A Combinatorial Abstraction of the Shortest Path Problem and its Relationship to Greedoids

by

E. Andrew Boyd

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Abstract

A natural generalization of the shortest path problem to arbitrary set systems is presented that captures a number of interesting problems, including the usual graph-theoretic shortest path problem and the problem of finding a minimum weight set on a matroid. Necessary and sufficient conditions for the solution of this problem by the greedy algorithm are then investigated. In particular, it is noted that it is necessary but not sufficient for the underlying combinatorial structure to be a greedoid, and three extremely diverse collections of sufficient conditions taken from the greedoid literature are presented.

0.1 Introduction

Two fundamental problems in the theory of combinatorial optimization are the shortest path problem and the problem of finding a minimum weight set on a matroid. It has long been recognized that both of these problems are solvable by a greedy algorithm — the shortest path problem by Dijkstra's algorithm [Dijkstra 1959] and the matroid problem by "the" greedy algorithm [Edmonds 1971]. Because these two problems are so fundamental and have such similar solution procedures it is natural to ask if they have a common generalization. The answer to this question not only provides insight into what structural properties make the greedy algorithm work but expands the class of combinatorial optimization problems known to be efficiently solvable.

The present work is related to the broader question of recognizing general conditions under which a greedy algorithm can be used to solve a given combinatorial optimization problem. This question has received attention scattered throughout the literature, but perhaps one of the most notable efforts in this direction was the introduction of *greedoids* by Korte and Lovász [1982a]. Since that time a wealth of articles have been published on greedoids, many of which are drawn upon in the present development. Papers of particular relevance include Korte and Lovász [1983a] [1983b] [1983c] [1984a] [1984c] [1985b] and Schmidt [1985a] [1985b]. Unlike the original work of Korte and Lovász which focused on developing compatible objective functions for a given combinatorial structure, namely greedoids, the present work focuses on developing compatible combinatorial structures for a given objective function, namely, the objective function defining the shortest path problem.

This paper is organized as follows. Sections 2 and 3 introduce necessary background material on greedoids. In Section 4 the generalization of the shortest path problem is introduced and in Section 5 it is noted that the structural conditions of greedoids are necessary but not sufficient for solution of this problem. Sections 6 and 7 prove some results that are interesting in and of themselves but that are introduced largely because they are needed for the developments in the following sections. In Sections 8, 9, and 10 a highly diverse set of structural conditions are developed that are sufficient for the greedy algorithm to solve the shortest path problem. The final section makes some concluding observations and mentions interesting open questions related to the shortest path problem.

0.2 Preliminary Definitions

We begin by presenting some definitions from the theory of greedoids necessary for the development in the following sections. For a more detailed treatment, the reader is directed to Korte and Lovász [1982a] [1983c] or to Björner and Ziegler [1986].

Let $(E, \mathcal{F})$ denote a set system comprised of a finite ground set E together with a family of sets $\mathcal{F} \subseteq 2^E$. A set system is *normal* if for every $x \in E$ there exists some set X such that $x \in X \in \mathcal{F}$. Sets contained in $\mathcal{F}$ are *feasible* with all other sets *infeasible.* A *word* $x_1 \ldots x_k$ consisting of a sequence of distinct elements of E will be called feasible if $\{x_1, \ldots, x_i\} \in \mathcal{F}$ for $i = 1, \ldots, k$. A set system is *accessible* if every feasible set can be ordered so as to form a feasible word or, equivalently, if for every $X \in \mathcal{F}$ there exists an $x \in X$ such that $X - \{x\} \in \mathcal{F}$. On occasion, words will be denoted by the lower case greek letters $\alpha, \beta, \gamma, \delta$. The concatenation of two words α, β will be denoted $\alpha\beta$, and the set of elements comprising a word α without reference to order will be denoted α^*.

One of many equivalent axiomatizations of greedoids is the following.

Definition: A *greedoid* is a nonempty set system $(E, \mathcal{F})$ satisfying the following two properties.

1. $\phi \in \mathcal{F}$
2. if $X, Y \in \mathcal{F}$ and $|X| > |Y|$ then there exists an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$

Note that the defining properties of a greedoid imply that all greedoids are accessible. It will be assumed that all greedoids considered throughout this paper are normal.

Many of the definitions used in the study of matroids generalize in a natural way to greedoids. The *restriction* of a set system $(E, \mathcal{F})$ to a set $A \subseteq E$ is the set system defined by

$$\mathcal{F}|_A = \{X \in \mathcal{F} : X \subseteq A\}$$

Clearly, if $(E, \mathcal{F})$ is a greedoid then the restriction of $(E, \mathcal{F})$ is also a greedoid.

The *rank* of a set $A \subseteq E$ is defined as

$$\rho(A) = \max\{|X| : X \subseteq A\}$$

The rank of a set system $(E, \mathcal{F})$, while properly denoted $\rho(E)$, will be denoted $\rho(\mathcal{F})$. Any feasible set $X \subseteq A$ with $|X| = \rho(A)$ is called a *basis* of A.

Closure does not generalize in such a natural way. Two different closure operators will prove useful. The *rank closure* of a set $A \subseteq E$ is defined as

$$\sigma(A) = \{x \in E : \rho(A \cup \{x\}) = \rho(A)\}$$

A set is called *closed* if $A = \sigma(A)$. The outstanding property of this closure operator that makes it undesirable is that it is not monotone; that is, $A \subseteq B \not\Rightarrow \sigma(A) \subseteq \sigma(B)$. The *monotone closure* of a set $A \subseteq E$ is defined as

$$\mu(A) = \bigcap\{B : A \subseteq B, B \text{ is closed }\}$$

The greedy algorithm has a natural definition for greedoids. The following formal definition is provided for reference.

Definition: *The Greedy Algorithm.*

Let $(E, \mathcal{F})$ be a set system with an associated function $W : \mathcal{F} \to \mathbf{R}$.

Let $X = \phi$.

Choose $x \in E - X$ such that

1. $X \cup \{x\} \in \mathcal{F}$
2. $W(X \cup \{x\}) \leq W(X \cup \{y\})$ for all y such that $X \cup \{y\} \in \mathcal{F}$

Let $X = X \cup \{x\}$ and repeat until X can no longer be augmented.

0.3 Properties of Greedoids

Important special classes of greedoids are characterized by additional properties beyond that of greedoids. For example, matroids are simply greedoids with the property that every subset of a feasible set is also feasible. By far the most important and well-studied class of greedoids is that of interval greedoids, recognized by Korte and Lovász as early as their original work on greedoids [1982a].

