Electromagnetic Propagation and Scattering in Spherically-Symmetric Terrestrial System-Models

by

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ELECTROMAGNETIC PROPAGATION AND SCATTERING IN SPHERICALLY-SYMMETRIC TERRESTRIAL SYSTEM-MODELS

by

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ABSTRACT

A study of the quantitative solut1onal approaches to boundary-value problems associated with terrestrial electromagnetic propagation is carried out, with particular attention given to spherical-system models and the frequency range below 3 MHz. The field solutions are determined from dyadic Green's functions, the elements of which are infinite Bessel-Legendre (zonal harmonic) series. The three classical approaches to evaluating these solutions -- mode theory, wave-hop, and summation of the zonal harmonic series -- are surveyed, and techniques to improve the rate of convergence of the latter series are investigated. Provided the field point is not too near the source, the more effective methods prove to be certain Shanks' transformations, an application of the generalized Euler transformation when an asymptotic expansion for the Legendre function is substituted, and a transformation developed using summation by parts that appears to be new. Rumer's transformation, Cesaro summation, and repeated-averaging are also considered. The groundwave (two-media) series solutions are shown to be summable with an order of magnitude fewer terms than the number required by earlier researchers.
I should like to express both my thanks and appreciation to Dr. David Middleton for his patience and persistence throughout this thesis effort, and my gratitude to Dr. Rui dafigueiredo and the Mathematical Sciences department for providing me with the opportunity to work with Dr. Middleton.

I am most grateful to Dr. Robert L. Durfee and my colleagues (particularly J. R. Vaill) at Field Research and Engineering for their encouragement and support. The necessity of coordinating the typing of the text with the mathematical entries made both tasks all the more difficult, and I am indebted to Nancy Downie and Sally Gravely for bearing the brunt of this burden.

Finally, though words are not entirely adequate, I wish to thank my wife, Janell Elschleig McKay, and my parents for their encouragement during my studies.
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I. INTRODUCTION

The intent of this investigation is the quantitative solution of selected boundary-value problems associated with terrestrial electromagnetic propagation, with particular attention given to the frequency bands below 3 MHz. This is a classical problem of long standing. Nevertheless, despite the many important results obtained by earlier researchers, standard field-calculation techniques at these frequencies still do not provide uniform coverage. While this is due in part to the inherent complexity of adequate media models, the numerical solutions for even the simpler earth/ionosphere structures are sufficiently complicated to render present methods either inadequate or extremely difficult to apply over certain frequencies and range domains.

We address this latter problem, advancing a largely neglected technique: summation of the zonal harmonic (Bessel-Legendre) series solutions for spherical systems. Using analytical and numerical methods (investigated and developed herein) we mitigate the principal difficulty associated with this approach — namely the slow convergence of the associated series. Spherical systems are chosen because they best model the terrestrial environment, particularly in the case of "transhorizon" propagation.

In the present investigation, the channel portion of terrestrial communication or detection systems is of primary concern; that is, assuming the source current to be known and a system model specified, our problem is to obtain the fields at the receiver. The capability to calculate such fields over a wide band of frequencies and at most ranges has important applications in system analysis and design.
1.1 PRESENT METHODS

There are three, essentially classical, approaches to evaluating the series solutions to the spherical boundary-value problem described above. The first, of course, is that of directly summing the zonal harmonic series forms — an approach that will receive much attention herein. The second method involves the use of "mode theory" (Watson 1919, 1919; Budden 1961; Watt 1962), by which the original series is transformed into a contour integral which can be evaluated by the residue theorem. Finally, there is the "wave-hop" formulation which is typically viewed as deriving from either an approximate geometrical-optic (ray theory) analysis or (more accurately) from an expansion of the denominator of the integral representation of the series solutions (Brunner 1949; Watt 1961, Berry and Chrisman 1965).

The latter two methods are in common use, with few researchers having evidenced any interest in summing the original series. Present techniques for field-evaluation in the waveguide formed by the earth and ionosphere usually involve the use of wave-hop approximations at ranges of less than 2000-4000 km in the LF (low frequency, 30-300 kHz) and upper VLF (very low frequency, 3-30 kHz) bands, and waveguide mode theory beyond 1000-2000 km at frequencies up to, perhaps, 60 kHz (Agrawal et al. 1971, Burgess and Jones 1975, Morfit and Shellman 1976, CCIR 1982: 695). The International Radio Consultative Committee (CCIR) gave the following assessment of calculational methodologies in their 1982 quadrennial report:

The theory of radio wave propagation at VLF, LF and MF, necessary to estimate field strengths is now becoming fairly well understood. Two competing theories are used to describe propagation in the VLF and LF bands: the wave-hop theory and the wave-guide mode theory. The solutions obtained from the wave-hop theory are generally most suitable for short distances and the higher (LF and MF) frequencies, while the wave-guide mode theory is most suitable for longer distances and the lower (VLF) frequencies.

(CCIR 1982: 265)

The CCIR did not address direct summation of the zonal harmonic series, but did go on to state explicitly the desirability of improving methods for predicting field strength below about 1.7 MHz (1982: Study program 310/10). In particular, encouragement was given to the development of mathematical models that allow the calculation of phase as well as amplitude at long distances, and to studies of present calculational methods "with a view to achieving improved accuracy and extending the frequency range." That direct summation was apparently not considered is not surprising; the convergence of the original series is notoriously slow, and it is not unusual for such a technique to be dismissed as impractical. Nevertheless, not only are the other methods considered inadequate for certain ranges and frequencies, but each involves its own set of difficulties (see Chapter IV).

In 1962, after a lapse of some forty years following Watson's refinement of mode theory, the summation of the zonal harmonic series was reconsidered by Johler and Berry. They used Kummer's transformation (subtracting a series with a known sum from the original series) to accelerate convergence, with Johler and Lewis (1969) and Lewis and Johler (1976) later evaluating the series at ELF (the frequency band between 3 Hz and 3000 Hz) using Kummer's transformation and repeated averaging of partial sums. They still required 10 (ka) to 15 (ka) terms (a being the earth radius and k the propagation constant of free-space which is proportional to frequency) with even more terms needed near the source or antipode. It was observed that this number could be reduced to 1.5 (ka) to 2 (ka) if the "groundwave" component (i.e., the two-media solution) were separated out and calculated by some other means (Lewis and Johler 1976). Similar findings prevail at VLF with Johler (1970) reporting that groundwave calculations required 15 (ka) terms, and Jones and Howorth (1982) reporting 10 (ka) terms required for the convergence of the three-media series solution at VLF. [Note that ka = 2/15 \pi, where \pi is the frequency in Hz; so, at 30 kHz, 10 (ka) terms is approximately 4.10^6 per field point.]
1.2 NEW RESULTS

The difficulties associated with conventional calculational methods for certain ranges and frequencies have prompted us to consider direct summation of the zonal harmonic series and the acceleration of their convergence.

We find that the number of terms required to evaluate the groundwave, or two-media solution, can be reduced by an order of magnitude over that previously reported with relatively few terms beyond $\alpha$ [e.g., about $3(\alpha)_{1/2}$] required when the source and receiver are near the earth's surface. This result applies for ranges not too near the source, and also holds for the three-media problem when the groundwave solution is not separated out.

The more effective methods prove to be certain Shanks' transformations (section V.1), an application of the generalized Euler transformation (section V.B, equation V.13), and a transformation developed using the Christoffel-Darboux formula for Legendre polynomials in conjunction with summation by parts which, so far as the author can determine, is new (section V.B, equation V.84). Shanks' transformations have previously been applied to solutions of propagation problems in planar and cylindrical systems, but not to the Bessel-Legendre series of the spherical system. Our application of the generalized Euler transformation to the series when the Legendre polynomial is replaced by its asymptotic expansion has not previously been exploited. The series associated with the new transform, V.24, is given a least squares interpretation. Also, since the planar system solution is a limiting case of the spherical system solution (as shown in section IV.1.B) we obtain an integral analogue of V.04 applicable to plane-earth solutions.

We also investigate the effectiveness of Kummer's transformation (section V.C), Cesàro summation (section V.D.), and repeated averaging (section V.F) in accelerating series convergence. We apply Kummer's transformation, used by earlier investigators to remove the leading $n$-factor as $n \to \infty$ (in the series index), to remove several of the leading factors in the case of a perfectly conducting earth. We discuss the possibility of extending this approach to lossy conductors using the surface impedance concept common to mode theory. The application of Cesàro summation to this problem has not previously been made. We find that the series convergence can be accelerated with Cesàro sums, but not as markedly or as efficiently as with the techniques mentioned earlier. In the particular case of a radial dipole, source and a receiver on the surface of a sphere, we show that the desired field value ($E_\theta$) is the "Abelian" sum of a divergent series (section V.A), and the series is Cesàro summable (C, 2) but not (C, 1). Using V.04, we obtain asymptotic error estimates for finite term approximations to (C,j), $j \geq 2$. We find repeated-averaging to be an effective method for certain angles between the source and the receiver but not to possess the generality of the more useful methods.

Some additional results are mentioned in the next section.
I.3 ORGANIZATION OF THESIS

The thesis is organized as follows. In Chapter II we give the general equations governing electromagnetic propagation in a material medium. Media characteristics are then reviewed, and their significance in various propagation problems examined.

In Chapter III we develop the general integral-operator solution applicable to an inhomogeneous, possibly random, bounded medium in terms of dyadic Green's functions. Volume inhomogeneities and surface effects can be seen to constitute secondary sources; and this development, which follows that of Tai (1971), provides a framework linking most terrestrial electromagnetic problems. We then obtain the free space Green's dyadic appropriate to spherical systems. Again we follow Tai, reducing, however, the resulting series summations from two to one.

In Chapter IV, we derive the Green's dyadic series for the two and three-media spherical boundary-value problem, discuss historical developments in terrestrial propagation analysis, and review classical solutional approaches in the special case of a radial Hertzian source. We note an error in Wait's (1962) corresponding solution (see Appendix A2; subsequent approximations introduced by Wait bring his equation into accord with the correct results). The planar-earth solution is shown to be a limiting case of the series solution for a spherical earth. We note that the waveguide mode equation, as usually formulated and solved, is not the most general equation as it fails when the outer medium is given the properties of the second medium.

In Chapter V, we examine the zonal harmonic series solutions, develop new methods for accelerating their convergence, and investigate the applicability of methods used by earlier researchers on related series or integrals. In Chapter VI, we present our conclusions.

*Though the author has not seen the three-media dyadic solutions given elsewhere, the problem is a basic one with equivalent solutional formations widely available.

CHAPTER II: ELECTROMAGNETIC PROPAGATION IN TERRESTRIAL MEDIA

II.1 FUNDAMENTAL RELATIONS

The mathematical system which describes macroscopic electromagnetic phenomena in material media consists of Maxwell's equations, the Lorentz force law, and a set of media-dependent constitutive relations which serve to connect the field-vectors and, effectively, defines the (macroscopic) electrophysical properties of the medium. Maxwell's equations are also valid in random media, but should then be regarded as stochastic partial differential equations.

II.1.A MAXWELL'S EQUATIONS

In rationalized MKSC units, Maxwell's equations are written:

\[
\begin{align*}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
\n\nabla \cdot \vec{D} &= \rho_f \\
\n\nabla \cdot \vec{B} &= 0
\end{align*}
\]

where

- \(\vec{E}\) = electric field (Volts/m)
- \(\vec{H}\) = magnetic field (Amperes/m)
- \(\vec{D}\) = electric displacement (Coulombs/m\(^3\))
- \(\vec{B}\) = magnetic induction (Webers/m\(^2\))
- \(\vec{J}\) = current density (Amperes/m\(^2\))
- \(\rho_f\) = free charge density (Coulombs/m\(^3\))

*\(\vec{J}(s,t)\) and \(\rho(s,t)\) are macroscopically-averaged quantities taken to be pointwise continuous (Stratton 1941).
and the Lorentz force, \( F \), on a point charge, \( q \), moving with velocity, \( v \), through a region of electric field, \( E \), and magnetic induction field, \( B \), is given by

\[
F = q \left( E + (v \times B) \right). \tag{II.1.5}
\]

A subsidiary relation — implicit in (II.1.2) and (II.1.3) — is the continuity equation,

\[
\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{II.1.6}
\]

which expresses the conservation of charge in the neighborhood of a point.

In a vacuum, we define

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{II.1.7}
\]
\[
\mathbf{B} = \mu_0 \mathbf{H} \tag{II.1.8}
\]

where \( \mu_0 \) and \( \varepsilon_0 \) are, respectively, the free-space permeability and permittivity with

\[
\mu_0 = 4\pi \times 10^7 \text{Tm/A},
\]
\[
\varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{4\pi} \times 10^{-7} \text{F/m}. \tag{II.1.9}
\]

for \( c \) the velocity of light in free-space. Substitution of these relations into the Maxwell set yields:

\[
\nabla \times \mathbf{D} = \frac{\partial \mathbf{B}}{\partial t}, \tag{II.1.10}
\]
\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{E}, \tag{II.1.11}
\]
\[
\varepsilon_0 \nabla \cdot \mathbf{E} = \rho, \tag{II.1.12}
\]
\[
\nabla \cdot \mathbf{B} = 0. \tag{II.1.13}
\]

This system would completely describe all macroscopic electromagnetic phenomena for material media if \( \rho \) and \( \mathbf{J} \) could be specified for all charges, free and bound.

Generally, however, this is neither possible nor practical; moreover, we are here concerned with the macroscopic fields (and their macroscopic fluctuations) rather than variations which take place on atomic scales. Consequently, in equations (II.1.1) through (II.1.4), \( \mathbf{E} \) and \( \mathbf{B} \) should be viewed as macroscopic averages of the corresponding fields in (II.1.10) through (II.1.13), with \( \mathbf{D} \) and \( \mathbf{H} \) taken as those related averages which specifically account for the aggregate effects of bound charge and current densities (Robinson 1973, Jackson 1975).

The vectors \( \mathbf{D} \) and \( \mathbf{H} \) can be expressed in terms of \( \mathbf{E} \) and \( \mathbf{B} \) through "constitutive" relations which for actual media are probably best regarded as essentially empirical constructions reflecting the results of measurements (Robinson 1973).
II.1.8 CONSTITUTIVE RELATIONS AND BOUNDARY CONDITIONS

Though simple in some important cases, the constitutive relations satisfied by actual media can be complex and varied. In general, $\varepsilon$ and $\mu$ can be written in terms of $\sigma$ and $\eta$ as:

$$
\mathbf{D}(r, t) = \tilde{\sigma} \cdot \mathbf{E}(r, t) + \tilde{\mu} \cdot \mathbf{H}(r, t)
$$

(II.1.14)

$$
\mathbf{B}(r, t) = \tilde{\sigma} \cdot \mathbf{E}(r, t) + \tilde{\mu} \cdot \mathbf{H}(r, t)
$$

(II.1.15)

where the tilde denotes an operator and the double underbar denotes a second rank tensor, and characterization of the medium is then made according to the functional dependence of the tensor elements. Thus, a deterministic medium is said to be homogeneous if no element is a function of position, stationary if no element is a function of time, nondispersive if no element contains either spatial or temporal derivatives (or exhibits nonlocal properties), linear if no element is a function of the field-vectors, bianisotropic unless both $\tilde{\sigma}$ and $\tilde{\mu}$ are identically zero (in which case the medium is simply anisotropic), and isotropic if both $\tilde{\sigma}$ and $\tilde{\mu}$ are scalar operators. These tensor elements can also involve integrodifferential operators (e.g., when nonlocal effects must be treated) as well as chemical, thermodynamical, or continuum-mechanical variables (Penfield and Haus 1967, Kong 1975), and, in random media, the elements become random functions of space and time.

Most media can be taken to be spatially local - an approximation that is valid for bound charges when the spatial variation of the applied fields is large relative to atomic dimensions (Jackson 1975). "Nonlocal" time dependence is characteristic of plasmas (e.g., the ionosphere) and marks a medium as temporally dispersive (i.e., permittivity is a function of frequency), although a moving medium with local time dependence will also exhibit temporal dispersion.

The forms of the constitutive relations applicable to terrestrial propagation problems are discussed in sections II.2.A - II.2.C.

We associate constitutive relations with a medium whose properties are pointwise continuous. When there are abrupt changes, as at the boundary between two distinct media, we can consider the properties as continuous through a thin transition layer. Maxwell's equations (in integral form) then provide boundary-conditions on the fields in the two media as the layer is allowed to become infinitesimally small:

$$
\nabla \times (\varepsilon_{1} \mathbf{E} - \varepsilon_{2} \mathbf{E}) = 0
$$

(II.1.16)

$$
\nabla \times (\mathbf{D}_{1} - \mathbf{D}_{2}) = \mathbf{J}_{s}
$$

(II.1.17)

$$
\varepsilon_{1} \mathbf{D} - \varepsilon_{2} \mathbf{D} = \mathbf{J}_{s}
$$

(II.1.18)

$$
\mu_{1} \mathbf{H} - \mu_{2} \mathbf{H} = \mathbf{M}_{s}
$$

(II.1.19)

where $\eta$ is the unit normal to the surface directed into medium 1, $\mathbf{J}_{s}$ is a surface current, and $\mathbf{M}_{s}$ is the surface charge density.
II.1.C RANDOM MEDIA

The problems of classical electrodynamics generally require some calculation of the fields which result when a given source is placed in a given system of one or more known (deterministic) media. In practice, however, it is often the case that the medium of interest is not so well known — that its spatial and temporal characteristics vary in a complicated, essentially nondeterministic manner (e.g., atmospheric turbulence or the seasurface) — and, when the required fields are significantly affected by such fluctuations, a statistical approach to the problem becomes necessary. In such cases, it is usually of greater value to have some measure of the average field (or its intensity) than it is to know its actual magnitude at any given instant; and different questions become important: How rapidly is the field fluctuating? To what extent does the field deviate from its mean-value? The random nature of the medium variation requires Maxwell’s equations (which still govern the field-physics) to be viewed as stochastic partial differential equations and the fields to be considered as stochastic vector functions (Frisch 1968, Tatarski 1971). The solutions are the various statistics of the field-values (Middleton 1980).

Now, by a random continuum, we distinguish a medium in which the constitutive relations involve continuous random functions of space and time. With the introduction of a random function, we assume the existence of a probability space, \((\Omega, B, \mathbb{P})\), where \(\Omega\) is an abstract space, \(B\) is a sigma field of subsets (events) in \(\Omega\), and \(\mathbb{P}\) is a probability measure on the sigma field (Frisch 1968, Pfeiffer 1965). Then, \(\mathbb{M}(r,t|w)\), \(w \in \Omega\), is a random function of space and time if, for fixed \(r\) and \(t\), \(\mathbb{M}(r,t|w)\) is a random variable. A random function is completely specified when its \(n\)th order distribution function is known; this is rarely determined, most descriptions relying almost completely upon first and second-order moments alone (Tatarski 1971, Yaglom 1962).

In most terrestrial propagation problems, random media are taken to be statistically stationary and homogeneous, or (perhaps more realistically) to possess stationary increments and exhibit statistically-local homogeneity.

A function, \(f(t)\), is said to be statistically stationary if its probability distribution function is invariant under time translation — a consequence being that the mean, \(\langle f(t) \rangle\), is constant, and the covariance,
\[
\rho(t_1, t_2) = \langle [f(t_1) - \langle f(t_1) \rangle][f(t_2) - \langle f(t_2) \rangle] \rangle
\]

is a function only of the time difference \([i.e., B(t_1, t_2) = B(t_1 - t_2)]\).

Statistical homogeneity is the spatial analogue for a function \(f(r)\); that is, the mean is constant and \(B(r,r+\xi) = B(\xi)\). If, moreover, the covariance depends only upon the distance between two points, \([i.e., B(r,r+\xi) = B(|\xi|)]\), then the medium is said to be statistically isotropic.

When \(f(t)\) is nonstationary, sometimes the difference \([f(t+\tau) - f(t)]\), is stationary; and, analogously, the function \([f(r+\xi) - f(r)]\) may be homogeneous when \(f(r)\) is not. Then \(f(t)\) is said to be a random process with stationary increments, and \(f(r)\) a locally homogeneous random function. [Such processes are believed to provide better descriptions of atmospheric turbulence (Tatarski 1971, Ishimaru 1978).] The correlation function associated with a process with stationary increments is called a structure function,
\[
D(\tau) = \langle [f(t+\tau) - f(t)]^2 \rangle
\]

For a locally homogeneous process,
\[
D(\xi) = D(\tau, |\xi|) = \langle [f(r+\xi) - f(r)]^2 \rangle
\]

is the structure function which, for zero means, reduces to \(D(\xi) = \langle [f(r)]^2 \rangle\). Finally, for a locally homogeneous and statistically-isotropic process, \(D(\xi) = D(\xi^2)\).
A spectral representation applicable for random functions has been developed (Yaglom 1962, Tatarski 1971) and is important in the analysis of propagation in random media. If \( f(t) \) is a complex stationary function with a zero mean, the following Fourier transform relations hold between correlation and spectral density functions (Wiener-Khinchin theorem),

\[
\Phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt,
\]

and

\[
W(\omega) = \int_{-\infty}^{\infty} \phi(t) e^{-i\omega t} dt.
\]

for a statistically homogeneous and isotropic medium, II.2.25 and II.1.26 reduce to

\[
\Phi(k) = \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt,
\]

and

\[
\Phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt.
\]

Analogous relations can be obtained for processes with stationary increments and for locally homogeneous functions.

Two important spectral densities are Kolmogorov's (developed from turbulence theory) and the Booker-Gordan formula (1950a, b) derived for an exponential correlation function (chosen for mathematical simplicity). The former is on firmer theoretical ground, but predictions by the two approaches differ only slightly (Gage and Balsley 1980), and the Booker-Gordan formula is easily extended to statistically anisotropic media (Ishimaru 1978).

The Kolmogorov formula assumes locally homogeneous and isotropic turbulence, with refractive index \( n = \langle n \rangle \), \( \eta \) random. Then

\[
\Phi_s(k) = \int \phi(t) e^{-i2\pi k t} dt,
\]

while, if \( f(t) \) is a complex stationary function with a zero mean,

\[
\phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt.
\]

For a statistically homogeneous and isotropic medium, II.2.25 and II.1.26 reduce to

\[
\Phi(k) = \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt,
\]

and

\[
\Phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi k t} dt.
\]

The Booker-Gordan formula assumes homogeneous and isotropic turbulence with refractive index \( n = \langle n \rangle \), \( \eta \) random.

\[
\Phi_s(k) = \int \phi(t) e^{-i2\pi k t} dt.
\]

where \( l \) is called the correlation distance and represents the scale size of the turbulence. Ishimaru (1978) hypothesizes tropospheric \( l \) values on the order 20-130m.
II.1.0 SECOND-ORDER EQUATIONS

For simple media satisfying the constitutive relations \( \mathbb{B}(\mathbf{r}, t) = \varepsilon \mathbb{E}(\mathbf{r}, t) \), \( \mathbb{q}(\mathbf{r}, t) = \mu \mathbb{H}(\mathbf{r}, t) \), and \( \mathbb{J}(\mathbf{r}, t) = \sigma \varepsilon \mathbb{E}(\mathbf{r}, t) \) for \( \sigma, \mu, \) and \( \varepsilon \) constants, the following second-order Fourier-transformed equations hold:

\[
\nabla \times \nabla \times \hat{\mathbb{E}}(\mathbf{r}) = -\frac{\omega^2}{c^2} \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) + \frac{\omega}{c} \frac{\partial \hat{\mathbb{E}}}{\partial t}(\mathbf{r})
\]

\[
\nabla \times \nabla \times \hat{\mathbb{H}}(\mathbf{r}) = -\frac{\omega^2}{c^2} \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) + \frac{\omega}{c} \frac{\partial \hat{\mathbb{H}}}{\partial t}(\mathbf{r})
\]

where the caret (') notation is used to denote Fourier-transformed quantities dependent on the frequency. In II.1.33 and II.1.34, \( \sigma \) is the conductivity of the medium, \( \hat{\mathbb{E}} \) and \( \hat{\mathbb{J}}_{\text{a}} \) are respectively, the (Fourier-transformed) conduction current-density and the current-density of the impressed source, while

\[
\omega = 2\pi f
\]

is the "propagation constant" of the medium for \( \varepsilon, \mu, \) and \( \varepsilon_0 \) as defined in section IV.1.A and

\[
k_0 = \frac{\omega}{c}
\]

the propagation constant of free space. The fields must also satisfy the divergence equations

\[
\nabla \cdot \hat{\mathbb{E}}(\mathbf{r}) = 0
\]

\[
\nabla \cdot \hat{\mathbb{H}}(\mathbf{r}) = 0
\]

with \( \hat{\mathbb{p}} \) the transformed charge density.

we now consider a more general medium with

\[
\mathbb{B}(\mathbf{r}, t) = [\varepsilon + \varepsilon_m(\mathbf{r})] \mathbb{E}(\mathbf{r}, t)
\]

\[
\mathbb{q}(\mathbf{r}, t) = \frac{1}{\mu} \varepsilon \mathbb{H}(\mathbf{r}, t)
\]

\[
\mathbb{J}(\mathbf{r}, t) = \sigma \varepsilon \mathbb{E}(\mathbf{r}, t)
\]

where \( \varepsilon, \mu, \) and \( \sigma \) are again constants but \( \varepsilon_m(\mathbf{r}) \) is a (deterministically) inhomogeneous component of the permittivity or, alternatively, a random component with a zero mean. Then the transformed vector wave equations are

\[
\nabla \times \nabla \times \hat{\mathbb{E}}(\mathbf{r}) = -\frac{\omega^2}{c^2} \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) + \frac{\omega}{c} \frac{\partial \hat{\mathbb{E}}}{\partial t}(\mathbf{r})
\]

\[
\nabla \times \nabla \times \hat{\mathbb{H}}(\mathbf{r}) = -\frac{\omega^2}{c^2} \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) + \frac{\omega}{c} \frac{\partial \hat{\mathbb{H}}}{\partial t}(\mathbf{r})
\]

where \( k^2 \) is again defined by II.1.36, and \( \hat{\mathbb{E}} \) and \( \hat{\mathbb{H}} \) must also satisfy the divergence equations

\[
\nabla \cdot \hat{\mathbb{E}}(\mathbf{r}) = 0
\]

\[
\nabla \cdot \left[ (\varepsilon + \varepsilon_m(\mathbf{r})) \hat{\mathbb{E}}(\mathbf{r}) \right] = \hat{\mathbb{p}}(\mathbf{r})
\]

Equation II.1.44 can be rewritten as

\[
\nabla \cdot \hat{\mathbb{E}}(\mathbf{r}) = \frac{\hat{\mathbb{p}}(\mathbf{r})}{(\varepsilon + \varepsilon_m(\mathbf{r}))}
\]

\[
\nabla \cdot \left[ \varepsilon \hat{\mathbb{E}}(\mathbf{r}) \right] = \frac{\hat{\mathbb{p}}(\mathbf{r})}{\varepsilon}
\]


\[
\nabla \cdot \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) = \frac{\hat{\mathbb{p}}(\mathbf{r})}{\varepsilon}
\]

for \( n(r) = [(\varepsilon + \varepsilon_m(r))/\varepsilon_0]^{1/2} \) (which is the index of refraction if the medium is nonpermeable). Observe that \(-i\varepsilon_m(\mathbf{r})\hat{\mathbb{E}}\) can be viewed as an effective (Fourier-transformed) current density and \(-\nabla \cdot \hat{\mathbb{J}}_{\text{a}}\) as an effective (Fourier-transformed) charge density. In rectangular coordinates, II.1.41 can be rewritten as

\[
\nabla \cdot \hat{\mathbb{E}}(\mathbf{r}) + k^2 \hat{\mathbb{E}}(\mathbf{r}) = -i\omega \left[ -i\omega \varepsilon_m(r) \hat{\mathbb{E}}(\mathbf{r}) + \hat{\mathbb{J}}_{\text{a}}(\mathbf{r}) \right]
\]

\[
-\nabla \left[ \frac{\hat{\mathbb{p}}(\mathbf{r})}{\varepsilon_0} \right] + 2\nabla \left[ \frac{\varepsilon_m(r)}{\varepsilon_0} \hat{\mathbb{E}}(\mathbf{r}) \right]
\]
with $\hat{\mathbf{E}}$ also required to satisfy II.1.45. In problems involving propagation through a random medium at high frequencies for which the wavelength is very much less than the scale-size of the random medium, the term $V \left( \frac{\partial^2 \mathbf{E}}{\partial \mathbf{r}^2}, \mathbf{E}(\mathbf{r}) \right)$ is usually assumed to be negligible for high-gain line-of-sight paths. The fields then decouple and each satisfies a scalar wave equation.

II.2 TERRESTRIAL MEDIA

II.2.A EARTH MODELS

Below LF (wavelengths > 10 km), earth models invariably entail "smooth sphere" approximations with radii equal to the mean earth-radius, $6.36765 \times 10^6$ m (NBS Handbook 1964), Barrick (1972). Above HF (wavelengths < 10 m), most problems are either line-of-sight, with plane-earth approximations modified for surface irregularities being generally sufficient for most practical applications, or over-the-horizon tropospheric scatter problems with earth effects (aside from siting) ignored.

The various media which comprise the earth's crust are usually taken as isotropic when concerned with above surface propagation — though anisotropic effects have been attributed to surface-media stratification (Galejs 1972) and (over higher frequency bands) to forested surface regions (Dence and Spence 1973). Excepting those cases where ferromagnetic media are predominant, earth-permeability is considered relatively constant at the free-space value ($\mu_e = 4\pi \times 10^{-7}$ H/m), whereas the permittivity and conductivity are (for terrestrial media) frequency-dependent above about 30 MHz (see CCIR 1978: 527). Often, relative permittivity and electrical conductivity are considered transversely homogeneous over a fixed path of propagation, though "mixed path" effects have been considered. Surface inhomogeneities such as sea waves, snow-ice fields, and vegetation have been modeled as random media (Barrick and Peake 1960, Dence and Spence 1973, Zuniga and Kong 1980, Middleton 1980).
By “troposphere”, we distinguish that portion of the atmosphere just above the
earth’s surface which extends (radially) from approximately 9 km at the poles, to 17 km
at the equator (Hall 1979). The troposphere is usually taken to be isotropic,
nonporameric (\( \nu = 0 \)), and negligibly ionized (\( \omega = 0 \)), with spatially-nonlocal
effects being neglected in all tropospheric characterizations (Bean and Dutton 1966,
Tatarski 1971, Hall 1979, Gage and Balsley 1980).

Tropospheric permittivity, \( \varepsilon(\xi, t) \), is generally related to the index of
refraction,
\[
v(\xi, t) = \left[ \frac{\varepsilon(\xi, t)}{\varepsilon_0} \right]^{1/2} = \sqrt{\frac{\varepsilon(\xi, t)}{\varepsilon_0}}.
\]
(II.2.1)
which is found to differ from unity only on the order of several parts per million.

Defining the refractivity, \( N \), by
\[
N = 1 + n - 1,
\]
(II.2.2)
measurement has led to the semi-empirical relationship,
\[
N = (1.4) \left( \frac{p}{T} \right) + (3.7 \times 10^5) \left( \frac{\ell}{T} \right).
\]
(II.2.3)
where
- \( \ell \) : partial pressure of water vapor (millibars)
- \( p \) : atmospheric pressure (millibars)
- \( T \) : temperature (°K),
and this is generally held to be valid for frequencies at least as high as 30 GHz (Bean
and Dutton 1966), and possibly, through 10^8 GHz, excluding molecular absorption bands
(Hall 1979, Kirby 1982).

The frequency-independence of II.2.3 — and, hence, of \( \varepsilon(\xi, t) \) — indicates that
refractive field-effects are temporally local and this is believed to be true, at least
outside of the molecular absorption bands above 30 GHz. Inhomogeneities are then to be
expected only as a result of temporal or spatial variations (or random fluctuations) in
the values of atmospheric water vapour pressure and temperature.

Measurements indicate the mean refractivity, \( N \), generally decreases with height,
although not necessarily monotonically. The CCIR now recommends a “reference”
atmosphere with

\[
N = (315) \exp (-0.136h)
\]
(II.2.4)
for \( h \) the height above sea level in kilometers (CCIR 1978: 269). Nevertheless,
observations vary widely. Regions with strong negative gradients commonly occur over
the sea surface and at elevated heights in trade-wind regions (Purves 1974, Dougherty
1979, Hitney et al 1978, Skolnick 1980). Horizontal variations are less significant and
are generally ignored in tropospheric modeling.

Mean variations aside, small-scale atmospheric turbulence is generally held to be
responsible for the random fluctuations in atmospheric refractivity. Regions of
significant fluctuation are presently believed to be horizontally-stratified in layers
several meters thick (Lane 1968, Van Zandt et al 1978, Hall 1979), and locally “patchy”
within such layers (Van Zandt et al 1978, Gage and Balsley 1980). For turbulent eddies
larger than the “outer scale” of turbulence, there appears to be general agreement that
the turbulence is anisotropic (Tatarski 1971, Ishimaru 1978). For turbulence in the
“inertial” range, i.e., eddy sizes larger than the inner scale and smaller than the
outer scale* of turbulence, statistically-local isotropy is believed to prevail

* The inner scale is generally thought to be on the order of millimeters. There is
less agreement as to the outer scale which is often defined as the longest size for
which the turbulence can be considered isotropic; it is thought to be on the order
of meters near the earth and to increase with height (Clifford 1977) with typical
values perhaps of order 10 m (Brooker 1970, Van Zandt et al 1978).
The structure constant $C_n^2$ (see section II.1.C), associated with the spectral power density of isotropic fluctuations in refractivity, has proven difficult to predict at heights greater than the first few hundred meters.* Aside from an overall tendency for its mean value to decrease with increasing height, the data appear to be random (Clifford 1977, VanZandt et al 1978). Typically, surface values are found between $10^{-14}$ and $10^{-13}$ m$^{-2/3}$, and decrease to values between $10^{-17}$ and $10^{-16}$ m$^{-2/3}$ near the middle of the troposphere (Ishimaru 1978, CCIR 563).

Summarizing, the troposphere is considered nonpermeable ($\mu = \mu_0$) and has a conductivity that is assumed to be negligibly small. In deterministic models, the permittivity is generally assumed to be constant or, particularly above 1 MHz, a function of height or radius. In models of random media, the index of refraction is almost always considered to be either statistically stationary, homogeneous, and isotropic, or a process with stationary increments which also exhibits statistically-local homogeneity and isotropy (Booker and Gordon 1950b, Wheelon 1959, Tatarski 1971, Ishimaru 1978). A "frozen-in" assumption, viz.

$$\varepsilon_\infty(f) = \varepsilon_\infty(f-t)$$  \hspace{1cm} (II.2.5)

has been used by Tatarski (1971) and Ishimaru (1978) to investigate the effects of moving random media.

* See, however, Van Zandt et al. (1978) in which vertical profile of $C_n$ are modeled on the bases of wind, temperature, and humidity data.

II.2.C IONOSPHERIC MODELS

The ionosphere is distinguished as the region where free ions exist in sufficient numbers to affect propagating electromagnetic fields, and the term therefore usually references the region between approximately 50 km and 2000 km above the surface of the earth. Ionization of the constituent species is generally attributed to incident cosmic radiation at altitudes below 60 to 65 km and to solar radiation above that height (Davis 1966, Volland 1980). Ionospheric domains are further characterized by temperature (generally decreasing from 50 to 85 km, and increasing above 85 to 100 km) and dynamical processes (turbulent mixing below about 100 km, and diffusive equilibrium from 120 km to 500-700 km) (Davis 1966) -- but, to date, no wholly satisfactory electrophysical model of the ionosphere has been established (Agard 1980: No. 295).

The following electrical characterizations are generally believed to be valid for the ionosphere:

(i) magnetic isotropy ($\mathbf{B} = \mu \mathbf{H}$) (Davis 1966, Ginzburg 1970);

(ii) magnetic homogeneity and nonpermeability ($\mu = \mu_0$) [Ginzburg (1970) estimates deviations from $\mu_0$ to be on the order of $10^{-12}$ Henrys/m];

(iii) geomagnetic anisotropy (Ratcliffe 1959) due to Lorentz forces (sec. II.1.A) on electrons moving in the earth's magnetostatic field;

(iv) field effects are temporally nonlocal (Ratcliffe 1959; Ginzburg 1970), hence the ionosphere is dispersive;

(v) field effects are spatially local -- this common assumption is strictly true only for a cold plasma according to Ginzburg (1970).

Principally, it is the variation in electron density and collision frequency (with position and time) which marks the effective ionospheric permittivity as inhomogeneous. Several recurring and important regions are distinguished by their electron density profiles (Fig. II.1); the variation of collision frequency with height is indicated in Fig. II.2.
The D layer extends from 50 to 90 km during the day and essentially disappears at night. The neutral atmosphere, however, still relatively dense and turbulently mixed, with the result that the collision frequency remains high \((v \approx 10^6 \text{sec}^{-1})\) (Galejs 1972, CCIR 1978: 125).

The E layer extends from 90-130 km, and, though ionization is greatly reduced at night, this layer becomes the effective lower edge of the ionosphere in the absence of the D-layer.

The F layer extends above 130 km and contains two distinct strata (the F1 and F2 layers) which merge at night and in the winter (Rush 1981, Galejs 1972). The F1 layer extends from about 130-210 km with F2 above. The electron density in F2 is strongly influenced by neutral-air winds, electrodynamic drifts, and diffusion processes (Rush 1981, CCIR 1978: 125).

Turbulence, which exists principally below 100 km, causes a randomly fluctuating component in the electron density in the D and lower E layers (Villas and Weiskopf 1955, Wheelon 1960, Davis 1966). At greater heights, ionization irregularities in the equatorial F layer (sometimes referred to as spread F) arise in the evening and persist at night. Patches of these irregularities can be spread over 100 km to possibly 3000 km in the north-south or east-west directions with scale-sizes between 3 m and 10 km reported (CCIR 1978: 125).

Ionospheric scattering also results from nonturbulent statistical fluctuations — the thermal motion of electrons — in a medium in thermal equilibrium. This is termed thermal or Thompson scattering, and is associated with frequencies above HF.

We now give the Fourier-transformed relations of the more commonly used ionospheric models. Let \(e, m, N, \) and \(\nu_c\) denote electron charge, electron mass, electron density, and electron-neutron collision frequency, respectively. The latter two quantities may be functions of position. If we define the "plasma frequency" as

\[
\omega_p = \left(\frac{\omega_n^2}{\omega_n^2 + \nu_c^2}\right)^{1/2}
\]

and the cyclotron frequency, for \(B_0\) the magnetic field strength, as

\[
\omega_c = \frac{e B_0}{m}
\]

then:

(1) simple plasma

\[
\hat{\rho}(\xi) = \hat{\rho}(0) e^{\xi} e^{\xi},
\]

\[
\hat{\rho}(\xi) = \xi \left[ 1 - \left(\frac{\omega}{\omega_p}\right)^2 \right]^{-1/2}
\]
(2) plasma with collisions

\[ \hat{E}(\xi) = \hat{E}(\zeta, \hat{E}(\zeta)) \]

\[ \hat{\varepsilon}(\xi) = \hat{\varepsilon}(\zeta) \left( 1 - \frac{\omega^2}{\omega_0^2} \right) + i \frac{\omega_{0}^2}{\omega(\omega^2 - \omega_0^2)} \]

(II.2.9)

(3) plasma in a magnetic field \((B - B_0\hat{z})\) without collisions

\[ \hat{E}(\xi) = \hat{E}(\zeta, \hat{E}(\zeta)) \]

\[ \hat{\varepsilon}(\xi) = \begin{pmatrix} \epsilon - i \omega \mu_0 & 0 \\ i \xi \omega \mu_0 & \epsilon \\ 0 & 0 & \epsilon_0 \end{pmatrix} \]

(II.2.10)

where

\[ \omega = \epsilon_0 \left( 1 - \frac{\omega^2}{\omega_0^2} \right), \]

(II.2.11)

\[ \xi = \epsilon_0 \left[ \frac{\omega^2}{\omega(\omega^2 - \omega_0^2)} \right], \]

(II.2.12)

\[ \epsilon_0 = \epsilon_0 \left[ 1 - \left( \frac{\omega_{0}^2}{\omega} \right)^2 \right], \]

(II.2.13)

II.3 PROPAGATION EFFECTS

II.3.A LOWER-BAND FREQUENCIES \((f < 300 \text{ KHz})\)

Earth Effects

At VLF \((3 \text{ KHz to } 30 \text{ KHz})\) and ELF \((3 \text{ Hz to } 3 \text{ KHz})\), earth-surface irregularities are usually much smaller than the associated wavelengths \((\lambda > 10 \text{ km})\), and field-measurements are not reported to differ from those predicted via smooth earth models. However, at LF \((30 \text{ KHz to } 300 \text{ KHz})\), increased groundwave phase delays of several microseconds over distances of a few wavelengths are attributed to surface irregularities (Doherty 1974). At both VLF and LF, mixed propagation paths \((\text{e.g. seawater to earth})\) produce significant variations in field-strength.

Tropospheric Effects

Measurements of groundwave at LF indicate that meteorological factors, particularly, the refractive index gradient, affect LF propagation (Doherty 1974, Doherty and Johler 1975). The CCIR, relying on model calculations for an exponential tropospheric index of refraction, concludes that effects are negligible below 10 KHz and become more noticeable at higher frequencies.

