [1974] related to polyhedra associated with independence systems.

In this section we present a procedure for lifting facets of $P_S$ with $S \in \mathcal{L}$ to facets of $P$. The development is very much in the spirit of the results mentioned above. In fact, the contribution of the present work is the recognition that a facet of $P$ is generated if the sequential lifting procedure respects the underlying partial order on $V$.

Theorem 1 The constraint generated by Algorithm 1 is a facet of $P$.

Proof: Clearly, $\sum_{i \in V} b_i x_i \leq \beta$ is valid at the beginning of Algorithm 1 by choice. As $b_i$ is chosen at each iteration so as to maintain feasibility, the constraint generated by Algorithm 1 is valid at the conclusion of the algorithm.
Input: A facet $\sum_{i \in B} d_i x_i \leq \beta$ of $P_B$ where $B \in \mathcal{L}$.

Note: It is assumed the set $V - B$ is indexed from 1 to $|V - B|$ and satisfies Property 1.

Output: A facet $\sum_{i \in V} b_i x_i \leq \beta$ of $P$.

Algorithm 1

begin
let $b_i = d_i$ for $i \in B$, $b_i = 0$ otherwise
for $i = 1, \ldots, |V - B|$ do
  \[ b_i = \beta - \max_{\{S \in \mathcal{C} : i \in S, a(S) \leq \alpha\}} b(S) \]
end

To prove that the constraint is a facet of $P$ it remains to provide $|V|$ affinely independent points in $P$ satisfying this constraint at equality. To this end, let $L(i) \in \mathcal{L}$ be some set achieving the optimal value in the maximization problem of Algorithm 1 at iteration $i$, noting that $i \in L(i)$. Since $b_j = 0$ for $j > i$ at iteration $i$, the only reason $L(i)$ might need to contain some vertex $j > i$ would be that $j \in \ell(k)$ for some $k \in B \cup \{1, \ldots, i\}$. However, by Property 1 and the fact that $B \in \mathcal{L}$ this cannot be the case. Thus, we can further assume $L(i) \subseteq B \cup \{1, \ldots, i\}$.

By the choice of the value of $b_i$ at iteration $i$ of Algorithm 1, it follows that $b(L(i)) = \beta$ for all iterations after iteration $i$, and in particular that $b(L(i)) = \beta$ for $i = 1, \ldots, |V - B|$ at the termination of Algorithm 1. As $\sum_{i \in B} d_i x_i \leq \beta$ is a facet of $P_B$, there exists some collection of $|B|$ sets $B(i) \in \mathcal{L}_B$ with $a(B(i)) \leq \alpha$ such that the vectors $x^{B(i)}$ are affinely independent and $b(B(i)) = \beta$. Thus, to complete the proof it remains only to show that the vectors $x^{B(i)}$ and $x^{L(i)}$ are affinely independent.

Consider the matrix $M_0$ constructed as follows. Let row $i$ correspond to vertex $i \in V - B$ and let the remaining $|B|$ rows correspond to vertices in $B$ (the order being arbitrary). In the same way, let column $i$ correspond to $z^{(i)}$ and let the remaining $|B|$ columns be the vectors $x^{B(i)}$. Let $M_1$ be the $|V| \times |V| - 1$ matrix formed by subtracting column $|V|$ from each column in $M_0$ and deleting column $|V|$.
Since $B(i) \subseteq B$, the $|V - B| \times |B| - 1$ upper-right submatrix of $M_1$ is the zero matrix. Further, since $L(i)$ was chosen so that $i \in L(i)$ and $L(i) \subseteq B \cup \{1, \ldots, i\}$ the $|V - B| \times |V - B|$ upper-left submatrix of $M_1$ is upper triangular with nonzero elements on the diagonal. It follows that any vector $y$ satisfying $M_1y = 0$ must have its first $|V - B|$ elements equal to 0. As the remaining $|B| - 1$ columns corresponding to the vectors $x^{B(i)}$ are linearly independent the proof is complete.