Definition: A greedoid has the *interval property* if $X, Y, Z \in \mathcal{F}$ with $X \subseteq Y \subseteq Z$ and $X \cup \{x\} \in \mathcal{F}$, $Z \cup \{x\} \in \mathcal{F}$ implies $Y \cup \{x\} \in \mathcal{F}$. A greedoid with the interval property is called an *interval greedoid.*

Another important characterization of interval greedoids is the following, proved by Korte and Lovász [1983a].

Definition: A greedoid has the *local union property* if $X, Y, Z \in \mathcal{F}$ with $X, Y \subseteq Z$ implies $X \cup Y \in \mathcal{F}$.

Proposition 1: A greedoid has the interval property if and only if it has the local union property.

The interval property and the local union property provide alternative perspectives on interval greedoids. Arguably, both properties are so fundamental that either could be used to define interval greedoids. As the local union property plays a more fundamental role in the following development, interval greedoids are sometimes referred to as greedoids with the local union property in order to emphasize this property.

Another extremely important class of greedoids is that of *antimatroids.* Antimatroids are intimately related to recent notions of abstract convexity and have been studied under a variety of names. Korte and Lovász [1984c] credit Edelman [1980] and Jamison [1982] with first introducing antimatroids although the first development of these structures by Korte and Lovász [1982a] was independent of these two authors.

As with interval greedoids, there are a host of ways to define antimatroids. For the purposes of this paper, the following common definition will prove most useful.

Definition: A greedoid has the *union property* if $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$.

Definition: A greedoid with the union property is called an *antimatroid.*

While it is suggested by symmetry, it is not the case that matroids are greedoids with the *intersection property.*

Definition: A greedoid has the *intersection property* if $X, Y \in \mathcal{F}$ implies $X \cap Y \in \mathcal{F}$. A greedoid with the intersection property is called an *intersection greedoid.*

However, antimatroids and matroids are linked via the interval property in the following way, as noted by Korte and Lovász [1982a].

Definition: A greedoid has the *interval property without upper bounds* if $X, Y \in \mathcal{F}$ with $X \subseteq Y$ and $X \cup \{x\} \in \mathcal{F}$ implies $Y \cup \{x\} \in \mathcal{F}$. A greedoid has the *interval property without lower bounds* if $X, Y \in \mathcal{F}$ with $X \subseteq Y$ and $Y \cup \{x\} \in \mathcal{F}$ implies $X \cup \{x\} \in \mathcal{F}$.

Proposition 2: A greedoid is an antimatroid if and only if it has the interval property without upper bounds.

Proposition 3: A greedoid is a matroid if and only if it has the interval property without lower bounds.

Greedoids with the intersection property are, in fact, a proper generalization of matroids as is easily seen by the example of *poset greedoids* [Korte and Lovász 1983b]. It will be shown in what follows that the intersection property is more fundamental to the solution of the shortest path problem than the properties of matroids.

Completing the list of properties that will prove to be of interest is the following.

Definition: A greedoid has the *local intersection property* if $X, Y, Z \in \mathcal{F}$ with $X, Y \subseteq Z$ implies $X \cap Y \in \mathcal{F}$.

0.4 Support, Paths, Antipaths, and the Shortest Path Problem

Given a directed graph $G = (V, E)$, a distinguished vertex r, and a function $w : E \to \mathbf{R}^+$ assigning weights to edges, the shortest path problem is to find a directed path of minimum weight from r to every other vertex of the graph. It is well-known that if there exists a directed path to every vertex that there exists a solution to the shortest path problem that is a directed spanning tree rooted at r. Dijkstra's algorithm [Dijkstra 1959] works by constructing a nested sequence of directed trees rooted at r until a directed spanning tree is constructed. Dijkstra's procedure is greedy in that the new edge chosen at each iteration is the edge with the shortest path from r through the tree constructed to that point in the algorithm.

Among many other observations, Korte and Lovász noted that while the collection of directed trees rooted at r does not form a matroid, this collection does form a greedoid called a *directed branching greedoid* [1983b]. It is this example and the closely related example of *undirected branching greedoids* [Korte and Lovász 1983b] that motivates the following definitions.

Consider the differences between a (normal) graphic matroid and a directed branching greedoid. While every edge in the graphic matroid is a feasible set by itself this is not true in the directed branching greedoid. This dependence of elements is intuitively one of the fundamental characteristics that differentiates greedoids from matroids. In fact, for interval greedoids it is not difficult to verify the following proposition.

Proposition 4: An interval greedoid $(E, \mathcal{F})$ is a matroid if and only if $\{x\} \in \mathcal{F}$ for all $x \in E$.

As an example, note that a directed branching greedoid with $\{x\} \in \mathcal{F}$ for all $x \in E$ is a head partition matroid. Element dependence is so fundamental to the following development that it is valuable to introduce the following.

Definition: Given a set system $(E, \mathcal{F})$, a set X is said to *support* x if $x \in X \in \mathcal{F}$.

The concept of support suggests a very natural function assigning weights to subsets of E. Rather than summing the individual weights of the elements in a set, sum the values of each element's minimum value support set.

Definition: Let $(E, \mathcal{F})$ be a set system with an associated weight function $w : E \to \mathbf{R}^+$. The *value* function $v : E \times 2^E \to \mathbf{R}^+$ is defined as

$$v(x, A) = \min\{\sum_{y \in X} w(y) : x \in X \in \mathcal{F},\ X \subseteq A\}$$

where $v(x, A) = 1 + \sum_{y \in E} w(y)$ if no such X exists. $v(x, A)$ is called the *value* of $x \in A$, and any X attaining $v(x, A)$ is called a *minimum value support set for x in A*.

It is easy to see that in a directed branching greedoid $v(x, A)$ is the length of a shortest path to the vertex at the head of x in the subgraph defined by the edges in A. In a matroid, $v(x, A) = w(x)$ as long as $x \in A$.

Support sets that achieve $v(x, A)$ are special by virtue of the fact that as minimum weight sets they tend to be of small cardinality. It is trivial to prove the following.

Definition: Let $(E, \mathcal{F})$ be a set system. A set $X \in \mathcal{F}$ is a *path* for x if $x \in X$ and there exists no $Y \subset X$ such that $Y \in \mathcal{F}$ and $x \in Y$.

Proposition 5: For any set system $(E, \mathcal{F})$ there exists a path $X \subseteq A$ for x achieving $v(x, A)$.

Schmidt [1985b] was the first to introduce the concept of a path. While a minimum value support set need not be a path, the existence of a minimum value support set implies the existence of a minimum value path contained within it. The following alternative characterization of paths for greedoids is easily verified and will prove useful in the development to follow.

Proposition 6: Let $(E, \mathcal{F})$ be a greedoid. A set $X \in \mathcal{F}$ with $x \in X$ is a path for x if and only if $X - \{y\} \notin \mathcal{F}$ for $y \neq x$.