Ionospheric Effects

The electrical conductivity of the atmosphere is very small at lower altitudes, but increases rapidly with height above 50 km — the number density of electrons increasing from essentially zero \((\text{at } 50 \text{ km})\) to as much as \(10^6 \text{ m}^{-3}\) at 60 km. This very large conductivity-gradient separates the atmospheric and ionospheric domains, reflects almost all the energy of incident lower-band fields, and thus forms the outer boundary of an effective spherical waveguide or cavity with the earth as the central element. The system is complicated, however, by the existence of the \((\text{static})\) geomagnetic field.

For systems relying on "groundwave" \((\text{the field propagating from the source to the receiver along the ground without intermediate ionospheric reflection})\) for communication or navigation purposes, ionospheric properties are of interest in ascertaining the region over which groundwave will dominate the field returned from the ionosphere.
and the degree of interference to be expected. Groundwave typically dominates
skywave within several hundred kilometers of the source at VLF (Malt and Croftc 1964,
Thrane 1978), and can possibly be distinguished as far out as 1000 to 2000 km at LF
(CCIR 1982). For VLF and LF daytime propagation, an interference pattern
(called a Hollingsworth pattern) is observed outside of about a hundred kilometers and
extending to perhaps 700 km; this pattern is attributed to interference between
groundwave and a singly-reflected ionospheric wave (Thrane 1978). Beyond about 700 km,
the field-structure becomes more complex. In the antipodal region — the region halfway
around the earth from the source (= 20,000 km) — a standing wave pattern, due to
waves traveling in the opposite directions around the earth, is observed (e.g., at
16 kHz the field is observed to increase beyond 16,000 km).

At VLF, isotropic and sharply-bounded homogeneous models account for many
experimental results (Malt and Spies 1964, Burgess and Jones 1975, Al'pert and Fligel
1970). The CCIR (1982: 895) reports that the magnetic field has little effect on
daytime propagation. At night, greater attenuation is observed for propagation to the
magnetic west than to the magnetic east. The propagation effects are latitudinally
dependent for propagation to the magnetic west but not for propagation to the east.
These nighttime observations are attributed to ionospheric anisotropy. The field
attenuation difference is about 1 db/1000 km (Davis 1966), while the phase velocity does
not appear to depend on propagation direction and is well described by simple models
(Al'pert 1973). The CCIR has called for further experimental observations of field
strength and phase velocity versus distances for different directions relative to the
earth's magnetic field.

At higher LF frequencies, the utility of a sharply-bounded ionospheric model
becomes increasingly suspect and analyses which include the effects of ionospheric
stratification may prove more accurate. Still, though experimental and analytical
results are more limited than those at VLF, sharply-bounded models yield a consistent
interpretation of long range propagation in daytime (Malt 1967). Anisotropic effects
are observed primarily in propagation near the magnetic meridian where polarization
losses can exceed 1 db at upper LF and MF, and depend upon propagation direction
(CCIR 1978: 575). However, a number of experimental studies in various parts of the
world have found little directional dependence, with most data following the results of
north-south propagation (CCIR 1978: 431). Skywave signal strength at LF varies
continually, but fade-depths remain below those observed for mid-band frequencies (UF or
HF), and Knight (1971) reports a "quasi-maximum" about 1.7 db above the median-value for
daytime paths and 3 db above the median for nighttime propagation.
II.3. B MID-BAND FREQUENCIES (300 KHz < f < 30 MHz)

Earth Effects

At both MF (300 KHz to 3 MHz) and HF (3 MHz to 30 MHz), estimates of transhorizon groundwave fields based on smooth-earth models have been found to agree with measurement [with the effects of surface irregularities accounted for by modifying the effective "surface impedance" (sec. IV.2.A) at the higher frequencies]. Along line-of-sight paths, surface irregularities become increasingly important as wavelength decreases (scattering the field and thus reducing the effective reflection coefficient for the specular component) and skywave returns are similarly scattered out of the great-circle path between source and receiver (CCIR 1978: 726). Mixed-path effects become more pronounced at mid-band frequencies, and surface motion over seawater paths can result in (small) shifts of the received frequencies.

Tropospheric Effects

At mid-band frequencies, atmospheric propagation paths are refracted toward regions of greater mean refractive index, and steep radial (vertical) gradients in the atmospheric index of refraction can result in ducted propagation (Pappert and Goodhart 1979). Random fluctuations about the mean refractive index appear to have little effect upon propagation at frequencies below 30 MHz but experimental evidence concerning the relationship between observed fields and meteorological conditions is inconsistent (Al'pert 1973).

Ionospheric Effects

Over all mid-band frequencies, but particularly at MF, the lower (D-layer) ionosphere is strongly absorptive under daylight conditions. As a result, skywave returns are substantially reduced in amplitude — contributing to fading and interference at long-range — and groundwave propagation can predominate out to $2 \times 10^3$ km (Kirby 1982, CCIR 1978: 368). Under nighttime conditions, the D-layer is greatly reduced, the E-layer becomes partially reflective to HF signals, and skywave returns are subject to continual fading (about a median-value). Observed anisotropic effects are similar to those at upper LF.

At HF, the daytime D-layer absorption is less pronounced than at MF, and HF fields are observed to reflect from both the E-layer (primarily under nighttime conditions) and the F-layer. Sporadic E-layers will scatter and reflect HF signals — though "reflection" at these frequencies is generally the result of a continuous refraction of the propagating field toward regions of greater refractive index until, ultimately, the field is returned earthward. High antipodal fields and, occasionally, around-the-world fields exhibiting low dispersion and minimal attenuation have also been observed. Explanations for such events include ionospheric ducting, successive ionospheric reflections without intermediate earth-contact, and waveguide propagation along ionospheric layers (CCIR 1978: 250). In general, significant amplitude-fading accompanies F-layer reflections at HF.
II.3.C UPPER-BAND FREQUENCIES (50 MHz < f)

Earth Effects

At upper-band wavelengths (10 m > λ > 1 cm), surface irregularities strongly scatter the incident fields, sea surface motion gives rise to "clutter" and Doppler-shift in received signals, and mixed-path effects produce significant variations in field-strength. At VHF (300 MHz to 3 GHz), depolarization (i.e., the appearance of an electric field-component orthogonal to that of the original wave) is observed in broadcast transmissions, and such effects are believed to be due to terrain irregularities along the propagation path (CCIR: 1978: 238). Nevertheless, Norton (1941) indicates that the model of a smooth spherical earth still has utility in this frequency range; citing experimental measurements presented to the FCC (Chapin and Norton 1940) he comments that "... the departures from these idealized conditions, such as hills, buildings, trees, etc., cause large variations from the calculated values at particular distances but the theory does provide an excellent guide to the average fields encountered in practice".

Atmospheric Effects

At most upper-band frequencies, tropospheric scattering due to random fluctuations in the refractive index produces slowly-attenuating (~0.1 dB/km) median field-strengths (after diffusive losses) over pathlengths up to 16 km (CCIR 1978: 238). Amplitude-fading is both rapid (from several fades/minute at VHF to a few per second at UHF) and severe. Measurements at UHF are largely supportive of locally isotropic scattering theory (Gage and Balsley 1980). And, though instances of statistical anisotropy may also exist, Hall (1979) remarks that "... radar and refractometer studies have not produced convincing evidence of marked nonisotropy as a normal feature."

Refraction is particularly important at lower angles of elevation and can result in enhanced, transhorizon fields. Furthermore, when sufficiently steep vertical gradients in refractive index are present, ducting (usually for frequencies above 1 GHz) can also yield high over-the-horizon fields.

The major effects observed with line-of-sight propagation paths include refraction, multipath-fading, attenuation, cross-polarization, and scintillation. Multipath effects are the principal cause of transmission loss below 5 GHz and, together with attenuation due to precipitation, contribute to most of the losses at higher frequencies. Generally, multipath phenomena are attributed to time-variations of the refractive index, coupled with diffraction and reflection at the earth's surface (CCIR 1978: 338). Direct attenuation results from both molecular absorption processes (sec. II.2.B) and precipitation scatter. Absorption by atmospheric gases has a negligible effect at frequencies below 1 GHz, but is increasingly important above 10 GHz (CCIR 1978: 719), while the effects of precipitation are believed to be negligible except for frequencies above about 3 GHz (Hall 1979). Scintillation is the rapid (and random) variation of amplitude and phase in received signals, generally attributed to fluctuations of the refractive index which cause partial focusing and defocusing of the line-of-sight beam. Though present at all upper-band frequencies, it can become particularly important above 10 GHz (Hall 1979, CCIR 1978: 338); the associated time-delays are much smaller than those attributed to multipath effects. According to Strobl and Clifford (1961), for optical wavelengths, the received power in the depolarized component is typically 160 dB weaker than the polarized component; in the millimeter range it is typically 100 dB weaker.

Ionospheric Effects

At VHF (30 MHz to 300 MHz) and above, the ionosphere is non-reflective, although ionospheric scatter for frequencies between 30 and 60 MHz results in weak (but persistent) fields over 800 to 2000 km ranges, and sporadic E-layer scattering has been observed for frequencies less than 90 MHz.
Thermal scattering, while having little impact on terrestrial propagation, has proved extremely valuable in providing information on the electron density profile of the ionosphere. At low VHF and below 100 km, thermal scatter is typically dominated by turbulent scatter. However, for frequencies above 100-200 MHz and heights above 50 km — and, more particularly, 100 km — thermal scatter can be significant (Gage and Balsley 1980).

III INTEGRAL OPERATOR FORMULATION AND FREE-SPACE GREEN'S DYADIC

III.1 GENERAL INTEGRAL-OPERATOR FORMULATION

In section II.1.D, we found that, for a medium in which the permittivity was a function of position (or fluctuated randomly), the Fourier-transformed inhomogeneous vector wave equations satisfied by the electromagnetic fields* can be written,

\[ \mathbf{\nabla} \times \mathbf{\nabla} \cdot \mathbf{\hat{E}(r)} - \mathbf{\nabla} \cdot \mathbf{\hat{J}(r)} = i\omega \mu \mathbf{\hat{J}_a(r)} + \mathbf{\hat{J}_m(r)} \] (III.1.1)

\[ \mathbf{\nabla} \times \mathbf{\nabla} \cdot \mathbf{\hat{H}(r)} - \mathbf{\nabla} \cdot \mathbf{\hat{J}(r)} = \mathbf{\nabla} \times [ \mathbf{\hat{J}_a(r)} + \mathbf{\hat{J}_m(r)} ] \] (III.1.2)

where \( \mu \) is independent of position, and \( \mathbf{\hat{J}_a} \) and \( \mathbf{\hat{J}_m} \) are the current-densities associated with an impressed source and a source attributable to media inhomogeneities. In classical problems, the media are assumed to be homogeneous, stationary, and nonrandom; \( \mathbf{\hat{J}_m} \) is then zero and, outside of the impressed source, the homogeneous vector wave equations are satisfied. A formulation involving the fields is sought which, for homogeneous media, reduces to the classical solutions (of primary concern in our present investigation), but which also provides a framework for the extension of such solutions to the case of inhomogeneous media. It is further desirable to have the effects of boundaries exhibited explicitly.

Such a formulation presently exists: Tai (1971) applies dyadic Green's functions to obtain solutions for equations of the general form (III.1.1) and (III.1.2) for various homogeneous and bounded systems. We follow Tai's development which, when we interpret medium inhomogeneity as a distributed source (as in section II.1.D), will result in an integral-operator formulation of the above problem.

*Or, in the case of random media, the equations from which the field statistics are obtained.
we consider the existence of a number of continuous, possibly inhomogenous media whose boundaries are marked by abrupt discontinuities in permittivity or conductivity. Let the impressed source, $\dot{E}(r,\omega)$, be situated in a medium we denote as $V$, with the boundary between $V$ and all other media denoted by $S$.

A useful vector Green's theorem is obtained by substituting into Gauss' theorem:

$$\int_V \left( \nabla \times \mathbf{E} \times \mathbf{B} - \mathbf{E} \times \nabla \times \mathbf{B} \right) \cdot d\mathbf{V} = \oint_S \left( \mathbf{E} \times \nabla \times \mathbf{B} - \mathbf{B} \times \nabla \times \mathbf{E} \right) \cdot d\mathbf{S}$$

(III.1.3)

where $\mathbf{n}$ is the unit normal to the closed surface $S$.

Identifying $\mathbf{P}$ with the Fourier-transformed electric field, $\mathbf{P} = \hat{E}(k)$, and setting $\mathbf{Q} = \hat{G}(k) \times \mathbf{a}$ where $\mathbf{a}$ is a constant but arbitrary vector and $\hat{G}$ is a dyadic Green's function to be defined, we obtain

$$\int_V \left\{ \hat{G}(k) \cdot \nabla \times \mathbf{E} \times \mathbf{B} - \mathbf{E} \times \nabla \times \hat{G}(k) \cdot \mathbf{B} \right\} \cdot d\mathbf{V}$$

$$= \oint_S \left\{ \mathbf{E} \times \nabla \times \hat{G}(k) \cdot \mathbf{B} - \hat{G}(k) \times \nabla \times \mathbf{E} \cdot \mathbf{B} \right\} \cdot d\mathbf{S}$$

(III.1.4)

But assuming $\hat{G}$ satisfies III.1.1, we require $\hat{G}(k,\mathbf{r}')$ to satisfy

$$\nabla \times \mathbf{E} \times \hat{G}(k,\mathbf{r}') - \hat{G}(k,\mathbf{r}') \times \nabla \times \mathbf{E} = -\frac{1}{V} \delta(k,\mathbf{r}').$$

(III.1.5)

where $\frac{1}{V}$ is the unit dyadic.

Substituting III.1.1 and III.1.5 in III.1.4, interchanging primed and unprimed variables, and noting that the relation must hold for all $\mathbf{a}$, we obtain

$$\hat{G}(k,\mathbf{r}) = \frac{1}{V} \int_V \hat{G}(k,\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}',\omega) \, d\mathbf{r}'$$

$$+ \frac{1}{V} \int_V \hat{G}(k,\mathbf{r}') \cdot \mathbf{E}(\mathbf{r},\omega) \, d\mathbf{r}$$

$$- \oint_S \left\{ \mathbf{E}(\mathbf{r},\omega) \times \nabla \times \hat{G}(k,\mathbf{r}') + \hat{G}(k,\mathbf{r}') \times \nabla \times \mathbf{E}(\mathbf{r},\omega) \right\} \cdot d\mathbf{S}.$$

(III.1.6)

Using Maxwell's equations, we can substitute $\hat{M}(k,\mathbf{r}')$ for $\mathbf{Q} \times \hat{G}(k,\mathbf{r}')$. Also, if $\hat{G}$ is the free space Green's dyadic ($\hat{G}_0$) satisfying the radiation condition,

$$\int_{\mathbf{r}'} \left[ \nabla \times \hat{G}_0(k,\mathbf{r}') - \hat{G}_0(k,\mathbf{r}') \times \nabla \times \hat{G}_0(k,\mathbf{r}') \right] = 0,$$

(III.1.7)

then the transpose of $\hat{G}_0(k,\mathbf{r}') = \hat{G}_0^T(k,\mathbf{r}')$ is $\hat{G}_0^T(k,\mathbf{r}')$. Hence for the free space Green's function (and for the spherical Green's functions in the source and receiver regions considered in this work (Tai 1971: p.65))

$$\hat{G}(k,\mathbf{r}) = \int_V \hat{G}_0^T(k,\mathbf{r}') \cdot \left[ \hat{M}(k,\mathbf{r}') - \hat{M}^T(k,\mathbf{r}') \right] \, d\mathbf{r}'$$

$$- \oint_S \left\{ \hat{M}(k,\mathbf{r}) \times \nabla \times \hat{G}_0^T(k,\mathbf{r}') + \hat{G}_0^T(k,\mathbf{r}') \times \nabla \times \hat{M}(k,\mathbf{r}') \right\} \cdot d\mathbf{S}.$$

(III.1.8)

The corresponding evaluation for the magnetic field gives us

$$\hat{H}(k,\mathbf{r}) = \int_V \hat{G}_0^T(k,\mathbf{r}') \cdot \left[ \nabla \times \hat{M}(k,\mathbf{r}') - \hat{M}(k,\mathbf{r}') \times \nabla \times \hat{H}(k,\mathbf{r}') \right] \, d\mathbf{r}'$$

$$- \oint_S \left\{ \nabla \times \hat{M}(k,\mathbf{r}) \cdot \hat{H}_0^T(k,\mathbf{r}') - \hat{H}_0^T(k,\mathbf{r}') \cdot \nabla \times \hat{M}(k,\mathbf{r}) \right\} \cdot d\mathbf{S},$$

(III.1.9)

where $\epsilon$ is the principal (constant) value of the permittivity from which the medium's inhomogeneities are assumed to deviate.
For certain bounded systems, including concentric, spherically-layered, media, solutions of \( v_{1.5} \) can be obtained which will satisfy the boundary conditions required to make the surface integrals in III.1.7 and III.1.8 vanish.

For volume scatter from random inhomogeneities, \( J_0 \) depends on the electric field (see section II.1.0). In the Born or single scatter, approximation to scattering from the troposphere, most analyses assume the medium to be unbounded and propagation to take place along the line of sight. In such cases, the surface integrals in III.1.7 and III.1.8 vanish and \( G \) is the free space Green’s function. Then the field (assumed unperturbed by the inhomogeneities) is calculated at points in the scattering volume to estimate \( J_0 (r') \), which is then substituted into III.1.7 and III.1.8. Wheelon (1959) has indicated that a second approach is possible, one that recognizes the role of both line-of-sight and diffracted primary and/or scattered waves. In this case, one takes \( E_0 \) [the field incident on the scattering volume] to be the actual (series) solution for a dipole radiating above a spherical conducting earth. This automatically includes line-of-sight propagation to blobs in the common volume, as well as blobs below the horizon which are reached only by the diffracted component of \( E_0 \). Subsequent propagation of the scattered radiation must also recognize the earth-screening effect, which it is to say that the Green’s function \( G(R,r) \) must be that appropriate to the exterior of a conducting sphere. However, this is just the field due to a unit dipole placed at the scattering point, and is therefore given again by a series solution of the form used for \( E_0 \).

He also notes that the series summations that result must then be evaluated using “Watson’s transformation.” In chapter IV we discuss this transformation as well as other approaches to evaluating the series sum (though herein we are primarily concerned with lower frequencies than those associated with troposcatter).

III.2 GENERAL DYADIC SOLUTION FOR AN INFINITE HOMOGENEOUS MEDIUM

The complete solution of the vector wave equation for electromagnetic boundary-value problems is known only for certain separable system-geometries. In each such system, the complete field can be obtained as the sum of solutions to the vector wave equation, each derivable from a solution to the scalar wave equation (Stratton 1941). For a spherical system, \( \nabla \times \nabla \times E = -\kappa^2 E \) and \( \nabla \cdot E = 0 \) are satisfied by the independent vector solutions, \( \mathbf{J} = \nabla \times \mathbf{A} \) and \( \mathbf{A} = \frac{1}{k} \nabla \psi \), where \( \psi \) is a solution of the scalar wave equation \( \nabla^2 \psi + \kappa^2 \psi = 0 \) and \( g \) is a constant vector (or \( g = r \hat{r} \)). A third independent solution, \( \mathbf{E} = \nabla \psi \), satisfies the vector wave equation but not the demand for zero divergence associated with homogeneous media.

To obtain the free space dyadic in terms of spherical coordinates and unit vectors directly from a Cartesian formulation is a laborious undertaking. Tal (1971) advanced a different, simpler, solutional route which he referred to as the "Olm-Rayleigh" method. We will illustrate the method for a spherical system. The solution, so obtained, is then simplified significantly, the result to be used in Chapter IV.

Solutions to the scalar wave equation, \( \nabla^2 \psi + \kappa^2 \psi = 0 \), can be written

\[
\psi_\alpha (\rho r) = \mathbf{Z}_\alpha (\rho r) \mathbf{P}_\ell^m (\cos \theta) \mathbf{e}_n (\omega t - \rho) \quad \text{III.2.1}
\]

where \( \mathbf{Z}_\alpha (\rho r) \) is a spherical Bessel function and \( \mathbf{P}_\ell^m (\cos \theta) \) is the associated Legendre function of order \( (n,m) \). These latter functions are called spherical harmonics when multiplied by \( \cos(m \theta) \) or \( \sin(m \theta) \). In the \( m = 0 \) case, the function is termed a zonal harmonic: since it depends on \( \theta \) only, the zeros of the function divide a sphere into zones for fixed \( \kappa \).

The corresponding spherical vector wave functions are

\[
\mathbf{E}_\alpha (\rho r) = J_x \left[ \mathbf{Z}_\alpha (\rho r) \mathbf{e}_n \right] \quad \text{III.2.2}
\]

\[
= \mathbf{Z}_\alpha (\rho r) \mathbf{J}_\ell (\ell \cos \theta) \mathbf{e}_n (\omega t - \rho) \quad \text{III.2.2}
\]
Substituting this result into the dyadic equation along with the general Green's dyadic expansion

\[ \mathbf{G}(\mathbf{r}, \mathbf{r}') = \int_0^{\infty} \int_{S^2} \mathbf{G}(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}, \mathbf{r}') \, dS \]

we obtain \( a(k) = b(k) = (k^2 - k^2) \).

Then using the integral relation (Lal 1971)

\[ \int_{-\pi}^{\pi} \frac{\sin \theta_1}{\sin \theta_2} \sin \theta_1 \, d \theta_1 = \frac{\pi}{2} \sin \theta_2 \]

with \( r > \) and \( r' < \) as the greater or lesser of \( r \) and \( r' \), respectively, we get

\[ \mathbf{G}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi r^3} \frac{\partial}{\partial r} \left[ \frac{1}{4\pi r'^3} \frac{\partial}{\partial r'} \right] \left[ \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \right]
\]

Now, define the operators

\[ \mathbf{\Lambda}^{(1)}(\mathbf{r}, \mathbf{r}') = \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \]

\[ \mathbf{\Lambda}^{(2)}(\mathbf{r}, \mathbf{r}') = \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \]

Then,

\[ \mathbf{\Lambda}^{(1)}(\mathbf{r}, \mathbf{r}') = \mathbf{\nabla}_{\mathbf{r}} \cdot \mathbf{\nabla}_{\mathbf{r}'} \left[ \frac{1}{4\pi r^3} \frac{\partial}{\partial r} \right] \left[ \frac{1}{4\pi r'^3} \frac{\partial}{\partial r'} \right]
\]

Using the Legendre polynomial expansion (Jackson 1975)

\[ \mathbf{P}_m(\cos \theta) = \mathbf{P}_m(\cos \theta) \mathbf{P}_m(\cos \theta) + \sum_{n=0}^{\infty} \left[ \mathbf{P}_m(\cos \theta) \mathbf{P}_m(\cos \theta) \right] \mathbf{P}_n(\cos \theta)
\]

and analogous relations hold for \( \mathbf{\Lambda}^{(2)}(\mathbf{r}, \mathbf{r}') \) and \( \mathbf{\Lambda}^{(3)}(\mathbf{r}, \mathbf{r}') \).

For

\[ \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') = \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') + \sum_{n=0}^{\infty} \left[ \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \mathbf{\nabla}(\mathbf{r}, \mathbf{r}') \right] \mathbf{P}_n(\cos \theta)
\]

where \( \delta_0 \) is unity for \( n = 0 \) and zero for \( n \neq 0 \).
we see that III.2.8 then simplifies to

\[
\hat{G}(r, r') = \frac{i}{4\pi} \sum_{n=0}^{\infty} \left[ \frac{e^{in\theta}}{r} + \frac{e^{-in\theta}}{r'} \right] \mathcal{E} \left[ \tan \theta' \right], \quad r > r'
\]

which is our series representation for the free-space Green's function.

\[
\text{III.2.13}
\]

CHAPTER IV. HOMOGENEOUS SPHERICAL SYSTEM SOLUTIONS — CALCULATIONAL APPROACHES

IV.1. SPHERICAL SYSTEM SOLUTIONS

IV.1.A GREEN'S DYADICS FOR TWO AND THREE-MEDIA SYSTEMS

We solve for the Green's dyadic associated with a concentric spherical system of three, homogeneous, isotropic media with a source in the second medium. The first medium occupies \( \{ (r, \theta, \phi) : 0 < r < a_1 \} \), the second \( \{ (r, \theta, \phi) : a_1 < r < a_2 \} \), and the third \( \{ (r, \theta, \phi) : r > a_2 \} \) for the coordinate geometry of Figure IV.1.

![Figure IV.1 Spherical System](image)

The electrophysical properties — permittivity, permeability, and conductivity — of the three media are specified as follows: \( \varepsilon_1, \mu_1, \) and \( \sigma_1 \) for medium 1; \( \varepsilon_2, \mu_2, \) and \( \sigma_2 \) for medium 2; and \( \varepsilon_3, \mu_3, \) and \( \sigma_3 \) for medium 3. The propagation constants are then

\[
\begin{align*}
\kappa_1^2 &= \left( \frac{\varepsilon_1}{\mu_1} \right)^2 \left( \frac{\varepsilon_2}{\mu_2} \right)^{1/2} \left( \frac{\varepsilon_3}{\mu_3} \right)^{1/2} \\
\kappa_2^2 &= \left( \frac{\varepsilon_2}{\mu_2} \right)^2 \left( \frac{\varepsilon_3}{\mu_3} \right)^{1/2} \\
\kappa_3^2 &= \left( \frac{\varepsilon_3}{\mu_3} \right)^2
\end{align*}
\]

\( \text{(medium 1)} \), \( \text{(medium 2)} \), \( \text{(medium 3)} \)  

where \( \omega = 2\pi f \), \( f \) is the frequency, and \( \kappa_0 = \omega/c = \omega(\mu_0 \varepsilon_0)^{1/2} \) is the free-space
propagation constant with \( c \) the velocity of light in vacuum, and \( \varepsilon_0 \) and \( \mu_0 \) the permittivity and permeability of free space. After obtaining the general solution, we will specialize to the case \( k = k_0 \) (i.e., \( \mu = \mu_0 \), \( \varepsilon = \varepsilon_0 \), \( \sigma = 0 \)) and to nonpermeable media (\( \mu_0 = \mu = \mu_0 \)).

The two-media system solution (for a source in the outer medium) can be obtained from the three-media solution either by equating the electrical properties of the third medium with those of the second or by letting \( b = + \infty \).

The field solutions can be expressed in terms of the Green's dyadic (sec. III.1):

\[
\hat{\mathbf{E}}(r, r') = i \omega \int_0^\infty \hat{\mathbf{E}}_0 \left( \frac{a}{r} \right)^2 \hat{\mathbf{E}}_{i0} \left( \frac{a}{r'} \right)^2 \sin \left( \frac{k r}{c} \right) \sin \left( \frac{k r'}{c} \right) \frac{c}{c^2 + k^2} \frac{\sin k r}{k r} dr'
\]

where

\[
\hat{\mathbf{E}}_0 \left( \frac{a}{r} \right)^2 \hat{\mathbf{E}}_{i0} \left( \frac{a}{r'} \right)^2 = \hat{\mathbf{E}}(r, r') \quad a \times r \quad \omega \quad \omega
\]

with boundary conditions (for \( r \rightarrow + \infty \))

\[
\begin{align*}
\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} &= \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \\
\psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} &= \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}
\end{align*}
\]

and

\[
\begin{align*}
\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} &= \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \\
\psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} &= \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}
\end{align*}
\]

The first dyadic superscript indicates the receiver domain for which the dyadic is a solution; the second indicates the domain in which the source is located. Note that source coordinates are primed and field (or receiver) coordinates are unprimed.

From III.2.13, the Green's dyadic for a source in an unbounded medium with propagation constant \( k \) is

\[
\hat{\mathbf{E}}_0(\mathbf{r}, r') = \frac{i \omega}{2 \pi} \int_{\mathbb{R}^3} \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} [\mathbf{1} - \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}] dx_0
\]

where \( \omega \), \( \varepsilon_0 \), \( \mu_0 \), and the \( \mathbf{H} \) and \( \mathbf{H} \) vector quantities, which are also operators, are defined in III.2.9 and III.2.10. The superscript on \( \mathbf{H}(x) \) and \( \mathbf{H}(x) \) indicates the type of Bessel function associated with the vector — specifically, 0, 1, and 2 denote \( j_0(x), h_1^{(1)}(x), \) and \( h_2^{(1)}(x) \), respectively.

Requiring only outwardly-propagating waves in region 3 and finite fields at \( r = 0 \), we can write the general Green's dyadics for the three medium problem as:

\[
\hat{\mathbf{E}}_0(\mathbf{r}, r') = \frac{i \omega}{2 \pi} \int_{\mathbb{R}^3} \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \left[ 1 - \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \right] dx_0
\]

where \( \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \) and \( \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \) are vector coefficients (and operators) independent of the field coordinates and determined by the boundary conditions at \( r = a \) and \( r = b \) [cf. Tai (1971) for the two media case]. Solving for the these coefficients, we find (Appendix A1)

\[
\hat{\mathbf{E}}_0 = \frac{1}{2} \left[ \frac{\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} + \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}}{1 - \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}} \right] - \psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}
\]

where

\[
\begin{align*}
\psi^+ &= \left[ \frac{\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} + \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}}{2} \right] \left[ \frac{\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} + \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}}{2} \right]^{-1} \\
\psi^- &= \left[ \frac{\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} + \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}}{2} \right] \left[ \frac{\psi^+ \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu} + \psi^- \left[ \hat{\mathbf{E}}_{i0} \right]_{\nu}}{2} \right]^{-1}
\end{align*}
\]
and, to obtain the two-medium system solution, the \( S_{\phi}(n) \) are set equal to zero in IV.1.10b.

Henceforth, we assume all media to be nonpermeable.

Special Case: Radial Hertzian Source

We substitute into IV.1.4 the (Fourier-transformed) current density for a radial Hertzian source located at \((r_0, \theta_0, \phi_0)\), viz.,

\[
\hat{J}(\xi') = \frac{2}{(r_0^2 - \xi')^{3/2}} \hat{E}(r_0, \theta_0, \phi_0), \ 
\hat{E}(r, \theta, \phi) = \hat{E}_r(r, \theta, \phi), \hat{E}_\theta(r, \theta, \phi), \hat{E}_\phi(r, \theta, \phi)
\]

and obtain \( \hat{E}(\xi, \xi_0) = \frac{2}{r_0^3} \frac{r}{(\xi')^{3/2}} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu} \).

Substituting for \( \hat{J}_r \) from IV.1.10a and using the identities \( \hat{E}_r(\xi, \xi_0) \to \hat{E}(\xi, \xi_0) \cdot \gamma_{\nu} \) and \( \hat{E}_\theta(\xi, \xi_0) \to \hat{E}(\xi, \xi_0) \cdot \gamma_{\nu} \), \( \nu = 0, 1, 2, \ldots \)

\[
\hat{E}(\xi, \xi_0) = \left\{ \begin{array}{l}
\frac{\hat{E}_r(\xi, \xi_0)}{r_0^3} \frac{r}{(\xi')^{3/2}} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu} \\
\frac{\hat{E}_\theta(\xi, \xi_0)}{r_0^3} \frac{r}{(\xi')^{3/2}} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu}
\end{array} \right.
\]

where the upper series is valid for \( a < r_0 < r < b \), the lower for \( a < r < r_0 < b \), and

\[
\cos(\xi') = [\cos(\xi_0) \cos(\theta_0) + \sin(\xi_0) \sin(\theta_0) \cos(\phi_0 - \phi_0)]
\]

replaces \( \cos(\xi') \) since \( r_0 \) replaces \( r' \). In particular, the radial electric field in the second medium can then be written

\[
\hat{E}_r(r, \theta, \phi) = \hat{E}_r(r_0, \theta_0, \phi_0) \frac{r_0}{r} \frac{r}{(\xi')^{3/2}} \frac{1}{r_0^3} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu}
\]

Substituting equations III.2.13 and IV.1.10-13 into equation IV.1.8, we see that

\[
\hat{S}_r(r, \theta, \phi) = \hat{S}_r(r_0, \theta_0, \phi_0) \frac{r_0}{r} \frac{r}{(\xi')^{3/2}} \frac{1}{r_0^3} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu}
\]

and

\[
\hat{S}_\theta(r, \theta, \phi) = \hat{S}_\theta(r_0, \theta_0, \phi_0) \frac{r_0}{r} \frac{r}{(\xi')^{3/2}} \frac{1}{r_0^3} \hat{E}(r_0, \theta_0, \phi_0) \cdot \gamma_{\nu}
\]

The superscripts, TM and TE, distinguish the two sets of vector quantities necessary to describe an arbitrary electromagnetic field. TM denotes transverse magnetic fields (i.e., fields having no radial magnetic component) and TE denotes the analogous (transverse) electric fields.
with \( r_\geq \) and \( r_\leq \) the greater and lesser of \( r \) and \( r_0 \), respectively. This series is convergent for \( r \neq r_0 \), but an interchange of differentiation and summation (implicit in the derivation of (IV.1.21)) is not, in a strict sense, permissible for \( r = r_0 \) as the series that results is then divergent. The case \( r = r_0 \) is analyzed in some detail in section V.A; the correct field value is equivalent to the "Abelian" sum of the divergent series that results when \( r = r_0 \) is directly substituted into (IV.1.21).

The analogous field for the two-media system can be obtained from the above by setting \( \xi_3 = \eta_2 \), in which case \( R_1^\text{TM} = 0 \):

\[
\hat{\psi}_r = \frac{c_\eta}{2\pi \xi_3} \sum_{n = -\infty}^{\infty} \frac{L_1^{\text{TM}}(\xi_2 \lambda_n)}{L_2^{\text{TM}}(\xi_3 \lambda_n)} \frac{L_3^{\text{TM}}(\xi_4 \lambda_n)}{\xi_4^{\text{TM}}(\xi_1 \lambda_n)}
\]  

(IV.1.25)

If \( r_\leq \), (IV.1.25) reduces to

\[
\hat{\psi}_r = \frac{c_\eta}{2\pi \xi_3} \sum_{n = -\infty}^{\infty} \frac{L_1^{\text{TM}}(\xi_2 \lambda_n)}{L_2^{\text{TM}}(\xi_3 \lambda_n)} \frac{L_3^{\text{TM}}(\xi_4 \lambda_n)}{\xi_4^{\text{TM}}(\xi_1 \lambda_n)} \frac{\xi_2^{\text{TM}}(\xi_1 \lambda_n)}{\xi_2(\xi_1 \lambda_n)}
\]  

(IV.1.26)

where

\[
D_3^{\text{TM}} = \frac{c_\xi}{\xi_3} \frac{\xi_1^{\text{TM}}(\xi_2 \lambda_n)}{\xi_1(\xi_2 \lambda_n)} - \frac{c_\xi}{\xi_2} \frac{\xi_2^{\text{TM}}(\xi_1 \lambda_n)}{\xi_2(\xi_1 \lambda_n)}
\]  

(IV.1.27)

In Appendix A2, we indicate alternative series formulations and compare the results of IV.1.21 with those of Wait (1962, 1970: IV.10.13) for the associated wavefunction. Our two solutions differ and we show Wait's result to be in error. However, his subsequent analysis is unaffected by the error inasmuch as he introduces approximations which bring the fields into accord with the correct results.
IV.1.B Plane Earth Limit

Setting \( r = 0 \) in A2.2 of Appendix A2, we have an alternative form for the two-media spherical-system solution for \( \hat{E}_r \),

\[
\hat{E}_r = - \frac{C_0}{4} \int \Phi_{\text{inc}} \left( (2\pi a) \right) \left( \frac{\omega}{c} \right)^2 \left[ \frac{k a}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)} \, \text{P}_1(\cos(\theta)) \tag{IV.1.28}
\]

where, from A2.3.,

\[
G_{\text{inc}} = \left\{ \begin{array}{l}
\left[ \frac{k a^2}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)} \\
\left[ \frac{k a^2}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)}
\end{array} \right. \tag{IV.1.29}
\]

and \( \theta \) is the polar angle between the radially Hertzian source (taken collinear with the polar axis) and the receiver. If we now let the earth-radius, \( a \), increase without bound, we can recover the two-media plane-earth solution for an elevated Hertzian radiator and, consequently, obtain some insight into the physical significance of the associated series-terms.

Taking \( \rho = (\sin \theta) \) as the (cylindrically) radial distance from the \( z \)-axis, and choosing the origin \((z = 0)\) of the cylindrical reference frame as the intersection of the \( z \)-axis with the earth's surface, we have:

\[
\begin{align*}
\theta &\rightarrow \frac{\rho}{a}, \quad (p = \infty) \\
\zeta &\rightarrow \zeta + \alpha, \quad \zeta = \text{source height} \\
r &\rightarrow \zeta + \alpha, \quad \zeta = \text{receiver height} \\
\hat{E}_r &\rightarrow \hat{E}_g
\end{align*} \tag{IV.1.30a}
\]

\[
\hat{E}_g = - \frac{C_0}{4} \int \Phi_{\text{inc}} \left( \frac{\omega}{c} \right)^2 \left[ \frac{k a}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)} \, \text{P}_1(\cos(\theta)) \tag{IV.1.30b}
\]

Since the series is convergent, there exists an integer \( N \) for every \( \delta > 0 \) such that

\[
| G_{\text{inc}} | \leq \frac{C_0}{4} \int \Phi_{\text{inc}} \left( \frac{\omega}{c} \right)^2 \left[ \frac{k a}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)} \, \text{P}_1(\cos(\theta))
\]

for every \( j \geq N \), \( Q \), defined as the summand of IV.1.30b.

Inspection of the series terms indicates that \( N \) must be greater than \( k a \) before the terms of IV.1.30b begin to decrease. Consequently, if we define \( k = n/a \), \( k_n \) will take on values between zero and infinity when \( a \) to \( n \) (with \( k_n \) between 0 and some value greater than \( k \), possibly much greater, being important). But \( k_n/a \) will go to zero in the limit as \( a \to \infty \) since, for some positive number \( M \),

\[
0 \leq \frac{k_n}{\alpha} = \frac{n}{\alpha} \leq \frac{M a}{\alpha} = \frac{M}{\alpha} \to 0 \tag{IV.1.31}
\]

as \( a \to \infty \).

It then follows that

\[
\lim_{a \to \infty} \text{P}_1(\cos(\theta)) = \frac{1}{\pi} \int_0^\pi \left( \frac{\xi}{\epsilon} \right) \text{E}_0(\epsilon \cos(\theta)) \, d\theta
\]

or

\[
= \frac{1}{\pi} \int_0^\pi \exp\left\{ -i \left( \frac{\xi}{\epsilon} \right) \cos(\theta) \right\} \, d\theta
\]

or

\[
= \text{J}_0(\xi) \tag{IV.1.32}
\]

Substituting IV.1.32 with \( k = n/a \) and \( \mu = k a \), into IV.1.30b, we get

\[
\hat{E}_g = - C_0 \int \Phi_{\text{inc}} \left( \frac{\omega}{c} \right)^2 \left[ \frac{k a^2}{\sqrt{1-k^2}} \right] \frac{j_0(k a)}{j_0(k a)} \, \text{P}_1(\cos(\theta)) \tag{IV.1.33}
\]

The limits of the Bessel functions require knowledge of their asymptotic expansions, and these are detailed in Appendix A3. For spherical Bessel solutions of real argument, \( x \), and order \( \mu = (\nu - 1/2) \), there are separate expansions for the regions

(i) \( \frac{\sqrt{x}}{\delta} < 1 \)

(ii) \( 1 < x < \frac{\sqrt{x}}{\delta} \)

(iii) \( x > \frac{\sqrt{x}}{\delta} \)
where \( \beta \) is a positive number greater than or approximately equal to one. If \( x = k \alpha \), the first expansion is valid when \( k < k - \beta k^{1/2} \alpha^{2/3} \) or \( k < \sqrt{x} (\alpha + m) \), the second when \( \sqrt{x} - \beta k^{1/2} \alpha^{2/3} \leq k < k + \beta k^{1/2} \alpha^{2/3} \) or \( k = \sqrt{x} (\alpha + m) \), and the third when \( k > k + \beta k^{1/2} \alpha^{2/3} \) or \( k > \sqrt{x} (\alpha + m) \). Analogous results hold if \( x = kr - k \) or \( x = kr + k \). Use of the expansion valid for region (ii) can be avoided by assuming \( k \) has a vanishingly small positive imaginary part. This takes the medium to have a nonzero electrical conductivity (true of all physical media) and requires that we reconsider the Bessel expansions for complex argument, \( z \). Again, there are three regions of possible concern for real \( \nu \):

(i): \( |x - z| > |z|^2 \) (in region 1 of the complex plane of Figure A.5.1)
(ii): \( |x - z| > |z|^2 \) (in region 3 of the complex plane of Figure A.5.1).