The facet generated by Algorithm 1 will generally be affected by the indexing of the set $V - B$. All that is required of the indexing is that it must satisfy Property 1, and different choices for the indices will generally lead to different facets.

It is also interesting to note that the maximization problems solved in Algorithm 1 are instances of the problem PK itself. The difficulty of actually solving these problems is reduced in practice by recognizing that at iteration $i$ the maximization problem can be solved on $L_{B \cup \{1, \ldots, i\}}$. Further, these problems are generated in a special fashion and are not indicative of the most general problems PK. An interesting open question is whether an efficient algorithm exists for solving the maximization problems encountered in Algorithm 1.

5 Two Classes of Facets

The present section develops two classes of inequalities that can be lifted into facets of $P$ using the results of the previous section. The following property represents a natural assumption for extending known results from the simple knapsack problem, and it will prove fundamental in the proofs presented in this section. Section 6 presents a general method for relaxing this property.

Property 2 Given $S \in \mathcal{L}$, if $i, j \in H(S)$ then $\ell(i) \cap \ell(j) = \emptyset$.

The following result provides the foundation for the results presented in this section.
Theorem 2 Assume $V$ satisfies Property 2 and let $K$ be an integer with $1 \leq K \leq |H(V)| - 1$. If the knapsack constraint (4) is of the form

$$x(H(V)) \leq K$$

then the set of constraints (1), (2), (3), and (6) provide a complete description of $P$.

Proof: Denote the set of vertices in $H(V)$ by $v(1), \ldots, v(|H(V)|)$. Let $M$ be the constraint matrix defined by (1), (2), (3), and (6) after multiplying the constraints (2) by $-1$ so that the problem is defined completely by $\leq$ constraints. Due to Property 2 this matrix is block diagonal with a single complicating constraint, specifically, constraint (6). Let $N_1, \ldots, N_{|H(V)|}$ denote the block matrices along the diagonal with matrix $N_i$ corresponding to the constraints associated with vertices in $\ell(v(i))$.

Consider the dual of the linear program formed by optimizing an arbitrary objective function subject to the constraints defined by $M$. The constraint matrix $M^T$ of the dual is very nearly the constraint matrix of $|H(V)|$ disjoint network flow problems. The complications are that each matrix $N_i^T$ has columns with a single $-1$ corresponding to constraints (2), a single 1 corresponding to constraints (3), and there is a complicating column in $M^T$ corresponding to constraint (6).

These complications can be alleviated as follows. Note that if the rows of matrix $N_i^T$ are summed, the resultant vector has the property that it has a $-1$ in entries corresponding to constraints (2), a 1 in entries corresponding to constraints (3), and a 0 in entries corresponding to constraints (1). Thus, adding the rows of $M^T$ that contain rows of $N_i^T$ and subtracting the result from the row corresponding to the vertex $v(i + 1)$ has the following effect. The 1 in the column corresponding to constraint (6) in row $v(i + 1)$ is eliminated. Further, all columns of $M^T$ that contain columns of $N_i^T$ have a single 1 and a single $-1$. All other columns remain the same. Performing this row operation for $i = 1, \ldots, |H(V)| - 1$, the resultant matrix has a single 1 and a single $-1$ in every column except the column corresponding to constraint (6), which has a single 1 in the row corresponding to $v(1)$.

By a well-known result (cf. Nemhauser and Wolsey [1988 p. 542]) the transformed constraint matrix is totally unimodular, and it follows that this dual problem always has an integer optimal
solution if its right-hand-side is integral. Thus the original system of constraints defined by $M$ is totally dual integral and it follows that all of the extreme points of $P$ are integral. □

The irredundancy of the constraints in Theorem 1 depends upon whether or not the constraints (3) satisfy the conditions of Proposition 3. The constraints (1) and (2) are facets of $P$ by Proposition 2 while the constraint (6) can never be redundant as it always renders $x^Y$ infeasible. In fact, the irredundancy of (6) implies it must be a facet of $P$ which in turn implies the following corollary.