The optimization problem to be considered can now be defined as follows.

Definition: The *cumulative linear function* or *shortest path function* is a function of the form $W : 2^E \rightarrow \mathbf{R}^+$

$$W(A) = \sum_{x \in A} v(x, A)$$

Definition: *The Shortest Path Problem.* Given a set system $(E, \mathcal{F})$ and a shortest path function W, find $X_k \in \mathcal{F}$ with $|X_k| = k$ such that

$$W(X_k) = \min\{W(Y) : Y \in \mathcal{F},\ |Y| = k\}$$

It is easily verified that when $(E, \mathcal{F})$ is a directed branching greedoid the above definition is equivalent to the usual graph-theoretic definition of the shortest path problem. In addition, this definition of the shortest path problem also captures the problem of finding a minimum weight set on a matroid. The unification of these two problems is interesting in and of itself, although it follows completely from definitions.

We conclude this section with a definition that will prove useful for later developments.

Definition: Let $(E, \mathcal{F})$ be a set system. A set $X \in \mathcal{F}$ is an *antipath in* A for x if $x \notin X$ and there exists no $Y \supseteq X$ such that $Y \in \mathcal{F}$, $Y \subseteq A$, and $x \notin Y$.

For greedoids, the following alternative characterization of antipaths is easily verified.

Definition: Let $(E, \mathcal{F})$ be a greedoid. A set $X \in \mathcal{F}$ with $x \notin X$ is an antipath for x in A if and only if $X \cup \{y\} \notin \mathcal{F}$ for $y \in A - X$, $y \neq x$.

0.5 Necessary Conditions

Most of the remaining sections are devoted to developing sufficient conditions under which the greedy algorithm solves the shortest path problem. As will be seen, the sufficient conditions are so varied that no good set of characterizing conditions seem to suggest themselves. It remains an interesting open question to determine a set of structural conditions that are both necessary and sufficient for the greedy algorithm to solve the shortest path problem.

The following necessary condition presents the restrictions on the relevant class of combinatorial structures. The proof is not difficult as it follows largely from the definition of the shortest path problem and it is therefore left to the reader.

Theorem 1: Let $(E, \mathcal{F})$ be a set system. If the greedy algorithm solves the shortest path problem for an arbitrary weight function $w : E \to \mathbf{R}^+$ then $(E, \mathcal{F})$ is a greedoid.

Interval greedoids are so fundamental that it is reasonable to ask if it is further necessary for the underlying combinatorial structure to be an interval greedoid. This is not the case, however, as is demonstrated by the following example: $E = \{a, b, c\}$, $\mathcal{F} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. For any weight function $w : E \to \mathbf{R}^+$ the greedy algorithm solves the shortest path problem.

0.6 The Ordered Path Problem

Given the close relationship between the shortest path problem and the special case of the shortest path problem on a directed branching greedoid, there is a certain intuition that this problem should succumb to solution by the greedy algorithm. While this is not in general true, even for greedoids, there is a sense in which intuition proves correct. This section develops a related problem for which a greedy algorithm yields an optimal solution for all accessible set systems. This result is interesting in and of itself, but it also provides foundational results that will prove necessary for later developments.

We begin by introducing the following problem that is closely related to the shortest path problem.

Definition: *The Ordered Path Problem.* Given a set system $(E, \mathcal{F})$ find an ordering $x_1 \ldots x_{|E|}$ of E such that $v(x_i, E) \leq v(x_j, E)$ for $i \leq j$.

The ordered path problem seeks to order the $x \in E$ by their minimum value support sets in E. This is similar to but clearly not the same as the shortest path problem. The ordered path problem is introduced because the following version of the greedy algorithm solves the ordered path problem.

Definition: *The Autonomous Greedy Algorithm.* Let $(E, \mathcal{F})$ be a set system with an associated function $f : E \times 2^E \rightarrow \mathbf{R}^+$.

Let A initially be ϕ.

Choose $x \in E - A$ such that $f(x, A \cup \{x\}) \leq f(y, A \cup \{y\})$ for all y.

Let $A = A \cup \{x\}$ and repeat until $A = E$.

The difference between the autonomous greedy algorithm and the greedy algorithm is that choices are not required to maintain feasibility. Each element is chosen for its autonomous function value independent of whether or not the resultant set is feasible. The algorithm is stated for a general function f, but the case of specific interest is when f is the value function v. The ultimate goal of this section is to demonstrate that the autonomous greedy algorithm solves the ordered path problem, but some preliminary results that are interesting in and of themselves are needed first. We begin with a trivial but nonetheless important result.

Proposition 7: Let $(E, \mathcal{F})$ be a set system. If $A \supseteq B$, then $v(x, A) \leq v(x, B)$.

Proof: Any support set for x contained in B is contained in A since $A \supseteq B$. $\square$

Proposition 8: If $(E, \mathcal{F})$ is an accessible set system then $v(x, A \cup \{x\}) \leq v(y, A \cup \{y\}) \Rightarrow v(x, A \cup \{x\} \cup \{y\}) = v(x, A \cup \{x\})$.

Proof: Suppose $v(x, A \cup \{x\} \cup \{y\}) \neq v(x, A \cup \{x\})$. Then there exists a path $P \subseteq A \cup \{x\} \cup \{y\}$ for x with value $v(x, A \cup \{x\} \cup \{y\}) < v(x, A \cup \{x\})$. Clearly, $y \in P$, and since $\mathcal{F}$ is accessible $P - \{x\} \in \mathcal{F}$ so that $P - \{x\}$ is a support set for y. Certainly, $v(y, P - \{x\}) \geq v(y, A \cup \{y\})$ since $P - \{x\} \subseteq A \cup \{y\}$. However, it then follows that

$$v(y, A \cup \{y\}) \leq v(y, P - \{x\}) \leq v(x, P) = v(x, A \cup \{x\} \cup \{y\}) < v(x, A \cup \{x\})$$

This contradiction completes the proof. $\square$

Proposition 9: If $(E, \mathcal{F})$ is an accessible set system then $v(x, A \cup \{x\}) \leq v(y, A \cup \{y\}) \Rightarrow v(x, A \cup \{x\} \cup \{y\}) \leq v(y, A \cup \{x\} \cup \{y\})$.

Proof: Let $P \subseteq A \cup \{x\} \cup \{y\}$ be any minimum cost path for y in $A \cup \{x\} \cup \{y\}$. If $x \notin P$ then P is a minimum cost path for y in $A \cup \{y\}$ so that $v(y, A \cup \{x\} \cup \{y\}) = v(y, A \cup \{y\}) \geq v(x, A \cup \{x\}) = v(x, A \cup \{x\} \cup \{y\})$ with the inequality following by assumption and the second equality following from Proposition 8. If $x \in P$ then P is a support set for x in $A \cup \{x\} \cup \{y\}$ so that again $v(x, A \cup \{x\} \cup \{y\}) \leq v(y, A \cup \{x\} \cup \{y\})$. $\square$.