But (ii) is now never satisfied in the limit as \( \alpha \to \infty \) since \( |kn - k| > 0 \) for all real \( k_n \), and we shall find that, in this limit, the expansions appropriate to (i) and (ii) give identical results at \( k_n = k \). In Appendix A.4 we show (for \( \nu = k_n \alpha \))

\[
\left[ \frac{L_{\nu}^{(1)}(kn)}{L_{\nu}^{(1)}(k)} \right] = \exp \left\{ i \left( \sqrt{x - z} \right) (k_n^2 - k^2)^{1/2} \right\} \label{eq:4.1.34}
\]

\[
\left[ \frac{L_{\nu}^{(2)}(kn)}{L_{\nu}^{(2)}(k)} \right] = \exp \left\{ i \left( \sqrt{x - z} \right) (k_n^2 - k^2)^{1/2} \right\} \label{eq:4.1.35}
\]

\[
\left[ \frac{L_{\nu}^{(3)}(kn)}{L_{\nu}^{(3)}(k)} \right] = \exp \left\{ i \left( \sqrt{x - z} \right) (k_n^2 - k^2)^{1/2} \right\} \label{eq:4.1.36}
\]

Equation IV.1.33 then reduces to

\[
\hat{\omega}_1 = -c_n \int_0^\infty \frac{\alpha^{1/2}}{k} \left( (k_n^2 - k^2)^{1/2} \right) \exp \left\{ i \left( \sqrt{x - z} \right) (k_n^2 - k^2)^{1/2} \right\} \left( \frac{L_{\nu}^{(1)}(kn)}{L_{\nu}^{(1)}(k)} \right) \frac{J_\nu(k_{n,\nu} \alpha)}{k_{n,\nu} \alpha} \, dk_n \label{eq:4.1.37}
\]
Maxwell's (1873) complete formulation of the laws of electrodynamics suggested that electromagnetic waves could propagate without the aid of wires, that visible light was, in fact, electromagnetic radiation. Wireless transmission of electromagnetic energy was confirmed experimentally by Hertz (1887) using decimeter wavelengths, though, as the twentieth-century began, it was only at frequencies less than 1 kHz (i.e., wavelengths longer than 30 meters) that substantial power could be radiated. Marconi, using progressively longer wavelengths, transmitted a signal as far as two kilometers in 1895, but it was his success (on December 12, 1901) in detecting signals sent across the Atlantic that captured scientific attention and greatly stimulated research in terrestrial propagation (Booker 1974, Carassa 1982).

Some early investigators, including Marconi, appear to have believed that long-range propagation could be explained by considering the earth to act as a giant conductor, not unlike a large telephone wire (Booker 1975). Following this general idea, Zenneck (1907) and Sommerfeld (1909, 1926) pioneered work on propagation over a plane half-space, with Sommerfeld obtaining a formal integral-solution (equivalent to IV.1.37) for a vertical dipole radiating over a "plane earth." These plane earth solutions could not adequately account for the experimental results; hence attention turned to calculations based on a spherical earth model. The formal, zonal harmonic series solution (eq. IV.1.21) to electromagnetic scattering by a sphere was obtained by Mie (1908) and Debye (1909). The slow convergence of this series for terrestrial propagation (e.g., over 8000 terms at 50 kHz) made its direct use virtually impossible at that time.

Consequently, a simplified solution was sought, and early investigations (e.g., MacDonald, 1910, 1914; Toilcaren 1910; Nicholson 1910, 1911; March 1912; von Rybezynski 1913; Love 1915) culminated with Watson's work (1918, 1919).

Watson's transform* (sec.IV.2.A) converted the original series into a contour integral which, via deformation of contour, could then be evaluated by a second "residue" series once these residues (or modes) had been determined. In contrast with the experimental (largely LF) data, which showed attenuation to be inversely proportional to distance, the theoretical solution predicted an exponential roll-off for distances well-beyond the radio-horizon. This showed that the two-media system model was inadequate, and Watson (1919) then introduced an upper conducting-layer (the existence of which had been proposed by Heaviside around 1900), finally obtaining results in agreement with the existing experimental data.

During this same period, a number of VLF and LF transmitters were placed in operation. Propagation characteristics were found to improve as the frequency was lowered; however, antenna size, cost, and power requirements also increased. By the late 1920's evidence that the ionosphere was reflective, the advent of directional antennas (wire arrays), and the development of the thermionic vacuum tube (which permitted efficient radiation at frequencies above LF) resulted in mid-band systems supplanting lower-band systems for long-range communications. This changeover was predicted as early as 1920 by David Sarnoff in a letter to the then president of RCA:

> While dealing with the subject of transmission and reception on shortwaves, I might record a hunch which I have held for some time and which I have discussed with a number of radio engineers, who, I should say in justification of their opinions, definitely disagree with me. I refer to the possibility of employing shortwaves for long-distance communications and, perhaps eventually, transoceanic communications. The obvious answer to this is that daylight absorption makes impractical the use of shortwaves over long distances, but I doubt whether a careful and exhaustive research has been made on this point. Perhaps extreme

* See Logan (1965) for an account of work preceding these papers. Lord Rayleigh in 1904 obtained the zonal harmonic solution to the scalar problem of acoustic scattering by a sphere.

* Watson's analysis closely followed that of Poincare (1910) and Nicholson (1910) whose work he characterized as "substantially sound" but "lacking in rigour in some points of detail."
amplification, such as is possible with the Armstrong amplifier, or even greater amplification, which should be possible, may detect radio signals from shortwaves where present-day amplification fails to do so.

Marconi (with C.S. Franklin), again at the experimental forefront, conducted systematic measurements between 1921 and 1924.*

The theoretical analysis of long-distance propagation from directed sources at mid-band did not make use of Watson's sophisticated analysis (which was largely neglected until the late 1930s, Booker 1975); rather, earth effects were ignored and the approximations of ray theory were employed. Booker (1975) credits Eccles (1912) with the earliest effort to analyze the effects of the ionosphere on propagating waves, and Larmor (1924) with first understanding that, because the dielectric constant of the ionospheric plasma is less than unity, a wave incident on the ionosphere can be continuously refracted from the vertical until returned to earth. The anisotropic nature of the ionosphere — due to the earth's magnetic field — was then recognized by Appleton (1925), Nichols and Schelleng (1925), and Lassen (1927).

It was soon realized that propagating waves of frequencies above 30 MHz (i.e., upper-band) were not significantly reflected by the ionosphere. Consequently, use of upper-band frequencies was generally believed to be restricted to line-of-sight applications (which were to receive increased attention beginning in the 1930s).

Marconi was an exception:

Electromagnetic waves of less than one metre in wavelength are usually called quasioptical waves; the general belief is that these waves do not permit radio communication unless the two terminals of the circuit are optical to each other, and that in consequence, their usefulness is limited by this condition. Long experience suggests to me that the limits determined from purely theoretical considerations, or by calculation, should not always be believed, since these often depend, as is well known, on an incomplete knowledge of all the factors involved. For my part, I believe it is necessary for research to venture in new directions, even though the forecast may be pessimistic and there is little to encourage the early stages. In what concerns the limiting distances for the propagation of these waves, the last word has not yet been spoken.

Marconi succeeded in maintaining communication links of 150-210 km at 500 MHz when the optical horizon was about 30 km (Marconi 1933 a,b; Carassa 1962). Recent extensions using the "surface impedance" concept of Leontovich and Fock (1946) have been made by Hill and Wait (1980).

* Daytime links of 2500 km were established at 3.25 MHz using antennas far shorter than those required at VLF or LF, and two orders of magnitude less power (Carassa 1962). Marconi had experimented with mid-band links as early as 1916 (Ramsey 1981).
Evidence accumulated during systematic studies of FM radio and television interference indicated that another phenomenon must also be operative. Beyond the horizon, field-strengths were measured at levels which significantly exceeded those predicted by diffraction theory even in the absence of ducting; moreover, the intensity fluctuated. These findings were explained by Booker and Gordon (1950) as resulting from the scattering effects of atmospheric turbulence. Their calculations also suggested the possibility of ionospheric forward scattering at VLF, which was experimentally confirmed.

In the 1950s Keller (e.g., 1953, 1958, 1962) extended ray theory (geometrical-optics) to include diffractive effects of scatterers; his techniques and their subsequent extensions are collectively referred to as the geometrical theory of diffraction (GTD). These methods were first applied to high frequency (e.g., radar) scattering from objects with dimensions orders of magnitude less than the earth's radius. Since scattering from spheres and long cylinders is invariably a function of the product ka, where a is the (spherical or cylindrical) radius and k is proportional to frequency, GTD solutions for these canonical problems often reproduced results obtained earlier in studies of lower frequency terrestrial propagation:

I considered the introduction of surface diffracted waves to be a nice feature of the theory. Therefore I was somewhat disappointed when the paper of Franz and Deppeman (1952) appeared, in which they introduced creeping waves on a circular cylinder. However, I later realized that similar waves on a sphere had been found long ago by Watson (1918) in his work on radio wave propagation around the earth... (Keller 1985)

In addition to reproducing classical results, GTD has found further application in bounded or inhomogeneous system problems for which exact solutions are not possible (Felson and Marcuvitz 1973).

Interest in lower-band propagation was also renewed during the 1950s, prompted by problems related to navigation, long-range strategic communication, storm-tracking, earth-ionosphere cavity resonances, communication in (underground) mines, and geological prospecting. Fields at these frequencies were known to attenuate slowly over distances of thousands of kilometers, and experiments during the 1930s had indicated phase and amplitude stability to have been maintained over great distances (Burgess and Jones 1975). Furthermore, these fields could be detected at relatively great depths in comparison to fields at higher frequencies. Again, most theoretical efforts began with the mode-series formulation of Watson (1919). Budden (1953, 1957) analyzed the mode-series for a planar-waveguide, later (1961) introducing an inhomogeneous atmosphere to "effectively" account for the spherical nature of the waveguide formed by the earth and ionosphere. This approach was subsequently developed by scientists at the Naval Ocean Systems Center (e.g., Pappart, Gossard, and Rothmueller 1967, Pappart and Shockey 1976). Alpert (1955, 1956, 1961) also considered the planar-waveguide mode-series modified for curvature by a geometric factor. This correction was later determined to be adequate only for ELF and lower VLF (Malt 1962). Malt analyzed the mode-series for a spherical-waveguide (e.g., 1957, 1962). These early studies typically modeled the ionosphere as homogeneous and sharply-bounded and were usually calculated only one or

---

* Booker and Walkinshaw (1946) used the "phase integral" approach developed by Eckersley and Willington, (1938) which extended Watson's analysis to inhomogeneous media.

** Transhorizon troposcatter telephone links of several hundred kilometers began operation as early as 1953 and continue in use today, though the development of satellite communication systems has lessened their importance (Collins 1985).
two modes (often sufficient at ELF or lower VLF) owing in part to the difficulty of solving the mode-determining equation (i.e., locating the poles in the complex-plane).

Whereas Nicholson, in the early twentieth-century, characterized the numerical solution of the zonal-harmonic series as one of the hardest problems confronting the mathematical-physical scientist of his day (Bremner 1949: p. 7), Wait in 1962 characterized the solution of the three-media mode-determining equation as one of the most difficult then confronting analysts. It remains a problem (Pappert 1981).

In the 1960's, interest in VLF and LF propagation and systems increased as the U.S. Navy initiated studies of a proposed ELF system for submarine communication. Generalization of the ionospheric model to include the effects of inhomogeneity and anisotropy was undertaken (e.g., Budden 1962, Johler and Harper 1962, Wait and Spies 1964, Pitteway 1965), and iterative routines for mode-determination were developed (Wait and Spies 1964, Sneddy et al 1968, Morfitt and Shellman 1976). Three analytically equivalent, but computationally distinct, solutional approaches to the waveguide problem emerged: the mode-series, the "wave-hop" method which involved the computation of a series of contour integrals, and the zonal harmonic series (secs. IV.2.C.D and E respectively). The wave-hop expansion, suggested by Rydbeck (1944) and Bremner (1949), was shown to reproduce the ray-hop series of geometrical optics (long utilized at HF) when a saddle-point approximation of the integrals was substituted (Wait 1961); that is, the first integral corresponded to groundwave (i.e., no ionospheric reflection), the second to a ray once reflected from the ionosphere, the third to a ray twice reflected from the ionosphere, etc. As the saddle-point approximation was inadequate at and beyond the caustic, Wait (1961) suggested evaluating the wave-hops either by numerical integration or by summing their equivalent residue-series, an approach developed by Berry and Chrisman (1965). The third approach, direct summation of the zonal harmonic series, was reconsidered by Johler and Berry (1962) given the availability of a high-speed computer. They conclude that:

Whereas the number of terms in the series could become quite large, the speed with which these terms can be summed on a large scale computer offsets the complications introduced by the Watson transformation. (Johler and Berry 1962: p. 768).

Johler and Lewis (1969) calculated the ELF field for a spherically-stratified ionosphere. Unfortunately, a programming error, not discovered until much later (Lewis and Johler 1976), led them to predict lower fields-strengths than the corresponding modal solution for a nonstratified ionosphere. Controversy ensued, mostly because of the significance attached to detectable field-strength levels for an ELF system. Experimental data were collected, which supported the modal (and zonal-harmonic) solution for a homogeneously-modeled ionosphere (Burrows 1978). Though Johler (1970) used the zonal-harmonic series in calculations at VLF through HF, other researchers appear not to have actively pursued this approach. It is one which is studied in some detail herein.
IV.2. CALORIATION: A/I/IW£S

IV.2.A. MODE THEORY: TWO-MEDIA SYSTEM

The solution for a two-media system with a radial Hertzian source in the outer medium is given by the series in IV.1.25. Early twentieth-century efforts to obtain a more tractable form culminated with Watson's 1918 paper, in which the residue series formulation was rigorously derived (section IV.1.C).

In terms of residues, \( \int_{\gamma_{v_1}} \frac{2 \pi (v_1)}{A_{v_1}} \, dv_1 = -2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\pi}{2} \), where \( A_{v_1} \) is the integration contour of Figure IV.2 that encloses the zeros of \( \sin(v_1) \), but no singularities of \( g(v) \). Noting that \( \int_{\gamma_{v_1}} \frac{\pi}{2} \, dv_1 = \pi \) for an integer, we obtain from IV.1.25:

\[
\Phi_{v_1}(x, \theta) = -\frac{\pi}{4\pi} \int_{A_{v_1}} \frac{\pi}{2} \left[ \sin(v_1 - \theta) \right] \, dv_1,
\]

where

\[
\pi(v) = v(v + 1) \frac{\sqrt{\pi} e^{i \theta}}{\sqrt{\pi} e^{-i \theta} + i} \left[ 1 + \frac{\sqrt{\pi} e^{i \theta}}{\sqrt{\pi} e^{-i \theta}} + \frac{\sqrt{\pi} e^{i \theta}}{\sqrt{\pi} e^{-i \theta}} \right],
\]

with \( \cos(\theta), C_0, \) and \( R_{TM}^0(v) \) as defined in IV.1.20, IV.1.22, and IV.1.23, respectively.

The poles of the integrand (excluding the \( \sin v_1 \) factor) correspond to the zeros of

\[
C_{v_1} = \left( \omega_{0v} \right) \left[ \frac{\pi}{2} \left( \omega_{0v} \right) \right] - \left( \frac{i}{\omega_{0v}} \right) \left[ \frac{\pi}{2} \left( \omega_{0v} \right) \right],
\]

the denominator of \( R_{TM}^0(v) \), and lie away from the real axis in the second and fourth quadrants of Figure IV.2 (Watson 1918, Van der Pol and Bremner 1937). Then, deforming \( A_{v_1} \) to the new contour \( (C_0 + C_2 + C_3) \) that results when \( R = m \) in Figure IV.2, we obtain

\[
\Phi_{v_1} = -\frac{\pi}{4\pi} \int_{C_0} \frac{\pi}{2} \left[ \sin(v_1 - \theta) \right] \, dv_1
\]

where \( C_2 \) is a curve enclosing all the "residues" \( v_m \) satisfying

\[
C_{2m} = 0.
\]

The line integral is widely interpreted as the contribution of waves that reach the observation point after traveling through the earth (Bremner 1949, Berry 1964), and such waves are strongly attenuated at all points not close to the transmitter. The integral, neglected in practice, would be identically zero if the term, \( k/K_1 \ln[(K_1 a) / (K_1 a)] \), in \( R_{TM}^0(v) \) were an even function of \( (v + 1/2) \) — a condition not strictly satisfied except for a perfect conductor. Still, if \( K_1 \), has a large imaginary component, the asymptotic Debye expansions will be valid for all values of \( v \), thus yielding approximations that are all even functions of \( v + 1/2 \), the most commonly used being

\[
\left[ \frac{\pi}{2} \left( \omega_{0v} \right) \right] \left[ \frac{\pi}{2} \left( \omega_{0v} \right) \right] = \frac{\pi}{2} \left( \omega_{0v} \right) \left( \frac{\pi}{2} \left( \omega_{0v} \right) \right)^2
\]

(Bremner 1949, Wait 1962).
For a "good" conducting medium, the important modes lie near \( \nu - ka \ll |K_a| \) and, for these, IV.2.7 is approximately 
\[
- \left[ 1 - (\frac{K_a}{a})^2 \right]^{1/2},
\]
a constant. Making this substitution, the ratio of the field components, \( E_z/E_y \), will equal a constant, 
\[
- \frac{2}{a},
\]
where \( E_z \) is identified as the surface impedance. This has prompted several analysts to assume \textit{ad initio} a constant surface impedance [Leonovich and Fock 1946, Walt 1962]; direct application of the surface impedance concept to the original zonal harmonic series is discussed in section V.C.

Interestingly, if the Debye approximation is made prior to the deformation of contour \( A \rightarrow (C_1 + C_2 + C_3) \), a branch-point singularity is introduced at \( (\nu + 1/2) = IT_a \). The deformation to the new path then introduces a branch-cut contribution not previously present, even though the line integral \( C_3 \) now vanishes.

In the limit as \( a \to \infty \), the leading Debye approximation holds exactly and IV.2.1 reduces to the plane earth result (IV.1.37). Again, a branch point exists at \( k_n = K_1 \) and its contribution has been shown to decay as \( (k_1 x \text{ distance})^{-2} \), with an additional exponential loss-factor as a result of the conductivity in the subsurface medium (Felson and Marcuvitz 1973). Were the source to be located in the subsurface medium, the case for neglecting the branch-cut contribution in the plane-earth problem and the line integral in the spherical problem would not be so straightforward.

We now direct our attention to the mode series of equation IV.2.5. For \( \xi \) not too near \( 0 \) or \( x, \left( |\xi|, \sin \xi \gg 1 \right) \),
\[
\begin{align*}
\frac{H_{\nu}^m(\xi)}{\nu H_{\nu}(\xi)} & \approx \frac{1}{\nu^{1/2}} \left\{ e^{-i\nu^{1/2} \frac{\xi}{2}} + e^{-i\nu^{1/2} \frac{\xi}{2}} \right\}, \\
& \approx \left\{ e^{-i\nu^{1/2} \frac{\xi}{2}} + e^{-i\nu^{1/2} \frac{\xi}{2}} \right\}.
\end{align*}
\]
(IV.2.10)
This involves approximating \( \left[ 1 - e^{i\nu^{1/2} \xi} \right]^{-1} \) as \( 1 \), which is justified because \( \Im(\nu^{1/2}) > 0 \); indeed, we will find subsequently that \( \Im(\nu^{1/2}) \propto \nu^{1/2} \), where \( \nu \) is positive and increases with \( m \). Clearly the most important modes are those with the smallest imaginary component.

The modes are those values of \( \nu \) which satisfy IV.2.6. A number of expansions, appropriate for different regions of the complex \( \nu \)-plane and dependent on the argument, exist for the various Bessel functions. The more general (viz, accurate) the approximation used, the more difficult the analytic solution becomes. For the two-media problem, methods exist for obtaining accurate values for the modes. For good conductors, IV.2.7 is usually an excellent approximation for the Bessel term in the mode equation. Approximation of the Hankel functions of real argument, \( \nu \), is more difficult because the important modes, \( \nu \), lie close to the argument.

Let \( x = ka \) and \( u = \nu + 1/2 \). Then from Appendix A3, a uniform approximation to the spherical Hankel function, \( W_{\nu}^m(x) = \sqrt{\frac{\pi}{2x}} H_{\nu}^{(1)}(u) \), is given by
\[
W_{\nu}^m(x) \approx 2^{1/2} \left( \frac{\nu}{x-\nu} \right)^{1/2} \phi(x)(\nu^{\nu + \frac{1}{2}}),
\]
where
\[
\nu^{\nu + \frac{1}{2}} \phi(x) \equiv \nu^{1/2} \left( \frac{\nu}{x} \right)^{1/2}.
\]
(IV.2.11)
and, if \( |u - x| \ll |u| \),
\[
\nu^{\nu + \frac{1}{2}} \phi(x) \approx \nu^{1/2} \left( \frac{\nu}{x} \right)^{1/2}.
\]
(IV.2.12)
For \( |u^{1/2} \xi | \gg 1 \) (or, correspondingly, \( |u - x| \gg \xi^{1/3} \)), IV.2.9 is asymptotically equivalent to the leading Debye approximation appropriate to region 2 of Figure A3.1.

The mode solution using the Debye form is called the "tangent" or second-order approximation and indicates that the modes lie along curve \( C_2 \) of Figure A3.1. However, the most important modes lie close to the argument, where the Debye expansion is inadequate; IV.2.9 must then be used.
Substituting IV.2.11 in IV.2.9 and defining $\mu = \sqrt{v + \frac{1}{2} = x + u}$, we get

$$I \left[ x, \frac{v}{\sqrt{v + \frac{1}{2}}} \right] \approx -z^2 x^2 \exp \left( \frac{2}{3} \right) \frac{A_1'(e^{(v + \frac{1}{2})x})}{A_1(e^{(v + \frac{1}{2})x})},$$

(IV.2.12)

and the mode determining equation IV.2.6 can then be approximated as

$$e^{\frac{2}{3} x} \frac{A_1'(e^{(v + \frac{1}{2})x})}{A_1(e^{(v + \frac{1}{2})x})} \approx -\frac{1}{2} \frac{d}{dx},$$

(IV.2.13)

where

$$d = \frac{\gamma \left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)}{\left( v \sqrt{v + \frac{1}{2}} \right)}.$$

(IV.2.14)

Use of Airy function identities (NBS 1964: 10.4.9, 10.4.23) leads to

$$\frac{A_1'(e^{(v + \frac{1}{2})x})}{A_1(e^{(v + \frac{1}{2})x})} = e^{-\frac{1}{3} \left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)} \frac{H_1'\left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)}{H_0'\left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)},$$

(IV.2.15)

which, when substituted in IV.2.13, gives

$$e^{\frac{2}{3} x} \frac{H_1'\left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)}{H_0'\left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right)} \approx -\frac{1}{2} \frac{d}{dx} x - 2\xi,$$

(IV.2.16)

and this is the mode equation used by Van der Pol and Bremmer (1937), Bremmer (1949), Johler et al (1959), and Wait (1962).

The limiting case $\delta_e \rightarrow 0$ is of a perfectly conducting earth; the solutions are

$$\xi_0 = \frac{1}{2} \left( \frac{3}{2} \right)^{\delta_e} e^{\frac{v}{\sqrt{v + \frac{1}{2}}}}.$$

(IV.2.17)

and the line integral has been neglected.

A physical interpretation is commonly given the summand terms in IV.2.15 when the source or the receiver is at a distance from the earth such that $x - v_m \gg x^{1/3}$ for $x = kr$ (or $kr_e$) and $|v_e|$ is approximately $ka$ for $m$ not too large (Jones 1979).

Substituting IV.2.17 into IV.2.5, gives us the "classical" two-media mode solution (Johler and Berry 1964):

$$\frac{d}{dx} \exp \left( \frac{2}{3} x \right) \approx \left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right) \left( \frac{1}{d_e} - 2\xi \right).$$

(IV.2.18)

Substituting IV.2.19 into IV.2.12, gives us the "classical" two-media mode solution

$$f(kr) \mid f(kr_0) \approx \exp \left( \frac{2}{3} x \right) \exp \left( i \cdot \frac{3}{2} \xi \right) \exp \left( i \cdot \frac{v}{\sqrt{v + \frac{1}{2}}} \right).$$

(IV.2.19)

can again be written as in IV.2.16, but now $x$ satisfies $J_3^3 (\xi_0) - J_3^3 (\xi_e) = 0$.

The solution for arbitrary $\delta_e$ can be derived from $\frac{d}{dx} \exp \left( \frac{2}{3} x \right) - 2\xi$ as a perturbation series involving $\xi_0$ for small $\delta_e$ or $\xi_0$ for large $\delta_e$. The values of $x$ will be positive and increasing in magnitude with $e$.

When $(ka) \gg 1$, then

$$\frac{d}{dx} \exp \left( \frac{2}{3} x \right) \approx \left( \frac{v}{\sqrt{v + \frac{1}{2}}} \right) \left( \frac{1}{d_e} - 2\xi \right).$$

(IV.2.11)

Assuming both $kr$ and $kr_0$ satisfy this condition, the product $\left[ f(kr) \! f(kr_0) \right]$ will have an exponential dependence given by the Debye approximation of $\exp \left( i \cdot \frac{3}{2} \xi \right)$.

This factor, coupled with the Legendre function approximation in IV.2.18, indicates that the summand in IV.2.18 has an exponential dependence of the form

$$\exp \left( i \cdot \frac{3}{2} \xi \right) \exp \left( i \cdot \frac{v}{\sqrt{v + \frac{1}{2}}} \right).$$

(IV.2.20)
where

\[ Q = \left( \frac{c_2}{c_1} \right)^2, \]
\[ Q_s = \left( \frac{c_{s2}}{c_{s1}} \right)^2, \]
\[ Q^\prime = \left[ a(e^{I - a}) - a\cos^I(\frac{\xi}{c_{s1}}) - \cos^I(\frac{\xi}{c_{s2}}) \right], \]
\[ Q^\prime = \left[ a(e^{I - a}) - a\cos^I(\frac{\xi}{c_{s1}}) - \cos^I(\frac{\xi}{c_{s2}}) \right]. \] (IV.2.21)

Thus the summand can be interpreted as two "creeping waves" that reach the receiver after traveling in opposite directions around the sphere as in Fig. IV.3. The exponential decay that results because \( \text{Im}(v_m) > 0 \) can be associated with that portion of the sphere traversed by the waves. The imaginary part of \( v_m \) is proportional to \( (ka)^{1/3} \) and increases with \( a \); hence, the residue series will be rapidly convergent whenever \( (ka)^{1/3} (\xi) \gg 1 \), \( v^2 \ll \kappa^2 \).

Figure IV.3 Ray Paths

We can divide the scattering region as in Fig. IV.4, where the penumbral region occupies the space bounded by tangents to the sphere at points an angular distance \( d = a(ka)^{-1/3} \) on either side of the geometric horizon. For receiver locations in the deep shadow region, the number of modes that need to be summed increases greatly as the penumbral region is approached.

Figure IV.4 Scattering Regions
IV.2.B. GEOMETRICAL OPTICS -- TWO-MEDIA SYSTEM

If the line integral, \( C_1 \), of IV.2.4 is negligible (or zero) and \( C_2 \) is the contour around the poles as shown in Fig. IV.2, then

\[
\hat{\mathcal{E}}_\nu = \frac{2\pi\nu}{4\pi\nu\nu} \int_{C_2} (2
\nu + 1) \frac{1}{\cos(\nu\pi)} R_\nu \left[ \cot(\nu \pi - \frac{\pi}{2}) \right] d\phi. 
\]

(IV.2.22)

For observation points in the illuminated region of Fig. IV.4, the mode series representation for the above integral converges extremely slowly. A modified treatment for this case was suggested by Fock (1946) [and, earlier, by White (1922)] who substituted

\[
R_\nu \left[ \cot(\nu \pi - \frac{\pi}{2}) \right] = -i e^{-i\nu\pi} R_\nu \left[ \cot(\nu \pi) \right] \cos(\nu \pi) Q_v \left[ \cot(\nu \pi) \right]. 
\]

(IV.2.23)

where \( Q_v \) is the Legendre function of the second kind (NBS 1964). The integral of the first term can then be transformed into a rapidly-converging mode series which will have a negligible contribution within the illuminated region when compared with the contribution of the integral of the second term -- at least for \( \nu > 1 \) (Fock 1946, Felsen and Marcuvitz 1973).

We retain only the integral associated with the second term of IV.2.23 and write

\[
\hat{\mathcal{E}}_\nu = -\frac{2\pi\nu}{4\pi\nu\nu} \int_{C_2} (2\nu + 1) \frac{1}{\cos(\nu\pi)} Q_v \left[ \cot(\nu \pi) \right] d\phi. 
\]

(IV.2.24)

Note that the poles at positive integer values of \( \nu \) have been eliminated, though \( Q_v(\cos \xi) \) does introduce poles at the negative integers. This final integral can be evaluated by steepest-descent methods with the resultant dominant saddle point contributions identifiable as the direct and reflected waves of geometrical-optics in the illuminated region (Nesseveig 1965, Jones 1979).

In the penumbral region of Fig IV.4, a simple saddle point analysis proves inadequate. The saddle points associated with the direct and reflected waves coalesce near the lowest order residues of the integrand. The Debye approximations to \( h_{\nu}^{(1,2)}(ka) \), useful for \( |\nu - ka| > O(ka)^{1/3} \), must then be replaced by uniform approximations in terms of Airy functions (Appendix A3) and a numerical integration carried out (White 1922, Pekeris 1947, Wu 1956, Beckman and Franz 1957, Jones 1979).
IV.2.C MODE THEORY: THREE-MEDIA SYSTEM

The mode series for the three-media spherical system is derived from the zonal-harmonic series solution (IV.1.21) in a manner paralleling the two-media mode series derivation; i.e., the zonal harmonic series can be rewritten as a mode sum and a line integral. The line integral, though identically zero only if the earth is a perfect conductor, is invariably neglected as insignificant for long range propagation (Berry 1964). Then, for \( K_3 \neq k \)

\[
\left[ 1 + \frac{\tilde{I}_0(\nu_k)}{\nu_0} \left( \frac{\nu_k}{\nu_0} \right)^{2} \right] \frac{i \nu_0 (\nu_0 \pi)}{2 \pi} \left( 2 \nu_0 \pi \right) \]

(IV.2.25)

where, now,

\[
\phi_{\nu_0} = \left[ 1 + \frac{\tilde{I}_0(\nu_k)}{\nu_0} \left( \frac{\nu_k}{\nu_0} \right)^{2} \right] \frac{i \nu_0 (\nu_0 \pi)}{2 \pi} \left( 2 \nu_0 \pi \right) \]

(IV.2.26)

and the modes, \( \nu_0 \), are determined by

\[ d_{\nu_0}^{TM} = 0 \]

(IV.2.27)

This is the classical mode series solution for a three-media system, and IV.2.27, with \( d_{\nu_0}^{TM} \) as defined in IV.2.26, is the standard mode-determining equation (Wait 1962, Juhler and Berry 1964, Galejs 1912, Morfit and Shellman 1976). Observe, however, that in the limit as \( K_3 \) goes to \( k \), \( R_1^{TM} \) becomes unity, and the mode equation, IV.2.27, clearly fails. A more general form of the mode equation (that the author has not found noted in earlier work) is

\[
\tilde{I}_0(\nu_k) \left( \frac{\nu_k}{\nu_0} \right)^{2} \]

(IV.2.26)

\[ = \left( \frac{\nu_k}{\nu_0} \right)^{2} \frac{i \nu_0 (\nu_0 \pi)}{2 \pi} \left( 2 \nu_0 \pi \right) \]

If \( K_3 \neq k \), we can divide through by the first term and recover IV.2.27. If \( K_3 = k \), the second term is zero and IV.2.27 reduces to IV.2.26, the mode equation for a two-media system, as it should. Use of the exact mode equation would require a modification of IV.2.25 and would complicate the already difficult mode-determination process. Mode programs based on the less general equations IV.2.25-IV.2.27 can be expected to have difficulty when the outer medium is taken to be poorly conducting and has a dielectric constant close to unity (i.e., when \( K_3 \) approaches \( k \)); furthermore, there must be separate mode routines for two and three-media problems.

We assume \( K_3 \) is not close in value to \( k \) and address the solution of \( d_{\nu_0}^{TM} = 0 \).

As late as 1962, it did not seem possible to solve for the modes without significant approximation (Wait 1962). In general, a complicated root-finding procedure is now employed. An early iterative solution, described by Wait and Spies (1964), used the leading Airy approximation to the Hankel functions and took as a first approximation to each mode the value determined for a perfectly-conducting earth and ionosphere. More accurate routines that exhaustively search the complex plane for modes have since been developed, \( d_{\nu_0}^{TM} \) requiring evaluation at each "searched" point. Morfit and Shellman (1976) wrote one such program, "Modesearch" (for what is now the Naval Ocean System Center), which is applicable between 10 kHz and 60 kHz. This program corrected some difficulties of an earlier routine (Shedd et al 1968) which would occasionally miss modes at frequencies above 30 kHz. "Modesearch" is described in CCIR Rep. 295 (1982), and is still considered not completely adequate at LF where many modes may be required (Pappert 1981).

Simplification of the modal analysis is sometimes possible. The Bessel functions can be approximated by their Debye expansions if \( (ka) \gg 1 \) (i.e., \( f > 50 \) Hz), \( (\nu_0 + 1/2) \gg 1 \) (Appendix A3). This last equation is not satisfied for frequencies above about 10-15 kHz; results using the approximation for the frequency range 10-15 kHz may be in reasonable agreement with more accurate calculations for distances of two or three
thousand kilometers, but not at greater distances (Wait 1962:p.157, Berry 1964, Al'pert 1973:p.134). Berry (1964a) found that the attenuation rate of the first mode (as calculated with the Debye approximation) was in error by 40 percent at 16.6 kHz, 25 percent at 10 kHz, and 15 percent at 8 kHz when compared with the results obtained using the more accurate Airy function approximation.

When only the leading term of the Debye expansion is substituted, the spherical reflection coefficients, \( R^T \) and \( R^T \), reduce to the Fresnel reflection coefficients

\[
\begin{align*}
R^T_\theta (\omega) &= \frac{[\sin(\omega \theta) - (\lambda_+ \theta)]^2 - (\lambda_+ \theta)}{[\sin(\omega \theta) + (\lambda_+ \theta)]^2 + (\lambda_+ \theta)} \\
R^T_\phi (\omega) &= \frac{[\sin(\omega \phi) - (\lambda_+ \phi)]^2 - (\lambda_+ \phi)}{[\sin(\omega \phi) + (\lambda_+ \phi)]^2 + (\lambda_+ \phi)}
\end{align*}
\]

(IV.2.29) (IV.2.30)

for complex angles of incidence \( \sin^{-1} \left( \frac{\lambda_+ \theta}{ka} \right) \) and \( \sin^{-1} \left( \frac{\lambda_+ \phi}{ka} \right) \) (cf Wait 1962).

The mode equation is then

\[
\begin{align*}
\left( \begin{array}{l}
\frac{\partial^2}{\partial \xi^2} - \frac{m^2}{\sin^2 \phi} \\
\frac{\partial^2}{\partial \eta^2} - \frac{m^2}{\sin^2 \phi}
\end{array} \right) F_{m}^T (\nu, \xi, \eta) &= \left( \begin{array}{l}
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \sin \phi \frac{\partial}{\partial \phi} - \frac{m^2}{\sin^2 \phi} \\
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \sin \phi \frac{\partial}{\partial \phi} - \frac{m^2}{\sin^2 \phi}
\end{array} \right) F_{m}^T (\nu, \xi, \eta)
\end{align*}
\]

(IV.2.25)

and, associated with each \( m \), will be a mode \( v_m \). If, for all the modes (or all the "important" ones), \( \left[ 1 - \left( \frac{\lambda_+ \phi}{ka} \right)^2 \right] \gg \left( \frac{\lambda_+ \phi}{ka} \right)^2 \), then IV.2.25 has been further approximated as

\[
\begin{align*}
\left( \begin{array}{l}
\frac{\partial^2}{\partial \xi^2} - \frac{m^2}{\sin^2 \phi} \\
\frac{\partial^2}{\partial \eta^2} - \frac{m^2}{\sin^2 \phi}
\end{array} \right) F_{m}^T (\nu, \xi, \eta) &= \left( \begin{array}{l}
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \sin \phi \frac{\partial}{\partial \phi} - \frac{m^2}{\sin^2 \phi} \\
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \sin \phi \frac{\partial}{\partial \phi} - \frac{m^2}{\sin^2 \phi}
\end{array} \right) F_{m}^T (\nu, \xi, \eta)
\end{align*}
\]

(IV.2.26)

with \((\sin \phi)^{-1} = (ka)^{-1}\) used in the calculation of \( F_{m}^T (\nu, \phi) \). This is simply the equation for modes in a planar waveguide (Al'pert 1955, 1956, 1973). Moreover, when the dominant asymptotic term for \( P_{\nu} \cos(\nu \phi) \sin(\nu \phi) \) in IV.2.8, valid for \( v_m \sin \phi >> 1 \), it is substituted into the mode series IV.2.19 (and the above simplifying assumptions are introduced) the resulting expression is equivalent, except for a (geometric) factor \( [(\nu \sin \phi)^{1/2}] \) to the planar waveguide solution at large distances. Consequently, plane-earth solutions have sometimes been "adapted" to spherical systems by multiplication by this factor. Burrows (1978) confirms the utility of this procedure at ELF, though the usefulness of such a procedure is limited above 10 kHz.
IV.2.0. WAVE-HOP: THREE MEDIA SYSTEM

The zonal harmonic series solution for a three-media system (IV.1.21) can be reformulated as the sum of a contour integral and a line integral, as in the two-media problem. If the line integral is taken as negligible (Berry 1964b), we have

$$\mathfrak{E}_r = \frac{i e_0}{4\pi}\int_{C_2} \left(\frac{2\pi i}{\omega} + s(v)\right) P_v[\omega(\pi - \chi)] \, dv,$$  \hspace{1cm} (IV.2.33)

where path $C_2$ is a contour enclosing the poles of $s(v)$, for

$$s(v) = \mathcal{U}(v+1) \mathcal{L}_{u,v}(k\xi_v) l_{u,v}(k\xi_v)$$

$$\times \left[ \frac{1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi)}{l_{u,v}(k\xi) l_{u,v}^*(k\xi) - 1} \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

with $C_0$, $R^1$, and $R^2$ as given in IV.1.22-24. If, further, both source and receiver are situated on the surface of the earth ($r = r_0 = a$) and we define

$$P_v = \frac{l_{u,v}(k\xi) l_{u,v}^*(k\xi)}{l_{u,v}(k\xi) l_{u,v}^*(k\xi) - 1},$$  \hspace{1cm} (IV.2.35)

then IV.2.34 reduces to

$$s(v) = \mathcal{U}(v+1) \mathcal{L}_{u,v}(k\xi_v) l_{u,v}(k\xi_v)$$

$$\times \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

$$\times \left[ 1 - l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right] \left[ 1 + l_{u,v}(k\xi) l_{u,v}^*(k\xi) \right]$$

It is typically assumed that a contour $C$ can be found along which $|p\delta| < 1$ (Bremner 1949, Wait 1961), and Berry (1964b) found this to be true in the lower-band problems he investigated. The denominator of IV.2.36 can then be expanded in a convergent geometric series; in particular, substituting for

$$\left(1 + \frac{1}{\xi} \right)^{ jos} \left(1 - \frac{1}{\xi} \right)^{jo}$$

we see that the field, with an interchange of summation and integration, becomes

$$\mathfrak{E}_r = \frac{i e_0}{4\pi}\int_{C_2} \left(\frac{2\pi i}{\omega} + s(v)\right) P_v[\omega(\pi - \chi)] \, dv$$

$$+ \frac{i e_0}{4\pi}\int_{C_2} \left(\frac{2\pi i}{\omega} + s(v)\right) P_v[\omega(\pi - \chi)] \, dv$$

$$+ \left\{ \frac{\pi_0}{\lambda_0} \left[ \left(1 + \frac{1}{\xi} \right)^{jos} \left(1 - \frac{1}{\xi} \right)^{jo} \right] P_v[\omega(\pi - \chi)] \right\} \, dv.$$

The first integral is identical to that encountered in the two-media problem when $r = r_0 = a$ [equation IV.2.4 with $s(u)$ given by IV.2.2 and the line integral neglected] and is identified with the groundwave. The remaining integrals can be evaluated by the saddle point method if the Debye approximations to the Bessel functions are substituted along with A3.43 for the Legendre function. Retention of only the leading Debye term reproduces the geometrical-optics solution, each term of which can be associated with a ray that reaches the receiver after $j$ ionospheric reflections (Bremner 1949). Hence, each such integral is designated a wave-hop.

Difficulties with the saddle point evaluation arise when one approaches (or goes beyond) the caustic — that point at which the first impinging ray is tangent to the earth's surface. Near the caustic, use of the Debye approximation proves to be inadequate; analysts have instead substituted the leading Airy function approximations (Appendix A3).

For field points well-beyond the caustic, Wait (1961) suggested that the integrals could be evaluated by a rapidly converging mode sum. After expansion of the denominator in the original integrand, the poles of each resultant integrand are simply those of the
two-media problem rather than the more complicated three-media poles of section IV.2.C. Nonetheless, additional difficulty is encountered because the order of the pole of the $j^{th}$ hop is $(j + 1)$. Hence, this calculational approach, as detailed by Berry and Chrisman (1965), is quite tedious and computer-intensive. For example, they report that a residue calculation corresponding to a single pole for the 5th hop required the sum of 18 distinct terms, and one factor in one of those terms required the summation of 73 terms involving Airy-type approximations to Hankel functions. Berry and Chrisman (1965) summarize the evaluation of the wave-hop integrals thusly:

The saddle point approximation should be used whenever possible because it is simplest, but it is not valid near, or beyond, the caustic. The residue series is accurate and efficient deep in the shadow region, but converges very slowly in the lit region. If these two methods do not overlap in the caustic region, [the contour integrals] must be integrated numerically. (p. 1475)

This approach is referred to by the CCIR (1982) as the "full wave-wave-hop" method, and its most advanced development is probably given in the work of Berry and Hermann (1971) or, according to the Air Force Geophysics Laboratory (1984), in that of R.L. Lewis (1970).