**Corollary 1** Assume $V$ satisfies Property 2 and let $K$ be an integer with $1 \leq K \leq |H(V)| - 1$.
There exist $|V|$ affinely independent vectors $x^{Q(i)}$ such that $Q(i) \in \{Q \in \mathcal{L} : x^Q(H(V)) = K\}$.

Theorem 1 and Corollary 1 contain the essence of the proofs for a number of important inequalities. Following the development of polyhedral results for the knapsack problem, we define a cover as any set $C \in \mathcal{L}$ such that $a(C) > \alpha$. A $K$-cover is a cover $C$ with $a(S) \leq \alpha$ for every $S \in \mathcal{L}_C$ such that $x^S(H(C)) \leq K - 1$ but $a(S) > \alpha$ for every $S \in \mathcal{L}_C$ such that $x^S(H(C)) \geq K$. The following theorem is an immediate consequence of the definition of a $K$-cover and Corollary 1.

**Theorem 3** Given any $K$-cover $C$ satisfying Property 2 the constraint

$$x(H(C)) \leq K - 1$$

is a facet of $P_C$.

An important special case of a $K$-cover $C$ arises when $K = |H(C)|$. As there exists no set $S \in \mathcal{L}_C$, $S \not= C$ that is a cover, a $|H(C)|$-cover is also called a minimal cover. Clearly, every $K$-cover contains a minimal cover. More important, however, is that every $K$-cover can be generated from some minimal cover by the lifting procedure described in the previous section. General $K$-covers are thus simply instances of lifted minimal covers.

Another class of inequalities is a generalization of the class of 1-configurations introduced by Padberg [1979]. Let $D \in \mathcal{L}$ be a cover such that for some fixed $k \in H(D)$, $a(D - \{k\}) \leq \alpha$. If $a(\ell(k) \cup S) \leq \alpha$ for every $S \in \mathcal{L}_{H(D) - \{k\}}$ such that $x^S(H(D) - \{k\}) \leq J - 1$ but $a(\ell(k) \cup S) > \alpha$
for every $S \in \mathcal{L}(H(D) - \{k\})$ such that $x^S(H(D) - \{k\}) \geq J$ then $D$ is a 1-configuration. Note that if $J = |H(D)| - 1$ then a 1-configuration is simply a minimal cover.

**Theorem 4** Let $D$ be a 1-configuration satisfying Property 2 and let $HK(D)$ denote any subset of $H(D) - \{k\}$ of cardinality $K$. The constraints

$$
(K - J + 1)x_k + x(HK(D)) \leq K
$$

are facets of $P_D$ for all $K$ with $J \leq K \leq |H(D)| - 1$.

**Proof:** The validity of constraints of the form (8) follows from the definition of a 1-configuration. To prove that any such constraint is a facet of $P_D$ we demonstrate that it is a facet of $P_{\ell(k) \cup \ell(HK(D))}$.

Since it is easily verified that the properties of a 1-configuration are such that lifting a constraint (8) from $P_{\ell(k) \cup \ell(HK(D))}$ to $P_D$ does not change the constraint, it follows that any constraint (8) is in fact a facet of $P_D$.

Let $\ell(k) \cup \ell(HK(D))$ be indexed so as to satisfy Property 1 with the additional condition that if $i \in \ell(k)$ and $j \in \ell(HK(D))$ then $i < j$; that is, the first $|\ell(k)|$ indices belong to the vertices in $\ell(k)$.

Construct an $|\ell(k) \cup \ell(HK(D))| \times |\ell(k) \cup \ell(HK(D))|$ matrix $M_0$ as follows. Let column 1 be the vector $x^{\ell(HK(D))}$; and let columns $i = 2, \ldots, |\ell(k)|$ be $x^{\ell(i-1) \cup \ell(HK(D))}$. Let the remaining $|\ell(HK(D))|$ columns be $x^{\ell(k) \cup Q(i)}$, where $Q(i) \in \mathcal{L}(HK(D))$, $x^{Q(i)}(HK(D)) = J - 1$, and the vectors $x^{Q(i)}$ are affinely independent. The existence of such a collection of vectors $x^{Q(i)}$ follows from Corollary 1. Note that the columns of $M_0$ are contained in $P_{\ell(k) \cup \ell(HK(D))}$ and satisfy (8) at equality.