Propositions 8 and 9 are local properties. Using induction, these local properties yield the following global result.

Theorem 2: Let $(E, \mathcal{F})$ be an accessible set system with an associated value function v and let $x_1 \ldots x_{|E|}$ be a word generated by the autonomous greedy algorithm. Then

(a) $v(x_i, \{x_1, \ldots, x_i\}) = v(x_i, \{x_1, \ldots, x_k\}) \qquad k \geq i$

(b) $v(x_i, \{x_1, \ldots, x_k\}) \leq v(x_k, \{x_1, \ldots, x_k\}) \qquad k \geq i$

Proof: The proof is by induction on k for an arbitrary i. Suppose that for some i and for $j = i, \ldots, k$, $v(x_i, \{x_1, \ldots, x_i\}) = v(x_i, \{x_1, \ldots, x_j\})$ and $v(x_i, \{x_1, \ldots, x_j\}) \leq v(x_j, \{x_1, \ldots, x_j\})$. This is certainly true for $k = i$.

Since $x_1 \ldots x_{|E|}$ was generated by the autonomous greedy algorithm, $v(x_k, \{x_1, \ldots, x_{k-1}, x_k\}) \leq v(x_{k+1}, \{x_1, \ldots, x_{k-1}, x_{k+1}\})$. Thus, by Proposition 8, $v(x_k, \{x_1, \ldots, x_{k+1}\}) = v(x_k, \{x_1, \ldots, x_k\})$ and by Proposition 9, $v(x_k, \{x_1, \ldots, x_{k+1}\}) \leq v(x_{k+1}, \{x_1, \ldots, x_{k+1}\})$. Consequently,

$$v(x_i, \{x_1, \ldots, x_{k+1}\}) \leq v(x_i, \{x_1, \ldots, x_k\}) \leq v(x_k, \{x_1, \ldots, x_k\}) =$$
$$v(x_k, \{x_1, \ldots, x_{k+1}\}) \leq v(x_{k+1}, \{x_1, \ldots, x_{k+1}\}).$$

This completes the proof of (b). To complete the proof of (a), note that given $v(x_i, \{x_1, \ldots, x_k\}) \leq v(x_{k+1}, \{x_1, \ldots, x_{k+1}\})$ it follows by Proposition 8 that $v(x_i, \{x_1, \ldots, x_{k+1}\}) = v(x_i, \{x_1, \ldots, x_k\})$. This completes the proof. □

The theorem we seek to prove follows immediately.

Theorem 3: Let $(E, \mathcal{F})$ be an accessible set system. Then the autonomous greedy algorithm solves the ordered path problem.

Proof: By Theorem 2, for $k \geq i$, $v(x_i, E) = v(x_i, \{x_1, \ldots, x_i\}) \leq v(x_k, \{x_1, \ldots, x_k\}) = v(x_k, E)$. □

0.7 The Minmax Path Problem

A problem more closely related to the shortest path problem than the ordered path problem is the following.

Definition: The *maximum path function* is a function $W : 2^E \rightarrow \mathbf{R}^+$ of the form

$$W(A) = \max_{x \in A}\{v(x, A)\}$$

Definition: *The Minmax Path Problem.* Given a set system $(E, \mathcal{F})$ and a maximum path function W, find $X_k \in \mathcal{F}$ with $|X_k| = k$ such that $W(X_k) = \min\{W(Y) : Y \in \mathcal{F}, |Y| = k\}$.

Unlike the ordered path problem, the maximum path problem seeks to generate a feasible set in the same way as the shortest path problem. In fact, the main reason for introducing the maximum path problem is captured in the following theorem.

Theorem 4: For an interval greedoid, if the greedy algorithm solves the minmax path problem then it solves the shortest path problem.

The proof of this theorem is postponed to the end of this section. As the minmax path problem proves to be somewhat easier to work with than the shortest path problem, the remaining developments will focus on sufficient conditions for the solution of the minmax path problem.

The observation elaborated upon in this section is that the ordered path problem and the minmax path problem are virtually identical when the underlying combinatorial structure is an antimatroid. This observation is formalized in the following results, which once again will be used for developments in later sections.

Proposition 10: Let $(E, \mathcal{F})$ be an accessible set system with an associated value function v. If $x_1 \ldots x_m$ is a word generated by the autonomous greedy algorithm then for any $i \leq \rho(\mathcal{F})$ there exists a support set $P_i \subseteq \{x_1, \ldots, x_i\}$ for x_i.

Proof: Suppose that for some index i there does not exist a support set $P_i \subseteq \{x_1, \ldots, x_i\}$. Then by definition, $v(x_i, \{x_1, \ldots, x_i\}) = 1 + \sum_{x \in E} w(x)$, and by Theorem 2 for $j \geq i$, $v(x_j, E) = v(x_j, \{x_1, \ldots, x_j\}) \geq v(x_i, \{x_1, \ldots, x_i\}) = 1 + \sum_{x \in E} w(x)$. But this implies that x_j has no support set in E, for otherwise $v(x_j, E) \leq \sum_{x \in E} w(x)$. Thus, $\rho(\mathcal{F}) < i \leq \rho(\mathcal{F})$, which is a contradiction. □

Proposition 11: Let $(E, \mathcal{F})$ be an antimatroid with an associated value function v. Then any word $x_1 \ldots x_m$ can be generated by the greedy algorithm applied with respect to the maximum path function if and only if it can be generated by the autonomous greedy algorithm.

Proof: Note that both the greedy algorithm and the autonomous greedy algorithm begin with ϕ. To complete the inductive argument, assume that for some $k < m$, $x_1 \ldots x_k$ can be generated by the greedy algorithm applied with respect to the maximum path function and by the autonomous greedy algorithm. Let S be the set of possible choices for the greedy algorithm and T the set of possible choices for the autonomous greedy algorithm; that is,

$$\begin{aligned} S &= \{x : \{x_1, \ldots, x_k, x\} \in \mathcal{F} \text{ and for all } y \text{ such that} \\ &\quad \{x_1, \ldots, x_k, y\} \in \mathcal{F}, \\ &\quad \max\{v(x_i, \{x_1, \ldots, x_k, x\}), v(x, \{x_1, \ldots, x_k, x\})\} \leq \\ &\quad \max\{v(x_i, \{x_1, \ldots, x_k, y\}), v(y, \{x_1, \ldots, x_k, y\})\} \ \} \\ T &= \{x : v(x, \{x_1, \ldots, x_k, x\}) \leq v(y, \{x_1, \ldots, x_k, y\})\} \end{aligned}$$