An alternative approach to wave-hop evaluation has been advanced by Johler (1964, 1970). He expanded the denominator of the zonal harmonic series in IV.1.21 in a geometric series to obtain

$$\hat{\psi}_r = \left(\hat{\psi}_r\right)_{\text{two-media}} + \sum_{j=1}^{\infty} \hat{\psi}_r[j]$$

where $\left(\hat{\psi}_r\right)_{\text{two-media}}$ groundwave is the two-media solution and

$$\hat{\psi}_r[j] = -\frac{\partial}{\partial \omega} \sum_{n=1}^{\infty} \alpha_{n,j}^{+} (r_j) \psi_n^{(j)}(\omega)$$

for $r = r_0 = a$.

Again, the $j^{th}$ term can be associated with the $j^{th}$ hop — and, through the convergence of the groundwave series remains slow, the remaining series converge much more rapidly than the original series.
IV.2.E ZONAL HARMONIC SOLUTIONS: TWO AND THREE MEDIA SYSTEMS

Examples of the two and three-media zonal harmonic series solutions for the radial electric field due to a radial Hertzian source operating in spherical systems are given in IV.1.25 and IV.1.21, respectively. Historically, once series solutions had been obtained for spherical scattering problems (see section IV.1.C), the investigators naturally attempted to sum them.

In the case of plane-wave scattering by a sphere, the associated series were found to converge quite rapidly after about $ka$ terms had been summed ($k$ the propagation constant of the wave and $a$ the radius of the sphere). Consequently, such solutions could be evaluated directly provided $ka$ was sufficiently small ($ka < 10$) as, for example, in Rayleigh scattering. With a source at an infinite distance from a spherical scatterer (thus yielding incident planewaves), Rayleigh (1904) calculated the scattered acoustic field at angles of $0^\circ$, $90^\circ$, and $180^\circ$ for $ka = 10$. Following Rayleigh's work, the intensity and related field quantities of an electromagnetic plane wave scattered by a sphere were plotted as a function of scattering angle for the case of $ka = 10$ (Proudman, Dooodson, and Kennedy 1911), an effort that took three years due to the large number of hand calculations required to ensure accuracy. Nevertheless, since the source was effectively at infinity, accurate approximations were possible by truncating a direct summation after a number of terms not much larger than $ka$.

A problem which has proven less amenable to evaluation via direct summation is that which results when both source and receiver are taken on (or close to) the surface of the sphere. Rayleigh addressed this problem in his 1904 paper (for an acoustic source), obtaining values for the surface field at the two special angles, $90^\circ$ and $180^\circ$. To carry out the calculation, he accelerated series convergence by repeated-averaging of the partial sums—a method which, for these particular angles, is a straightforward application of Euler's transformation (see section V.F.). Love (1915) treated the electromagnetic analogue of the problem, and accelerated the series sum for a number of specifically-chosen $\theta$-values by selectively grouping terms and using a technique similar to Rayleigh's (section V.F). However, Bremner (1949) and Johler and Berry (1962) report that the method was not considered completely satisfactory as a consequence of numerous estimates and graphical construction. Finally, Watson's (1918, 1919) transformational reformulation of the solution as a residue series made summation of the zonal harmonic forms unnecessary—at least for the propagation problems of interest at that time.

Johler and Berry's 1962 evaluation of the zonal harmonic series sums—made possible by the advent of the high-speed computer—marked the revival of this approach. Series convergence was improved by use of Kummer's transformation (sec. V.C.) and, for the groundwave field (i.e., the two-media solutions), by a complicated procedure involving the series for the vector potential (which converges faster by a factor $1/n^2$). Later papers coupled a Kummer transformation with an averaging process (Johler and Lewis 1969; Lewis and Johler 1976), with separation of the groundwave component from the total field solution improving the convergence of the remaining series. Johler (1970), exploiting an earlier finding (Johler 1964), showed that, for frequencies at and above 20 kHz, this remaining series could be further accelerated by expanding its summand denominator and re-ordering the resulting sum.

Generally, series calculations at ELF for three-media terrestrial system models have been found to require approximately $12(ka)$ to $15(ka)$ terms ($k$ the propagation constant, $a$ the earth radius) to be summed, with more required for field points close to the source or its antipode (Johler, Lewis 1969; Lewis, Johler 1976; Galejs 1972). However, if the groundwave fields are calculated separately, the remaining series have been evaluated with $1.5(ka)$ to $2(ka)$ terms (Lewis and Johler 1976). At VLF and above, the groundwave calculation has also been found to require on the order of $15(ka)$ terms (Johler, 1970).
In three-media system solutions, if groundwave is calculated separately or, alternatively, if the groundwave contributions are negligible, roughly \(2(ka)\) (but possibly fewer) terms may be necessary for reasonable accuracy (Xait 1962; Johler, Berry 1962). If, however, the denominator is expanded in a geometric series, each resulting series has been found to require only a few terms beyond \(ka\) — at least for LF and above (Johler 1970), and this may also be true at upper VLF frequencies. However, two later papers (Jones and Howorth 1982, Howorth and Jones 1983) indicate that, for a 20 kHz example, each resulting series of the geometric expansion requires about \(10(ka)\) terms to converge. It should be noted that this expansion of the term-denominator and subsequent calculation of the resulting geometric series corresponds to the exact calculation of the wave-hop integrals (sec. IV.2.D).

Techniques to accelerate convergence are investigated in Chapter V.

IV.2.F SUMMARY OF COMPUTATIONAL APPROACHES

Two-Media Systems

A terrestrial system model consisting of a lossy sphere in free space is useful at mid and lower-band frequencies for "short" propagation paths (hundreds of kilometers), at longer ranges when the groundwave component can be distinguished from skywave components, and at upper-band frequencies (sec. II.3.C) whenever the skywave components are negligible.

With reference to Fig. IV.4, field calculations in the shadow region are, with rare exceptions, made via the mode series as developed in sec. IV.2.A. At mid and upper-band frequencies, a surface-impedance (sec. IV.2.A) is sometimes introduced to account for the effects of surface irregularities or long-range propagation (Barrick 1972, Xait and Hill 1980) and an "effective" earth-radius (\(4/3\) times the mean radius) is typically employed to account for refractive atmospheric effects.

In the illuminated region, fields are most readily calculated via geometrical-optics (sec. IV.2.B) — though this (approximate) method begins to lose accuracy as the caustic is approached. Similarly, as this transition (penumbral) regime is approached from the shadow region, either an ever-increasing number of modes must be summed, or the fields must be obtained via numerical integration (Jones 1979).

Worton (1941) reduced these mode and geometrical-optic analyses to graphical methods for obtaining the groundwave intensity. For fields in the penumbral domain, he suggested drawing a smooth curve (an approach still recommended, Collin 1985: p. 369) between the values calculated in the illuminated and shadow regions, noting that:

The alternative to the above approximate graphic method of solution for the field in this intermediate range of distances consists of a long and tedious process of computing and adding (in proper phase) long series of terms, this latter method constituting the only means of arriving at an accurate solution.
The calculation of groundwave can also be accomplished by summing the zonal harmonic series directly (with some form of convergence acceleration usually necessary). Though this method is rarely pursued for the groundwave series, it has been summed (at ELF) by Johler and Berry (1962), Johler and Lewis (1976), and the author. As noted above, earlier efforts required the summation of 10 to 20 times $ka$ terms ($ka = 2/15 f$, $f$ the frequency) to obtain convergence, with even more required for field points close to the source or its antipode. At ELF, the total field (not just the groundwave) is the quantity of interest, and the appropriate terrestrial model is therefore that of a three-media system (sec. II.3.A). According to Johler (1970), the number of terms required at VLF is also (roughly) 15 times $ka$, and he still believed zonal harmonics to be an attractive alternative to mode theory for three-media problems (with groundwave extracted), though he also states that the groundwave could be more efficiently calculated via mode theory.

Indeed, the principal objections to the zonal harmonic approach have always been the large number of terms that must be summed and the complexity of the method relative to geometrical-optics or mode summation. A secondary objection appears to be that the zonal harmonic sum provides little physical insight into the propagation process -- though in the penumbral region neither geometrical-optics, numerical integration, nor mode theory offers either physical insight or freedom from complexity.

Three-Media Systems

The three-media terrestrial system model has applicability at lower and mid-band frequencies where the ionosphere affects propagation over paths greater than several tens of kilometers. Sharply-bounded, homogeneous, ionospheric models have utility below LF and (perhaps) at lower IF (secs. II.3.A and B) while, at higher frequencies, the inhomogeneous -- and sometimes the anisotropic -- nature of the ionosphere must be taken into account. We have not addressed the modifications to the methods of secs. IV.2.C-E if the medium is to be considered inhomogeneous or anisotropic, but, in each instance, such modification is possible (Johler and Berry 1964, Berry and Chrisman 1965, Johler 1970).

Since Watson, field calculations at lower band frequencies have usually made via mode theory (sec. IV.2.C) or a wave-hop approach (sec. IV.2.D). The CCIR (1982) reports that wave-hop theory is most applicable at higher (IF) frequencies and short distances whereas mode theory is most suitable for longer distances and lower (VLF) frequencies. Burgess and Jones (1975) and Morfitt and Shellman (1976) also indicate that, in practice, mode theory is generally used below 30kHz (outside of 1000-2000 km) and wavehop above 30 kHz. (The latter authors sought to extend the applicability of the mode approach to 60 kHz.) As mentioned earlier, field calculation can also be accomplished by summing the zonal harmonic series directly, but this approach has been employed by only a few researchers to date (sec. IV.2.E).

As indicated, "waveguide" mode theory (distinguished from mode theory as applied to the two-media problem), is the principal method presently in use at VLF -- at least at lower VLF and distances beyond 1000 to 2000 km from the transmitter, for many problems of interest, one to three modes can sometimes provide sufficient accuracy. When only a few modes are necessary, the approach does yield physical insight via waveguide analogies: the number of important modes is roughly proportional to the waveguide height divided by the wavelength, with only one required at ELF, perhaps ten to fifteen at 30 kHz, and as many as twenty-five at 60 kHz (Morfitt and Shellman 1976). Since determination of the modes is independent of source or receiver locations, once the modes are known, field calculation can proceed rapidly for any number of source and receiver locations; indeed, modes can be precalculated for system models that will be reused.
There are, however, several disadvantages and difficulties associated with the method. Determination of the waveguide modes involves pole-location, which necessitates an exhaustive search of the complex plane and generally requires a large computer (CCIR 1982: decision 9-3). Modes can be missed, especially above 30 kHz where they evidently lie close together (Jones and Haworth 1982, Morfitt and Shellman 1976), and degenerate solutions may also exist (Budden and Eve 1976). Kelly et al (1984) notes that the need to check the results for completeness and accuracy is the "laborious and expensive part of these predictions". Moreover, it is extremely difficult to bound the error of a waveguide mode calculation. A weakly-conducting ionospheric model may cause computational difficulties inasmuch as root-finding programs are usually developed from the mode equation IV.2.27 (valid for \( \kappa_3 \ll k \)) rather than the more general form given by IV.2.28. The flexibility of the technique is also limited by the increasing difficulty of the root-finding process as concentric layers (of finite thickness) of differing media are added to the system model (Johler and Lewis 1969).

The wave-hop approach involves an expansion of the denominator in the integral form of the full wave solution (IV.2.33). An interchange of integration with summation yields a series of integrals, the poles of which are then those of the simpler two-media equation (IV.2.6) rather than the waveguide mode equation (IV.2.27). These poles increase in order with each successive integral. If the integrals are evaluated asymptotically, the geometrical-optic solution for a waveguide results, and each integral can be interpreted as a ray reaching the field point after one or more ionospheric reflections. Geometrical-optic methods, first derived by considering simplified propagation equations ab initio, have been used extensively at MF and HF, and a number of useful engineering approximations have resulted. These methods are generally applicable to situations where only a few rays are important and are often difficult to apply when the source or receiver is varying its position — the ray geometry then requiring continual modification (Morfitt and Shellman 1976). At mid-band the CCIR (1982: 415 and 435) recommends the use of empirical relations (formulas partially based on geometrical-optic analyses and partially on measured data). Below MF, the methods of geometrical-optics become inadequate at and beyond the caustic (sec. IV.2.20). The wave-hop integrals can then be more accurately evaluated by numerical integration (Walt 1961, Berry 1964b), by their corresponding mode series for points beyond the caustic (Berry and Chrisman 1965), or by their corresponding zonal harmonic series (Johler 1970).

The wave-hop approach does not, therefore, circumvent the need for either an asymptotic (or numerical) analysis, a mode sum, or a zonal harmonic sum (a point sometimes overlooked). Nevertheless, if an asymptotic calculation of the series of integrals is made, the calculation of each resultant integral is simplified relative to the original integral: if the mode approach is taken, the mode calculation of each resultant series is simplified relative to the waveguide mode series (as long as not too many hops are required); if the zonal harmonic approach is taken, the convergence of each skywave series (i.e., groundwave excepted) is simplified relative to the original zonal harmonic series (see, however, Chapter V). Wave-hop programs in present use generally employ geometrical-optic or saddle-point analyses before the caustic region, numerical integration through the caustic region, and mode theory beyond the caustic following the work of Berry and Chrisman (1965). At VLF, the CCIR (1982: 285) indicates the need to sum many wave-hops for distances beyond one or two thousand kilometers. The three-media solution can also be obtained by a direct sum of IV.1.21. While the large number of required terms forced investigators in the early twentieth century to alternative approaches, high-speed computation now permits such calculations to be made. Several investigators have summed the series at ELF and have found 10(\(ka\)) to 20(\(ka\)) terms necessary for convergence — the resulting fields in agreement with those calculated via mode theory (Galejs 1972, Lewis and Johler 1976, Burrows 1978). The mode approach may have some advantage at ELF when only one or two modes are required and
simplifications can be introduced in the mode-determining equation (sec. IV.2.C). The zonal harmonic series has also been summed at VLF and LF, though generally only after the denominator has been expanded in a geometric series (Johler 1970, Jones and Mawforth 1982). As indicated above, this corresponds exactly to the wave-hop series of integrals with each integral replaced by its zonal harmonic representation. However, at VLF, mode theory may still be computationally more efficient and afford greater physical insight into the propagation process if only a few modes are significant. But, as the number of significant modes increases, the zonal harmonic series may well become more efficient and, in the wave-hop-equivalent form, offer even greater physical insight (Johler 1970).

A trade-off is involved: use of the zonal harmonic series (usually) necessitates the summation of a greater number of terms than the residue series of mode theory, but each term of the zonal harmonic sum is readily calculated. In contrast, the residue series may require relatively few terms, but each term of this series requires the execution of a computationally-intensive iterative routine to determine the mode values. With zonal harmonics, changes in source or receiver location may require a series recalculation; duplicate calculations can be minimized if the desired source and receiver locations are known or vary in a prescribed manner. This problem does not arise in modal field calculations though certain source and receiver locations may require more modes than others. In its favor, zonal harmonic summation provides a uniform approach to calculations over the whole lower-band, is readily extended to systems involving additional layers of finite thickness, and has an error associated with neglected terms that can at least be approximated (Chapter V).

CHAPTER V: SUMMATION OF THE ZONAL HARMONIC SERIES

V.A INTRODUCTION

In this chapter we investigate summation techniques for the zonal harmonic (spherical wave or Legendre) series, which we can write as 
\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \frac{1}{n!} \frac{1}{m!} \frac{1}{(2n+1)!} \frac{1}{(2m+1)!} P_n^m(y),
\]
for \( y = \cos(\theta) \). Our immediate interest in such series derives from their importance in terrestrial electromagnetic propagation, although they are encountered in other important problems: Regge pole analysis in high energy physics (Squires 1964), acoustic scattering in spherical systems, electromagnetic scattering by small spheres at optical frequencies — indeed at all frequencies — (Born and Wolf 1965, Kerker 1969), heat conduction (Hildebrand 1966), seismology (Magase 1956), geophysics (Knopoff and Gilbert 1961), and elastic wave propagation (Einspruch, Witterholt, and Truell 1960).

In the past, these series have been examined with the purpose of ascertaining what functions have convergent Legendre polynomial expansions (e.g., Hobson 1931, Sansone 1959, Szego 1955). Herein, however, we assume that a zonal harmonic series exists, having specifically arisen out of an electromagnetic boundary-value problem, and we want to accelerate its convergence. (In an important special case, described below, we even need to evaluate a divergent series.) While the notoriously slow convergence of the zonal harmonic series representation of terrestrial field solutions has resulted in the development of other solutional approaches (Chapter IV), the advent of high-speed computers has led to renewed interest in numerical series summation. In fact, we find that zonal harmonic series, — widely believed to require the summation of 10 to 20 times \( ka \) terms (a being the radius of the scatterer and \( k \) the propagation constant) — can be summed with as few as \( (ka) - O(ka) \) terms.

The radial electric field component for the problem of a radial Hertzian dipole located at \( (r, \theta, \phi) \) in the second medium of a three-media spherically symmetric
system is given in IV.1.21 and repeated here for convenience:

$$\mathcal{E}_r(r, \theta, \phi | r_0, \theta_0, \phi_0) = - \frac{c_0}{2r_0} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \left( \frac{r}{r_0} \right)^n \mathcal{E}_n^{(2)}(r_0 \theta, \phi),$$  \hspace{1cm} (V.1)

where

$$c_0 = \frac{\omega a}{\sin \theta_0},$$  \hspace{1cm} (V.2)

$$\mathcal{E}_n^{(2)} = \left[ \left( \frac{\omega a}{\sin \theta_0} \right) - (h/\kappa) \right] \mathcal{E}_n \left[ \left( \frac{\omega a}{\sin \theta_0} \right) - (h/\kappa) \right],$$  \hspace{1cm} (V.3)

$$\mathcal{E}_n^{(2)} = \left[ \left( \frac{\omega a}{\sin \theta_0} \right) - (h/\kappa) \right] \mathcal{E}_n \left[ \left( \frac{\omega a}{\sin \theta_0} \right) - (h/\kappa) \right],$$  \hspace{1cm} (V.4)

with $a$ being the radius of the inner sphere (earth radius), $b$ the radius of the boundary between the second and third medium (ionospheric radius), and $r_0$ as the greater or lesser of $r$ and $r_0$, respectively. The propagation constants of the second and third media (assumed nonpermeable) are $K_1$ and $K_3$ (IV.1.1 and IV.1.3, respectively), and the medium containing the source is assumed to be free-space with a real propagation constant $k_0$, $v/c$, with $v$ the source frequency and $c$ the speed of light in vacuum. The solution for the corresponding two-medium problem with the source in the outer medium is obtained from the above if $R_1(n) = 1$ is substituted (or, equivalently, $K_3$ is set equal to $k_0$).

We first consider the question of series convergence. In the limit as $n$ increases without bound, the series summand in V.1 can be viewed as the product of the following three expressions, the limits of which are obtained in Appendix A.3:

1. \( \left( \mathcal{E}_n \right) = \left( \mathcal{E}_n \right) \left( \frac{r}{r_0} \right)^n \to \mathcal{E}_n \left( \frac{r}{r_0} \right)^n \)
2. \( \left[ 1 + \frac{h}{\kappa} \right]^{(2n+1)} \mathcal{E}_n \left( \frac{r}{r_0} \right)^n \to 1 + \frac{h}{\kappa} \left( \frac{r}{r_0} \right)^n \)
3. \( \mathcal{E}_n \left( \frac{\omega a}{\sin \theta_0} \right) \left[ \cos \left( \left( \frac{\omega a}{\sin \theta_0} \right) \phi \right) \right] \)

For the two-medium problem, expression (ii) is identically unity. If $r \neq r_0$, the series is absolutely (though often slowly) convergent. The series solution is not strictly valid if $r = r_0$ — direct substitution would yield a divergent series. This is a result of the solutional development [see III.2.1 - III.2.6 and note in III.2.3 that $c_n(\mu, \kappa) = c_n(\mu, \kappa)$] for $c_n(\mu, \kappa)$ a Bessel function in which there is an implicit interchange of differential operators with a summation sign valid only when the resultant series converges (i.e. when $r \neq r_0$). In the case $r = r_0$, V.1 needs to be modified to

$$\mathcal{E}_r(r, \theta, \phi | r_0, \theta_0, \phi_0) = \lim_{r \to r_0} \left\{ \left( \frac{\omega a}{\sin \theta_0} \right) \mathcal{E}_n \left[ \left( \frac{\omega a}{\sin \theta_0} \right) - (h/\kappa) \right] \right\}$$  \hspace{1cm} (V.5)

The new series is convergent for $r = r_0$, but again $r_0$ cannot be directly substituted for $r$ because of the need to take derivatives (with respect to $r$) of the result.* Unfortunately, except in the particular case of unbounded free space, a closed-form solution for the series in V.5 is not known to exist. We can, however, take advantage of field continuity, which requires that the field at $r = r_0$ be continuous with the solutions for $r < r_0$ and $r > r_0$ (except at the source point, $0 = 0$, or, if $r_0$ is situated at a boundary, for $r > r_0$ or $r < r_0$, whichever is applicable).

*Johler and Berry (1962) actually evaluated the series in V.5 at a number of points, the results of which were then used to calculate the derivatives and thus the field at a single point by approximation. This approach was subsequently abandoned (e.g., Johler and Lewis 1969).
Consequently, $Q = E(r_0, \Theta, \psi | r_0, 0, 0)$ can be written as the limit as $r$ goes to $r_0$ of the right hand side of V.1. In a strict sense, the limit cannot be interchanged with the summation process, but the series may be treated as an absolutely-convergent series (e.g., series terms may be rearranged) prior to taking the limit. Should a transformation result in a second series representation equivalent to the first for $r \neq r_0$, and convergent in the limit $r$ goes to $r_0$, we can then interchange the limit and summation processes and identify the sum for $r = r_0$ as $Q$.

This sum thus corresponds to the "Abelian" sum of the divergent series that arises if $r = r_0$ is substituted directly in V.1: viz., if $\sum_{n} a_n$ is convergent for $0 < x < 1$, $f(x)$ is its sum, and $\lim f(x) = 0$, then $Q$ is the Abelian sum of $\sum_{n} a_n$ (Hardy 1949).

In this chapter, all series will be treated as absolutely convergent with the above limiting process understood to apply when $r = r_0$, i.e., when $r = r_0$, we seek the Abelian sum of the divergent series obtained by direct substitution of $r = r_0$ in V.1. This aspect of convergence is rarely addressed (see however Watson 1918) inasmuch as direct summation is usually avoided by use of the Watson transformation (section IV.2.A) which leads to formulations that converge for $r = r_0$. Direct summation with truncation is manifestly impossible for $r = r_0$ using V.1 and, since $r$ is typically close to $r_0$ for terrestrial problems, the series can be expected to converge very slowly for most problems of practical interest.

In section IV.2.E it was noted that the three-media solution could be written as the sum of the two-media solution (groundwave) and an ionospheric contribution. The summand of this latter involves the product of the summand of the former and a factor that approaches zero as $(r/a)^2n$ as $n$ increases (see limit (iii)). Since it is the groundwave portion of the solution that converges more slowly, we consider the two-media series solution which, with the source at the earth's surface, can be written

$$\hat{E}_n(r, \Theta, \psi | r_0, 0, 0) = -\frac{-i C_0}{2 \pi r_0} \sum_{n} \left\langle \frac{r_0}{r} \right\rangle (2n + 1) \left\langle \frac{r_0}{r} \right\rangle \left\langle \frac{1}{r_0} \right\rangle \left\{ \mathrm{d} \left[ \cos \Theta \right] \right\} \right.$$

where

$$D_n = \left\langle \cos \Theta \right\rangle \left\{ \left[ \cos \Theta \right] \right\} \left\{ \left[ \cos \Theta \right] \right\} .$$

The same solution is obtained (with $r_0$ replacing $r$) if, instead, the receiver is taken to be on the earth's surface and the source at $r_0 = a$ (again, if $r_0 = r = a$, the Abelian sum of the resulting divergent series is implied).

For most earth conductivities $(\gamma/m)_n$ can be adequately approximated as $-i \left( \frac{\gamma/m}{m} \right) \sqrt{1 - \left( \frac{\gamma/m}{m} \right)^2}$ for all $n$ (Mait 1962) and is zero for a perfectly conducting scatterer $(k_1 = \infty)$. However, the quantities involving Hankel functions of real argument, $x$, change character in a region of $O(x^{-1/2})$ about $x$. It is in this "transition" region about $x = ka$ that the factor $\left\{ \left[ \frac{r_0}{r} \right] \right\}$ typically reaches its maximum value for terrestrial problems. From Appendix A, if $n - kr >> (kr)^2$ and $n - ka >> (ka)^2$, use of the dominant Debye expansions (with $\nu = n + 1/2$) lead to the approximations

$$\left\langle \frac{r_0}{r} \right\rangle \left\{ \left[ \cos \Theta \right] \right\} \left\{ \left[ \cos \Theta \right] \right\} .$$
while, if \(|n - ka| < o(ka)^\frac{3}{2}\) and \(|n - kr| < o(kr)^3\), use of Olver's uniform asymptotic expansions (with \(v = n + \frac{1}{2}\)) leads to the approximations

\[
\frac{L_{v+1}^{(m)}(kr)}{L_{v}^{(m)}(kr)} \approx \frac{\sqrt{n}}{2} \left\{ A_v \left[ \frac{(y_0)^{1/2}(v+1)}{(v+1)(v+2)} \right] - i \beta_v \left[ \frac{(y_0)^{1/2}(v-1)}{(v+1)(v+2)} \right] \right\} \tag{V.10}
\]

\[
L_v^{(m)}(ka) \approx \frac{1}{2^{1/2}} \left\{ A_v \left[ \frac{(y_0)^{1/2}(v+1)}{(v+1)(v+2)} \right] - i \beta_v \left[ \frac{(y_0)^{1/2}(v-1)}{(v+1)(v+2)} \right] \right\} \tag{V.11}
\]

When \((kr - n) \gg (kr)^3\) and \((ka - n) \gg (ka)^3\), the Debye expansions for the Hankel functions have exponential dependences that are purely imaginary.

Those (relatively few) analysts who have actually summed zonal harmonic series associated with the earth scattering problem have indicated the need to sum 10 to 15 times \((ka)^2\) terms to ensure groundwave series convergence to several place accuracy, with more terms required for field points within several degrees of the source or antipode (Sec. IV.2.2: Johler and Lewis 1969, Johler 1970, Jones and Nowforth 1982). In past efforts, Kummer's transformation, sometimes in conjunction with an averaging process, has typically been used to accelerate series convergence and these methods are considered in sections V.C and V.F, respectively. Cesaro summation, a classical technique for accelerating divergent series, is considered in section V.D. The Shanks transformation has found recent application to evaluating series arising in cylindrical boundary-value problems (Richards et al 1983) and to evaluating the integrals arising in planar boundary-value problems (Bubenik 1977, Burke et al 1981); its application to the spherical-wave series is considered in section V.E. Section V.B develops new approaches using summation by parts.

The effectiveness of these techniques will be compared in the particular case of a perfectly conducting scatterer — more specifically, in the case of the real part of the associated series solution:

\[
F = \sum_{n=1}^{\infty} (2n+1) \Re_n \left( \frac{1}{\sin \theta} \right) \tag{V.12}
\]

where

\[
d_n = \Re \left\{ \frac{w_n(0, m)}{n \sin \theta} \right\} \tag{V.13a}
\]

and, in the case \(r = a\),

\[
f_n = \Re \left\{ \frac{w_n(0, m)}{n \sin \theta} \right\} \tag{V.13b}
\]

The summand, while in general complex, has an imaginary component that vanishes rapidly beyond \(n = ka\). Indeed, the imaginary part of the series can be calculated via series truncation and requires relatively few terms beyond the transition zone. If we define

\[
F(n, m) = \sum_{n=1}^{N} (2n+1) \Re_n \left( \frac{1}{\sin \theta} \right) \tag{V.14}
\]

and

\[
F(N+1, m) = \sum_{n=1}^{\infty} (2n+1) \Re_n \left( \frac{1}{\sin \theta} \right) \tag{V.15}
\]

then \(F = F(n, m) + F(N+1, m)\), and we seek to accelerate the convergence of \(V.15\) for \(N\) an integer greater than \(ka\). The extension to non-perfect conductors will be immediate in most cases.

The Legendre polynomials can be computed by recursion; however, when an approximation is required and \(n(\sin \theta) \gg 1\), we use the asymptotic expansion in A3.35. When \(L\) terms are retained,

\[
P\left( \cos \theta \right) \sim U_L(\cos \theta) \cos \left( \frac{1}{2} \right) - V_L(\cos \theta) \sin \left( \frac{1}{2} \right) \tag{V.16}
\]

with \(U_L(\cos \theta)\) and \(V_L(\cos \theta)\) given in A3.40 and A3.41, respectively. An approximation to \(F(N+1, m)\) can then be defined by

\[
F(N+1, m) = \sum_{n=1}^{\infty} C_n(\cos \theta) \cos \left( \frac{1}{2} \right) - \sum_{n=1}^{\infty} D_n(\cos \theta) \sin \left( \frac{1}{2} \right) \tag{V.17}
\]
with
\[ C_n(l) = (2n+1) \pi \xi_n \cdot \big( \xi_n \big) \]  
(V.18)
\[ S_n(l) = (2n+1) \pi \xi_n \cdot \big( \xi_n \big) . \]  
(V.19)

Note that \( N \) must be sufficiently large that \( N (\sin \theta) \gg 1 \) if the approximation is to have any accuracy.

We shall also be interested in improving the convergence of a zonal harmonic series which arises in conjunction with mode-theory analysis (section IV.2.A):
\[ \sum_{n=0}^{\infty} \frac{(2n+1)^2}{2n+1} \sin^2 \theta \left( \frac{\pi}{2} \right) \left( \frac{\sin \theta}{\sin \left( \frac{\pi}{2} \right)} \right) \]
(V.20)

where \( \nu \) is possibly complex and \( \nu=0,1,2,\ldots \) (Nicholaenko and Rabinowitsch 1972, Galejs 1972 p. 149, O. O. Jones 1974b). In mode theory calculations at ELF only one mode (i.e., one value of \( \nu \)) may be required but \( \vert \nu (\sin \theta) \vert \) may not be large enough to use the asymptotic Legendre expansion of A3.38 in the calculation of the left hand side of V.20.

Lastly, we collect below some results that will be used in the ensuing sections. We define \( \Delta \) to be the forward difference operator,
\[ \Delta \nu_n = \nu_{n+1} - \nu_n \]  
(V.21)

Then
\[ \Delta \Delta \nu_n = \Delta (\Delta \nu_n) \]  
(V.22)

and, from Milne-Thompson (1940, p. 35),
\[ \Delta^k \left( \nu_n \nu_n \right) = \frac{k!}{\pi^k} \left( \frac{k}{\nu} \right) \Delta^k \nu_n \Delta^k \nu_{n+1} \]  
(V.23)

If we have
\[ \nu_n = \nu_n^\Delta \left( \nu_n + \frac{\partial \nu_n}{\partial y} + \ldots \right) \]  
(V.24)

\( a_i \) constant for all \( i \) and \( a_0 \neq 0 \), then (Wimp 1981 p. 21),
\[ \Delta \nu_n = \begin{cases} \frac{n^2}{2} (\nu + \frac{\partial \nu}{\partial y} + \ldots), & \nu > 0 \\ \frac{n^2}{2} (\nu + \frac{\partial \nu}{\partial y} + \ldots), & \nu < 0 \end{cases} \]  
(V.25a)
\[ \Delta^2 \nu_n = \begin{cases} \frac{n^2}{2} (\nu + \frac{\partial \nu}{\partial y} + \ldots), & \nu > 0 \\ \frac{n^2}{2} (\nu + \frac{\partial \nu}{\partial y} + \ldots), & \nu < 0 \end{cases} \]  
(V.25b)

where \( (a)^n \) is Pockhammer's symbol
\[ (a)^n = a (a+1) (a+2) \cdots (a+n-1) \]  
(V.26)

and all subscripted Greek letters are taken as constants. If \( \nu_n \) can be expressed as an asymptotic series with the form in V.24, then
\[ \Delta^k \nu_n \sim \begin{cases} \frac{n^k}{(k-1)!} \left( \nu_n + \frac{\partial \nu_n}{\partial y} + \ldots \right), & \nu > 0 \\ \frac{n^k}{(k-1)!} \left( \nu_n + \frac{\partial \nu_n}{\partial y} + \ldots \right), & \nu < 0 \end{cases} \]  
(V.27)

(cf. Wimp 1981 p. 19, and note the sign correction for the second result). The first result of V.27 also holds for complex values of \( \nu \) for which \( \nu = 1 \) and \( \nu \neq 1 \), as can readily be ascertained using the generalized Euler transform of V.47 or referencing Olver (1974 p. 297) provided that, when the original series is divergent, the Abelian sum (Introduction above) is implied. The ascending series forms for the Bessel functions (NBS 1964, 10.1.2, 10.1.3) can be used to show that
\[ \frac{\Delta^k \nu_n}{\Delta^k \xi_n} \sim \left( \frac{\xi_n}{\nu_n} \right)_{\nu_n}^{\nu_n} \left( \nu_n + \frac{\partial \nu_n}{\partial y} + \ldots \right) \]  
(V.28)

Use of the Dernby asymptotic expansions for the functions indicates that V.28 becomes a useful approximation for finite \( n \) when \( n > k \nu + O(\nu^2) \) and \( n > k \sqrt{r-a} \). The ascending series forms can also be used to show that
\[ \Delta \nu_n \left[ \log \left( \nu_n \nu_n \right) \right] \sim \pi \left[ 1 - \frac{(k+1)!}{(k+1)^{k+1}} \right] \]  
(V.29)

\( (k-1)! \)
It follows that \( z_n \), defined in V.13a, has an asymptotic expansion

\[
\frac{z_n}{\vartheta} = (n+1) \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots, \quad (n \to \infty)
\]  

(V.30a)

that becomes a useful approximation for finite \( n \) when \( n > kr + O(\vartheta) \) and \( n > k^2 r (r-a) \).

If \( r = a \),

\[
\frac{z_n}{\vartheta} \sim (n+1) \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots, \quad (n \to \infty)
\]

(V.30b)

which becomes a useful approximation for finite \( n \) when \( n > ka + O(\vartheta) \). Furthermore, we obtain from Appendix A3,

\[
\begin{align*}
U_n(L) & \sim \frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 0, 1, 2, \ldots \\
V_n(L) & \sim \left\{ \begin{array}{ll}
0, & L = 0 \\
\frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 1, 2, 3, \ldots
\end{array} \right.
\end{align*}
\]

(V.31)

(V.32)

implying that \( C_n(L) \) and \( S_n(L) \) defined in V.18 and V.19 have asymptotic expansions of the form

\[
\begin{align*}
C_n(L) & \sim \frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 0, 1, 2, \ldots \\
S_n(L) & \sim \left\{ \begin{array}{ll}
0, & L = 0 \\
\frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 1, 2, 3, \ldots
\end{array} \right.
\end{align*}
\]

(V.33)

(V.34)

which are useful approximations for finite \( n \) when \( n (\sin \theta) \gg 1 \), \( n > kr + O(\vartheta) \), and \( n > k^2 r (r-a) \). If \( r = a \),

\[
\begin{align*}
C_n(L) & \sim \frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 0, 1, 2, \ldots \\
S_n(L) & \sim \left\{ \begin{array}{ll}
0, & L = 0 \\
\frac{1}{2} \left[ \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \quad L = 1, 2, 3, \ldots
\end{array} \right.
\end{align*}
\]

(V.35)

(V.36)

which are again useful approximations for finite \( n \) when \( n (\sin \theta) \gg 1 \) and \( n > ka + O(\vartheta) \). Note that the leading terms of the asymptotic expansions V.31 - V.36 are independent of \( L \) while higher orders are dependent on \( L \).

Equations V.33, V.34, and V.27 can be used to obtain an asymptotic estimate of the relative error of truncating the series \( F(N+1,m) \) in V.15 or \( F(L,N,m) \) in V.17 after \( M \) terms. Substituting in V.17 for \( C_n(L) \) and \( S_n(L) \) (from V.33 and V.34) gives us

\[
\begin{align*}
F(L,N+1,m) & \sim \exp \left\{ \left[ \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\} \\
& + \sum_{k=1}^{M} \exp \left\{ \left[ \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\} \\
& \times \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots
\end{align*}
\]

(V.37)

These series can be evaluated using V.27, with the result

\[
\begin{align*}
F(L,N+1,m) & \sim \exp \left\{ \left[ \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\} \\
& + \sum_{k=1}^{M} \exp \left\{ \left[ \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\} \\
& \times \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots
\end{align*}
\]

(V.38)

We define

\[
\Phi = \text{arctan} \left( \frac{C_n(L)}{S_n(L)} \right) \cos \left( \frac{\vartheta}{2} \right)
\]

(V.39)

so that

\[
F(L,N+1,m) = \left( 1 - \vartheta \right) \sin \left( \frac{\vartheta}{2} \right) \exp \left\{ \left[ \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right] \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\}
\]

(V.40)

Provided \( (N+1)\theta + \pi/4 = \pi \neq 0 \), the leading term of V.38 is

\[
\begin{align*}
F(L,N+1,m) & \sim \frac{1}{2} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots \right\} \left( \frac{\vartheta}{\pi} \right)^{1/2} \left( \frac{\vartheta}{\pi} \right)^{1/4} \ldots
\end{align*}
\]

(V.41)
and since $y_0$ is independent of $L$, so too is the leading asymptotic term of $F_L$. Then the relative error of truncating the series sum for $F_L(N+1,m)$ in V.17 after $M$ terms is

$$
\frac{F_L(N+1,m) - F_L(M+1,m)}{F_L(N+1,m)} = O\left[\left(\frac{M}{N}\right)^M\right], \quad M \to \infty \quad (V.42)
$$

provided $(N+1)\delta + \pi/4 - \delta \neq 0$. This gives a useful estimate (for large but finite $N$) of the rate at which the relative error declines (or increases) with increasing $M$ when $N > kr + O(kr)^{1/2}$ and $N > N_{r}(r-a)$ [from the conditions on the usefulness of the asymptotic expansion of $C_n(L)$ for large $n$]. Furthermore, $F(N+1,m) - F_L(N+1,m)$ (at $M = \infty$) and, since the leading term of the asymptotic expansion for $F_L(N+1,m)$ is independent of $L$, the above results (V.41 and V.42) hold for $F$ as well as $F_L$.

### V.B Transformations Obtained Using Summation by Parts

Summation by parts is the series analogue of integration by parts. As noted by Olver (1974), summation by parts can be useful in approximating series whose summands involve the product of a slowly varying function and an oscillatory function such as $(-1)^n$ or a trigonometric function (just as integration by parts has proved useful in approximating Fourier integrals). A colleague, J. R. Vaill (Field Research and Engineering), suggested to the author that summation by parts might be used to accelerate the zonal harmonic sums that arise in electromagnetic propagation work: the Legendre polynomial, $P_n(y)$, is, for $-1 \leq y < 1$, an oscillatory function of $n$ and its multiplying factor in V.6 is a slowly varying function of $n$ provided that $n \gg ka$ and that the receiver (or source) height is small relative to the earth's radius.

The equation for summation by parts can be readily derived. Given that

$$
A(N,n) = u_n + u_{n+1} + \ldots + u_N, \quad n \geq N, \quad \text{we have}
$$

$$
\lim_{n \to \infty} A(N,n) v_n = \sum_{n=0}^{N} A(n,m) v_n + \sum_{n=0}^{N} \frac{d}{dn}[A(n,m) - A(n,m-1)] v_n
$$

$$
= \left(\frac{d}{dn} A(n,m) \right) v_n + \sum_{n=0}^{N} \frac{d}{dn} A(n,m) v_n = \sum_{n=0}^{N} \frac{d}{dn} A(n,m) v_n
$$

$$
= \sum_{n=0}^{N} A(n,m) \left(v_n - v_{n+1}\right) + A(n,m) v_p. \quad (V.43)
$$

When $\sum_{n=0}^{N} u_n v_n$ is a convergent series and $\lim_{n \to \infty} A(N,n) v_n = 0$, we can let $p$ go to infinity and obtain

$$
\sum_{n=0}^{N} u_n v_n = \sum_{n=0}^{N} A(n,m) \left(v_n - v_{n+1}\right). \quad (V.44)
$$

If the partial sums also form a bounded sequence (i.e., $|A(N,n)| \leq Q$ for $Q$ positive and $n \geq M$) and $(v_n)$ is a decreasing sequence, then $\sum_{n=0}^{N} u_n v_n \leq Q v_p$; this is Dirichlet's convergence test for $\sum_{n=0}^{N} u_n v_n$ and one approach to bounding such series.