Let $M_1$ be the $|\ell(k) \cup \ell(HK(D))| \times |\ell(k) \cup \ell(HK(D))| - 1$ matrix obtained from $M_0$ by subtracting column $|\ell(k) \cup \ell(HK(D))|$ from all other columns. The resultant $|\ell(k)| \times |\ell(k)|$ upper-left submatrix of $M_1$ is lower triangular with nonzero entries on the diagonal while the upper-right $|\ell(k)| \times |\ell(HK(D))| - 1$ submatrix is the 0 matrix. It follows that any vector $y$ satisfying $M_1y = 0$ has $y_i = 0$ for $i = 1, \ldots, |\ell(k)|$. As the remaining columns of $M_1$ are linearly independent, it follows that the columns of $M_0$ are affinely independent. □
It is not difficult to verify that except for the case where \( K = J \), constraints of the form (8) cannot arise as liftings of minimal cover inequalities so that 1-configurations are indeed fundamentally different than cover inequalities.

### 6 Rooted Facets

While many of the \( K \)-covers and 1-configurations encountered in an arbitrary precedence-constrained knapsack problem may satisfy Property 2, many may not. In this section we present simple conditions under which more general \( K \)-covers and 1-configurations give rise to homogeneous variants of the facets (7) and (8). In fact, the following theorem has a far broader scope as it provides conditions under which very general facets can be homogenized.

**Theorem 5** Let \( S, T \in \mathcal{L} \) with \( T \subseteq S \) and assume \( \sum_{i \in S - T} c_i x_i \leq \gamma \) is a facet of \( P_{S - T} \). If there exists a vertex \( k \in H(T) \) such that \( k \in \ell(i) \) for all \( i \in S \) with \( c_i > 0 \), then the constraint

\[
\sum_{i \in S - T} c_i x_i \leq \gamma x_k
\]

is a rooted facet of \( P_S \) with root \( k \).

**Proof:** The validity of a constraint of the form (9) follows immediately from the assumptions of the theorem. To prove that any such constraint is a facet, let \( S \) be indexed so as to satisfy Property 1 with the additional condition that if \( i \in T \) and \( j \in S - T \) then \( i < j \). Also, let the vertex \( k \) have index \( |T| \).

Construct an \(|S| \times |S|\) matrix \( M_0 \) as follows. Let column 1 be the 0 vector and let columns \( i = 2, \ldots, |T| \) be the vectors \( z^{(i-1)} \). Let the remaining columns \( i \) be \( z^{T_{u(Q(i)}} \), where \( Q(i) \in \mathcal{L}_{S - T} \), \( c(Q(i)) = \gamma \), and the vectors \( z^{Q(i)} \) are affinely independent. The existence of such a collection of vectors \( z^{Q(i)} \) follows from the assumption that \( \sum_{i \in S - T} c_i x_i \leq \gamma \) is a facet of \( P_{S - T} \). Note that the columns of \( M_0 \) are contained in \( P_S \) and satisfy (9) at equality. As the columns of \( M_0 \) are effectively the same as those in the matrix \( M_0 \) of Theorem 4 it follows that any constraint of the form (9) is
indeed a facet of $P_S$. □

An example of a rooted minimal cover is depicted in Figure 1.
\[ V = S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \]

\[ T = \{1, 2, 3\} \]

Knapsack Constraint: \[ \sum_{i=1}^{12} x_i \leq 11 \]

Facet of \( P_{S-T} \): \[ x_6 + x_7 + x_{12} \leq 2 \]

Facet of \( P_S \): \[ x_6 + x_7 + x_{12} \leq 2x_3 \]

Figure 1
Example of a Rooted Minimal Cover
References