Since $x_1 \ldots x_k$ can be generated by the autonomous greedy algorithm, Theorem 2 implies that $v(x_i, \{x_1, \ldots, x_k, x\}) \leq v(x, \{x_1, \ldots x_k, x\})$. It follows that an alternative definition for S is the following.

$$\begin{aligned} S &= \{x : \{x_1, \ldots, x_k, x\} \in \mathcal{F} \text{ and for all } y \text{ such that} \\ &\quad \{x_1, \ldots, x_k, y\} \in \mathcal{F}, \\ &\quad v(x, \{x_1, \ldots, x_k, x\}) \leq v(y, \{x_1, \ldots, x_k, y\}) \ \} \end{aligned}$$

Clearly, $S \subseteq T$ by the definition of these two sets. By Proposition 10, for any $x \in T$ there exists a support set $P \subseteq \{x_1, \ldots, x_k, x\}$ for x. Thus, by the union property of antimatroids, for any $x \in T$, $\{x_1, \ldots, x_k\} \cup P = \{x_1, \ldots, x_k, x\} \in \mathcal{F}$, implying $T \subseteq S$. Thus, the set of choices is always the same for both algorithms, completing the proof. □

Since any word generated by the greedy algorithm can be generated by the autonomous greedy algorithm, the following is an immediate corollary of Theorem 2.

Proposition 12: Let $(E, \mathcal{F})$ be an antimatroid with an associated value function v. Then any word $x_1 \ldots x_m$ generated by the greedy algorithm applied with respect to the maximum path function satisfies

(a) $v(x_i, \{x_1, \ldots, x_i\}) = v(x_i, \{x_1, \ldots, x_k\}) \qquad k \geq i$

(b) $v(x_i, \{x_1, \ldots, x_k\}) \leq v(x_k, \{x_1, \ldots, x_k\}) \qquad k \geq i$

Finally, the following important result is closely related to Proposition 12.

Theorem 5: Let $(E, \mathcal{F})$ be an interval greedoid with an associated value function v. Then any word $x_1 \ldots x_m$ generated by the greedy algorithm applied with respect to the maximum path function satisfies

(a) $v(x_i, \{x_1, \ldots, x_i\}) = v(x_i, \{x_1, \ldots, x_k\}) \qquad k \geq i$

(b) $v(x_i, \{x_1, \ldots, x_k\}) \leq v(x_k, \{x_1, \ldots, x_k\}) \qquad k \geq i$

Proof: Let $X = \{x_1, \ldots, x_m\}$. Since $x_1 \ldots x_m$ was generated by the greedy algorithm applied to $\mathcal{F}$, it can clearly be generated by the greedy algorithm applied to $\mathcal{F}|_X$. Since $\mathcal{F}$ is an interval greedoid, $\mathcal{F}|_X$ is an antimatroid. Thus, the result follows from Proposition 12. □

The proof of Theorem 4 can now be completed.

Proof of Theorem 4: Let $x_1 \ldots x_k$ be a word generated by the greedy algorithm for the minmax path problem so that by assumption, for all $j \leq k$ it follows that $\max_{i \leq j}\{v(x_i, \{x_1, \ldots, x_j\})\} \leq \max_{i \leq j}\{v(z_i, \{z_1, \ldots, z_j\})\}$ for any feasible word $z_1 \ldots z_j$. Let $y_1 \ldots y_k$ be a feasible word satisfying $\sum_{i=1}^k v(y_i, \{y_1, \ldots, y_k\}) \leq \sum_{i=1}^k v(z_i, \{z_1, \ldots, z_k\})$ for any feasible word $z_1 \ldots z_k$. Finally, for $j \leq k$ let $X_j = \{x_1, \ldots, x_j\}$ and let $Y_j = \{y_1, \ldots, y_j\}$. Assume that the specific ordering $y_1 \ldots y_k$ represents an ordering of Y_k generated by applying the greedy algorithm with respect to the maximum path function to $\mathcal{F}|_{Y_k}$. By Theorem 5 and the assumption that the greedy algorithm solves the minmax path problem it follows that

$$v(x_j, X_k) = v(x_j, X_j) = \max_{i \leq j}\{v(x_i, X_j)\} \leq$$
$$\max_{i \leq j}\{v(y_i, Y_j)\} = v(y_j, Y_j) = v(y_j, Y_k)$$

Thus, $\sum_{j=1}^k v(x_j, X_k) \leq \sum_{j=1}^k v(y_j, Y_k)$ and therefore $x_1 \ldots x_k$ is a solution to the shortest path problem. □

One implication of Theorem 5 is that the maximum path function and the shortest path function look identical to the greedy algorithm when the underlying greedoid is an interval greedoid. Since the value of the previously chosen elements remains unchanged and since the value of any augmented element is at least as large as the value of these elements, the set of greedy choices for either function is the same. Henceforth, we will take advantage of this fact when considering interval greedoids and simply refer to "the greedy algorithm" without specifying which objective function is under consideration.

0.8 Antimatroids: The Union Property

As might be suspected, antimatroids provide sufficient structure for the greedy algorithm to solve the shortest path problem. This was indirectly alluded to in the previous section but never formally proved. We now provide a formal proof.

Theorem 6: If $(E, \mathcal{F})$ is an antimatroid then the greedy algorithm solves the minmax path problem.

Proof: Let $x_1 \dots x_k$ be a word generated by the autonomous greedy algorithm with $X_k = \{x_1, \dots, x_k\}$ and let $y_1 \dots y_k$ be any feasible word with $Y_k = \{y_1, \dots, y_k\}$. Let $y'_1 \dots y'_k$ be a reordering of Y_k such that $v(y'_i, Y_k) \leq v(y'_j, Y_k)$ for $i \leq j \leq k$. Certainly, by Theorem 2 it follows that $v(x_i, X_k) \leq v(y'_i, Y_k)$. Thus, if X_k is feasible the greedy algorithm solves the minmax path problem. However, this is clearly true by Proposition 11. □

Theorem 7: If $(E, \mathcal{F})$ is an antimatroid then the greedy algorithm solves the shortest path problem.

Theorem 7 follows as an immediate corollary of Theorem 6 taken together with Theorem 4. Essentially, the greedy algorithm solves the shortest path problem because it is equivalent to the autonomous greedy algorithm on antimatroids. Thus, in some sense, antimatroids are the most natural combinatorial structure for which the greedy algorithm solves the minmax path problem.

0.9 The Intersection Property

As demonstrated in the previous section, the union property is a very natural structural condition for the solution of the shortest path problem. In complete contrast with this result is the fact that for interval greedoids the intersection property provides sufficient structure as well. The proof is actually quite straightforward.