Summation by parts has frequently been applied to series $\sum_{n=0}^{N} u_n v_n$ for which
\(u_n = q^n\), \(q\) a possibly complex constant, and \(v_n\) is a slowly varying function of \(n\) (e.g., Hardy 1949, Hamming 1973, Dahlquist and Bjorck 1974, Olver 1974):

\[
\sum_{n=0}^{\infty} \frac{1}{(1-q^n)} \frac{v_n}{v_{n+1}} + \frac{1}{(1-q^n)} \sum_{n=0}^{\infty} \gamma^n \Delta v_n, \quad \gamma \neq 1
\]

(V.45)

where we have defined \(\Delta\) to be the forward difference operator

\[
\Delta v_n = (v_{n+1} - v_n)
\]

(V.46)

and, for subsequent use, \(\Delta^k v_n = \Delta(\Delta^{k-1} v_n)\) with \(\Delta^0 v_n = v_n\). When summation by parts is applied \(k\) times and \(\lim_{\Delta v_n} = 0\) for \(j = 0, 1, \ldots, k-1\),

\[
\sum_{n=0}^{\infty} \frac{1}{(1-q^n)} \frac{v_n}{v_{n+1}} + \frac{1}{(1-q^n)} \sum_{n=0}^{\infty} \gamma^n \Delta v_n, \quad \gamma \neq 1
\]

(V.47)

Letting \(k = 0\) also, we have

\[
\sum_{n=0}^{\infty} \frac{1}{(1-q^n)} \frac{v_n}{v_{n+1}} + \frac{1}{(1-q^n)} \sum_{n=0}^{\infty} \gamma^n \Delta v_n, \quad \gamma \neq 1
\]

(V.48)

provided the original series converges (Hamming 1973). The last equality corresponds to the generalized Euler series transform and the special case \(q = -1\) is commonly referred to as Euler's transform (NBS 1964, Olver 1974: p. 538). The application of Euler's transformation by summing together the alternately positive and negative series terms is analyzed in section V.F. In this section we consider the application of the generalized Euler transform to summing zonal harmonic series (an application that appears to have been made previously), but then develop and investigate a more directly applicable transform which, so far as the author can determine, is new.**

One approach is to substitute for the Legendre polynomial several terms of its asymptotic expansion (which consists of trigonometric functions) and then apply the generalized Euler transform. Though a straightforward application of a well-known transform, this approach has not specifically been used with the zonal harmonic series. (See, however, Wimp, 1974, on evaluation of series of orthogonal polynomials using their asymptotic expansions.) Thus, if \(L\) terms are carried, we obtain the approximation of V.17 to the zonal harmonic series \(F(N+1, \ldots)\):

\[
C_L(m, \theta) = \sum_{n=0}^{L} \frac{\cos(n \theta)}{n} \left\{ \frac{\exp \left[ i (m+n+1) \frac{\pi}{2} \right]}{\cos \left( \frac{\pi (m+n+1)}{2} \right)} \right\}
\]

(V.49)

where \(C_L(m, \theta)\) and \(S_L(m, \theta)\) are defined in V.18 and V.19, respectively, and \(\theta \neq \pi\) is assumed throughout. Then, using V.47 with \(q = e^{i \theta}\), we get

\[
\sum_{n=0}^{\infty} \frac{1}{(1-e^{i \theta})} \frac{v_n}{v_{n+1}} + \frac{1}{(1-e^{i \theta})} \sum_{n=0}^{\infty} \gamma^n \Delta v_n, \quad \gamma \neq 1
\]

(V.50)

and, similarly,

\[
\sum_{n=0}^{\infty} \frac{1}{(1-e^{-i \theta})} \frac{v_n}{v_{n+1}} + \frac{1}{(1-e^{-i \theta})} \sum_{n=0}^{\infty} \gamma^n \Delta v_n, \quad \gamma \neq 1
\]

(V.51)

where \(R_c\) and \(R_s\) are "remainder" series given by

\[
C_L(m, \theta) = \sum_{n=0}^{L} \frac{\cos(n \theta)}{n} \left\{ \frac{\exp \left[ i (m+n+1) \frac{\pi}{2} \right]}{\cos \left( \frac{\pi (m+n+1)}{2} \right)} \right\}
\]

(V.52)

\[
S_L(m, \theta) = \sum_{n=0}^{L} \frac{\cos(n \theta)}{n} \left\{ \frac{\exp \left[ i (m+n+1) \frac{\pi}{2} \right]}{\cos \left( \frac{\pi (m+n+1)}{2} \right)} \right\}
\]

(V.53)

Hence, we have \((L \neq 0, m = 1, 2, \ldots)\)

\[
F_L(m, \theta) = \sum_{n=0}^{L} \frac{\cos(n \theta)}{n} \left\{ \frac{\exp \left[ i (m+n+1) \frac{\pi}{2} \right]}{\cos \left( \frac{\pi (m+n+1)}{2} \right)} \right\}
\]

(V.54)
Examining the remainder terms, first consider the case $r = a$ (for which the original series is, in a strict sense, divergent). With the understanding that subscripted Greek letters designate constants, we have, from V.36 and V.37,

$$C_{n}(\omega) = \frac{\phi^{2}}{2\pi} \left( \frac{\omega + \frac{\Omega}{2}}{\omega + \frac{\Omega}{2}} \right) \left( 1 - \frac{\Omega}{\omega} \right)$$  \hspace{1cm} (V.55)

$$B_{n}(\omega) = \frac{\phi^{2}}{2\pi} \left( \frac{\omega - \frac{\Omega}{2}}{\omega - \frac{\Omega}{2}} \right) \left( 1 - \frac{\Omega}{\omega} \right)$$  \hspace{1cm} (V.56)

assuming $r = a$, and these provide useful approximations for finite $n$ when $n > ka + O(ka)^{1/3}$. If $L = 0$, $S(L) = 0$, but we shall proceed to assume $L > 0$; also, while the leading coefficient is independent of $L$ in each series, higher order coefficients are not (section V.A). Equation V.25b, applied to V.55 and V.56, gives us

$$\Delta C_{n}(\omega) = a_{0} (\frac{\omega}{\omega + \frac{\Omega}{2}})^{m} \left( 1 - \frac{\Omega}{\omega} \right)$$  \hspace{1cm} (V.57)

$$\Delta B_{n}(\omega) = b_{0} (\frac{\omega}{\omega - \frac{\Omega}{2}})^{m} \left( 1 - \frac{\Omega}{\omega} \right)$$  \hspace{1cm} (V.58)

which, in conjunction with V.27, lead to asymptotic estimates of the remainder terms valid for $N > ka + O(ka)^{1/3}$:

$$R_{C}(\omega) = a_{0} (\frac{\omega}{\omega + \frac{\Omega}{2}})^{m} \left[ 2\sin(\frac{\Omega}{2}) \right] \left( 1 - \frac{\Omega}{\omega} \right) \left[ (\frac{\omega}{\omega + \frac{\Omega}{2}})^{m} \left( \omega + \frac{\Omega}{2} \right) \frac{\omega}{2} \right]$$  \hspace{1cm} (V.59)

$$R_{B}(\omega) = b_{0} (\frac{\omega}{\omega - \frac{\Omega}{2}})^{m} \left[ 2\sin(\frac{\Omega}{2}) \right] \left( 1 - \frac{\Omega}{\omega} \right) \left[ (\frac{\omega}{\omega - \frac{\Omega}{2}})^{m} \left( \omega - \frac{\Omega}{2} \right) \frac{\omega}{2} \right]$$  \hspace{1cm} (V.60)

Though the remainder terms do not vanish in the limit $M \to \omega$, note that the difference, $[R(\omega) - R(M)]$, is of the same order of magnitude as the summand of V.54 evaluated for $M = M$. It follows that $F_{l}(N+1,m)$ is asymptotic to the $m$-series in V.54, i.e.,

$$F_{l}(N+1,m) \approx \sum_{n=0}^{\infty} \left[ 2\sin(\frac{\Omega}{2}) \right] \left( \frac{1}{\omega} \left[ (n + \frac{1}{2}) \frac{\omega}{2} \right] \Delta C_{n}(\omega) \right)$$

$$+ \sum_{n=0}^{\infty} \left[ 2\sin(\frac{\Omega}{2}) \right] \left( \frac{1}{\omega} \left[ (n + \frac{1}{2}) \frac{\omega}{2} \right] \Delta B_{n}(\omega) \right)$$

$$\text{for } r = a$$  \hspace{1cm} (V.61)

with the error of an order not exceeding the first neglected term. Though the original series V.49 is, in a strict sense, divergent for $r = a$, the Abel sum of the series (which is sought) is of order not exceeding that of the $s = 0$ term (see V.41 with $r = a$). Consequently the relative error after summing $M$ terms of the $m$-series in V.54 or V.61 is

$$\left| \frac{R_{C}(\omega)}{R(M)} \right| \approx O \left\{ \left( \frac{\omega}{\omega + \frac{\Omega}{2}} \right)^{m} \right\}$$  \hspace{1cm} (V.62)

for $r = a$ and $N > (ka) + O(ka)^{1/3}$. Hence, for $\delta$ not too close to zero (i.e., for the receiver not too close to the source), the $m$-series in V.54 and V.61 converge very rapidly when $N > ka$ (though the relative error of these asymptotic series will, of course, begin to diverge for some $M$). If the inequality

$$2(ka)^{1/2} \left( \frac{\omega}{\omega + \frac{\Omega}{2}} \right) \gg 1$$  \hspace{1cm} (V.63)

is satisfied, we can evaluate the zonal harmonic series of V.12 by directly summing $F(1,M)$ for $N = (ka) + O(ka)^{1/3}$ and approximating $F(N+1,m)$ by several terms of the transformed series of V.61. Thus in the important case of $r = r_{c} = a$, it is possible to evaluate the field with $ka + O(ka)^{1/3}$ terms — in contrast to the generally supposed 10-20 times $ka$ terms.

Now suppose that $r > a$. The original series for $F_{l}(N+1,m)$ in V.11 (also in V.15 and V.49) is now convergent and could be directly summed until the desired precision was obtained. An asymptotic estimate of the relative error of truncating the "tail" of the original series after $M$ terms is given by V.42. For the purpose of comparison, we obtain asymptotic estimates of the remainder series in the transformed representation V.54,
but then introduce a second Eulerian-transformed series representation (again new) that is more applicable for \( r > a \).

We show in Appendix A6 that for \( C_n(L) \) given by V.33 and \( r \neq a \),

\[
\Delta^2 C_n(L) = \left( \frac{q}{\pi} \right)^{n/2} (-a) \sum_{j=0}^{\infty} \left( \frac{2j}{n} \right)^n \left( \frac{2j}{n} \right)^n \left( -1 \right)^j \left( \frac{2j}{n} \right)^n + \cdots.
\]

which is also an approximation for large, but finite, \( n \) when \( n > kr + O(kr)^1/3 \) and \( n > k^2r(a-r) \). This result, with V.27, gives us an estimate of the remainder series of V.52:

\[
\Delta^2 C_n(L) = \left( \frac{q}{\pi} \right)^{n/2} (-a) \sum_{j=0}^{\infty} \left( \frac{2j}{n} \right)^n \left( \frac{2j}{n} \right)^n \left( -1 \right)^j \left( \frac{2j}{n} \right)^n + \cdots
\]

where \( \psi \) is as defined in V.39, and this is an approximation for finite \( N \) when \( N > kr + O(kr)^1/3 \) and \( N > k^2r(a-r) \). A similar expansion exists for \( C_n(L) \) with \( M/2 \) replacing \( M^2/2 \) and a sine function replacing the cosine function, but \( C_n(L) \) generally dominates \( C_n(L) \). Hence, from V.65, we have

\[
\Delta^2 C_n(L) = \left( \frac{q}{\pi} \right)^{n/2} (-a) \sum_{j=0}^{\infty} \left( \frac{2j}{n} \right)^n \left( \frac{2j}{n} \right)^n \left( -1 \right)^j \left( \frac{2j}{n} \right)^n + \cdots
\]

The relative error of the \( M \)-sum of V.54 \( (r > a) \) is then given by

\[
Z = \frac{\left( \frac{R_M(C_n)}{R_M(-C_n)} \right)^M}{\left( \frac{R_M(L)}{R_M(-L)} \right)^M} \approx \left( \frac{-a}{n} \right)^M.
\]

which decreases rapidly with increasing \( M \) only if

\[
Z = \left( \frac{-a}{M} \right)^M \gg (r-a).
\]

(see, however, the subsequent discussion) and decreases more rapidly than the original series, V.15 or V.49, only if

\[
Z = \left( \frac{-a}{M} \right)^M \gg (r-a).
\]

which, for \( N \) large, is essentially equivalent to the condition

\[
Z = \left( \frac{-a}{M} \right)^M \gg (r-a).
\]

The analysis of V.56-V.69, while asymptotically correct as \( N \to \infty \), requires closer scrutiny for finite \( N \) since other terms in V.65 besides \( j = 0 \) can then be dominant. (In any case, the relative error decreases with increasing \( M \) no faster than that indicated in V.67.) Inspection of V.65 indicates that V.66 provides a valid approximation to \( [R_M(n) - R_M(0)] \) for \( N \) large but finite if \( (-\frac{2}{k})^M \) is not too large for \( j = 1, 2, \ldots, M \) and \( N(r-a)/r \gg 1 \). [The \( j = M \) term dominates when \( N(r-a)/r \gg 1 \), and, if we set \( r = a \), this term reproduces the results of the analysis for \( r = a \).] If we take \( N \) finite and just large enough to insure that the asymptotic approximation \( (N = M) \) of V.64 is valid, \( N \) will be (approximately) the greater of \( kr + O(kr)^1/3 \) and \( k^2r(a-r) \).

Hence, \( N(r-a)/r \gg 1 \) will hold only if \( kr + O(kr)^1/3 \) and \( k^2r(a-r) \). Hence, \( N(r-a)/r \gg 1 \) will hold only if \( kr + O(kr)^1/3 \) and \( k^2r(a-r) \).

We now use the generalized Euler transform to obtain a second series transform for \( F_n(n+1, m) \) which is applicable to our problem when \( r > a \) and reduces to the series transform of V.54 in the case \( r = a \). We define two new quantities.

\[
\gamma_n = (-1)^n \gamma_n(L) \quad \text{(V.70)}
\]

and substitute these into V.49 to obtain

\[
\gamma_n(L) = \sum_{j=0}^{\infty} \left( \frac{2j}{n} \right)^n \left( \frac{2j}{n} \right)^n \left( -1 \right)^j \left( \frac{2j}{n} \right)^n + \cdots
\]

Then, using V.47 with \( q = ae^{i\theta}/r \) (rather than \( e^{i\theta} \) as used in the derivation of V.49),
we get (Appendix A5)
\[
\psi_n^{(m)}(w, \omega) = \left( \frac{i}{\omega} \right)^{\frac{m}{2}} \sum_{k=0}^{\infty} \left[ -2(\frac{\omega}{\omega}) \sin\theta + (\frac{\omega}{\omega}) \right]^{2k} \left( \frac{\omega}{\omega} \right)
\]
\[
\times \left\{ \sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)] \right\} R^m_{\omega}(\psi_n)
\]
\[
- \frac{\sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)]}{R^m_{\omega}(\psi_n)}
\] (V.73)

where \( \psi \) is as defined in V.39 and \( R^m(\psi) \) and \( R^m_{\omega}(\psi) \) are the new reminder series.

\[
Q_m^{(m)}(w, \omega) = \left( \frac{i}{\omega} \right)^{\frac{m}{2}} \sum_{k=0}^{\infty} \left[ -2(\frac{\omega}{\omega}) \sin\theta + (\frac{\omega}{\omega}) \right]^{2k} \left( \frac{\omega}{\omega} \right)
\]
\[
\times \left\{ \sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)] \right\} R^m_{\omega}(\psi_n)
\]
\[
- \frac{\sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)]}{R^m_{\omega}(\psi_n)}
\] (V.74)

\[
Q_m^{(m)}(w, \omega) = \left( \frac{i}{\omega} \right)^{\frac{m}{2}} \sum_{k=0}^{\infty} \left[ -2(\frac{\omega}{\omega}) \sin\theta + (\frac{\omega}{\omega}) \right]^{2k} \left( \frac{\omega}{\omega} \right)
\]
\[
\times \left\{ \sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)] \right\} R^m_{\omega}(\psi_n)
\]
\[
- \frac{\sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)]}{R^m_{\omega}(\psi_n)}
\] (V.75)

Observe that we can let \( \theta = 0 \) provided \( r \neq a \). If \( r = a \), then \( \psi = 0 \),

\( C_n^{(m)}(\omega) = C_n^{(m)}(\omega), S_n^{(m)}(\omega) = S_n^{(m)}(\omega), \left[ -2(\frac{\omega}{\omega}) \sin\theta + (\frac{\omega}{\omega}) \right]^{2k}, \left[ -2(\frac{\omega}{\omega}) \sin\theta + (\frac{\omega}{\omega}) \right]^{2k} \),

and V.13 becomes identical to V.54. However, if \( r > a \), \( C_n^{(m)}(\omega) \) and \( S_n^{(m)}(\omega) \) will have the same asymptotic form as \( C_n^{(m)}(\omega) \) and \( S_n^{(m)}(\omega) \) in the case \( r = a \) (V.55 and V.56, respectively). Hence, for \( r > a \), we can follow the earlier analysis and use the results of V.57 and V.58 in conjunction with V.27 to asymptotic estimates of the reminder terms of V.13. We find

\[
Q_m^{(m)}(w, \omega) = \left[ \frac{1}{2} \right] (\omega^2 - \omega^2 - 1) \left( \frac{\omega}{\omega} \right) \left( \frac{\omega}{\omega} \right) \left( \frac{\omega}{\omega} \right)
\]
\[
\times \left\{ \sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)] \right\} R^m_{\omega}(\psi_n)
\]
\[
- \frac{\sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)]}{R^m_{\omega}(\psi_n)}
\] (V.76)

\[
Q_m^{(m)}(w, \omega) = \left[ \frac{1}{2} \right] (\omega^2 - \omega^2 - 1) \left( \frac{\omega}{\omega} \right) \left( \frac{\omega}{\omega} \right) \left( \frac{\omega}{\omega} \right)
\]
\[
\times \left\{ \sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)] \right\} R^m_{\omega}(\psi_n)
\]
\[
- \frac{\sin [\omega (\omega^2 - \omega^2 - 1)] \cos [\omega (\omega^2 - \omega^2 - 1)]}{R^m_{\omega}(\psi_n)}
\] (V.77)

and these are useful approximations for finite \( w \) when \( w > (kr) \) and \( w > (kr)(r-a) \). The remainder terms do not vanish in the limit \( M \to \infty \). However, \( \psi_{\omega}^{(m)}(w, \omega) \) is asymptotic to the \( m \)-series of V.13 for arbitrary \( M \) since the error produced by truncating the series after \( M \) terms is of an order not exceeding the first neglected term (i.e., the \( \psi^{(m)}(w, \omega) \)-term). This can be seen by comparing the remainder sums in V.76 and V.77 with the term that would be obtained by setting \( m = M \) in the summand of V.13. The relative error associated with the \( M \)-sum of V.13 is then given by \( \sim (1/M) \) for \( r > a \), \( M \geq 1 \), and this provides a good indication of the decrease in relative error with increasing \( M \) when \( M \) \( \rightarrow (kr) \) and \( M \) \( \rightarrow (kr)(r-a) \) are both satisfied. This reproduces the relative error estimate of V.62 for \( r \) \( \approx a \) and is smaller than that found in V.67 for \( r > a \). For finite \( N \), V.67 was found to provide a useful estimate for \( M \) not too large if \( k(r-a) \geq 1 \).

We now develop an approach that is not based on the generalized Euler transform. Exploiting the Christoffel-Darboux formula for Legendre polynomials (HBS 1964: 22.12.1, Sansone 1959),

\[
(w-y) \sum_{n=0}^{\infty} P_n(w) P_n(y) = (w-y) \left[ P_n(w) P_n(y) - P_n(w) P_n(y) \right]
\]

(V.79)

we obtain a new transform directly applicable to the zonal harmonic series. In particular setting \( w = 1 \), we have

\[
(w-y) \sum_{n=0}^{\infty} P_n(w) P_n(y) = (w-y) \left[ P_n(w) P_n(y) - P_n(w) P_n(y) \right]
\]

(V.80)

where \(-1 \leq y \leq 1\), a condition manifestly satisfied for \( y = \cos(\theta) \).
Next we substitute \( u_n = (2n+1)P_n(y) \) and \( v_n = s_n \) into V.43 to obtain

\[
\frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} = \sum_{n=0}^{j-1} \left[ \frac{1}{j!} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right] s_n
\]

\[
+ \frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} v_n
\]

\[
\frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} s_n
\]

\[
= \frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} v_n
\]

\[
= \frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} v_n
\]

\[
= \frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} v_n
\]

where \( R_j(n) \) is the remainder series given by

\[
R_j(n) = \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} v_n
\]

As an application of V.84 to a series for which a closed-form solution is known, we seek the Abelian sum of the divergent series \( \sum_{n=0}^{\infty} (2n+1)P_n(y) \), \( y \neq 1 \). By definition (section V.A), the Abelian sum is

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

with \( s_n^{(1)} = \sum_{n=0}^{\infty} P_n(y) \). We can take the limit as \( j \) goes to infinity. Then (\( y \neq 1 \))

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.83 \))

where \( s_n^{(1)} = \sum_{n=0}^{\infty} P_n(y) \). Repeating this transform \( M \) times, we obtain (\( y \neq 1 \))

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.84 \))

Next, we substitute \( u_n = (2n+1)P_n(y) \) and \( v_n = s_n \) into V.43 to obtain

\[
\frac{1}{j!} \left\{ \left( \frac{1}{2} \right) \sum_{n=0}^{j-1} \left( \frac{2n+1}{2} \right) \binom{j}{n} \Delta P_n(y) \right\} s_n
\]

(\( V.81 \))

\[
\sum_{n=0}^{\infty} P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \binom{n}{0} \Delta P_n(y)
\]

(\( V.82 \))

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.83 \))

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.84 \))

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.85 \))

As an application of V.84 to a series for which a closed-form solution is known, we seek the Abelian sum of the divergent series \( \sum_{n=0}^{\infty} (2n+1)P_n(y) \), \( y \neq 1 \). By definition (section V.A), the Abelian sum is

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.86 \))

where we have used V.80 and the generating function for Legendre polynomials. With \( s_n^{(1)} = \sum_{n=0}^{\infty} P_n(y) \) in V.84, \( s_n^{(1)} = 0 \), and the result follows immediately.

Next, we define \( s_n \) as in V.13 — \( F(n+1,m) \), defined in V.15, is then given by

\[
V.84 - \sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \binom{n}{0} \Delta P_n(y)
\]

(\( V.86 \))

As an application of V.84 to a series for which a closed-form solution is known, we seek the Abelian sum of the divergent series \( \sum_{n=0}^{\infty} (2n+1)P_n(y) \), \( y \neq 1 \). By definition (section V.A), the Abelian sum is

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.87 \))

where we have used V.80 and the generating function for Legendre polynomials. With \( s_n^{(1)} = \sum_{n=0}^{\infty} P_n(y) \) in V.84, \( s_n^{(1)} = 0 \), and the result follows immediately.

Next, we define \( s_n \) as in V.13 — \( F(n+1,m) \), defined in V.15, is then given by

\[
V.84 - \sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \binom{n}{0} \Delta P_n(y)
\]

(\( V.86 \))

As an application of V.84 to a series for which a closed-form solution is known, we seek the Abelian sum of the divergent series \( \sum_{n=0}^{\infty} (2n+1)P_n(y) \), \( y \neq 1 \). By definition (section V.A), the Abelian sum is

\[
\sum_{n=0}^{\infty} (2n+1)P_n(y) = \frac{1}{1-y} \sum_{n=0}^{\infty} \left( \frac{2n+1}{2} \right) \binom{n}{0} \Delta P_n(y)
\]

(\( V.87 \))

where we have used V.80 and the generating function for Legendre polynomials. With \( s_n^{(1)} = \sum_{n=0}^{\infty} P_n(y) \) in V.84, \( s_n^{(1)} = 0 \), and the result follows immediately.
which also provides a useful approximation for finite $n$ when $n > k a + O(k a)^{1/2}$, a condition we assume is satisfied in our subsequent analysis. In Appendix A6 we show

(by induction) that

$$
\sum_{n=0}^{k a} \Delta \left( n \Delta \frac{F_n}{a} \right) = \frac{1}{z_n} \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right) = \frac{1}{z_n} \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

where

$$
\sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right) = \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

This result can then be used to establish that

$$
\sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right) = \frac{1}{2} \Delta \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

[Note that $R_0(0) = F(k a, 0)$ and that, if $(-3)! n = 1$, V.91 gives the leading asymptotic term for $F(k a, 0)$] Consequently, $(\gamma \neq 0, \gamma \to \infty)$

$$
\frac{1}{z_n} \Delta \left( n \Delta \frac{F_n}{a} \right) \approx \frac{1}{z_n} \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

since V.91, the error after summing $M$-terms ($M \geq 1$), is of an order not exceeding that of the $(M+1)$-term. The relative error of an $M$-term approximation of V.91 is, from V.91 with $F(k a, 0) = R_0(0)$, $(\gamma \neq 0, \gamma \to \infty)$

$$
\frac{1}{z_n} \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right) \approx \Delta \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

where we have substituted for $y = \cos(\theta)$. Observe that, if we increase the number of terms summed by one to $M+1$, the relative error changes by a factor, $(\Delta \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right))^2$, for $M$ small relative to $N$. This transform is, also, directly applicable to the series in V.20.

We now take the case for $r > a$ with $s_n$ in V.85 equal to $s_0$ defined in V.13a.

From V.30a, we have

$$
E_n = - \left( -1 \right)^n \frac{1}{n} \gamma_0 \quad (n \to \infty)
$$

with $\gamma_0$ a constant and

$$
\frac{1}{n} \gamma_0 = \frac{1}{n} \gamma_0.
$$

Using $n a \gamma_0 = n a \gamma_0 + v_n (\Delta \gamma_0)$, we find that

$$
E_n = - \frac{1}{n} \Delta \left( \frac{1}{n} \Delta \frac{1}{n} \Delta \gamma_0 \right)
$$

$$
= - \frac{3}{2} \left( \frac{1}{n} \right)^2 \left( \frac{1}{n} \right) \frac{1}{n} \gamma_0.
$$

and it follows immediately that

$$
E_n = - \frac{3}{2} \left( \frac{1}{n} \right)^2 \gamma_0, \quad (n \to \infty)
$$

Hence, $R_0(0)$, defined in V.85, satisfies

$$
R_0(0) \to \frac{1}{z_n} \sum_{n=1}^{k a} \left( \frac{1}{z_n} \right)^2 \Delta \left( n \Delta \frac{F_n}{a} \right), \quad (n \to \infty)
$$

where we have used V.91 and substituted $\cos(\theta)$ for $y$ and $2 \sin^2(\theta/2)$ for $(1 - y)$. 

\[V.94\]
If we now substitute the leading term of the asymptotic expansion for \( P_n(\cos \theta) \) in \( V.98 \), writing it as

\[
\frac{2}{\pi} \left[ \frac{\xi}{\sin(\xi)} \right]^{1/2} Q_\xi \exp \left\{ \left( \frac{\xi}{2} - y \right) \right\},
\]

we obtain

\[
T(y) = \int \frac{2}{\pi} \left[ \frac{\xi}{\sin(\xi)} \right]^{1/2} Q_\xi \exp \left\{ \left( \frac{\xi}{2} - y \right) \right\} \sin^n(\xi) \, d\xi.
\]

(V.99)

and an asymptotic evaluation of this series can be found using \( V.27 \); viz.,

\[
T(y) \approx \frac{2}{\pi} \left[ \frac{\xi}{\sin(\xi)} \right]^{1/2} \left\{ \frac{1}{2} - y \right\} \left( \frac{1}{2} + y \right) \left( \frac{1}{2} - y \right) \left( \frac{1}{2} + y \right) \ldots \left( \frac{1}{2} - y \right) \left( \frac{1}{2} + y \right)
\]

(V.100)

With the substitution of \( t = x^2 \) and \( (1 - \xi \xi^2 + y) \), \( x \) as defined in \( V.39 \), \( V.101 \) can be rewritten as

\[
T(y) \approx \frac{2}{\pi} \left[ \frac{\xi}{\sin(\xi)} \right]^{1/2} \left\{ \frac{1}{2} - y \right\} \left( \frac{1}{2} + y \right) \left( \frac{1}{2} - y \right) \left( \frac{1}{2} + y \right) \ldots \left( \frac{1}{2} - y \right) \left( \frac{1}{2} + y \right)
\]

(V.101)

Since, from \( V.85 \),

\[
\sum_{n=0}^{\infty} \frac{1}{\pi^2} \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i = \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y),
\]

(V.102)

the relative error in neglecting the remainder series (after summing \( M \) terms of the first series in \( V.84 \)) is given by

\[
\frac{\alpha_n \beta_n \gamma_n}{\alpha_m \beta_m \gamma_m} = 0 \left\{ \left( \frac{\xi}{2} \right)^M \left( \frac{\xi}{2} \right)^{-M} \right\}.
\]

(V.103)

But, using the transform \( V.84 \), we also have

\[
S(y) = \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \frac{2}{\pi} \left[ \frac{\xi}{\sin(\xi)} \right]^{1/2} Q_\xi \exp \left\{ \left( \frac{\xi}{2} - y \right) \right\}.
\]

(V.104)

Thus, the approximation to \( S(y) \) produced by neglecting \( N \) in \( V.105 \) is equal to the \( (N + 1) \)-term Legendre expansion in \( V.107 \). [Note that, for \( N > 0 \), this still represents an infinite polynomial expansion in \( y \) due to the \( (1 - y)^N \) factor.]

From the orthogonality condition of the Legendre polynomials, \( V.105 \) implies

\[
T = \frac{2}{\pi} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \frac{2}{\pi} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y),
\]

(V.105)

and, also,

\[
S = \frac{2}{\pi} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \frac{2}{\pi} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y).
\]

(V.106)

Equation \( V.110 \) can also, be proved by induction on \( J \) from \( V.106 \) using \( V.109 \). This confirms the validity of \( V.106 \).

The \( N \)-th degree least squares polynomial approximation to a function \( g(y) \) over \([-1,1]\) (assuming a constant weighting function) is given by

\[
\sum_{n=0}^{\infty} \frac{1}{\pi} \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i = \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y),
\]

(V.107)

where \( \alpha_n = \frac{2}{\pi} \sum_{i=0}^{\infty} \alpha_i \beta_i \gamma_i = \frac{2}{\pi} \sum_{i=0}^{\infty} \alpha_i \beta_i \gamma_i = Q_\xi(y) \) (Hildebrand 1974). Consequently,

\[
\sum_{n=0}^{\infty} \frac{1}{\pi} \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i = \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y),
\]

(V.108)

which is equivalent to the product of \((1 - y)^N\) and

\[
\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{\pi} \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i = \frac{2}{\pi} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y).
\]

(V.109)

can be identified as the \((N + M)\)-th degree least squares polynomial approximation to \((1 - y)^M S(y)\). Alternatively, \( V.111 \) can be viewed as the \((N + M)\)-th degree least squares approximation to \( S(y) \) of the form \((1 - y)^N h(y)\), with

\[
h(y) = \sum_{n=0}^{\infty} \frac{1}{\pi} \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i = \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = Q_\xi(y)
\]

(V.107) (Hildebrand 1974 p. 331).
As in the case of the relative error of V.67 (obtained with the Eulerian transformation) V.104 gives an indication of the decrease of relative error to be expected for finite $N$ and increasing $N$, primarily when $(r - a)$ is not too small (see discussion following V.67).

**Least-squares interpretation**

We find that a "least squares" interpretation can be associated with the transform of V.84. As a preliminary step, we show that

$$
\Xi_0(\eta) = \sum_{n=0}^{\infty} \left( \frac{a}{n+1} \right)^{1/2} \left( \frac{2^{n+1}}{\pi} \right)^{3/2} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} P_n(\eta), \quad n = 0, 1, 2, \ldots
$$

(V.105)

where

$$
\Xi_2(\eta) = -\left( \frac{a}{2n+1} \right)^{1/2} \left( \frac{2^{n+1}}{\pi} \right)^{3/2} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} \left[ \sin\left( \frac{n\pi}{2} \right) \right]
$$

(V.106)

Note that V.61 can be rewritten as

$$
\Xi_0(\eta) = \frac{a}{n+1} \sum_{n=0}^{\infty} \left\{ \frac{2^{n+1}}{\pi} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} \left[ \sin\left( \frac{n\pi}{2} \right) \right] \right\}
$$

(V.107)

Repeating this transform $M$ times gives us the results in V.105 and V.106.

If we substitute for $R_n(\phi)$, defined in V.85, in V.105, we have

$$
\Xi_0(\eta) = \left( \frac{a}{n+1} \right)^{1/2} \left( \frac{2^{n+1}}{\pi} \right)^{3/2} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} P_n(\eta) + \Xi_2(\eta).
$$

(V.108)

**Plane-earth limit**

In section IV.1.B we derived the boundary-value solution for a vertical Hertzian source over a planar half-space as a limit of the corresponding two-media spherical solution. Similarly, we now seek the plane-earth analogue to the transform in V.84 which, dividing through by $a^2$ and substituting $y = \cos \theta$, we can rewrite as

$$
\sum_{n=0}^{\infty} \left( \frac{2^{n+1}}{\pi} \right)^{3/2} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} \left[ \sin\left( \frac{n\pi}{2} \right) \right] \left( \frac{2^{n+1}}{\pi} \right)^{3/2} \frac{\sin\frac{n\pi}{2}}{n^{1/2}} \left[ \sin\left( \frac{n\pi}{2} \right) \right]
$$

(V.109)

We again take $p = a(\sin \theta)$ as the (cylindrically) radial distance from the $z$-axis, and choose the origin ($z = 0$) of the cylindrical reference frame as the intersection of the $z$-axis with the earth's surface. Then, defining

$$
k_a = \frac{a}{2}
$$

we have, in the limit as $a \to w$,

$$
G \to p_a, \quad k_a \to \\infty,
$$

(V.110)

Also, in the limit, $k_n$ becomes a continuous quantity and the summation in $n$ goes over into an integral in $k_n$ (sec. IV.1.B). If $ka < \infty$ for some $Q > 0$, define

$$
q = \frac{ka}{\sin(\phi)} = k_a > k_a,
$$

(V.111)
and assume that, in the limit as \( a \to \infty \), \( s_n \) is a function of \( na \); that is

\[
\lim_{a \to \infty} s_n = S(na)
\]  

(V. 117)

(This is true, in particular, for the series summand of the two-media problem.) It then follows that

\[
\lim_{a \to \infty} \left\{ \frac{s^{(m)}(u)}{2s^{(m)}(\frac{u}{2})} \right\} = \frac{S^{(m)}(ua)}{2^{m}}.
\]  

(V. 118)

where

\[
\hat{S}(u) = \frac{1}{2} \frac{d}{du} \left\{ \log \left( \frac{\hat{S}(u)}{2^{m}} \right) \right\} , \quad \hat{S}^{(m)}(u) = \hat{S}^{(m-1)}(u) \frac{d}{du} \left( \frac{\hat{S}(u)}{2^{m}} \right) .
\]  

(V. 119)

as can be shown by induction. We prove it true in the case of \( m = 1 \):

\[
\lim_{a \to \infty} \left[ \frac{s^{(1)}(u)}{2s^{(1)}(\frac{u}{2})} \right] = \lim_{a \to \infty} \left\{ -\frac{\left[ \frac{(na)^{2}}{(na)^{2} - s^{(1)}(\frac{u}{2})} \right]}{2\left( \frac{(na)^{2} - s^{(1)}(\frac{u}{2})}{2} \right)} \right\} \quad \times \quad \frac{1}{2^{1}} \frac{d}{du} \left[ \frac{\hat{S}(u)}{2^{1}} \right] - \frac{1}{2^{1}} \frac{d}{du} \left[ \frac{\hat{S}(u)}{2^{1}} \right] .
\]

But \( k_n = \lim (n/a) \), \( \Delta k = 1/a + 0 \) as \( a \to \infty \), and \( \lim s_n = s(k_n) \); hence,

\[
\lim_{a \to \infty} \left[ \frac{s^{(1)}(u)}{2s^{(1)}(\frac{u}{2})} \right] = \frac{1}{2} \lim_{a \to \infty} \left\{ -\frac{1}{k_n} \left[ \frac{(na)^{2}}{(na)^{2} - s^{(1)}(\frac{u}{2})} \right] - \frac{1}{k_n} \left[ s^{(1)}(\frac{u}{2}) \right] \right\} \quad \times \quad \frac{1}{2^{1}} \frac{d}{du} \left[ \frac{\hat{S}(u)}{2^{1}} \right] - \frac{1}{2^{1}} \frac{d}{du} \left[ \frac{\hat{S}(u)}{2^{1}} \right] .
\]

by the definition of the derivative. The proof for \( m = k + 1 \), assuming the validity of V.118 for \( m = k \), follows the steps of the proof for \( k = 1 \) and is omitted here.

Thus, in the limit as \( a \to \infty \), V.112 becomes

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} \frac{d}{du} \left[ \frac{s^{(m)}(u)}{2s^{(m)}(\frac{u}{2})} \right] \frac{\hat{S}(ua)}{2^{m}} \, du = \frac{1}{2^{m}} \frac{S^{(m)}(ua)}{2^{m}} \frac{\hat{S}(ua)}{2^{m}}.
\]  

(V. 120)

as can be shown directly. We assume \( \lim [\log s^{(m)}(t)] = 0 \), for \( s^{(m)}(t) \) as defined in V.119, and integrate

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

using the parts rule twice

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

First, take

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]

\[
\int_{\frac{b}{2}}^{\frac{b}{2}} w \, du = \int_{\frac{b}{2}}^{\frac{b}{2}} w \, du
\]
Defining

\[ z_i(t) = \frac{1}{t} \int_{0}^{t} \left[ J_i(t_x) \right] dt_x \]

and, substituting, \( J_i(t_x) = \frac{1}{t_x} \int_{0}^{t_x} \left[ J_i(t_y) \right] dt_y \)

we see that

\[ z = \frac{1}{t} \int_{0}^{t} \left[ J_i(t_x) \right] dt_x + \frac{1}{t} \int_{0}^{t} \left[ J_i(t_x) \right] dt_x \]

which reproduces V.120 in the case of \( M = 1 \). Clearly, \( M \) repetitions of this process reproduces V.120.

While the subject of the present study does not specifically include planar-earth solutions, we note Burke et al (1981) on the significance of numerical integration to this problem and, in particular, to the case of a source near a planar surface:

A problem that has traditionally resisted a genuinely practical solution despite considerable longevity and study is that of modeling conducting structures (antennas or scatterers) located near a planar interface such as the earth's surface. More than 70 years ago, Sommerfeld (1909) worked out the basis for the rigorous solution of this problem in terms of Fourier integrals of cylindrical wave expansions. These Sommerfeld integrals have been studied extensively and numerous approximations have been developed for them involving various combinations of the problem's parameters [see Bokow (1964) for example]. However, the evaluation of the Sommerfeld integrals for the ranges of source and observation-point locations that typically must be covered for any self-consistent description of the structure's current - an integral equation for example - almost inevitably involves some parameter combinations to which such approximations do not apply. Consequently, the only feasible approach for obtaining a reliable, straightforward solution has been to integrate the Sommerfeld integrals numerically, which can be a computationally inefficient process.

Siegel and King (1970) integrated Sommerfeld integrals (section IV.1.B) along the real axis using Romberg integration coupled with Euler's transformation. Bubenik (1977) used the Shanks transformation along the real line as did Burke et al (1981) for integration along a complex contour. Johnson and Dudley (1983), integrating along the real line, subtract asymptotic terms to accelerate convergence. The application of these techniques to the spherical earth problem is investigated in the following sections.

V.C THE KUMMER TRANSFORMATION

The Kummer transformation allows the rate of convergence of a series to be accelerated by subtracting a second series with a known sum (NBS 1964: 3.6.26) and has been frequently applied to series (and integrals) encountered in propagation analysis. The rule is simple; if \( \sum_{n=0}^{\infty} a_n \) is a convergent series and if \( t = \sum_{n=0}^{\infty} b_n \) is the known sum of a second series for which \( \lim (a_n/b_n) = 1 \), then it follows that

\[ \sum_{n=0}^{\infty} a_n = t + \sum_{n=0}^{\infty} (a_n - b_n) \]

The transformation has been used to remove the principal limiting factor in zonal harmonic series solutions by Johler and Berry (1962), Johler and Lewis (1969), Galejs (1972), and Lewis and Johler (1976). Nicholaenko and Rabinowicz (1974) and Blokh, Nicholaenko, and Filippov (1978) employed it to accelerate the calculation of

\[ P_n(\xi) \]

with both Nicholaenko groups of authors removing several of the leading limiting factors. The technique has also been used to accelerate the numerical calculation of Sommerfeld integrals (i.e., integrals arising in plane-earth solutions) by Bokow and Alexopoulos (1961) and Johnson and Dudley (1963) among others, and as an aid in the summation of Cartesian and cylindrical series solutions by Morse and Feshbach (1953, p. 819), Wilton et al (1983), Wright et al (1983), and Richards et al (1983).

For the cases of interest here, we consider an infinite series of Legendre polynomials,

\[ S = \sum_{n=0}^{\infty} (2n+1) P_n(\eta), \quad \eta = \cos(v) \]
where $S$ is the series sum if the series is convergent or its Abelian sum (if it exists) if the series is divergent (see sec. V.A). If there exists an $a_n$ such that $s_n$ approaches $a_n$ as $n$ increases without bound, and if the infinite series with $a_n$ replacing $s_n$ has a known (direct or Abelian) sum, $A$, then Kummer's transformation allows us to write

$$S = \sum_{n=0}^{\infty} (2n+1) \left( x^{n} a_{n} \right) P_{n}(y) + A \quad (V.121)$$

where

$$A = \sum_{n=0}^{\infty} (2n+1) a_{n} P_{n}(y) \quad (V.128)$$

is assumed known in closed form.

With this, certain infinite expansions in Legendre polynomials can be evaluated exactly (see Appendix A6); specifically, those for which the coefficients are linear combinations of terms having the form $n^{m}$ (for $|z| < 1$ and $m$ a nonnegative integer) with terms of the form $\dfrac{z^{n}}{(n+p)(n+p+1) \cdots (n+p+q)}$ (for $|z| < 1$ and $p$ and $q$ nonnegative integers). We can also evaluate exactly the Abelian sum of divergent series for which the coefficients are a linear combination of terms of the form $n^{m}$ for $m$ a nonnegative integer. Consequently, we can now apply Kummer's transformation to the series of $V.13a$:

$$F = \sum_{n=0}^{\infty} (2n+1) x^{n} a_{n} P_{n}(y) + A' \quad (V.129)$$

with

$$A' = \sum_{n=0}^{\infty} (2n+1) a_{n} P_{n}(y) \quad (V.130)$$

and $w = kr$ and $x = ka$.

From the ascending series for the Bessel functions (NBS 1964: 10.1.2, 10.1.3), we obtain

$$\lim_{n \to \infty} \left[ \frac{(n+1)w^{n}}{(n+2)w^{n}} \right] \to \left( \frac{z}{x} \right)^{w} \quad (V.131)$$

and

$$\lim_{n \to \infty} \left\{ \sum_{n=0}^{\infty} \left( \frac{z}{x} \right)^{w} \right\} \to -w \left[ 1 - \frac{\pi}{(2n+1) \left( x^{n} \right) v} \right] \quad (V.132)$$

Several researchers (cited above) have used Kummer's transformation to cancel the leading term in the summand of V.130; that is, they substitute

$$a_{n}' = -\left( x^{n+1} \right) \left( \frac{z}{x} \right)^{w} \quad (V.133)$$

for $a_n$ in V.127 and evaluate the sum $A'$ of the associated series in V.128. Thus V.129 can then be transformed to

$$F = \sum_{n=0}^{\infty} (2n+1) \left( x^{n} - a_{n}' \right) P_{n}(y) + A' \quad (V.134)$$

where $a_{n}'$ is given by V.133 and, from Appendix A6,

$$A' = \sum_{n=0}^{\infty} (2n+1) \left( x^{n+1} \right) \left( \frac{z}{x} \right)^{w} P_{n}(y) \quad (V.135)$$

The algebraic term in V.134 [more particularly the term $(A' - 1)$], which dominates as $\theta \to 0$, has a physical interpretation noted by Nicholson (1911, p.520) and Johler and Lewis (1969), viz., that it is proportional to the quasi-static electric field of a dipole above the surface of a perfect conductor. Also note that, while the original series (V.129) is divergent if $r = a$ is substituted, the series in V.134 is convergent.
The series of V.129 can be further accelerated by other (more complex) choices of
\( a_n \), a technique that does not appear to have been pursued except by Nicholaenko and
Rabinowicz (1914) for the series in V.125.

We investigate additional applications of Kummer’s transform to V.129 in the case
\( r = a \) (i.e., \( u = x \)). In particular, for \( r = a \), \( z_n = f_n \) where
\[
\begin{align*}
F_n &= \frac{a_n \cdot \left( \frac{u}{a_n} \right)^{z_n}}{u^{z_n} \cdot z_n!} \\
\text{and, from V.132, we have}
\end{align*}
\]
\[
\begin{align*}
F_n &= \left( \frac{u}{a_n} \right)^{z_n} \left[ 1 + \frac{a_n x}{u} + \frac{3a_n x^2}{2u} + \cdots \right] \cdot O \left( \frac{x^j}{a_n} \right),
\end{align*}
\]

If we now define:
\[
\begin{align*}
a_n(1) &= -a_n(1), \\
a_n(x) &= \left[ (x + 1) + \frac{x}{a_n(1)} \right], \\
a_n(z) &= \left[ (z + 1) + \frac{x^2}{a_n(1)} \right],
\end{align*}
\]

then \( \{ f_n - a_n(j) \} \) is of progressively lower order in \( n (n \to \infty) \) as \( j \) increases.

Thus, with subscripted Greek letters denoting constants,
\[
\begin{align*}
\left[ F_n - a_n(j) \right] &= -\frac{x^j}{a_n(1)} \left( \frac{z}{a_n} \right)^{z_n}, \\
\left[ F_n - a_n(x) \right] &= -\frac{3x^2}{4a_n^2} \left( \frac{z}{a_n} \right)^{z_n}, \\
\left[ F_n - a_n(z) \right] &= -\frac{3x^3}{8a_n^3} \left( \frac{z}{a_n} \right)^{z_n}.
\end{align*}
\]

Furthermore, the \( a_n(j) \) have been selected so that the series in V.128 can be evaluated
exactly for \( a_n = a_n(j), \) \( j = 0, 1, 2; \) the resultant closed-form expressions, \( A(j) \),
being given in Appendix A6. Hence,
\[
\begin{align*}
F &= \sum_{n=1}^{\infty} \left( \frac{a_n \cdot \left( \frac{u}{a_n} \right)^{z_n}}{u^{z_n} \cdot z_n!} \right) \cdot P_n \left( \eta \right) \\
&= \sum_{n=1}^{\infty} \left( \frac{a_n \cdot \left( \frac{u}{a_n} \right)^{z_n}}{u^{z_n} \cdot z_n!} \right) \cdot P_n \left( \eta \right) + A(j),
\end{align*}
\]

and numerical comparisons of the rates of convergence obtained for the various choices
of \( a_n(j) \) are given in tables V.5-V.6.

We must now estimate the error that results from the truncation of the series of
V.144. Again, these series do not begin to converge until \( n > N > ka + O(ka)^{\frac{1}{3}} \)
(section V.A.), beyond which \( f_n \) begins to approach \( a_n(j) \). If we truncate at
\( n = N > ka, \) the error, \( E(N,j), \) will be given by
\[
E(N,j) = o \left( \frac{u}{a_n(1)} \right) \left[ F_n - a_n(j) \right] \cdot P_n(\eta), \quad \eta = \cos(\theta),
\]

Using V.63 with V.91 to estimate the asymptotic error, we have \( (\eta \to \infty) \)
\[
\begin{align*}
E(N,j) &= \left( \frac{u}{a_n(1)} \right) \left[ F_n - a_n(j) \right] \Delta P_n(\eta) - \left( \frac{ka}{2} \right)^2 \frac{u}{a_n^2(1)} \Delta \left[ F_n - a_n(j) \right] + \Delta \left[ F_n - a_n(j) \right] \cdot P_n(\eta).
\end{align*}
\]

for \( j = 1, 2, 3, \ldots \) and \( y \neq 1 \). Then, if \( y \neq 1, \) V.141 through V.143 (with \( x = ka \)) and
the asymptotic form of \( P_n \left[ \cos(\theta) \right] \) [valid for \( n(\sin \theta) \gg 1 \)] gives us
\[
\begin{align*}
E(N,1) &= O \left( \frac{ka}{a_n(1)} \right), \quad (\eta \to \infty) \\
E(N,2) &= O \left( \frac{ka}{a_n(1)} \right), \quad (\eta \to \infty)
\end{align*}
\]

and
\[
\begin{align*}
E(N,3) &= O \left( \frac{ka}{a_n(1)} \right) + O \left( \frac{ka}{a_n(1)} \right), \quad (\eta \to \infty)
\end{align*}
\]

and these quantities provide an indication of the decrease in error to be expected for
increasing (but finite) \( N \) when \( M(\sin \theta) \gg 1 \) and \( M > ka + O(ka)^{\frac{1}{3}} \).

The extension of Kummer’s transformation to the series solutions for lossy
conductors, while immediate, somewhat surprisingly proves to be less useful than its
application to those for perfectly conducting media. The series corresponding to V.129
for a scattering sphere that is lossy is, from V.6 (with \( r = a \))

\[
\langle n \rangle = \sum_{n=1}^{\infty} \left( 2n+1 \right) \frac{\mathcal{D}_n}{\mathcal{C}_n}, \quad \mathcal{C}_n = \text{csc}(\mathbb{C})
\]

(V.150)

where

\[
\mathcal{C}_n = \text{csc} \left\{ \text{csc}^{-1} \left( \frac{2n+1}{\mathcal{D}_n} \right) \right\}
\]

(V.151)

\( x = ka \) and \( z = K_1 a = k a \left( \sigma_1 / \omega \epsilon_0 \right) \) (from IV.3, where \( \sigma_1 \) is the conductivity of the scatterer). \( \mathcal{O} \) is the Abelian sum of the divergent series in V.150.

In the limit as the conductivity, \( \sigma_1 \), goes to infinity, \(|z| \to \infty\) and it can be shown that

\[
\lim_{|z| \to \infty} \left\{ \mathcal{C}_n \text{csc}^{-1} \left( \frac{2n+1}{\mathcal{D}_n} \right) \right\} = 0.
\]

(V.152)

Equations V.150 and V.151 then reduce to the solution for the perfectly conducting sphere considered earlier. For large but not infinite conductivity, the large-N expansion of the logarithmic derivative of the quantity in V.152 must be obtained. From the ascending series for the Bessel function, we have

\[
\mathcal{C}_n \left. j_n(x) \right|_{x=ka} \to (2n+1) - \frac{x^2}{(2n+1)(2n+3)} - \frac{x^4}{(2n+1)(2n+3)(2n+5)} - \ldots
\]

(V.153)

(Note that, once this expansion is made, we cannot subsequently let \(|z| \to \infty\).

This expansion provides a useful approximation for finite \( n \) only for \( n \gg |z| \).

Similarly, if we obtain several terms of an asymptotic expansion (\( n \to \infty \)) for \( g_n \) and denote their sum by \( a_n \), then \( g_n - a_n \) will not become small until \( n \gg |z| \).

For lower-band frequencies and the conductivity of typical earth-like media, \(|z| \gg ka\).

One approach to circumventing this difficulty (at least for field points far removed from the source) is to use the surface impedance concept associated with mode-theory (see section IV.2.A) and replace \( z \ln[z] \) with a constant quantity independent of \( n \); in particular with

\[
-i z \sqrt{\mathcal{O}} - \left( \frac{ka}{z} \right)^2
\]

(V.154)

In mode theory, researchers have justified this approach on the basis that the important modes, \( \psi \), in the corresponding quantity

\[
\mathcal{C}_n \left. j_n(x) \right|_{x=ka}
\]

(V.155)

all fall near \( ka \), and the Debye asymptotic expansion for V.155 with \( \psi = ka \) is given by V.154. But, the mode sum obtained by substituting V.154 for V.155 is exactly equivalent to the corresponding zonal harmonic sum with V.154 replacing \( z \ln[z] \), as can be shown by applying Watson's transform in reverse order of that in section V.2.A. Use of a surface impedance in conjunction with the zonal harmonic series has not previously been investigated (at least to the author's knowledge), but may well be useful in connection with the application of Kummer's transform to lossy conductors.
A series is said to be Cesàro summable \((C,1)\) with sum \(S\), if, for \(A_n = \sum_{x=0}^{\infty} a_n\),
\[
\lim_{n \to \infty} \left( \frac{A_0 + A_1 + \cdots + A_n}{n+1} \right) = S.
\]
Furthermore, a series is Cesàro summable \((C,j)\) with sum \(S\) if \(A_0 = A_n = \sum_{x=0}^{\infty} a_n\) and \(A^j_n = \sum_{x=0}^{\infty} A^{j-1}_n x^n\).

The (\(w^j\))-term approximation to the sum \((C,j)\), \(C^j_n + A^j_n / (w^j)\), can be expressed in terms of the original series of \(a_n\). Following Hardy's derivation (1949), but using summation by parts in reverse,
\[
C^j_n = \sum_{x=0}^{\infty} \left( \frac{1}{(w+1)} \right) (\frac{x^n}{w^n} - \frac{x^{n+1}}{w^{n+1}})
\]
\[
= \left(1 - \frac{1}{w+1}\right) \sum_{x=0}^{\infty} \frac{x^n}{w^n} \cdot \frac{x^n}{w^n} - \frac{x^{n+1}}{w^{n+1}}
\]
\[
= \left(1 - \frac{1}{w+1}\right) \sum_{x=0}^{\infty} \frac{x^n}{w^n} a_n x^n
\]
\[
= \frac{\sum_{x=0}^{\infty} (P^j_n) x^n}{\sum_{x=0}^{\infty} x^n a_n x^n}
\]
Equating equal powers of \(x\),
\[
C^j_n = \frac{\sum_{x=0}^{\infty} (1 - \frac{p}{e^{nx}}) (1 - \frac{p}{e^{nx}}) \cdots (1 - \frac{p}{e^{nx}})}{\sum_{x=0}^{\infty} x^n a_n x^n}
\]
Equation 156.

The Cesàro sum \((C,1)\) of a convergent series will agree with the series sum, and, if a series is summable \((C,j)\), then for \(j > j \geq 1\), the \((C,j)\) sum equals the \((C,j)\) sum (Hardy 1949: theorem 43). Hence, when the spherical wave series is convergent (i.e., when \(r \neq r_0\)), each Cesàro sum \(j \geq j\) converges to the actual series sum.

In the limiting case where \(r\) approaches \(r_0\), the series sum is the Abelian sum of the divergent series that results from direct substitution of \(r = r_0\), as discussed in Section V.A. If the Cesàro sum \((C,j)\) of a divergent series exists for some \(j\), this sum will agree with the Abelian sum (Hardy 1949: theorem 55); however, a divergent series can have an Abelian sum and not be Cesàro summable \((C,j)\). For \(r = r_0 = a\), we establish that the \((C,j)\) sum for the groundwave series of V.12 exists for \(j \geq 2\) and does not exist for \(j = 1\). This result also holds for the three-media problem, which can be written as the sum of the groundwave and an ionospheric (outer medium) contribution, since the latter is convergent when \(r = r_0 \neq b\) (section V.A). The proof (original) leads to new results that provide some insight into the rate of convergence to be expected as well as the error in an \((\infty^j)\)-term approximation to the \((C,j)\) sum.

From V.15, we have
\[
F_n = \sum_{x=0}^{\infty} \left( 2\pi x + \epsilon \right) e^{ixx} + \epsilon = \cos(\phi)
\]
and the substitution of \(r = r_0 = a\) reduces \(F_n\) to
\[
E_n = \sum_{x=0}^{\infty} \left( \frac{x}{w} \right) e^{ixx} + \epsilon
\]
Equation 157a.

with \(x = ka\), which, for \(n > x\), can be expanded as in V.30b. We note (for later use) that \(f_n = O(n)\), \(df = O(1)\), and \(e^{-\epsilon n} F_n = O(n) F_n = O(n)\); moreover, as the following proof requires only this information about \(f_n\), it may be applied to any Legendre series satisfying such conditions. The Cesàro series sum \((C,j)\) of \(F(N+1,\omega)\) approximated by \((\infty^j)\) terms is, from V.156,
\[
C^j_n = \sum_{x=0}^{\infty} \left( 2\pi x + \epsilon \right) e^{ixx} + \epsilon
\]
Equation 157b.

where
\[
F_n = \sum_{x=0}^{\infty} \left( 2\pi x + \epsilon \right) e^{ixx} + \epsilon
\]
Equation 158.

\[
F_n = \sum_{x=0}^{\infty} \left( 2\pi x + \epsilon \right) e^{ixx} + \epsilon
\]
Equation 159.

\[
F_n = \sum_{x=0}^{\infty} \left( 2\pi x + \epsilon \right) e^{ixx} + \epsilon
\]
Equation 160.
Suppressing the argument of the Legendre polynomials, we can use the transform V.81 (derived using summation by parts) to express \( C_n^{(m)} \) as:

\[
C_n^{(m)} = -\frac{1}{(2n+1)} \left[ (-1)^{n+1} P_n^{(m)}, \Delta F_n - (\Delta n) \Delta P_n^{(m)} \right] + \frac{1}{(2n-1)} \left[ (\Delta n) \Delta F_n, \Delta P_n^{(m)} - (\Delta n + 1) \Delta P_n^{(m)}, \Gamma_n^{(m-1)} \right] + \frac{1}{(2n+1)} \left[ \Delta F_n, \Delta P_n^{(m)} \right].
\]  

(V.161)

where we have defined:

\[
E_n^{(m)} = -\frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \Delta F_n, \Delta P_n^{(m)} \right]
\]

(V.162)

with

\[
E_n^{(m)} = -\frac{1}{(2n+1)} \left( \begin{array}{c} n+1 \\ n \end{array} \right) \Delta F_n
\]

(V.163)

\[
P_n^{(m)} = \frac{1}{(2n+1)} \left( \begin{array}{c} n+1 \\ n \end{array} \right) \Delta P_n^{(m)}
\]

(V.164)

\[
h_n = -\frac{1}{(2n+1)} \left[ \Delta F_n, \Delta P_n^{(m)} \right]
\]

(V.165)

But, from V.159 and V.160, we find

\[
F_n^{(m)} = F_n, \quad \Delta F_n = \Delta P_n^{(m)} - \frac{1}{2n+1} F_n^{(m+2)}.
\]

\[
F_n^{(m+1)} = \frac{1}{(2n+1)} \left( \begin{array}{c} n+1 \\ n \end{array} \right) F_n
\]

(V.166)

We now substitute for these quantities in V.161, adding and subtracting the series \( \frac{1}{(2n+1)} \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n \end{array} \right) \), to obtain

\[
C_n^{(m)} = \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n - (\Delta n) \Delta P_n^{(m)} \right] + \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n \right] + \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n \right].
\]

(V.167)

However, from V.83, V.157a (with \( z_n = f_n \)) is exactly equivalent to

\[
P_n^{(m)}, \alpha_n^{(m)}, \beta_n^{(m)} \quad \text{with} \quad n = 0, 1, 2, \ldots
\]

for \( f^{(1)} \) as defined in V.163, and we observe that this is also identical to the first term in V.166. Consequently, we have the useful (and new) result

\[
C_n^{(m)} = \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n \right] + \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n \right] + \frac{1}{(2n+1)} \left[ \left( \begin{array}{c} n+1 \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) P_n^{(m)}, \Delta F_n \right].
\]

(V.168)

The first term above, (a.), is \( \mathcal{O}\left(n^{1/2}\right) \) \((N = \infty)\) independent of \( j \), thus vanishing for all \( j \) in the limit. Since \( f_n \) is \( \mathcal{O}(n) \) and \( P_n \) is \( \mathcal{O}(n^{-1/2}) \), the second term, (b.), is \( \mathcal{O}\left[J_n^{(m+1)}J_n^{(m)}\right] (N = \infty) \), vanishing in the limit only if \( j \geq 2 \) (we assume \( j \) is an integer throughout). In V.91, we found (c.) to be \( \mathcal{O}(n^{1/2}) \) \((N = \infty)\).
hence, this term also vanishes in the limit independent of \( j \). The remaining two series

\( - (d.) \) and \( (e.) \) with \( g_n \) and \( h_n \) defined in V.164 and V.165, respectively — require additional analysis to show that they vanish in the limit, \( M \to \infty \), if \( j = 2 \).

From V.160, we obtain

\[
\Delta_k = \begin{cases} \frac{-1}{(2\pi)^2} j^2 & j \geq 2 \\ \frac{1}{(2\pi)^2} \left( \frac{m-n+2}{m-n+1} \right) \left( \frac{m-n+3}{m-n+2} \right) \ldots \left( \frac{m-n+2}{m-n+2} \right) & j \geq 3 \end{cases}
\]  

(V.169)

which, when substituted into V.164 and V.165 with \( j = 2 \), yield

\[
G_N = \frac{2}{(2\pi)^2} \left( \frac{m-n+2}{m-n+1} \right) \Delta_k \Gamma_N, \quad j = 2
\]  

(V.170)

Then,

\[
\Delta_N = \frac{2}{(2\pi)^2} \left[ 1 + \left( \frac{m-n+2}{m-n+1} \right) \Gamma_N \right] \Delta_k \Gamma_N, \quad j = 2
\]  

(V.171)

and

\[
\Delta(N, \Delta_N) = \frac{2}{(2\pi)^2} \left[ 1 + \left( \frac{m-n+2}{m-n+1} \right) \Gamma_N \right] \Delta_k \Gamma_N, \quad j = 2
\]  

(V.172)

If we now substitute for \( g_n \) in V.171 (from V.171), we find that \( A(N, \Delta_N) \to 0(1/M) \) if \( M \to \infty \), and decreases like \( N^{-1/2} / M \) for \( N \) and \( M \) large. Also, since the coefficients multiplying \( \pi_n \) in the summands of the three series of V.176 are \( O(1/\pi^2) \), \( O(1) \), and \( O(1/n) \), respectively, these series converge to constants in the limit as \( M \to \infty \).

The \( M \)-factor multiplying these series then dictates their behavior in the limit as \( M \to \infty \); the first being \( O(1/M) \) whereas the second and third are \( O(1/M^2) \). Consequently, the term given by V.168b, i.e., \( G(N, \Delta_N) \), is \( O(1/M) \times \Delta_N \to 0(1/M) \). If \( j > 2 \), \( g_n \) in V.171 is modified, but \( G(N, \Delta_N) \) will again be \( O(1/M) \), \( M \to \infty \).

We take the same approach with the last term in V.168b which we can rewrite as

\[
G(N, \Delta_N) = \frac{1}{(2\pi)^2} \left( \frac{m-n+2}{m-n+1} \right) \Delta_k \Gamma_N
\]

(V.173)

yields

\[
G(N, \Delta_N) = \frac{1}{(2\pi)^2} \left( \frac{m-n+2}{m-n+1} \right) \Delta_k \Gamma_N
\]

(V.174)

where \( A_N \) has the same form as \( A \) in V.177, but with \( h_n \) replacing \( g_n \). If we substitute for \( h_n \) in \( A_N \) we find that the term, \( -\left( \frac{m-n+2}{m-n+1} \right) \Delta_k \Gamma_N \),
dominates in the limit and is $O\left( \frac{1}{m} \right)$. Again, the coefficient of $p_n$ in the series summand is $O\left( \frac{1}{m} \right)$, and the series is convergent for $m > 1$. The series term is then $O\left( \frac{1}{m^2} \right)$, and we have $O\left( \frac{1}{m^2} \right)$.

If $j > 2$, the term $\frac{m+1}{2^n m^2}$ dominates for $m$ that dominated for $j = 2$ becomes $O\left( \frac{1}{m^2} \right)$, which is subdominant for $j > 2$. Also, for $j > 2$, the contribution of the series term will remain $O\left( \frac{1}{m^2} \right)$.

In summary, we find that the Cesàro sum $(C, j)$ of the divergent series $F(N, m)$, obtained by setting $r = a$, exists for $j \geq 2$. The error in replacing $(C, j)$, $j \geq 2$, by its $(j-1)$-term approximation, $C_{j-1}$, is given by $V.167$; the five terms contributing to the error, listed in $V.168$, are, respectively,

(a) $O\left( \frac{1}{m^2} \right)$

(b) $O\left( \frac{1}{m^3} \right)$

(c) $O\left( \frac{1}{m^4} \right)$

(d) $O\left( \frac{1}{m^5} \right)$

(e) $O\left( \frac{1}{m^6} \right)$

For $j = 2$, the terms given by (d), (c), and (e) in $V.168$ dominate as $m \to \infty$ and the error ultimately decreases as $O\left( \frac{1}{m^3} \right)$. Taking $j > 2$ only reduces the error ($m \to \infty$) of the terms in (b) and (e). Modest improvement in the error results if we can eliminate the term in $V.168b$. This is possible if only those $0$ and (integer) $M$ values are used for which $P_N(\cos \theta) = 0$. Note that, though the series diverges for $j = 1$, the error for finite $N$ can be small and can even decrease with increasing $N (M < N)$ since the actual series sum is $O(N^{3/2}) \to m = m)$ and the error is $O(N^{3/2}/N)$.}

---

V.E. THE SHANKS TRANSFORM

Let the sum of a convergent series be $S$ and the $n$th partial sum be given by $A_n$. A transform $e_1(A_n)$ is defined by:

$$e_1(A_n) = \frac{A_{n+1} A_{n+1} - A_{n+1}^2}{A_{n+1} A_{n} - 2A_{n+1}}$$

which generates a new sequence for successive values of $n$. If, for all $n$, $A_n = S \pm o(n)$, where $\alpha$ and $t$ are constants independent of $n$ and $|t| < 1$, then $e_1(A_n) = S$ holds for all $n$. When $A_n = S \pm o(n)$ fails to hold exactly, the sequence $\{ e_1(A_n) \}$ may still converge to $S$ more rapidly than the original sequence of partial sums. This is a nonlinear transform: i.e., although $e_1(cA_n) = c e_1(A_n)$, note that $e_1(A_n + B_n) \neq e_1(A_n) + e_1(B_n)$.

This transform is often called Shanks' or Aitken's transformation. Aitken (1926) used it to accelerate Bernoulli iterates in the solution of algebraic equations, though it is now also known to have been used in the earlier part of the 19th century by E.E. Kummer and C.G.J. Jacobi (Wimp 1981). This transform was generalized by Schmidt (1941) to solve linear equations by iteration. His methods were popularized — and their significance to series summation analyzed — by Shanks (1955), whose paper contains a more complete bibliography.

We now consider the generalized formulation; again, let $S$ denote the sum of a convergent series with partial sums, $A_n$.

If, for all $n$,

$$A_n = S + \frac{1}{\alpha} [a_{n+1} a_{n+1} - a_{n+1}^2]$$

$$+ \frac{1}{\alpha^2} [a_{n+1} a_{n+1} - a_{n+1}^2] + \cdots$$

$$+ \frac{1}{\alpha^N} [a_{n+1} a_{n+1} - a_{n+1}^2] + \alpha N a_{n+1} a_{n+1} - a_{n+1}^2] + \cdots$$

we may define $e_i(A_n)$ as:

$$e_i(A_n) = \frac{A_{n+1} A_{n+1} - A_{n+1}^2}{A_{n+1} A_{n} - 2A_{n+1}}$$

we may define $e_i(A_n)$ as:
where, for \( j = 1, 2, \ldots, r \), the \( t_j \) are distinct, each \( t_j \) being possibly complex but
\( t_j \neq 0 \) or 1, each \( m_j \) is a nonnegative integer, and \( a_j = 0 \), then an exact
solution can be written which requires only \( (2k+1) \) series terms, \( k = r + m + \ldots + m_r \).
\[
\omega_n(m, A_n) = \begin{bmatrix}
\omega_0 & \omega_1 & \ldots & \omega_{m-1} \\
\omega_1 & \omega_0 & \ldots & \omega_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{m-1} & \omega_{m-2} & \ldots & \omega_0
\end{bmatrix}
\]
the solution can be written
\[
S = \omega_n(m, A_n) = \begin{bmatrix}
\omega_0 & \omega_1 & \ldots & \omega_{m-1} \\
\omega_1 & \omega_0 & \ldots & \omega_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{m-1} & \omega_{m-2} & \ldots & \omega_0
\end{bmatrix}
\]
(W.V.182)
Schmidt (1941) obtained this solution for \( m_j = 0 \), \( j = 1, 2, \ldots, r \). When the series is
not known to have partial sums of a form that permits an exact solution, the sequence
\( e_n(A_n) \), with \( e_n(A_n) \) defined as above, can still be generated provided the
denominator is not zero. An equivalent (but more practical) algorithm for \( e_n(A_n) \)
was introduced by Wynn (1956). Let
\[
e_j,^{(\omega)} = e_j,^{(\omega-1)} + [e_j,^{(\omega-1)} - e_j,^{(\omega-2)}], \quad \omega j > \omega,
\]
(W.V.184)
with \( e_j,^{(0)} = 0 \) and \( e_j,^{(\omega)} = A_n \). Then
\[
e_n(A_n) = e_n,^{(\omega)} \quad \omega, \omega j > \omega
\]
(W.V.185)
The generalized transforms \( e_n(A_n) \), can be used to generate various sequences
that may converge more rapidly than the original series. We can consider the
convergence of the sequence formed for a fixed \( k \) and increasing values of \( n \) or,
alternatively, the sequence for fixed \( n \) and increasing values of \( k \). Yet another group
of sequences include those formed by the \( k \)th transform, \( e_k(A_n) \), applied \( m \) times for
increasing values of \( n \). Transforms of this latter type we denote as
\[
e_{e_k}(A_n) = e \left[ e_{e_k}(A_{n+1}) \right].
\]
(W.V.186)
Wynn (1957: pp 126-127) demonstrates that, if \( A_j \) has an asymptotic expansion
\[
A_j \sim \sum_{p=0}^{\infty} \frac{\kappa_p}{j^{p+1}} \quad \kappa_p \neq 0, \quad j \to \infty
\]
then, for \( t \neq 1, q \neq 0, 1, \ldots, k-1, \)
\[
S = \sum_{p=0}^{\infty} \kappa_p \left[ \frac{\left( \sum_{q=0}^{k-1} \begin{pmatrix} k-1 \\ q \end{pmatrix} \right)^p}{(q+1)^{2q}} \right] \left[ 1 + \cdots (j) \right].
\]
(W.V.187)
This result also holds if \( j \) is replaced on the right hand side of V.187 and V.188 by
\( N + j \). Inspection of V.73 indicates that \( F_{2}(N+1,m) \) can be written in the form
\[
F_{2}(N+1,m) \sim \left( \frac{q^2}{2} \right)^{N} \left[ 1 - 2 \left( \frac{q^2}{2} \right) \omega(N) \right] \left( \frac{q}{2} \right)^{N} \left[ 1 - \frac{q^2}{2} \right] \left[ 1 + \cdots (j) \right]
\]
(W.V.188)
where all subscriptsed Greek letters are parameters dependent on \( \theta \) and, except for
\( \delta_{\omega} \) on \( L \). Then, defining
\[
A_j = \sum_{N=0}^{\infty} (2N+1) \omega(N) \left[ \frac{q^2}{2} \right]^{N} \left( \frac{q}{2} \right)^{N} \left[ 1 + \cdots (j) \right]
\]
(W.V.190)
\[
S = \sum_{N=0}^{\infty} (2N+1) \omega(N) \left[ \frac{q^2}{2} \right]^{N} \left( \frac{q}{2} \right)^{N} \left[ 1 + \cdots (j) \right]
\]
(W.V.191)
for $z_n$ given in V.13a, we have

$$A_j = \mathbb{S} - \sum_{n=0}^{\infty} (2n+1) z_n P_n(y),$$

$$\mathbb{S} = F \left( w_j y, \infty \right),$$

$$\mathbb{S} = F_n \left( w_j y, \infty \right),$$

$$\mathbb{S} = \left( \frac{\beta}{\pi} \right)^{\frac{1}{2}} \left( w_j \right)^{\frac{1}{2}} \left[ 1 + 2 \left( \frac{\beta}{\pi} \right) \cos \left( \theta_j \right) \left( \frac{\beta}{\pi} \right)^{\frac{1}{2}} \right],$$

$$\times \left\{ \left( w_j \right)^{\frac{1}{2}} \left[ 1 - \frac{1}{(w_j)^{\frac{1}{2}}} \right], \ldots \right\}$$

$$= \left[ \left( w_j \right)^{\frac{1}{2}} - \frac{1}{(w_j)^{\frac{1}{2}}} \right] \left[ \sum_{n=0}^{\infty} \frac{1}{(w_j)^{\frac{1}{2}}} + \ldots \right]$$

(V.192)

for $(a/r \exp (i \theta)) \neq 1$.

Because V.191 involves two different exponential factors, equations V.187 and V.188 do not immediately exhibit the power of $e_k(A_j)$ when applied directly to the zonal harmonic series (recall that $e_k$ is nonlinear). We can, however, infer from these relations, that the Shanks transformations, $e_k(A_j)$ will be quite effective in evaluating the series for increasing $k$, if the asymptotic expansion of the Legendre polynomial is substituted and the transform applied to two separate series, each having a single exponential dependence. This is, of course, an unnecessarily complicated approach if there are Shanks transformations directly applicable to the original series.

We can show that $e_k(A_j)$ is effective in improving the convergence of the original series. From V.191 and V.192, we have

$$A_j = \mathbb{S} + \eta_j,$$

(V.193)

where $\eta_j$ has the form

$$\eta_j = \left( \eta_j \right)^{\frac{1}{2}} \left[ \eta_j \exp (i \theta_j) + \rho \exp (-i \theta_j) \right],$$

(V.194)

with

$$\eta_j = \left( \eta_j \right)^{\frac{1}{2}} \left[ \eta_j + \phi \eta_j \right],$$

(V.195)

$$\rho_j = \left( \eta_j \right)^{\frac{1}{2}} \left[ \rho_j + \phi \eta_j \right].$$

Defining,

$$j - \frac{\phi}{2} \exp (i \theta),$$

(V.196)

$$r = \left( \frac{\beta}{\pi} \right) \exp (i \theta),$$

(V.197)

we can write

$$\eta_j = \left( \beta_j + \rho_j \right).$$

(V.198)

Then, from V.183,

$$\mathbb{S} + \frac{\mathbb{S} \left( \eta_j, \eta_j \right)}{w_k (1, \eta_j)} \approx \mathbb{S} + \frac{\mathbb{S} \left( \rho_j, \eta_j \right)}{w_k (1, \eta_j)}.$$  

(V.200)

Operations on row and column elements demonstrate that

$$w_k (1, \eta_j) = w_k (1, \eta_j),$$

(V.201)

$$w_k \left( \rho_j, \eta_j \right) = w_k \left( \eta_j, \eta_j \right),$$

(V.202)

and hence,

$$\eta_j = \left( \beta_j + \rho_j \right).$$

(V.203)

We let

$$\delta_j = \rho_j,$$

(V.204)

and note that $\beta_j$, $\delta_j$, and $\delta^m j$ (m a positive integer) are all $O(N+j)^q$, while $\delta^m j$ is $O(N+j)^m$. If we define

$$\alpha_j = (\delta^m j) (\delta^m j) - (\delta^m j)^z,$$

$$\beta_j = (\delta^m j) (\delta^m j) - (\delta^m j)^z,$$

$$\gamma_j = (\delta^m j) (\delta^m j) - (\delta^m j)^z,$$

(V.205)

$$\delta_j = \delta_j (\delta^m j) - (\delta^m j)^z.$$
and evaluate the determinant for $W_2$ in V.203, we obtain

$$
\begin{align*}
\omega_2 \left[ (s_j - s), (l_j - l) \right] &= \left[ s_j A_j + (\Delta^2 s)_j \right] \\
&\quad + \left[ l_j A_j + (\Delta^2 l)_j \right]
\end{align*}$$

(V.206)

Expanding $\Delta^2 a_j = \Delta [r^2 p_j]$ as in V.23 and substituting in V.205, we get

$$
\begin{align*}
A_j &= r^5 e^4 (r - 1)^4 \left[ p_j A_j - \Delta p_j \right] + O \left( \omega_j^{(1,1)} \right) \\
B_j &= -2 r^6 e^4 (r - 1)^5 \left[ p_j A_j - \Delta p_j \right] + O \left( \omega_j^{(2,1)} \right) \\
C_j &= r^3 r^4 (r - 1)^4 \left[ p_j A_j - \Delta p_j \right] + O \left( \omega_j^{(1,1)} \right) \\
D_j &= O \left( \omega_j^{(1,1)} \right)
\end{align*}
$$

and

$$
\begin{align*}
\omega_2 \left[ (s_j - s), (l_j - l) \right] &= \left[ s_j A_j + (\Delta^2 s)_j \right] \\
&\quad + \left[ l_j A_j + (\Delta^2 l)_j \right]
\end{align*}$$

(V.207)

which, when substituted in V.206, yields

$$
\begin{align*}
\omega_2 \left[ (s_j + p r^2), (l_j + p r^2) \right] &= r^5 e^4 (r - 1)^4 \left[ p_j A_j - \Delta p_j \right] b_j \\
&\quad + r^6 (r - 1)^5 \left[ l_j A_j - \Delta l_j \right] b_j \\
&\quad + O \left( \omega_j^{(1,1)} \right).
\end{align*}

(V.208)

The terms that remain are $O(N+1)^{3q-2}$ as can be determined from V.195 and V.196.

$W_2(A^2 s_j, A^2 l_j)$ can be similarly evaluated and we find

$$
\begin{align*}
\omega_2 \left[ (s_j, l_j) \right] &= r^5 e^4 (r - 1)^4 \left[ s_j l_j + \Delta s_j l_j \right] b_j \\
&\quad + O \left( \omega_j^{(1,1)} \right).
\end{align*}

(V.209)
The cyclic nature of the Legendre polynomials allows the convergence of the zonal harmonic series to be accelerated by the selective grouping of terms and the repeated-averaging of their successive partial sums (a process equivalent to Euler's transformation, equation V.48 with \( q = -1 \), (Dahlquist and Björck 1974)). Indeed, in previous direct summation efforts (see section IV.2.E), this has been the principal acceleration technique (sometimes in conjunction with Kummer's transformation).

Rayleigh (1904) was possibly the first to use it in this connection, evaluating the two-media acoustic wavefunctions for a source and receiver at the surface of a sphere. Love (1915) applied the technique in determining the magnetic field, \( H_x \), at the earth's surface for the two-media problem with a radial Hertzian source also at the earth's surface. He chose five points: \( \theta \approx 30^\circ, \pi/20, \pi/15, \pi/12, \) and \( \pi/10 \).

Since the Legendre function satisfies
\[
\mathcal{P}_n^k(\cos \theta) = \left[ \frac{\sin \phi}{\sin \phi_{n}} \right]^k \sin \left[ (n+1)\phi - \frac{\pi}{2} \right] + \mathcal{O} \left( \frac{1}{n^4} \right), \tag{V.211}
\]
choices of \( \theta = \pi / m \), assured Love of \( m \) terms between each half-oscillation of the Legendre polynomial, and, for large \( n \), the polynomial would come close to repeating itself with opposite sign over consecutive half-oscillations. Furthermore, the coefficient in the summand that multiplies \( P_n(\cos \theta) \) is a decreasing function of \( n \) for \( n \) somewhat beyond \( k \alpha \) (note that the summand of the \( H_x \) decreases by an additional factor of \( O(1/n^2) \) relative to \( E_r \) in V.1.26). Thus, Love was able to argue that sums contributed by such half-oscillations "have alternate signs and diminish continually in absolute magnitude". Interest in summing the zonal harmonic series then lapsed until the 1960's (see section IV.1.C).

Repeated averaging, coupled with Kummer's transform, has since been used by J.R. Johler and R.I. Lewis; they subtract the leading asymptotic term (section V.C), and then average successive partial sums, truncating each sum near a minimum of the Legendre function "so that the difference \( \Delta n \) between the successive truncation points is given by \( 2\pi / \theta^* \)" (Johler and Lewis 1969, Lewis and Johler 1976). In general, for three-place accuracy, they report that 10(\( k\alpha \)) to 15(\( k\alpha \)) terms are needed with even more terms required in the vicinity of the source.

Herein, we shall investigate how best to group terms and the general applicability of this approach to improving the convergence of the series defined in V.12; that is, for
\[
E_{\text{L}}(\omega, \alpha) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \langle 2n+1 \rangle \mathcal{P}_n(\cos \theta) \mathcal{P}_n(\gamma), \tag{V.212}
\]
where \( f_n \) is defined in V.13b. Note that \( f_n \) has an asymptotic expansion given in V.30b, with \( \delta(\mathcal{P}_n) = O(f) \) and \( \delta^2(\mathcal{P}_n) = O(f/n) \)

If we define
\[
U_{j} = (-1)^j \sum_{n=0}^{\infty} \mathcal{P}_n(\cos \theta) \mathcal{P}_n(\gamma), \tag{V.213}
\]
for all nonnegative integer \( A_j \), with \( A_{-1} = 1 \) and \( A_{A_j} > A_j \) then
\[
F(u, \omega, \alpha) = \sum_{j=0}^{\infty} (-1)^j U_j, \tag{V.214}
\]
and Euler's transformation (V.48 with \( q = -1 \)) gives us
\[
F(u, \omega, \alpha) + \frac{1}{2} \sum_{n=0}^{\infty} (-\frac{1}{2})^n \Delta U_n. \tag{V.215}
\]

We can also obtain \( M \) terms of the series in V.215 by averaging the partial sums:
\[
0, U_0, U_0 + U_0, \ldots, \sum_{m=0}^{M-1} U_m, \ldots, M \text{ times}, \tag{Dahlquist and Björck 1974).
\]
we first consider two special cases, \( \theta = \pi \) and \( \theta = \pi/2 \), and compare a two-term Euler approximation with a single term of the new transformation in V.84 (obtained by summation by parts). When \( \theta = \pi \), \( P_n(\cos \pi) = P_{n-1}(\pi) = (-1)^n \).

Then

\[
F(\omega_1, \omega_1) = \frac{(-1)^{n+1}}{n} \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + O \left( \frac{F_n}{n} \right) \tag{V.216}
\]

and, from V.84, we also have

\[
F(\omega_1, \omega_1) = \frac{(-1)^{n+1}}{n} \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + O \left( \frac{F_n}{n} \right) \tag{V.217}
\]

where the bracketed terms in the above expressions are equivalent to within the order of the neglected terms.

In the case of \( \theta = \pi/2 \), \( P_n(\cos \pi/2) = P_n(0) = 0 \) if \( n \) is odd; otherwise, we can write \( P_{n+1}(0) = -\left( \frac{2+n}{2+n} \right) P_n(0) \) by the Legendre recursion relation, and, assuming \( N \) is odd,

\[
F(\omega_1, \omega_1) = P_n(0) \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + \cdots \tag{V.218}
\]

However, retaining only two terms of the Euler transformation,

\[
F(\omega_1, \omega_1) = \frac{(-1)^{n+1}}{n} \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + O \left( \frac{F_n}{n} \right) \tag{V.219}
\]

whereas, from V.84, we have

\[
F(\omega_1, \omega_1) = P_{n+1}(0) \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + O \left( \frac{F_n}{n} \right) \tag{V.220}
\]

Again, the two estimates agree to within the order of the neglected terms.