Proposition 13: Let $(E, \mathcal{F})$ be a greedoid with the intersection property. Then there exists a unique path X for every $x \in E$.

Proof: Suppose X is not unique so that there exists a path Y for x with $X \neq Y$. Clearly, $X \not\subset Y$ and $Y \not\subset X$ otherwise the set of greater cardinality would not be a path for x by definition. Thus, $X \cap Y \subset X$, $X \cap Y \subset Y$, $x \in X \cap Y$, and $X \cap Y \in \mathcal{F}$ by the intersection property. However, this implies neither X nor Y is a path for x by definition, completing the proof. □

Clearly, any feasible set Z with $x \in Z$ contains at least one path for x, and in general it will contain many. However, it follows from Proposition 13 that if $x \in Z \in \mathcal{F}$ and $(E, \mathcal{F})$ is a greedoid with the intersection property, then Z contains only the unique path for x. The following proposition is therefore an immediate consequence of Proposition 13.

Proposition 14: Let $(E, \mathcal{F})$ be a greedoid with the intersection property. If $X, Y \in \mathcal{F}$ with $x \in X, Y$ then $v(x, X) = v(x, Y)$.

Definition: If $(E, \mathcal{F})$ is a greedoid with the intersection property then the value of the unique path for $x \in E$ will be denoted $v(x)$.

It is now possible to complete the following theorems.

Theorem 8: If $(E, \mathcal{F})$ is a greedoid with the intersection property then the greedy algorithm solves the minmax path problem.

Proof: Let $x_1 \ldots x_k$ be a word generated by the greedy algorithm and suppose that $\max_{i \leq j}\{v(x_i)\} \leq \max_{i \leq j}\{v(z_i)\}$ for all feasible words $z_1 \ldots z_j$ and all $j \leq k$. However, suppose that $\max_{i \leq k+1}\{v(x_i)\} > \max_{i \leq k+1}\{v(y_i)\}$ for some feasible word $y_1 \ldots y_{k+1}$. Clearly, this implies $v(x_{k+1}) > v(y_i)$ for $i = 1, \ldots, k+1$. By the augmentation property of greedoids, there exists a $y \in \{y_1, \ldots, y_{k+1}\}$ such that $\{x_1, \ldots, x_k\} \cup \{y\} \in \mathcal{F}$. However, the fact that $v(y) < v(x_{k+1})$ contradicts the assumption that x_{k+1} was a greedy choice for $\{x_1, \ldots, x_k\}$. □

Theorem 9: If $(E, \mathcal{F})$ is an interval greedoid with the intersection property then the greedy algorithm solves the shortest path problem.

Theorem 9 follows as an immediate corollary of Theorem 8 with the interval property assumed so that Theorem 4 can be invoked. Interval greedoids with the intersection property were deemed *semi-poset greedoids* by

Chang [1986]. It is interesting to note that Theorem 8 does not require the underlying greedoid to be an interval greedoid — the intersection property alone is sufficient for the greedy algorithm to solve the minmax path problem.

0.10 Local Union, Local Intersection, and Union Closure

The previous two sections demonstrated that the union property and the intersection property together with the interval (or local union) property are sufficient structural conditions for the greedy algorithm so solve the shortest path problem. These conditions are sufficient to capture many of the greedoids that have been studied elsewhere in the literature including the most general instances, antimatroids and semi-poset greedoids, as well as *tree shelling greedoids* [Korte and Lovász 1982a], *convex shelling greedoids* [Korte and Lovász 1982a], *vertex branching greedoids* (*vertex search greedoids* in [Björner and Ziegler [1986]), *dense branching greedoids* (*line search greedoids* in [Korte and Lovász 1982a]), poset greedoids, matroids, and *distributive supermatroids* [Korte and Lovász 1985b] among others. Strikingly, however, neither of the two sets of conditions captures directed nor undirected branching greedoids, which in fact provided much of the motivation for studying the shortest path problem on general set systems! This section develops conditions that include these combinatorial structures.

Interestingly, *local poset greedoids* [Korte and Lovász 1983b] – which embody local manifestations of the union and intersection properties – form the foundation for a third set of sufficient conditions.

Definition: A greedoid is a *local poset greedoid* if it has the local union and local intersection properties.

Both the local union and local intersection properties play central roles in the proofs to follow. However, an additional condition is required that, while seemingly somewhat more innocuous than the local union and local intersection properties, is still necessary to complete the proof. Schmidt [1985b] introduced the following closure property in the study of directed branching greedoids.

Definition: The *union closure property* for a set system $(E, \mathcal{F})$ is defined as

$$\sigma(X) \cap \sigma(Y) \subseteq \sigma(X \cup Y)$$

In his paper, Schmidt referred to the union closure property as property *BR1*. He also proved the following, which provides an alternative interpretation of the union closure property.

Proposition 15: Let $(E, \mathcal{F})$ be a local poset greedoid. $(E, \mathcal{F})$ has the union closure property if and only if every path in $\mathcal{F}$ has a unique feasible ordering.

The main theorems to be proved in this section are the following.

Theorem 10: If $(E, \mathcal{F})$ is a local poset greedoid with the union closure property then the greedy algorithm solves the minmax path problem.

Theorem 11: If $(E, \mathcal{F})$ is a local poset greedoid with the union closure property then the greedy algorithm solves the shortest path problem.

Theorem 11 follows as an immediate corollary of Theorem 10 taken together with Theorem 4. The proof of Theorem 10 is somewhat lengthy and draws upon many earlier results in this paper. We begin by stating a number of facts that are easily verified.

Proposition 16: Let $(E, \mathcal{F})$ be a greedoid and let $X \in \mathcal{F}$ with $x, y \notin X$. If $X \cup \{x\}$, $X \cup \{y\} \in \mathcal{F}$ and $y \in \sigma(X \cup \{x\})$ then $x \in \sigma(X \cup \{y\})$.

Proposition 17: Let $(E, \mathcal{F})$ be an interval greedoid and let $X \in \mathcal{F}$ with $x, y \notin X$. If $X \cup \{x\}$, $X \cup \{y\} \in \mathcal{F}$ and $y \in \sigma(X \cup \{x\})$ then $x \in \mu(X \cup \{y\})$.

Proposition 18: Let $(E, \mathcal{F})$ be an interval greedoid and let $Y, B \in \mathcal{F}$ with $Y \subset B$. Further, suppose $x \notin \sigma(Y)$ but that for any $z \in B - Y$ with $z \notin \sigma(Y)$, $x \in \sigma(Y \cup \{z\})$. Then

(a) there exists a unique $y \in B - Y$ such that $A = B - \{y\} \cup \{x\} \in \mathcal{F}$

(b) Y is an antipath for y in B and for x in A

Proof: (a) Consider the set $Q = \{q \in (B \cup \{x\}) - Y : Y \cup \{q\} \in \mathcal{F}\}$. Clearly, any basis of $B \cup \{x\}$ containing Y must contain some $q \in Q$. By assumption, $x \in Q$. Further, since $Y, B \in \mathcal{F}$ and $Y \subset B$, the augmentation property of greedoids implies that there exists a $y \in B - Y$ such that $y \in Q$. We wish to show that y is unique and then to argue that this y satisfies the conditions stipulated in (a).