For these two angles, we (implicitly) grouped terms over half cycles of the Legendre polynomial. In the case of \( \theta = \pi \), we can consider the effect of averaging over a complete cycle, the general approach Jolley and Lewis (1969) indicate that they use. By repeated averaging, the leading term is given by

\[
\frac{(-1)^{n+1}}{n} \left[ (2\omega_1) \Delta F_{n+1} - (\omega_1) \Delta F_{n+2} \right] + \cdots \tag{V.221}
\]

This result is indeed smaller in value than that found by repeated averaging over half-cycles (V.216), but recall that our problem is not to minimize \( F(N+1,m) \), which is
fixed for a given value of $N$. In fact, if we include another term in the estimate for repeated averaging over complete cycles, we have

$$
\frac{(-1)^{j+1}}{K} \left[ \frac{2(2n^2 - 2n + 1)}{n} \frac{\Delta P_{n+1} - \Delta P_{n+1}}{2} + \frac{2(2n^2 - 2n + 1)}{n} \frac{\Delta P_{n+1} - \Delta P_{n+1}}{2} \right].
$$

(V.222)

The "correction" term can then be rewritten

$$
\frac{(-1)^{j+1}}{K} \left[ \frac{2(2n^2 - 2n + 1)}{n} \frac{\Delta P_{n+1} - \Delta P_{n+1}}{2} + \frac{2(2n^2 - 2n + 1)}{n} \frac{\Delta P_{n+1} - \Delta P_{n+1}}{2} \right].
$$

(V.223)

which is of the same order in $N$ as the first term. Though some acceleration of convergence has been achieved [the magnitude of the terms in the new series decreases as $\frac{f}{n^2} P_n^{(-1)}$ rather than $P_n^{(-1)}$], the improvement is much less than that obtained by averaging over half-cycles for this particular angle.

We now consider more general values of $\theta$. Supposing the argument of the Legendre polynomial, the finite sum rule in V.81,

$$
\int_{\sin^{-1} \frac{x}{n}}^{\sin^{-1} \frac{y}{n}} \frac{d\theta}{\sin \theta} \int_{\sin^{-1} \frac{x}{n}}^{\sin^{-1} \frac{y}{n}} = \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right] \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right]
$$

(V.224)

allows us to evaluate the leading term (in $N$) of $U_j$. Specifically, we obtain the useful result

$$
U_j = \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right] \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right]
$$

(V.225)

where

$$
R_n = \frac{1}{K} \sum_{m=1}^{n} (-1)^{m+1} \Delta P_{n+m},
$$

(V.226)

In the case $A_j = A_{j+1} + 1$, when only one point is in the $j$th group, V.225 holds exactly.

Substituting $[2\sin^{-1} \frac{x}{n}]$ for $(1 - y)$ in V.225, we find

$$
U_j = \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right] \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right]
$$

(V.227)

But, using V.211,

$$
\Delta P \left[ \sin \left( \frac{\pi}{n} \right) \right] = \psi \left[ \sin \left( \frac{\pi}{n} \right) \right] - \psi \left[ \sin \left( \frac{\pi}{n} \right) \right]
$$

(V.228)

so, the leading asymptotic factors, given in V.226 and V.227, become

$$
U_j \approx \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right] \left( \frac{1}{x^2 - 1} \right) \left[ (\sin^{-1} \frac{x}{n})^2 - (\sin^{-1} \frac{y}{n})^2 \right]
$$

(V.229)

$$
\Delta P \left[ \sin \left( \frac{\pi}{n} \right) \right] = \psi \left[ \sin \left( \frac{\pi}{n} \right) \right] - \psi \left[ \sin \left( \frac{\pi}{n} \right) \right]
$$

(V.230)

If we substitute V.229 and V.230 into V.215, we have

$$
\frac{1}{n} \sum_{m=1}^{n} (-1)^{m+1} \Delta P_{n+m},
$$

(V.231)

where

$$
R_n = \frac{1}{K} \sum_{m=1}^{n} (-1)^{m+1} \Delta P_{n+m},
$$

(V.232)
This series involves a term (and differences of terms) for which the leading asymptotic approximation is given by V.230. Note that we have yet to specify the values of $A_j$.

As in our analysis of the $\theta = \pi$ and $\theta = \pi/2$ cases, we can obtain the actual leading term in $N$ from the transform in V.84; substituting $[2m - \left(\frac{\ell}{2}\right)]$ for $[1-y]$ in V.84, we find

\[ F(N + 1, m) = \frac{(N + 1) \varphi(2m - \frac{\ell}{2})}{Z_{2m}} \]

\[ \sim (\frac{N + 1}{2m}) \left[ \frac{2}{\varphi(\ell)} \right]^{1/2} \left[ (2m - \frac{\ell}{2}) \right]^{1/2} \]

(V.233)

Observe that, if $A_0 = (2p+1)\pi$ (p integer), then the first term of V.230 agrees with V.232, but the contribution of $A_N$ will be of the same order unless $A_j = A_{j-1} + 2\pi m$, $m$ an integer. If $\theta$ has the form,

\[ \theta = \left(\frac{2\pi n + \pi}{m}\right) \]

(V.234)

we can satisfy both of the above conditions by setting

\[ A_j = \left(\frac{2\pi n + \pi}{m}\right) \]

(V.235)

Thus, for $\theta = \pi$, $p = 0$, $m = 1$, and V.234 indicates we should form partial sums after every term (as we did earlier). For $\theta = \pi/2$, $p = 0$, $m = 2$, and V.234 indicates we should form partial sums after every two terms (again, as previously). More generally, if $\theta = (2p + 1)\pi/m$ and $p \neq 0$, this analysis suggests convergence will be accelerated more rapidly by forming partial sums after every $m$ terms rather than after every cycle or half-cycle of the Legendre polynomial.

V.G. NUMERICAL RESULTS

Using the techniques in the proceeding sections of Chapter V, we evaluate $F(1,m)$ and $F(N+1,m)$ defined in V.12 and V.15 for $r = a$. To illustrate the associated rates of convergence, we consider each case for $ka = 1000$, with $\theta = 20^\circ$ and $\theta = 17^\circ$. The Shanks transformations and the transformation of equation V.84, are also considered when $ka = 10,000$.

In Tables V.1 - V.4, $F(N+1,m)$ and $F(1,m)$ are calculated using the transform of V.84 (obtained using summation by parts). The Nth partial sum is given (first number in last column), and corresponds to truncation of the original series after $N$ terms. The column headed $F(N+1,m)$ gives the $M$-term estimate $(M = 1, 2, 3, 4)$ of $F(N+1,m)$ using V.84. The adjacent figure in the column headed $F(1,m)$ provides the estimate of $F(1,m)$ obtained by adding the $-m$-term approximation of $F(N+1,m)$ to the finite sum $F(1,N)$.

In Tables V.5 and V.6, we use Kummer's transformation as formulated in V.144 to calculate $F(1,m)$. The asymptotic factors defined in V.144 to V.143 (i.e., $a_n(1)$, $a_n(2)$, and $a_n(3)$) are (separately) substituted into V.144. The first column of the tables indicates the number of terms summed, while the columns designated $(1)$, $(2)$, and $(3)$ give estimates of $F(1,m)$ resulting from the use of $a_n(1)$, $a_n(2)$, and $a_n(3)$, respectively.

In Tables V.7 and V.8, we use the Cesaro sums $C_j(N)$ (j = 1, 2, 3, and 4), given by V.158, to calculate $F(N+1,m)$. Here, we have fixed the value of $N$ and incremented $M$, and the results can be compared with those given in Table V.1 and V.2 (obtained using the transform in equation V.84) for $N = 1080$.

In Tables V.9 - V.16 we present results of the calculation of $F(N+1,m)$ using two Shanks' transforms designated A and B. In case A (Tables V.9 - V.12) we apply the transform $e_{2m}$ to the partial sums, $A_n$, for increasing $m$ -- in particular, for
In case B (Tables V.12 - V.15) we repeatedly apply $e_j$ (with $m$ applications designated $(e_j)^m$). In each table the results are given for the same three values of $N$ used in the corresponding calculation using the transform in V.84 (Tables V.1 - V.4).

In Tables V.17 - V.20 we use Kummer's transformation — specifically, we substitute $a_n(l)$ given by V.130 into V.144 — and calculate $F(1,\infty)$ using repeated averaging (V.215) over half-cycles and full-cycles of the Legendre polynomial. Tables 17 and 19 give the results for $\Theta = 20^\circ$ and $77^\circ$, respectively, when the averaging takes place over half-cycles. Tables 18 and 20 give the corresponding results when full cycles are used. Table 21 presents results for the case of $\Theta = 77^\circ$ when groups of 180 terms are taken (see equation V.234 et seq).

### Table V.1

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<thead>
<tr>
<th>$N$</th>
<th>Finite sum: $F(N+1,\infty)$</th>
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<tbody>
<tr>
<td>1020</td>
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</tr>
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### Table V.3

Summation by parts.  \( ka = 10000.0 \)  \( \Theta = 77.0 \)

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### Table V.5

Summation by parts.  \( ka = 10000.0 \)  \( \Theta = 20.0 \)

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### Table V.6

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 20.0 \)

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Table V.4

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 77.0 \)

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<td>2500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>4000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
</tbody>
</table>

Table V.7

### Table V.5

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 77.0 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( (1) )</th>
<th>( (2) )</th>
<th>( (3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>1.35402D+04</td>
<td>2.74200D+03</td>
<td>7.20250D+02</td>
</tr>
<tr>
<td>2500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>4000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
</tbody>
</table>

Table V.8

### Table V.6

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 20.0 \)

<table>
<thead>
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</thead>
<tbody>
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<td>10000</td>
<td>4.62203D+04</td>
<td>7.0874D+04</td>
<td>7.50958D+04</td>
</tr>
<tr>
<td>10500</td>
<td>9.6881D+04</td>
<td>7.62067D+04</td>
<td>7.62138D+04</td>
</tr>
<tr>
<td>11000</td>
<td>5.48689D+04</td>
<td>7.61577D+04</td>
<td>7.62130D+04</td>
</tr>
<tr>
<td>11500</td>
<td>5.48689D+04</td>
<td>7.61577D+04</td>
<td>7.62130D+04</td>
</tr>
<tr>
<td>12000</td>
<td>8.63292D+04</td>
<td>7.62235D+04</td>
<td>7.62131D+04</td>
</tr>
<tr>
<td>12500</td>
<td>9.74414D+04</td>
<td>7.62287D+04</td>
<td>7.62131D+04</td>
</tr>
</tbody>
</table>

Table V.9

### Table V.7

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 77.0 \)

<table>
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<tr>
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<th>( (2) )</th>
<th>( (3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>1.35402D+04</td>
<td>2.74200D+03</td>
<td>7.20250D+02</td>
</tr>
<tr>
<td>2500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>4000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
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</table>

Table V.10

### Table V.8

Kummer Transform:  \( ka = 10000.0 \)  \( \Theta = 77.0 \)

<table>
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<th>( (2) )</th>
<th>( (3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>1.35402D+04</td>
<td>2.74200D+03</td>
<td>7.20250D+02</td>
</tr>
<tr>
<td>2500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>3500</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
<tr>
<td>4000</td>
<td>6.23122D+04</td>
<td>2.74200D+03</td>
<td>7.50958D+02</td>
</tr>
</tbody>
</table>
### Table V.7

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<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>-3.21035D+05</td>
<td>-3.23480D+05</td>
<td>-3.24501D+05</td>
<td>-3.25517D+05</td>
</tr>
<tr>
<td>1200</td>
<td>-3.22811D+05</td>
<td>-3.21788D+05</td>
<td>-3.21957D+05</td>
<td>-3.22127D+05</td>
</tr>
<tr>
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<td>-3.24259D+05</td>
<td>-3.21527D+05</td>
<td>-3.21725D+05</td>
<td>-3.21817D+05</td>
</tr>
<tr>
<td>3200</td>
<td>-3.17126D+05</td>
<td>-3.21557D+05</td>
<td>-3.21758D+05</td>
<td>-3.21839D+05</td>
</tr>
<tr>
<td>4200</td>
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<td>-3.21512D+05</td>
<td>-3.21545D+05</td>
<td>-3.21578D+05</td>
</tr>
<tr>
<td>5200</td>
<td>-3.15290D+05</td>
<td>-3.21491D+05</td>
<td>-3.21513D+05</td>
<td>-3.21535D+05</td>
</tr>
<tr>
<td>6200</td>
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<td>-3.21497D+05</td>
<td>-3.21516D+05</td>
<td>-3.21546D+05</td>
</tr>
<tr>
<td>7200</td>
<td>-3.16531D+05</td>
<td>-3.21491D+05</td>
<td>-3.21513D+05</td>
<td>-3.21535D+05</td>
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</table>

Table V.7

### Table V.9

<table>
<thead>
<tr>
<th>$A_n$</th>
<th>$e_2$</th>
<th>$e_4$</th>
<th>$e_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3229B+05</td>
<td>-7.48095B+05</td>
<td>-6.812B6D+05</td>
<td>-6.90749D+05</td>
</tr>
<tr>
<td>5.2574D+05</td>
<td>-7.36334D+05</td>
<td>-6.85360D+05</td>
<td>-6.90275D+05</td>
</tr>
<tr>
<td>5.64738D+05</td>
<td>-7.21884D+05</td>
<td>-6.89126D+05</td>
<td>-6.8963D+05</td>
</tr>
<tr>
<td>4.52107D+05</td>
<td>-7.07046D+05</td>
<td>-6.9200D+05</td>
<td>-6.89021D+05</td>
</tr>
</tbody>
</table>

Case A:

Shanks transformation,

$A = 1000.0, \Theta = 20.0, N = 1020, 1050, 1080^*$

$e_2 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$e_4 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$e_6 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$\text{Table V.9}$

### Table V.10

<table>
<thead>
<tr>
<th>$A_n$</th>
<th>$e_2$</th>
<th>$e_4$</th>
<th>$e_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8075D+05</td>
<td>2.26062D+05</td>
<td>2.26379D+05</td>
<td>2.26383D+05</td>
</tr>
<tr>
<td>4.2114D+05</td>
<td>2.26539D+05</td>
<td>2.26384D+05</td>
<td>2.26383D+05</td>
</tr>
<tr>
<td>3.54962D+05</td>
<td>2.26726D+05</td>
<td>2.26385D+05</td>
<td>2.26383D+05</td>
</tr>
<tr>
<td>9.67987D+04</td>
<td>2.26321D+05</td>
<td>2.26384D+05</td>
<td>2.26383D+05</td>
</tr>
</tbody>
</table>

Case A:

Shanks transformation,

$A = 1000.0, \Theta = 77.0, N = 1020, 1050, 1080^*$

$e_2 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$e_4 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$e_6 = 6.46847D+04, 6.46847D+04, 6.46847D+04$

$\text{Table V.10}$

$^*N = 1020$ was used to obtain the first group of results in Tables V.9 and V.10, $N = 1050$ to obtain the second group, and $N = 1080$ to obtain the third group.
Shanks transformation, Case A:

\[ k_a = 10000.0, \ \theta_a = 20.0 \]

\[ N = 10035, 10105, 10175 \]

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.48764D+07</td>
<td>4.80148D+07</td>
<td>4.90640D+07</td>
<td>4.90859D+07</td>
</tr>
<tr>
<td>7.01710D+07</td>
<td>4.75815D+07</td>
<td>4.90995D+07</td>
<td>4.90833D+07</td>
</tr>
<tr>
<td>2.81701D+08</td>
<td>4.73864D+07</td>
<td>4.91249D+07</td>
<td>4.90835D+07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.17119D+07</td>
<td>5.12503D+07</td>
<td>5.12421D+07</td>
<td>5.12418D+07</td>
</tr>
<tr>
<td>2.70527D+07</td>
<td>5.12213D+07</td>
<td>5.12423D+07</td>
<td>5.12418D+07</td>
</tr>
<tr>
<td>4.69320D+07</td>
<td>5.11961D+07</td>
<td>5.12424D+07</td>
<td>5.12418D+07</td>
</tr>
<tr>
<td>6.68300D+07</td>
<td>5.11774D+07</td>
<td>5.12424D+07</td>
<td>5.12418D+07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.32750D+06</td>
<td>4.23241D+06</td>
<td>4.23076D+06</td>
<td>4.23076D+06</td>
</tr>
<tr>
<td>2.60130D+06</td>
<td>4.23174D+06</td>
<td>4.23076D+06</td>
<td>4.23076D+06</td>
</tr>
<tr>
<td>1.12830D+07</td>
<td>4.23098D+06</td>
<td>4.23076D+06</td>
<td>4.23076D+06</td>
</tr>
<tr>
<td>2.36610D+07</td>
<td>4.23021D+06</td>
<td>4.23076D+06</td>
<td>4.23076D+06</td>
</tr>
</tbody>
</table>

Table V.11

Shanks transformation, Case B:

\[ k_a = 10000.0, \ \theta_a = 20.0 \]

\[ N = 10105, 10175 \]

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.32298D+05</td>
<td>7.48095D+05</td>
<td>6.75625D+05</td>
<td>6.87560D+05</td>
</tr>
<tr>
<td>5.25748D+05</td>
<td>7.36134D+05</td>
<td>6.82807D+05</td>
<td>6.92662D+05</td>
</tr>
<tr>
<td>5.64783D+05</td>
<td>7.21684D+05</td>
<td>6.91013D+05</td>
<td>6.91978D+05</td>
</tr>
<tr>
<td>4.52107D+05</td>
<td>7.07046D+05</td>
<td>6.95599D+05</td>
<td>6.91663D+05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.91267D+04</td>
<td>8.37325D+05</td>
<td>8.15272D+05</td>
<td>8.14750D+05</td>
</tr>
<tr>
<td>2.53211D+05</td>
<td>8.19414D+05</td>
<td>8.18555D+05</td>
<td>8.18479D+05</td>
</tr>
<tr>
<td>4.99990D+05</td>
<td>8.80254D+05</td>
<td>8.81496D+05</td>
<td>8.81472D+05</td>
</tr>
<tr>
<td>7.88820D+05</td>
<td>8.78848D+05</td>
<td>8.81436D+05</td>
<td>8.81462D+05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( e_2 )</th>
<th>( e_4 )</th>
<th>( e_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.48127D+05</td>
<td>-3.20481D+05</td>
<td>-3.21455D+05</td>
<td>-3.21455D+05</td>
</tr>
<tr>
<td>-5.03150D+05</td>
<td>-3.20093D+05</td>
<td>-3.21454D+05</td>
<td>-3.21455D+05</td>
</tr>
<tr>
<td>-7.34265D+05</td>
<td>-3.19887D+05</td>
<td>-3.21461D+05</td>
<td>-3.21455D+05</td>
</tr>
<tr>
<td>-9.13837D+05</td>
<td>-3.19876D+05</td>
<td>-3.21467D+05</td>
<td>-3.21448D+05</td>
</tr>
</tbody>
</table>

Table V.13

*N = 1020 was used to obtain the first group of results in Tables V.11 and V.12, N = 1050 to obtain the second group, and N = 1080 to obtain the third group.
Shanks transformation, Case B:

\[ \text{ka} = 10000.0, \ \Theta = 20.0 \]

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
3.487640+07 & 4.801480+07 & 4.894310+07 & 4.908790+07 \\
7.017100+07 & 4.758150+07 & 4.900140+07 & 4.911470+07 \\
1.017010+08 & 4.738840+07 & 4.911870+07 & 4.911470+07 \\
1.258720+08 & 4.742360+07 & 4.920070+07 & 4.911460+07 \\
\end{array}
\]

Table V.15

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
1.117190+07 & 5.125030+07 & 5.124250+07 & 5.124210+07 \\
2.705270+07 & 5.122130+07 & 5.124230+07 & 5.124210+07 \\
4.569230+07 & 5.119610+07 & 5.124150+07 & 5.124210+07 \\
6.482800+07 & 5.117740+07 & 5.124190+07 & 5.124200+07 \\
\end{array}
\]

Table V.15

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
-1.327500+06 & 4.232410+07 & 4.230760+07 & 4.230770+07 \\
2.602130+06 & 4.231740+07 & 4.230770+07 & 4.230770+07 \\
1.128830+07 & 4.230980+07 & 4.230770+07 & 4.230770+07 \\
2.366100+07 & 4.230210+07 & 4.230770+07 & 4.230770+07 \\
\end{array}
\]

Table V.15

Shanks transformation, Case B:

\[ \text{ka} = 10000.0, \ \Theta = 77.0 \]

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
1.930830+07 & 4.119270+06 & 4.109280+06 & 4.109280+06 \\
1.476520+07 & 4.123810+06 & 4.109250+06 & 4.109280+06 \\
-5.865190+06 & 4.106660+06 & 4.109280+06 & 4.109280+06 \\
-1.057300+07 & 4.695510+06 & 4.109280+06 & 4.109280+06 \\
\end{array}
\]

Table V.15

Table V.15

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
1.112070+07 & 3.656100+06 & 3.655780+06 & 3.655780+06 \\
1.061970+07 & 3.656440+06 & 3.655780+06 & 3.655780+06 \\
-6.218280+05 & 3.655770+06 & 3.655780+06 & 3.655780+06 \\
-5.195830+06 & 3.655150+06 & 3.655780+06 & 3.655780+06 \\
\end{array}
\]

Table V.15

\[
\begin{array}{cccc}
A_n & e_2 & (e_2)^2 & (e_2)^3 \\
8.979030+06 & 3.961650+06 & 3.961600+06 & 3.961600+06 \\
1.015330+07 & 3.961780+06 & 3.961600+06 & 3.961600+06 \\
1.746920+06 & 3.961620+06 & 3.961600+06 & 3.961600+06 \\
-3.190400+06 & 3.961430+06 & 3.961600+06 & 3.961600+06 \\
\end{array}
\]

Table V.15

\[ N = 10035 \] was used to obtain the first group of results in Tables V.15 and V.16, \[ N = 10105 \] to obtain the second group, and \[ N = 10175 \] to obtain the third group.
Repeated averaging: \( ka = 1000.0 \) \( \Theta = 77.0 \)

<table>
<thead>
<tr>
<th>( N = 1050 )</th>
<th>( F(l, \Theta) )</th>
<th>( N = 1050 )</th>
<th>( F(l, \Theta) )</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>8.98350D+04</td>
</tr>
<tr>
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</tr>
<tr>
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<td>1.11867D+04</td>
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<tr>
<td>3</td>
<td>-1.45019D+04</td>
<td>3</td>
<td>4.99351D+03</td>
</tr>
<tr>
<td>4</td>
<td>-1.98424D+04</td>
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<td>2.35834D+03</td>
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<td>5</td>
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<tr>
<td>6</td>
<td>-1.34961D+04</td>
<td>6</td>
<td>6.48219D+02</td>
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<tr>
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<td>-6.19816D+03</td>
<td>7</td>
<td>3.98726D+02</td>
</tr>
<tr>
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<td>7.25475D+02</td>
<td>8</td>
<td>2.81342D+02</td>
</tr>
<tr>
<td>9</td>
<td>5.68000D+03</td>
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<td>2.25637D+02</td>
</tr>
<tr>
<td>10</td>
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<tr>
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<td>9.20903D+01</td>
<td>14</td>
<td>1.75670D+02</td>
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<td>15</td>
<td>-1.92308D+03</td>
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<td>1.74967D+02</td>
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Table V.19

Repeated averaging: \( ka = 1000.0 \) \( \Theta = 77.0 \)

Averaging over half-cycles

<table>
<thead>
<tr>
<th>( N = 1050 )</th>
<th>( F(l, \Theta) )</th>
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<tbody>
<tr>
<td>0</td>
<td>8.98350D+04</td>
</tr>
<tr>
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<td>5.31032D+04</td>
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<td>2</td>
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<td>3</td>
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<td>7</td>
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Table V.20

Averaging over full-cycles

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<th>( F(l, \Theta) )</th>
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<td>2.81342D+02</td>
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<td>1.99029D+02</td>
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Table V.21

Averaging over groups of 180 terms
VI. CONCLUSIONS

The components of the electromagnetic field produced by a known source operating in a spherically-symmetric system can be determined from a Green's dyadic function, the elements of which are infinite Bessel-Legendre series (referred to here as zonal harmonic series). Three general procedures exist for evaluating these series solutions to problems associated with terrestrial electromagnetic propagation at frequencies below HF: mode theory, wave-hop, and summation of the zonal harmonic series. In Table VI.1 we specify those frequencies and ranges over which the techniques are applicable (or in common use), comment on this use, and indicate references (and sections of the present work) where the method in question is discussed.

For field evaluation above ELF, waveguide mode theory and wave-hop are used almost exclusively, the exceptions apparently limited to those authors indicated in Table VI.1. The reason most commonly given for neglecting direct summation of the zonal harmonic series as a computational technique is its extremely slow convergence. However, as indicated in Chapter IV, the alternative approaches are not without their difficulties, the CCIR reporting (1982) that neither mode theory nor wave-hop is entirely adequate above VLF at the longer ranges and indicating regret that no consistent engineering technique for field calculation at all frequencies below HF appears to exist.

Thus motivated, we considered the possibility of accelerating the zonal harmonic series, showing that the number of terms required to sum the two-media solution (or the three-media solution — even without the removal of the groundwave component) can generally be reduced by a factor of ten from that found by earlier investigators. Three acceleration methods proved to be particularly useful: a generalized Euler transformation applied to the series when the Legendre polynomials can be replaced with their asymptotic expansion (V.73); certain Shanks' transformations (V.180, V.186); and a new series transformation (V.84) derived using summation by parts. While the first two methods are relatively well-known, their application to this problem appears to be new.

The third method, developed herein, has a least-squares interpretation (given in V.111 et seq) and, by taking the limit as the spherical scattering radius tends to infinity, a plane-earth analogue of this transformation is obtained which is applicable to the numerical evaluation of Sommerfeld integrals along the real line.

We apply the transformation of V.84 in our investigation of Cesàro summation and two methods (Kummer's transformation and repeated averaging) used by earlier authors to improve convergence. Cesàro summation (V.185 et seq) can be used to force series convergence, but its use in convergence acceleration is here shown to be limited, and Kummer's transformation (V.127-V.128) is most useful when the field point is near the source (where other methods are less effective). It may be possible to improve the utility of this latter method in the case of a lossy scatterer — at least for field points far from the source — by using a surface impedance (a concept common to mode theory). Finally, though repeated averaging (V.214) can rapidly accelerate series convergence for selected angles between the source and the receiver, it does not possess the generality of the more useful methods.
Table V1.1. Comparison of Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Frequency*</th>
<th>Surface Range (polar angle)</th>
<th>Remarks</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Waveguide</td>
<td>ELF to</td>
<td>w 10° - 180°</td>
<td>Rapidly convergent except at upper VLF and higher frequencies; difficult iterative solution of mode equation required for each series term; provides physical insight when few modes important; error difficult to bound; additional media layers cripple approach</td>
<td>Wait (1961); Pappert et al (1968); Galejs (1972); Mortfitt, Shellman (1976); CCIR (1982); (Thesis: IV.2.C., IV.2.F.)</td>
</tr>
<tr>
<td>mode-theory</td>
<td>Lower LF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wave-hop</td>
<td>Upper VLF</td>
<td>1° - 50°</td>
<td>Range and frequency limited new routines less restrictive but rely on mode (or zonal harmonic) summation; provides physical insight when few hops important</td>
<td>Wait (1961); Berry, Chrisman (1965); Berry, Herman (1971); Alpert et al (1972); Lewis 1970; (Thesis: IV.2.D., IV.2.F.)</td>
</tr>
<tr>
<td></td>
<td>through MF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zonal harmonic</td>
<td>ELF</td>
<td>w 1° to 180°</td>
<td>10(ka) terms widely believed required with groundwave calculation slowing convergence; mode search not required; error can be estimated; particularly applicable for layered media; uniform approach over surface range and frequency</td>
<td>Johler, berry (1962); Johler, Lewis (1969); Johler (1970); Jones, Newthorpe (1982); (Thesis: IV.2.E., IV.2.F., Chapter V)</td>
</tr>
<tr>
<td>summation</td>
<td>VLF</td>
<td>w 1° to 180°</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>LF:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>MF, but not in common use</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* ELF: 3-3000 Hz; VLF: 3-30 kHz; LF: 30-300 kHz; MF: 300-3000 kHz
** Herein, we accelerate the groundwave calculation and show that, for source and receiver near the surface, only (ka) + 0(ka) 1/3 teams are required.

APPENDIX A1: THREE-MEDIA SERIES COEFFICIENTS

The boundary conditions IV.1.5 a, b and IV.1.6 a, b lead to the following two sets of equations for the unknown vector coefficients of the Green's dyadics IV.1.7-IV.1.9 associated with the three-media electromagnetic boundary value problem. We distinguish these two sets by their superscripts, TE and TM. The TE coefficients are determined from

\[
\begin{align*}
\mathbf{l}_{\text{TM}}^{(1)} &= \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(3)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(2)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(3)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(3)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(4)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(3)}.
\end{align*}
\]

Similarly, the TM coefficients are found from

\[
\begin{align*}
\mathbf{l}_{\text{TM}}^{(1)} &= \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(3)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(2)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(3)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(3)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(4)}, \\
\mathbf{l}_{\text{TM}}^{(4)} &= \mathbf{l}_{\text{TM}}^{(1)} + \mathbf{l}_{\text{TM}}^{(2)} + \mathbf{l}_{\text{TM}}^{(3)}.
\end{align*}
\]

Consider first A1.1 - A1.4 for the TE coefficients. Inspection of III.2.9 for \[\mathbf{l}_{\text{TM}}^{(1)}\] indicates that the radial components of A1.1 - A1.4 are satisfied by setting the radial components of the individual vector constants to zero. Furthermore, defining

\[
\mathbf{x}_{\text{TM}}^{(i)} = \left[ \frac{\delta}{\text{TM}} \right],
\]

A1.1

and setting

\[ A_n^T = \frac{1}{(kr)^n} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] \]

\[ \sum_{n=1}^{\infty} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] = 0 \]

\[ \sum_{n=1}^{\infty} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] = 0 \]

reduces A1.1 - A1.4 to a scalar problem involving four unknowns and four equations. In A1.5 - A1.8, the form of \( Q \) given by III.2.10 indicates that the unknown TM vectors can each be written in a form involving only four unknowns. Inspection of the resulting equations then indicates that the TM solutions can be derived from the TE solutions. We follow this approach below.

If we define

\[ u^T = \left[ \frac{1}{kr} \left( \frac{1}{kr} \right) \right] \]

(A1.11)

then we can write the TM vector coefficients in terms of scalar quantities:

\[ A_n^T = \frac{1}{(kr)^n} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] \]

\[ A_n^T = \frac{1}{(kr)^n} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] \]

(A1.12a)

\[ A_n^T = \frac{1}{(kr)^n} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] \]

\[ A_n^T = \frac{1}{(kr)^n} \left[ - (n+1) (a_{n-1}^T) r \left( a_{n-1}^T \right) r + (a_{n-1}^T) r \right] \]

where

\[ u^T = \left[ \frac{1}{kr} \left( \frac{1}{kr} \right) \right] \]

(A1.12b)

Equation A1.12b indicates that we need only solve for \( a_{n-1}^T \), \( b_{n-1}^T \), \( c_{n-1}^T \), and \( \phi_{n-1}^T \).

Substitution of A1.12a into A1.5 - A1.8 gives four equations for these four unknowns.

The TE equations for the scalar TE quantities reduce to these TM equations when the following substitutions are made:

\[ c_n^T \rightarrow \frac{c_n^T}{kr} (c_{n-1}^T) r \]

\[ u_n^T \rightarrow \frac{u_n^T}{kr} (u_{n-1}^T) r \]

\[ \phi_n^T \rightarrow \frac{\phi_n^T}{kr} (\phi_{n-1}^T) r \]

\[ \lambda_n^T \rightarrow \frac{\lambda_n^T}{kr} (\lambda_{n-1}^T) r \]

(A1.13)

It follows that the TM unknowns can then be determined by the same substitutions in the TE solutions. Thus, our problem is reduced to finding the TE scalar quantities in A1.10. Defining,

\[ S_{\text{TE}} = - \left[ \frac{\lambda_{n+1} (\lambda_{n+1})}{\lambda_{n+2} (\lambda_{n+2})} \right] \]

\[ S_{\text{TE}} = - \left[ \frac{\lambda_{n+1} (\lambda_{n+1})}{\lambda_{n+2} (\lambda_{n+2})} \right] \]

(A1.14)

\[ S_{\text{TE}} = - \left[ \frac{\lambda_{n+1} (\lambda_{n+1})}{\lambda_{n+2} (\lambda_{n+2})} \right] \]

\[ S_{\text{TE}} = - \left[ \frac{\lambda_{n+1} (\lambda_{n+1})}{\lambda_{n+2} (\lambda_{n+2})} \right] \]

(A1.15)

\[ S_{\text{TE}} = - \left[ \frac{\lambda_{n+1} (\lambda_{n+1})}{\lambda_{n+2} (\lambda_{n+2})} \right] \]

(A1.16)
\[ S_2^{\text{TM}} = \left( \frac{1}{\bar{S}_2^{\text{TH}}} \right) \left[ \begin{pmatrix} \beta_2 \bar{S}_2 к \frac{\bar{S}_2 - \lambda_1 \bar{S}_1}{\bar{S}_2 - \lambda_1 \bar{S}_1} \\ \bar{S}_2 - \lambda_1 \bar{S}_1 \end{pmatrix} \bar{S}_1 \right] \] (A1.17)

(where \( S_2^{\text{TM}} \) can be obtained from \( S_1^{\text{TH}} \) and \( S_2^{\text{TH}} \) using A1.13), we obtain from the scalar versions of A1.1-A1.4

\[ b_2^{\text{TM}} = \frac{1}{2} \left[ \frac{1}{S_2^{\text{TM}}} \sum_{i=1}^{n} \left( \frac{u_2^{\text{TM}}(c_i) + u_2^{\text{TM}}(d_i)}{S_2^{\text{TM}-(c_i,d_i)}} \right) \right] \]
\[ \leq \frac{1}{2} \left[ S_2^{\text{TM}} \sum_{i=1}^{n} \left( \frac{u_2^{\text{TM}}(c_i) + u_2^{\text{TM}}(d_i)}{S_2^{\text{TM}-(c_i,d_i)}} \right) \right] \]

Then, using A1.13, \( c_n^{\text{TM}} \) and \( b_n^{\text{TM}} \) are similarly defined with \( S_1^{\text{TM}} \) and \( S_2^{\text{TM}} \) replacing the IE quantities, and, by A1.12b,

\[ (c_n^{\text{TM}})' = \frac{1}{2} \left[ \left( S_1^{\text{TM}} \right)^{-1} \sum_{i=1}^{n} \left( \frac{u_1^{\text{TM}}(c_i) + u_1^{\text{TM}}(d_i)}{S_1^{\text{TM}-(c_i,d_i)}} \right) \right] \]
\[ \leq \frac{1}{2} \left[ S_1^{\text{TM}} \sum_{i=1}^{n} \left( \frac{u_1^{\text{TM}}(c_i) + u_1^{\text{TM}}(d_i)}{S_1^{\text{TM}-(c_i,d_i)}} \right) \right] \]

Consequently, employing the notation of sec. III.2.1, we have from A1.10 and A1.12,

\[ S_2^{\text{TM}} = \frac{1}{2} \left[ \frac{1}{S_2^{\text{TM}}} \sum_{i=1}^{n} \left( \frac{u_2^{\text{TM}}(c_i) + u_2^{\text{TM}}(d_i)}{S_2^{\text{TM}-(c_i,d_i)}} \right) \right] \]

(A1.18)
Starting with equation IV.1.21,

\[ \psi_{\mu}(r, \phi) = -\frac{e_0}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \left[ L_{\mu}^{(1)}(\mathbf{k} \cdot \mathbf{r}) L_{\nu}^{(2)}(\mathbf{k} \cdot \mathbf{r}) \right] \mathbf{P}[\cos \phi] \]  

(A2.1)

where \( c, \mu, \tilde{c}, \) and \( \tilde{\mu} \) are defined in IV.1.20, IV.1.22, IV.1.23, and IV.1.24, respectively, we can substitute for \( L_{\mu}^{(1)}(\mathbf{k} \cdot \mathbf{r}) \) and \( L_{\nu}^{(2)}(\mathbf{k} \cdot \mathbf{r}) \), and obtain (after some manipulation)

\[ \psi_{\mu}(r, \phi) = -\frac{e_0}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \left[ \frac{1}{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})} \left[ 1 - \frac{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})}{L_{\mu}^{(1)}(\mathbf{k} \cdot \mathbf{r})} \right] \right] \mathbf{P}[\cos \phi] \]  

(A2.2)

with

\[ \tilde{\mu} = -\frac{1}{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})} \left[ \frac{L_{\mu}^{(1)}(\mathbf{k} \cdot \mathbf{r}) - L_{\nu}^{(2)}(\mathbf{k} \cdot \mathbf{r})}{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})} \right] \]  

(A2.3)

\[ \tilde{\mu} = -\frac{1}{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})} \left[ \frac{L_{\mu}^{(1)}(\mathbf{k} \cdot \mathbf{r}) + L_{\nu}^{(2)}(\mathbf{k} \cdot \mathbf{r})}{L_{\tilde{\mu}}^{(2)}(\mathbf{k} \cdot \mathbf{r})} \right] \]  

(A2.4)

for nonpermeable media.

Equations A2.1 and A2.2 were obtained for an assumed \( e^{\text{out}} \) time-dependence, i.e., \( e^{\text{out}} = e^{-i\omega t} \). While this time-dependence has been used by Van der Pol, Bremmer, Fock, Stratton, Galejs, and Tai among others, a number of researchers including Watson, Wait, Alpert, and Johler have utilized \( e^{\text{out}} \). A solution obtained for one time-dependence is the complex-conjugate of the other solution. The identities \( L_{\mu}^{(1)}(x) = L_{\mu}^{(1)}(x^*) \), \( L_{\mu}^{(2)}(x) = L_{\mu}^{(2)}(x^*) \), and \( j_{\mu}(x) = j_{\mu}(x^*) \) allow A2.1 and A2.2 to be appropriately modified when they are compared with solutions using \( e^{\text{out}} \).

Given a Hertzian source, the traditional approach has been to obtain one or (if necessary) both of the scalar wavefunctions associated with transverse magnetic (TM) or transverse electric (TE) solutions. For a radial element in a homogeneous medium, the fields can be determined from the TM wavefunction, \( \psi_{\text{TM}}^M \),

\[ \hat{E} = \nabla \times \nabla \times (\psi_{\text{TM}}^M) \]

and

\[ \hat{H} = \frac{1}{\mu_0} \nabla \times (\psi_{\text{TM}}^M) \]

where we have assumed \( e^{\text{out}} \) time-dependence and (see sec. II.1.0)

\[ \frac{\psi_{\text{TM}}^M}{\mu_0} = \frac{\mu_0}{c} \tilde{\psi}_{\text{TM}}^M \]

Choosing the \( z \)-axis of a spherical reference frame to be collinear with the Hertzian element establishes azimuthal symmetry, and the fields are then given by

\[ \hat{E}_r = (\mathbf{k}^2 + \frac{\mathbf{k}^2}{\mu_0}) \tilde{\psi}_{\text{TM}}^M \]

(A2.5)

\[ \hat{E}_\theta = \mathbf{k} \hat{\mathbf{e}}_\theta \tilde{\psi}_{\text{TM}}^M \]

(A2.6)

\[ \hat{E}_\phi = \mathbf{k} \hat{\mathbf{e}}_\phi \tilde{\psi}_{\text{TM}}^M \]

(A2.7)

with \( \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_{\text{TM}} = 0 \).

(A2.8)

Let \( K = k \) in a region in which the fields are to be determined. We can then determine the wavefunction associated with the solution A2.1 [note \( P_n(\cos \theta) = P_n(\cos \theta) \)] by our choice of the \( z \)-axis] by using A.2.5 and the spherical Bessel equation,

\[ (\mathbf{k}^2 + \frac{\mathbf{k}^2}{\mu_0}) \tilde{\psi}_{\text{TM}}^M = \frac{\mathbf{k}^2}{\mu_0} \tilde{\psi}_{\text{TM}}^M \]

(A2.9)

In particular, we find \((\mathbf{k} = \mathbf{r} = 0)\)
and, in the limit of an unbounded medium (cf. Stratton 1941: p 414)

\[ \psi_{\text{TM}}^{\text{new}} = - c r_0 \frac{\cos k z}{\cos k r} \sum_{m=0}^{\infty} (2m+1) \frac{P_m(z \cos \theta)}{P_{m+1}(z \cos \theta)} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \]

where \( r_0 = \left[ r^2 + 2r z \cos \theta + (z \cos \theta)^2 \right]^{1/2} \).