Suppose y is not unique; that is, there exist distinct $y_1, y_2 \in B - Y$ with $y_1, y_2 \in Q$. Since $x \in \sigma(Y \cup \{y_1\})$ and $x \in \sigma(Y \cup \{y_2\})$ by assumption, Proposition 16 implies that $y_1, y_2 \in \sigma(Y \cup \{x\})$, and since $\mathcal{F}$ has the interval property it follows by Proposition 17 that $y_1, y_2 \in \mu(Y \cup \{x\})$. Let Z be any basis of $B \cup \{x\}$ containing $Y \cup \{x\}$. Clearly, $y_1, y_2 \notin Z$ since $y_1, y_2 \in \mu(Y \cup \{x\})$. However, this implies $|Z| \leq |B \cup \{x\}| - 2 < |B|$ and since $B \in \mathcal{F}$, Z cannot be a basis of $B \cup \{x\}$. It follows that y is unique.

Thus, $Q = \{x, y\}$. As noted above, $y \in \mu(Y \cup \{x\})$ and using a duplicate argument $x \in \mu(Y \cup \{x\})$. It therefore follows that every basis of $B \cup \{x\}$ containing Y contains either x or y but never both. Further, since $Y \cup \{x\} \in \mathcal{F}$ and $Y \cup \{y\} \in \mathcal{F}$ there exists a basis containing $Y \cup \{x\}$ and a basis containing $Y \cup \{y\}$. This, and the fact that the bases of $B \cup \{x\}$ have cardinality $|B|$ yield the desired result.

(b) The proof of (b) follows from the proof in (a) that $Q = \{x, y\}, x \in \mu(Y \cup \{y\})$, and $y \in \mu(Y \cup \{x\})$. □

By Proposition 13, any element in a greedoid with the intersection property has a unique path. Thus, given a greedoid $(E, \mathcal{F})$ with the local intersection property and some $X \in \mathcal{F}$ it follows that $x \in X$ has a unique path in $\mathcal{F}|_X$. The following notation will prove useful.

Definition: Let $(E, \mathcal{F})$ be a greedoid with the local intersection property and let $X \in \mathcal{F}$. Then the unique path for any $x \in X$ in $\mathcal{F}|_X$ will be denoted $P(x, X)$.

Proposition 19: Let $(E, \mathcal{F})$ be a local poset greedoid with the union closure property. If $q \in A \in \mathcal{F}$ and Y is an antipath for q in A, then for any $z \in A$,

(a) $q \in P(z, A)$ if and only if $z \notin Y$

(b) $q \in P(z, A)$ implies $P(q, A) \subseteq P(z, A)$

(c) $q \in P(z, A)$ implies $P(z, A) \cap (Y \cup \{q\}) = P(q, A)$

Proof: (a) (only if) Suppose $q \in P(z, A)$ but $z \in Y$. Then $P(z, A) \cap Y \in \mathcal{F}$ by the local intersection property, $z \in P(z, A) \cap Y$, and $P(z, A) \cap Y \subset P(z, A)$ since $q \notin Y$, implying $P(z, A)$ is not a path for z.

(if) Suppose $z \notin Y$ but $q \notin P(z, A)$. By the local union property, $Y \cup P(z, A) \in \mathcal{F}$. However, since $q \notin Y \cup P(z, A)$ and $Y \subset Y \cup P(z, A) \subseteq A$, Y cannot be an antipath for q in A.

(b) Suppose $q \in P(z, A)$ but $P(q, A) \not\subseteq P(z, A)$. Then $P(q, A) \cap P(z, A) \in \mathcal{F}$ by the local intersection property, $q \in P(q, A) \cap P(z, A)$, and $P(q, A) \cap P(z, A) \subset P(q, A)$, implying $P(q, A)$ is not a path for q in A.

(c) By (b), $P(q, A) \subseteq P(z, A)$. Since $Y \cup \{q\} \in \mathcal{F}$ and $Y \cup \{q\} \subseteq A$ it follows that $P(q, A) \subseteq Y \cup \{q\}$. Thus, $P(q, A) \subseteq P(z, A) \cap (Y \cup \{q\})$.

Suppose that $P(z, A) \cap (Y \cup \{q\}) \not\subseteq P(q, A)$; that is, there exists a $p \in P(z, A) \cap (Y \cup \{q\})$ but $p \notin P(q, A)$. Certainly, since $p \notin P(q, A)$, p and q cannot be the same and it follows that $p \in (Y \cup \{q\})$ implies $p \in Y$. By (a) it thus follows that $q \notin P(p, A)$. By (b), $P(q, A) \subseteq P(z, A)$ and $P(p, A) \subseteq P(z, A)$.

Let α and β be feasible words with $\alpha^* = P(p, A)$ and $\beta^* = P(q, A)$. Further, let γ and δ be such that $\alpha\gamma$ and $\beta\delta$ are feasible words with $(\alpha\gamma)^* = (\beta\delta)^* = P(z, A)$. The proposed α, β, γ, and δ are certain to exist by the conclusions proved above and the properties of greedoids. Clearly, $p \in \alpha^*$ and $q \in \gamma^*$ while $q \in \beta^*$ and $p \in \delta^*$. However, this implies that the path $P(z, A)$ has two distinct feasible orderings, and so by Proposition 15 this contradicts the assumption that $\mathcal{F}$ has the union closure property. It follows that $P(z, A) \cap (Y \cup \{q\}) \subseteq P(q, A)$, completing the proof. □

Proof of Theorem 10: Let $x_1 \ldots x_m$ with $X = \{x_1, \ldots, x_m\}$ be a word generated by the greedy algorithm for the minmax path problem and further assume that for all $j \leq m$, $\max_{i \leq j}\{v(x_i, \{x_1, \ldots, x_j\})\} \leq \max_{i \leq j}\{v(z_i, \{z_1, \ldots, z_j\})\}$ for any feasible word $z_1 \ldots z_j$. Further, assume that for all $i > m$ there exists a minmax set of cardinality i containing X but that for some $k > m$ there does not exist a minmax set of cardinality k containing $X \cup \{x\}$, where x is a greedy choice for X. Let B be any minmax set of cardinality k. The proof will construct a set A of cardinality k containing $X \cup \{x\}$ such that $\max_{z \in A}\{v(z, A)\} \leq \max_{z \in B}\{v(z, B)\}$. Since any minmax set of any cardinality contains ϕ, it follows by induction that the greedy algorithm solves the minmax path problem.

Let Y be a maximum cardinality set such that $X \subseteq Y \subseteq B$ and $Y \cup \{x\} \in \mathcal{F}$. Suppose $Y = B$ so that $B \cup \{x\} \in \mathcal{F}$. Let $Z \subseteq B \cup \{x\}$ be a set generated by the greedy algorithm applied to $\mathcal{F}|_{B \cup \{x\}}$ with $|Z| = k$ and $X \cup \{x\} \subseteq Z$. Since $X \cup \{x\}$ can be generated by the greedy algorithm applied to $\mathcal{F}$, $X \cup \{x\}$ certainly can be generated by the greedy algorithm applied to $\mathcal{F}|_{B \cup \{x\}}$ and therefore such a Z is certain to exist. Since $\mathcal{F}$ has the interval property, $\mathcal{F}|_{B \cup \{x\}}$ is an antimatroid and so by Theorem 6, $\max_{z \in Z}\{v(z, Z)\} \leq \max_{z \in B}\{v(z, B)\}$; that is, $Z \in \mathcal{F}, X \cup \{x\} \subseteq Z$, and Z is at least as good as the minmax solution B, which is a contradiction.

Thus, suppose $Y \subset B$. By Proposition 18 there exists a unique y such that $A = B - \{y\} \cup \{x\} \in \mathcal{F}$, and Y is an antipath for y in B and for x in A. It follows from Proposition 19a that for $z \in Y$, $x \notin P(z, A)$ and $y \notin P(z, B)$. Thus, for $z \in Y$, $P(z, A) = P(z, B)$ and thus $v(z, A) = v(z, B)$. By Theorem 5, $v(x, A) \leq v(y, B)$. Therefore, the elements of interest are those $z \in Z = \{z : z \neq x, z \neq y, z \notin Y\}$. Certainly, if it can be demonstrated that for $z \in Z$, $v(z, A) \leq v(z, B)$ then the proof is complete.

Since Y is an antipath for y in B and for x in A, it follows by Proposition 19a that $Z = \{z : x \in P(z, A)\} = \{z : y \in P(z, B)\}$. Consider an arbitrary $z \in Z$. By the local union property, $(Y \cup \{y\}) \cup P(z, B) \in \mathcal{F}$. Since $P(y, B) \subseteq P(z, B)$ by Proposition 19b, $P(z, B)$ can be partitioned into $P(y, B)$ and $S(z) = P(z, B) - P(y, B)$, and by Proposition 19c, $S(z) \cap (Y \cup \{y\}) = \phi$. By Proposition 17, $y \in \mu(X \cup \{x\})$ and so it follows by augmenting $Y \cup \{x\}$ from $Y \cup \{y\} \cup S(z)$ that $Y \cup \{x\} \cup S(z) \in \mathcal{F}$.

Since Y is an antipath for x in A, it is clearly an antipath for x in $Y \cup \{x\} \cup S(z) \subseteq A$. By Propostion 13, $P(z, Y \cup \{x\} \cup S(z)) = P(z, A)$, and therefore $x \in P(z, Y \cup \{x\} \cup S(z))$. Thus, by Proposition 19c, $P(z, Y \cup \{x\} \cup S(z)) \cap (Y \cup \{x\}) = P(x, Y \cup \{x\} \cup S(z))$, and it follows that

$$
\begin{aligned}
& P(z,A) \\
= \; & P(z,A) \cap A \\
= \; & P(z, Y \cup \{x\} \cup S(z)) \cap (Y \cup \{x\} \cup S(z)) \\
= \; & [P(z, Y \cup \{x\} \cup S(z)) \cap (Y \cup \{x\})] \cup [P(z, Y \cup \{x\} \cup S(z)) \cap S(z)] \\
\subseteq \; & P(x, Y \cup \{x\} \cup S(z)) \cup S(z) \\
= \; & P(x,A) \cup S(z)
\end{aligned}
$$

with the last equality following from Proposition 13. Since by Theorem 5

$$\sum_{q \in P(x,A)} w(q) = v(x,A) \leq v(y,B) = \sum_{q \in P(y,B)} w(q)$$

it follows that

$$v(z,A) \leq \sum_{q \in P(x,A) \cup S(z)} w(q) \leq \sum_{q \in P(y,B) \cup S(z)} w(q) = v(z,B)$$

completing the proof. □

0.11 Conclusions

Theorem 11 provides a somewhat elaborate proof that the shortest path problem on directed or undirected branching greedoids is solvable by the greedy algorithm. Theorem 11 can be invoked due to the work of Schmidt [1985a][1985b] who proved that both directed and undirected branching greedoids are local poset greedoids with the union closure property. Schmidt also noted that both of these classes of greedoids have additional properties, and it is not difficult to show that local poset greedoids with the union closure property are a strict relaxation of directed and undirected branching greedoids. In fact, it is easy to see that matroids are local poset greedoids with the union closure property. Local union and local intersection follow from the hereditary property of matroids, and the union closure property follows from the monotonicity of the rank closure operator. Theorem 11 thus provides a unifying proof that Dijkstra's algorithm solves the usual graph-theoretic shortest path problem and that "the" greedy algorithm finds a minimum weight set on a matroid.

While the greedy algorithm does not solve the shortest path problem on arbitrary greedoids, it does on a variety of interesting greedoids that extend well-beyond directed and undirected branching greedoids. Certainly, it remains an interesting question as to whether or not there exist a set of structural conditions that are both necessary and sufficient for the greedy algorithm to solve the shortest path problem. The extremely diverse nature of the sufficient conditions presented in this paper, however, suggests that this may not be an easy question to answer. Some encouraging insight, though, is provided by dense branching greedoids.

Definition: Let $G = (V, E)$ be a directed graph with a distinguished vertex r and let $\mathcal{F} = \{X \subseteq E$: there exists a directed path from r to the tail vertex of every $x \in X\}$. $(E, \mathcal{F})$ defines a greedoid called a *directed dense branching greedoid.*

It is easily demonstrated that directed dense branching greedoids are antimatroids, and so the shortest path problem is solved by the greedy algorithm. In fact, the results of the greedy algorithm can be interpreted as the shortest path to each edge in the graph as opposed to each vertex. The point being raised is that behind each directed branching greedoid lurks an antimatroid, namely, a directed dense branching greedoid. It is the suspicion of the author that either the union or intersection property is somehow always responsible for the success of the greedy algorithm in solving the shortest path problem, but whether or not this is true and whether or not it can be formalized remains to be seen.

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