Comparing A2.10 with Wilt's (1962, 1970) wavefunction solution is somewhat complicated but, perhaps, worthwhile inasmuch as Wilt's text is an often-cited reference in propagation work. For a three-media spherical system with a radial source colinear with the z-axis and an \( e^{iut} \) time-dependence assumed, Wilt (1970: IV.10.13) obtains

\[ \psi_{\text{new}}(z, r, \theta) = - c_r \sum_{n=0}^{\infty} (2n+1) \frac{P_n(z \cos \theta)}{P_{n+1}(z \cos \theta)} \frac{\partial}{\partial r} \left[ r_{\text{TM}}(\theta, \phi) \right] \]

where

\[ c_r = \left[ 1 + \frac{l_1(z \cos \theta)}{l_2(z \cos \theta)} \right] \left[ 1 + \frac{l_3(z \cos \theta)}{l_4(z \cos \theta)} \right] \]

and

\[ c_{\text{TM}} = \left( 1 - \frac{l_1(z \cos \theta)}{l_2(z \cos \theta)} \right) \left( 1 - \frac{l_3(z \cos \theta)}{l_4(z \cos \theta)} \right) \]

\[ c_{\text{TM}} = \left( 1 - \frac{l_1(z \cos \theta)}{l_2(z \cos \theta)} \right) \left( 1 - \frac{l_3(z \cos \theta)}{l_4(z \cos \theta)} \right) \]

\[ c_{\text{TM}} = - i \frac{\ell}{r} \left( \sum_{m=0}^{\infty} (2m+1) \frac{P_m(z \cos \theta)}{P_{m+1}(z \cos \theta)} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \right) \]

\[ c_{\text{TM}} = + i \frac{\ell}{r} \left( \sum_{m=0}^{\infty} (2m+1) \frac{P_m(z \cos \theta)}{P_{m+1}(z \cos \theta)} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \right) \]

with the fields as given by A2.6 - A2.9 (\( k_m = k \)).

Our solution (taking the complex conjugate of A2.10) differs on two counts. We have an additional factor of \( 1/r_0 \) multiplying the series and, without this factor, Wilt's wavefunction will neither yield the correct units for the fields nor reduce to the wavefunction he determined in an earlier chapter (1970: V.1.5-10) for the two-media system. However, even with this correction, our solutions will agree only if

\[ z_{\text{TM}} = - i \frac{\ell}{r} \left( \sum_{m=0}^{\infty} (2m+1) \frac{P_m(z \cos \theta)}{P_{m+1}(z \cos \theta)} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \right) \]

\[ (\text{Thesis}) \] (A2.15)

and

\[ z_{\text{TM}} = - i \frac{\ell}{r} \left( \sum_{m=0}^{\infty} (2m+1) \frac{P_m(z \cos \theta)}{P_{m+1}(z \cos \theta)} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \right) \]

\[ (\text{Thesis}) \] (A2.16)

were defined in place of A2.13 and A2.14. A review (below) of the surface impedance calculations indicates that equations A2.15 and A2.16 are correct; moreover, Wilt's subsequent approximations for \( z_{\text{TM}}^1 \) and \( z_{\text{TM}}^2 \) (1970: VI.10.31-34 with VI.10.19 and IV.10.20) and calculational results are in accord with these definitions (A2.15, 16).

To substantiate A2.15 and A2.16, we review the surface impedance calculation following Wilt's analysis. He defines

\[ a_{\text{TM}} = - i \frac{\ell}{r} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \]

\[ (\text{Wilt}) \] (A2.17)

and

\[ a_{\text{TM}} = - i \frac{\ell}{r} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \]

\[ (\text{Wilt}) \] (A2.18)

and obtains the fields from equations A2.5-A2.8, with A2.7 modified by changing \( i \) to \( -i \) because of his use of \( e^{iut} \) time-dependence, i.e.,

\[ a_{\text{TM}} = - i \frac{\ell}{r} \frac{\partial}{\partial r} \left[ c_m(z \cos \theta) \phi_{\text{TM}}^m(\theta, \phi) \right] \]

\[ (\text{Wilt}) \] (A2.19)

Associated with each of our three media is a distinct propagation constant, \( k_{m'} \), and a distinct wave function. The wavefunctions in the three regions are proportional to the following series:

\[ \psi_{\text{TM}}(z, r, \theta, \phi) \propto \sum_{m=0}^{\infty} \frac{(2m+1)}{P_m(z \cos \theta)} L_{m-1}^1(\kappa r') P_m(\theta') \]

\[ (A2.20) \]

\[ \psi_{\text{TM}}(z, r, \theta, \phi) \propto \sum_{m=0}^{\infty} \frac{(2m+1)}{P_m(z \cos \theta)} L_{m-1}^1(\kappa r') P_m(\theta') \]

\[ (A2.21) \]

\[ \psi_{\text{TM}}(z, r, \theta, \phi) \propto \sum_{m=0}^{\infty} \frac{(2m+1)}{P_m(z \cos \theta)} L_{m-1}^1(\kappa r') P_m(\theta') \]

\[ (A2.22) \]
which agree with Wait's equations 10.7, 10.13, and 10.8, respectively (1970). Wait
determines the surface impedances using the wave functions for region 1 and region 3
respectively. When the wave function for medium 1 is used, $K_m = K_1$ should be
substituted in the field equations A2.6 and A2.18. The surface impedance $Z_T$ as
defined in A2.15 then results:

$$Z_T = \lim_{r \to \infty} \frac{V}{I} = \lim_{r \to \infty} \left[ \frac{-\frac{1}{r} \frac{\partial}{\partial r} \left( r \psi_r \right)}{\frac{1}{r^2} \frac{\partial}{\partial r} \left( r \psi_r \right)} \right]$$

$$= \lim_{r \to \infty} \left[ \frac{-\frac{1}{r} \frac{\partial}{\partial r} \left( r \psi_r \right)}{\frac{1}{r^2} \frac{\partial}{\partial r} \left( r \psi_r \right)} \right]$$

$$= \lim_{r \to \infty} \left[ \frac{\psi_r}{\psi_r} \right] = \frac{1}{r} \left. \frac{\partial}{\partial r} \psi_r \right|_{r=\infty}$$

$$= -i \psi_z(z) \frac{1}{\psi_z(z)} \left[ (\psi_z(z), j_z(z)) \right] = -i \left( \frac{1}{\psi_z(z)} \right) \left[ (\psi_z(z), j_z(z)) \right].$$

If $K_m = k$ instead of $K_1$ is substituted in equation A2.19 for $H_\phi$ then Wait's
equation A2.13 results. Analogous comments hold with respect to the calculation of
$Z_T$.

Finally, our definitions, A2.15 and A2.16, are in accord with those given by Galejs
(1972: p. 94).

### APPENDIX A3: SPECIAL FUNCTIONS

In this appendix, we present some background information relating to the spherical
Bessel functions and the Legendre functions used in the present work.

Spherical Bessel functions are solutions of the differential equation

$$\frac{d}{dr} \left( r \frac{d}{dr} \psi_r \right) + \left( \frac{\nu^2 - \frac{1}{4}}{r^2} \right) \psi_r = \frac{\nu^2}{r^2} \psi_r,$$

for $n = 0, \pm 1, \pm 2, \ldots$

The spherical Bessel function of the first and second kinds are designated $j_n(z)$ and
$y_n(z)$, with solutions of the third kind being given by $h^{(1)}_n(z)$ and $h^{(2)}_n(z)$,
also known as spherical Hankel functions of the first and second kind:

$$h^{(1)}_n(z) = j_n(z) + i y_n(z);$$

$$h^{(2)}_n(z) = j_n(z) - i y_n(z).$$

(Often $u = n + 1/2$, these functions can be expressed in terms of their cylindrical
Bessel representations:

$$l^{(1)}_n(z) = j_n(z) + i y_n(z);$$

$$l^{(2)}_n(z) = j_n(z) - i y_n(z).$$

All of which are defined for complex $v$ and $z$.

The cylindrical Bessels each satisfy the recurrence relation,

$$l^{(1)}_n(z) + l^{(2)}_n(z) = \frac{2}{z} l^{(0)}_n(z).$$

$$j_n(z) = \frac{2}{z} j^{(1)}_n(z),$$

$$j^{(2)}_n(z);$$

$$j^{(0)}_n(z);$$

$$j^{(1)}_n(z);$$

$$j^{(2)}_n(z);$$

$$l^{(1)}_n(z);$$

$$l^{(2)}_n(z);$$

$$l^{(0)}_n(z);$$

$$l^{(1)}_n(z);$$

$$l^{(2)}_n(z);$$

all of which are defined for complex $v$ and $z$.

The cylindrical Bessels each satisfy the recurrence relation,
and can be approximated by various asymptotic expansions in order and argument. In particular, the following (asymptotic) relations involving Airy functions hold, for \( v = \infty \), uniformly with respect to \( s \) for \( |\arg(s)| \leq \pi - \epsilon \), and \( \epsilon \) an arbitrarily small number [complete expansions are given in (NBS 1964)]:

\[
\begin{align*}
\text{(A3.8)} & \quad w_{0,1} \sim 2e^{-\frac{2\pi}{3} \left( \frac{v^{2/3}}{2^{2/3} \pi} \right)^{1/4} \text{Ai} \left( \frac{2v^{1/3}}{\pi} \right)} \\
\text{(A3.9)} & \quad w_{0,2} \sim e^{-\frac{2\pi}{3} \left( \frac{v^{2/3}}{2^{2/3} \pi} \right)^{1/4} \text{Ai} \left( \frac{2v^{1/3}}{\pi} \right)} \\
\text{(A3.10)} & \quad J_{v/2} \sim \left( \frac{v^{2/3}}{\pi} \right)^{1/4} \text{Ai} \left( \frac{2v^{1/3}}{\pi} \right) \\
\text{(A3.11)} & \quad J_{v/2} \sim -\frac{v^{1/3}}{4v^{2/3} \pi} \left( \text{Ai} \left( \frac{2v^{1/3}}{\pi} \right) \right)
\end{align*}
\]

where

\[
\frac{\sqrt[3]{v}}{s^{1/2}} = \frac{\sqrt{v - z^{1/2}}}{s^{1/2}} - \frac{z^{1/2}}{s^{1/2}},
\]

with the branch of the square root chosen so that \( \zeta \) is real when \( z/v > 0 \).

From A3.12, note that if

\[
|v - z^{1/2}| < v
\]

then

\[
\frac{\sqrt[3]{v}}{s^{1/2}} = \frac{z^{1/2}}{s^{1/2}} + \frac{(v - z^{1/2})^{1/3}}{s^{1/2}} + \cdots
\]

When this quantity is large, asymptotic (large argument) Airy function expansions can be substituted. The resultant Bessel function approximations will be asymptotically equivalent to those developed with the Debye asymptotic expansions. For real \( v \) and \( z \), A3.14 will be much greater than unity when

\[
(v - z^{1/2}) \gg \frac{z^{1/3}}{s^{1/2}}.
\]

In Chapter V, we use the relation

\[
\sqrt{v} > \frac{z}{\pi} + O \left( \frac{z^{3/2}}{v^{2}} \right)
\]

(A3.16)

to indicate those values of \( v > z \) (real) for which the Debye approximations are useful.

The direct derivation of the Debye expansions, extended to complex order and argument, is given by Watson (1944). For the regions of the complex plane (Fig. A3.1) of particular interest, the appropriate expansions are specified in Table A3.1 in terms of \( s^{(1)} \) and \( s^{(2)} \) of particular interest, the appropriate expansions are specified in Table A3.1 in terms of \( s^{(1)} \) and \( s^{(2)} \):
whereas, if \( \frac{v}{z} \) is real and \( v < z \),

\[
I_0^{(v)}(z) = \frac{\exp\left(\frac{z}{2} - \frac{v}{z}\right)}{\sqrt{\frac{v}{2}}} \left[ \sqrt{\left(\frac{z}{2}\right)^2 - 1} \right]^\frac{1}{2}
\]

(A3.21)

\[
I_0^{(v)}(z) = \frac{\exp\left(\frac{z}{2} - \frac{v}{z}\right)}{\sqrt{\frac{v}{2}}} \left[ \sqrt{\left(\frac{z}{2}\right)^2 - 1} \right]^\frac{1}{2}
\]

(A3.22)

We now use the above relations and obtain results needed in our present investigation. If \( w, x, \) and \( n \) are real, with \( n - w \gg 0(x^2/3) \) and \( n - x \gg 0(x^2/3) \), then, from A3.2, A3.18, and A3.19 (with \( v = n + 1/2 \)), we obtain

\[
\begin{align*}
\frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} &\sim \frac{\exp\left(\nu \left[1 - \left(\frac{1}{2}\right)^2\right]\right)}{\sqrt{\frac{1}{2}}} \left[1 + \frac{1}{\nu^2} \left(1 - \nu^2\right)^\frac{1}{2}\right] \\
&\sim \nu^{(w+1)}
\end{align*}
\]

(A3.23)

and

\[
\begin{align*}
I_n^{(x)} \left[ \frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} \right] &\sim -\frac{\nu}{\lambda} + \frac{\nu}{\lambda} \left[1 - \left(\frac{1}{2}\right)^2\right]^{\frac{1}{2}}
\end{align*}
\]

(A3.24)

Also in the limit as \( v = n + 1/2 + m \),

\[
\begin{align*}
\frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} &\rightarrow \frac{\nu}{\sqrt{2\nu}} \left(\frac{\nu}{\lambda}\right)^\nu \\
I_n^{(x)} \frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} &\rightarrow -\frac{\nu}{\sqrt{2\nu}} \left(\frac{\nu}{\lambda}\right)^\nu
\end{align*}
\]

(A3.25)

(A3.26)

And, for complex real or complex arguments, we obtain the following limits using the ascending series for the Bessel functions [NBS 1964: 10.1.2, 10.1.3]

\[
\begin{align*}
I_n^{(x)} \left[ \frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} \right] &\rightarrow \frac{\nu^{(w+1)}}{\lambda} \\
I_n^{(x)} \frac{L^{(w)}_{\nu}(x)}{L^{(x)}_{\nu}(w)} &\rightarrow -\frac{\nu}{\lambda}
\end{align*}
\]

(A3.27)

(A3.28)

The last limit follows directly from A3.24.

We define

\[
A_n = \nu \left[ l^{(w)}_{\nu}(x) \right] \left[ j^{(w)}_{\nu}(x) \right] \left[ l^{(x)}_{\nu}(w) \right] \left[ j^{(x)}_{\nu}(w) \right]
\]

(A3.29)

\[
B_n = \nu \left[ l^{(w)}_{\nu}(x) \right] \left[ j^{(w)}_{\nu}(x) \right] \left[ l^{(x)}_{\nu}(w) \right] \left[ j^{(x)}_{\nu}(w) \right]
\]

(A3.30)

where \( R^{(m)}(n) \) and \( R^{(m)}(n) \) are given by V.3 and V.4. The Wronskian for the Hankel functions can be used to show

\[
\begin{align*}
A_n &\sim \nu \left[ l^{(w)}_{\nu}(x) \right] \left[ j^{(w)}_{\nu}(x) \right] \left[ l^{(x)}_{\nu}(w) \right] \left[ j^{(x)}_{\nu}(w) \right] \\
A_n &\sim \nu \left[ l^{(w)}_{\nu}(x) \right] \left[ j^{(w)}_{\nu}(x) \right] \left[ l^{(x)}_{\nu}(w) \right] \left[ j^{(x)}_{\nu}(w) \right] \\
A_n &\sim \nu \left[ l^{(w)}_{\nu}(x) \right] \left[ j^{(w)}_{\nu}(x) \right] \left[ l^{(x)}_{\nu}(w) \right] \left[ j^{(x)}_{\nu}(w) \right]
\end{align*}
\]

(A3.31)

(A3.32)

The limiting forms \( n \rightarrow m \) of \( A_n \) and \( B_n \) are obtainable using A3.31 and A3.32.

First consider \( A_n \) which from A3.2 and A3.31 can be written

\[
A_n = \nu \left(\text{a}_n + \text{b}_n \right) \left(\text{c}_n + \text{d}_n \right) \left(\text{e}_n + \text{f}_n \right) \left(\text{g}_n + \text{h}_n \right)
\]

(A3.33)
In the limit, \( n = v - 1/2 + \infty \), the first term in the parenthesis on the right hand side of (3.33) is \( \Theta \left[ \left( \frac{v}{2} \right)^{1/2} \right] \), from A3.33 and A3.26; the second term is \( \Theta \left[ \left( \frac{v}{2} \right)^{1/2} \right] \), from A3.23, A3.27, and A3.28. Thus, for \( \alpha \) a constant, we have

\[
A_n \rightarrow \Theta \left( n + \frac{1}{2} \right) \left( \frac{v}{2} \right)^{1/2}.
\]

(A3.34)

Next consider \( B_n \). From A3.30,

\[
B_n = 1 + \frac{1}{\alpha} \left[ \left( \frac{v}{2} \right)^{1/2} \right] \Theta \left[ \left( \frac{v}{2} \right)^{1/2} \right]
\]

(A3.35)

Again, we can use A3.31 and A3.32 and the limits obtained earlier to show that the numerator in (3.35) is \( \Theta \left( \frac{v}{2} \right)^{1/2} \) and the denominator is \( \Theta \left( \frac{v}{2} \right)^{1/2} \). Thus, in the limit \( n = v - 1/2 + \infty \),

\[
B_n \rightarrow \Theta \left( v + \frac{1}{2} \right) \left( \frac{v}{2} \right)^{1/2}.
\]

(A3.36)

for \( \alpha \) a constant. We reference A3.34 and A3.36 in section V.A.

The Legendre functions satisfy the recurrence relation

\[
(\lambda + 1) P_{\lambda+1}(\lambda) = (2\lambda + 1) \lambda P_{\lambda}(\lambda) - \lambda P_{\lambda-1}(\lambda),
\]

(A3.37)

and the asymptotic relation

\[
\Phi_{\nu, \lambda}(z) = \frac{1}{i^n \Gamma(n+1) \lambda^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda + 1/2)_k}{k!} \left[ \left( \frac{\nu + 1/2}{\nu} \right)^{\lambda - k} (\nu + 1/2) \right]
\]

(A3.38)

We define

\[
A_{\nu}(\xi) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu + 1)} \left[ \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu + 1)} \right] \left[ \frac{1}{\xi^{1/2}} \right] \left[ \zeta \right]^{-\nu}.
\]

(A3.39)

\[
U_{\nu}(L) = \left[ \frac{1}{2} \ln \left( \frac{\nu + 1/2}{\nu} \right) \right] \frac{e^{-i\xi L}}{e^{i\xi L}} \left[ \frac{\nu + 1/2}{\nu} \right] \left[ \zeta \right]^{-\nu}.
\]

(A3.40)

\[
V_{\nu}(L) = \left[ \frac{1}{2} \ln \left( \frac{\nu + 1/2}{\nu} \right) \right] \frac{e^{-i\xi L}}{e^{i\xi L}} \left[ \frac{\nu + 1/2}{\nu} \right] \left[ \zeta \right]^{-\nu}.
\]

(A3.41)

Then

\[
\Phi_{\nu, \lambda}(z) = \frac{1}{i^n \Gamma(n+1) \lambda^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda + 1/2)_k}{k!} \left[ \left( \frac{\nu + 1/2}{\nu} \right)^{\lambda - k} (\nu + 1/2) \right]
\]

(A3.42)

for \( L = 0, 1, \ldots \).

Note that \( \Phi_{\nu, \lambda}(z) = 0 \) and the leading approximation is

\[
\Phi_{\nu, \lambda}(z) = \frac{1}{i^n \Gamma(n+1) \lambda^{1/2}} \frac{1}{\sqrt{\nu^{1/2} \left( \frac{\nu + 1/2}{\nu} \right)^{\lambda - k} (\nu + 1/2)}}.
\]

(A3.43)

for \( \epsilon \leq \theta \leq \pi - \epsilon, \epsilon > 0 \) (Magnus and Oberhettinger 1954).
A3.1. Bessel Function Expansions

<table>
<thead>
<tr>
<th>Function</th>
<th>Region</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0^{(1)}(z)$</td>
<td>1,3,4</td>
<td>$s_0^{(1)}(z)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$s_0^{(1)}(z) - s_0^{(2)}(z)$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$s_0^{(1)}(z) - \exp(-2\pi i\nu) s_0^{(2)}(z)$</td>
</tr>
<tr>
<td>$h_0^{(2)}(z)$</td>
<td>1,2,5</td>
<td>$s_0^{(1)}(z)$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$s_0^{(1)}(z) - s_0^{(2)}(z)$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$s_0^{(1)}(z) - \exp(2\pi i\nu) s_0^{(2)}(z)$</td>
</tr>
<tr>
<td>$2J_0(z)$</td>
<td>1</td>
<td>$s_0^{(1)}(z) + s_0^{(2)}(z)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$s_0^{(1)}(z)$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$s_0^{(2)}(z)$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$s_0^{(1)}(z) + s_0^{(2)}(z) - \exp(2\pi i\nu) s_0^{(1)}(z)$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$s_0^{(1)}(z) + s_0^{(2)}(z) - \exp(-2\pi i\nu) s_0^{(2)}(z)$</td>
</tr>
</tbody>
</table>
APPENDIX M. PLANE EARTH LIMITS

We are interested in the limits \( (r_k a - s_k, r_k a - s_k) \)

\[
\lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right] = \lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right]
\]

\[
\lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right]
\]

\[
\lim_{a \to \infty} \left[ \frac{s_k j_{-s_k}}{\sqrt{2} \pi} \right]
\]

where \( r_k^M(w) \) is given in IV.1.29 or A2.3 and we assume \( \text{Im}(s_k) > 0 \) and \( \text{Im}(k) > 0 \).

Using the recurrence and cross-product relations for the spherical Bessel functions (NBS 1964: 10.1.9-10.1.22 and 10.1.31), we find

\[
\left\{ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right\} \left\{ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right\}^{-1}
\]

Consequently, using A3.3-A3.6, we need only consider the two limits

\[
A(x, w) = \lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right]
\]

\[
A(x, w) = \lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right]
\]

\[
\lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right] = \lim_{a \to \infty} \left[ \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right]
\]

where \( x, w, \) and \( s \) are proportional to \( a \) and

\[
\lim_{a \to \infty} \left( \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right) = 1 = \lim_{a \to \infty} \left( \frac{\text{Im}(s_k)}{\text{Re}(s_k, j_{-s_k})} \right)
\]

Set \( v = \mu \pm \frac{1}{2} \) and let \( z \) denote one of the arguments \( x, w, \) or \( s \). Then,

\[
|v - z| > 0(2/3) \quad \text{holds in the limit at } a \to \infty \quad \text{since } n \text{ is real and } z \text{ has an imaginary component.}
\]

Consequently, in approximating the Bessel functions we need only consider regions 1 and 3, of the complex \( v-z \) plane (Fig. A3.1). From Table A3.1, we find

\[
J_\nu(z) = \frac{s_\nu(z)}{2} \quad \text{for } \nu > 0
\]

\[
J_\nu(z) = \frac{s_\nu(z)}{2} \quad \text{for } \nu > 0
\]

in region 3, and

\[
J_\nu(z) = \frac{s_\nu(z)}{2} \quad \text{for } \nu > 0
\]

\[
J_\nu(z) = \frac{s_\nu(z)}{2} \quad \text{for } \nu > 0
\]

in region 1, where \( s_\nu(1)(z) \) and \( s_\nu(2)(z) \) have the asymptotic expansions given in A3.17 and A3.18. But, for \( v > 0 \), the curve \( C_\nu \) in Fig. A3.1 separates those values of \( v \) for which the real part of the exponential exponent of \( s_\nu(1)(z) \) is positive from those where it is negative. In region 1, the real part is negative and, since the exponent is also proportional to \( a \), \( s_\nu(1) \) will vanish in the limit as \( a \to \infty \).

Therefore, the substitution of A4.7 and A4.8 for the Bessel functions in A4.5 and A4.6 will be valid for all \( v \) in the limit as \( a \to \infty \). We then have

\[
A(x, w) = \lim_{a \to \infty} \left[ \frac{s_\nu(z)}{2} \right] \left( \text{Re}(z) \right) \left( \text{Im}(z) \right)
\]

\[
A(x, w) = \lim_{a \to \infty} \left[ \frac{s_\nu(z)}{2} \right] \left( \text{Re}(z) \right) \left( \text{Im}(z) \right)
\]

Defining

\[
\delta_x = \left( \frac{1}{2} \right)^{1/2}
\]

\[
\delta_x = \left( \frac{1}{2} \right)^{1/2}
\]

\[
\delta_x = \left( \frac{1}{2} \right)^{1/2}
\]

*Along \( C_\nu \), the real part of the exponential exponent in \( s_\nu(1) \) and \( s_\nu(2) \) is zero.
for $z = x, w,$ or $s,$ and substituting for $S^{(1)}_u$ and $S^{(2)}_u$ from A3.17 and A3.18, we obtain

$$
\frac{(iwu - \frac{1}{nu})}{nu} \sim 0.25 \left(\frac{1}{nu} \right)^{\frac{1}{2}} \left(\cos nu + i \sin nu\right)^{\frac{1}{2}} \\
$$

where $u_n(t)$ is given in NBS (1964: 9.3.9, 9.3.10). In particular,

$$
\frac{(iw_n - \frac{1}{nu_n})}{nu_n} \sim 0.25 \left(\frac{1}{nu_n} \right)^{\frac{1}{2}} (\cos nu_n + i \sin nu_n)^{\frac{1}{2}} \\
$$

where $u_n(t)$, $m > 0$, and $v_n(t)$, $m > 1$, involve sums of terms, each having a $t$-dependent factor of the form $t^p$ where $p$ is a positive integer and $p > m$. Noting that $\varphi = \mu + 1/2$ and recalling (from IV.1.8) that

$$
\mathcal{L} = \frac{1}{\mathcal{L} - \mathcal{L}_m} (\frac{h}{\mathcal{L}}) = \mathcal{L}_m, \\
$$

for $\mathcal{L}_m$ assumed finite, we find

$$
\lim_{\mathcal{L} \to \mathcal{L}_m} (\mathcal{L} - \mathcal{L}_m)^{(\mu - 1)} \sim \mathcal{L}_m. \\
$$

It then follows that

$$
\lim_{\mathcal{L} \to \mathcal{L}_m} u_n(t) = 0, \quad t < 0, \\
\lim_{\mathcal{L} \to \mathcal{L}_m} u_n(t) = 0, \quad t > 0, \\
$$

We now consider

$$
\Lambda = \lim_{\mathcal{L} \to \mathcal{L}_m} \left(\omega_n - \frac{1}{nu_n}\right) - \frac{1}{nu_n} \left[\left(\frac{1}{nu_n} - 1\right)\right]^{\frac{1}{2}}, \\
$$

where $\Lambda$ is a constant, we can expand $\omega_n$ as

$$
\omega_n = \left(\mathcal{L} - \mathcal{L}_m\right)^{\frac{1}{2}} \left(\cos nu_n + i \sin nu_n\right)^{\frac{1}{2}} \\
$$

since

$$
\left(\mathcal{L} - \mathcal{L}_m\right)^{\frac{1}{2}} = \left(\mathcal{L} - \mathcal{L}_m\right)^{\frac{1}{2}} \left(\cos nu_n + i \sin nu_n\right)^{\frac{1}{2}} \\
$$

whereas

$$
\left(\mathcal{L} - \mathcal{L}_m\right)^{\frac{1}{2}} = \left(\mathcal{L} - \mathcal{L}_m\right)^{\frac{1}{2}} \left(\cos nu_n + i \sin nu_n\right)^{\frac{1}{2}} \\
$$

(recall that a small imaginary part is associated with $k$). An expansion analogous to A4.25 holds for $\omega_n$ assuming $w = k(a + z_w)$. With these restrictions on the form of $x$ and $w$, (recall that $a$ is assumed finite,

$$
\frac{(iw_n - \frac{1}{nu_n})}{nu_n} \sim 0.25 \left(\frac{1}{nu_n} \right)^{\frac{1}{2}} (\cos nu_n + i \sin nu_n)^{\frac{1}{2}} \\
$$

and A4.24 becomes

$$
\Lambda = \frac{1}{\mathcal{L} - \mathcal{L}_m} \left(\frac{h}{\mathcal{L}}\right)^{\frac{1}{2}} \left(\cos nu_n + i \sin nu_n\right)^{\frac{1}{2}} \\
$$

Thus, with the definitions

$$
\chi = \frac{1}{\mathcal{L} - \mathcal{L}_m} \left(\frac{h}{\mathcal{L}}\right)^{\frac{1}{2}} \left(\cos nu_n + i \sin nu_n\right)^{\frac{1}{2}} \\
$$

we find

$$
\Lambda = (z_w - \mathcal{L}_m) \left(\mathcal{L}_m - \mathcal{L}\right)^{\frac{1}{2}} \\
$$

where we have used the limit in A4.20 (recall $\varphi = \mu + 1/2$). Furthermore,

$$
\frac{1}{\mathcal{L} - \mathcal{L}_m} \left(\frac{h}{\mathcal{L}_m}\right)^{\frac{1}{2}} \\
$$

and

$$
\frac{1}{\mathcal{L} - \mathcal{L}_m} \left(\frac{h}{\mathcal{L}_m}\right)^{\frac{1}{2}}. \\
$$
For 
\[ \xi = \xi_0, \]
we also have
\[ \lim_{\zeta \rightarrow \xi_0} \left( \frac{\zeta - \xi_0}{\zeta - a} \right) = \left[ \left( \frac{\xi - \xi_0}{\xi - a} \right)^{1/2} \right]^2. \] (A4.36)

Substituting A4.15, with A4.18, A4.22, and A4.31-A4.35, into A4.11, we obtain
\[ A \left[ \frac{1}{(z-\xi_0)(\xi-\xi_0)^{1/2}} \right] \equiv \frac{1}{\xi_0} \left[ \frac{\xi \left( \frac{\xi - \xi_0}{\xi - \xi_1} \right)^{1/2}}{1 - \xi \left( \frac{\xi - \xi_0}{\xi - \xi_1} \right)^{1/2}} \right] \] (A4.37)
\[ \left( \text{with the branch of the square root taken as indicated in Appendix A3.} \right) \]

Substituting A4.16 and A4.17, with A4.18, A4.22, A4.31 (for \( z = 0 \)), A4.34, A4.36, and A4.37, into A4.12, we obtain
\[ \left( \frac{\zeta}{\xi_0, \xi_0^2} \right) = \left[ \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \right]^{1/2} \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \] (A4.38)
\[ \left( \text{with the branch taken as above.} \right) \]

We now modify the branch cut definitions so that the real part of the square root is positive; in particular, referencing our earlier definition (which is given following A3.18), we replace \(-i \left[ \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \right]^{1/2} \) with \(-i \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \) and \(-i \left[ \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \right]^{1/2} \) with \(-i \left( \frac{\xi_0}{\xi_0^2} \right)^{1/2} \).

So that,
\[ A \left[ \frac{1}{(z-\xi_0)(\xi-\xi_0)^{1/2}} \right] \equiv \frac{1}{\xi_0} \left[ \frac{\xi \left( \frac{\xi - \xi_0}{\xi - \xi_1} \right)^{1/2}}{1 - \xi \left( \frac{\xi - \xi_0}{\xi - \xi_1} \right)^{1/2}} \right] \] (A4.40)
\[ \left( \text{with the branch taken as above.} \right) \]

Using A4.40 and A4.41 and defining
\[ \gamma_0 = \frac{\alpha + z_1}{\alpha} \] (A4.42)
\[ \gamma_0^* = \frac{\zeta + z_1}{\zeta} \] (A4.43)
we obtain the desired limits in A4.1-A4.3
\[ \lim_{\zeta \rightarrow \xi_0} \left( \frac{\zeta - \xi_0}{\zeta - \xi_0^2} \right) = \exp \left[ i \left( \frac{\gamma_0 - \xi_0}{\gamma_0^* - \xi_0^2} \right)^{1/2} \right] \] (A4.44)
\[ \lim_{\zeta \rightarrow \xi_0} \left( \frac{\zeta - \xi_0}{\zeta - \xi_0^2} \right)^{i/2} = \frac{1}{2} \left( \frac{\gamma_0 - \xi_0^*}{\gamma_0^* - \xi_0^2} \right)^{1/2} \exp \left[ i \left( \frac{\gamma_0 - \xi_0}{\gamma_0^* - \xi_0^2} \right)^{1/2} \right] \] (A4.45)
\[ \lim_{\zeta \rightarrow \xi_0} \left( \frac{\zeta - \xi_0}{\zeta - \xi_0^2} \right)^{i/2} = \frac{1}{2} \left( \frac{\gamma_0 - \xi_0^*}{\gamma_0^* - \xi_0^2} \right)^{1/2} \exp \left[ i \left( \frac{\gamma_0 - \xi_0}{\gamma_0^* - \xi_0^2} \right)^{1/2} \right] \] (A4.46)
APPENDIX A5: DETAILS OF RESULTS IN SECTION V.B

(a) We establish V.64; i.e., for \( C_n(L) \) given by V.33 and \( r > a \), we show that

\[
\Delta^n C_n(L) = \left( \frac{9}{8} \right)^n \frac{V}{n!} \left( -1 \right)^n \left( 1 - \frac{t}{a} \right)^n \sum_{j=0}^{\infty} \left( \frac{9}{8} \right)^j \left( \frac{r}{a} \right) \left( -1 \right)^j \left( 1 - \frac{t}{a} \right)^j \left( 1 - \frac{t}{a} \right)^{-j}.
\]

where the \( L \)-dependence of the right-hand side is associated with the coefficient beyond \( Y_0 \); that is, the lower order (in \( n \)) terms.

From V.33, we have

\[
C_n(L) \sim \left( \frac{9}{8} \right)^n \frac{V}{n!} \left( 1 - \frac{t}{a} \right)^n \left( -1 \right)^n \left( 1 - \frac{t}{a} \right)^{-n}.
\]

Letting \( u_n \sim \Delta^n (Y_0 + Y_n + \ldots) \), \( Y_0 = t^0 \), and taking \( t = a/r \), we see that

\[ C_n(L) \sim u_n Y_0 \quad \text{and} \quad \Delta^n C_n(L) \sim \Delta^m u_n Y_0. \]

But, from V.25, we have

\[
\Delta^n u_n \sim \left( \frac{9}{8} \right)^n \left( 1 - \frac{t}{a} \right)^n \left( -1 \right)^n \left( 1 - \frac{t}{a} \right)^{-n}.
\]

Then, defining

\[ \Delta^n u_n \sim \left( \frac{9}{8} \right)^n \left( 1 - \frac{t}{a} \right)^n \left( -1 \right)^n \left( 1 - \frac{t}{a} \right)^{-n}, \]

which, with V.23, gives us

\[
\Delta^n u_n Y_n \sim \sum_{j=0}^{\infty} \left( \frac{9}{8} \right)^j \left( 1 - \frac{t}{a} \right)^j \left( -1 \right)^j \left( 1 - \frac{t}{a} \right)^{-j} \sum_{j=0}^{\infty} \left( \frac{9}{8} \right)^j \left( 1 - \frac{t}{a} \right)^j \left( 1 - \frac{t}{a} \right)^{-j}.
\]

Substituting for \( t = a/r \) thus establishes A5.1.

(b) We prove V.73 given V.72. Using the generalized Euler transform (V.47) with \( q = te^{i \theta} \), we find that

\[
\mathcal{R}(k) = \mathbb{E} \left\{ \exp \left[ i \left( \frac{k}{2} - \frac{r}{2} \right) \right] \sum_{n=0}^{\infty} \left( e^{i \theta} \right)^n C_n(L) \right\}
\]

implies

\[
\mathcal{R}(k) = \mathbb{E} \left\{ \exp \left[ i \left( \frac{k}{2} - \frac{r}{2} \right) \right] \sum_{n=0}^{\infty} \left( e^{i \theta} \right)^n C_n(L) \right\}.
\]

Then, defining

\[ \Phi = \frac{1}{2} \left[ \frac{1}{1 + e^{i \theta}} \right] \]

and writing

\[ \gamma = \left[ 1 - 2 \left( \cos(\theta/2) + i \sin(\theta/2) \right) \right]/2, \]

we have

\[ (1 - \gamma e^{i \theta}) = \frac{1}{2} \left[ (1 + \Phi - \frac{r}{2}) \right] \]

which, upon substitution into A5.3, gives us

\[
\mathcal{R}(k) = \mathbb{E} \left[ \left( \exp \left[ i \left( \frac{k}{2} - \frac{r}{2} \right) \right] \right) \frac{1}{2} \left( 1 + \Phi - \frac{r}{2} \right) \right] \frac{1}{2} \left( 1 + \Phi - \frac{r}{2} \right) \Delta^n C_n(L).
\]

Substituting for \( t = a/r \) thus establishes A5.1.
If we again define \( t = a/r \), \( \varphi \) is as defined in V.39, and
\[ \varphi = \left[ 1 - 2 \left( \frac{a}{r} \right) \cos \left( \frac{a}{r} \right) + \left( \frac{a}{r} \right) \right]^{1/2}. \]
Furthermore,
\[
\mathcal{I} \left( \frac{\varphi}{1 - \varphi} \right) = \frac{\left( \frac{1}{\varphi} \right)^{2/\varphi}}{\varphi^{1/\varphi}} \left[ 1 - 2 \left( \frac{a}{r} \right) \cos \left( \frac{a}{r} \right) + \left( \frac{a}{r} \right) \right]^{1/\varphi} 
\]

\[
\times \frac{\varphi_{\varphi}(\varphi)}{\varphi_{\varphi}(1 - \varphi)} \left[ (\varphi - 1)^{1/\varphi} - (\varphi - 1)^{1/\varphi} \right] 
\] 

\[
\Delta \left( \frac{\varphi}{1 - \varphi} \right) 
\]

where
\[
\varphi_{\varphi}(\varphi) = \left( \frac{a}{r} \right)^{1/\varphi} \left[ 1 - 2 \left( \frac{a}{r} \right) \cos \left( \frac{a}{r} \right) + \left( \frac{a}{r} \right) \right]^{1/\varphi} 
\]

\[
\times \frac{\varphi_{\varphi}(\varphi)}{\varphi_{\varphi}(1 - \varphi)} \left[ (\varphi - 1)^{1/\varphi} - (\varphi - 1)^{1/\varphi} \right] 
\] 

\[
\Delta \left( \frac{\varphi}{1 - \varphi} \right) 
\]

Using the generalized Euler transformation in an analogous manner, we find that
\[
I(\varphi) = J_{m-1} \left( \frac{1}{\varphi} \right) \left[ (\varphi - 1)^{1/\varphi} \right] \left( \frac{\varphi}{1 - \varphi} \right) \Delta \left( \frac{\varphi}{1 - \varphi} \right) 
\]

\[
\text{(AS.7)} 
\]

\[
\text{(AS.9)} 
\]

\[
\text{(AS.10)} 
\]

\[
\text{(AS.11)} 
\]

From V.72, we then have that
\[
\Delta \left( \frac{\varphi}{1 - \varphi} \right) = \varphi \left( \frac{\varphi}{1 - \varphi} \right) - \varphi \left( \frac{\varphi}{1 - \varphi} \right). 
\]

With \( R(a/r) \) and \( I(a/r) \) as defined in AS.2 and AS.9. But \( R(a/r) \) and \( I(a/r) \) are also given by AS.7 and AS.10; hence V.73 follows immediately.

(c) We establish V.80; i.e., for
\[
\mathcal{F}_{n} = \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) = \left( \begin{array}{c} \lambda_{n} \\ \lambda_{n + 1} \end{array} \right) = \left( \begin{array}{c} \lambda_{n} \\ \lambda_{n + 1} \end{array} \right) 
\]

we show (by induction) that \( (\cdots \cdots ) \)
\[
\mathcal{F}_{n}^{(m)} = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) = \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n+2} \end{array} \right) 
\]

\[
\left( \begin{array}{c} \cdots \\ \lambda_{n+2} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n+3} \end{array} \right) 
\]

\[
\cdots 
\]

\[
\text{(AS.12)} 
\]

with \( (2m-1)!! \) as defined in V.90.

For \( m=1 \),
\[
\mathcal{F}_{n}^{(1)} = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) 
\]

\[
\left( \begin{array}{c} \cdots \\ \lambda_{n+2} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n+3} \end{array} \right) 
\]

\[
\cdots 
\]

\[
\text{where we have used V.25b.} 
\]

We now assume AS.12 is true for \( n=p \) and prove it true for \( n=p+1 \).

In particular,
\[
\mathcal{F}_{n}^{(p+1)} = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n+1} \end{array} \right) 
\]

\[
\left( \begin{array}{c} \cdots \\ \lambda_{n+2} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n+3} \end{array} \right) 
\]

\[
\cdots 
\]

\[
\text{(AS.13)} 
\]

But, \( \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) \right) = \left( \frac{1}{(2m+1)} \right) \Delta \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdot \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) \left( \begin{array}{c} \cdots \\ \lambda_{n} \end{array} \right) 
\]

\[
\cdots 
\]

\[
\text{(AS.14)} 
\]

Substitution of AS.14 into AS.13 proves AS.12 true for \( n=p+1 \) and, by induction, for all \( m \).
We next establish V.91 and show that 

\[
R_{\nu}^{(\mu)}(\nu) = \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} R_{n+1}^{(\mu)} \Delta R_{n+1}^{(\mu)}(\nu) \\
= O \left\{ \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} \right\} \\
= O \left\{ \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} \frac{(1/2)_{2n+1}(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} \right\} \\
= O \left\{ \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} \frac{(1/2)_{2n+1}(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} \right\}, \quad (\nu \to \infty). \tag{AS.15}
\]

From V.95, \( R(\nu) \) is given by

\[
R_{\nu}^{(\mu)} = \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} \\
= \left( \frac{1}{1 - \frac{1}{2}} \right) \sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}}. \tag{AS.16}
\]

With the inequality,

\[
P_{\nu}[\cos(\theta)] < \left[ \frac{\pi}{\nu \sin(\theta)} \right]^{1/2}, \tag{AS.17}
\]

(Magnus and Oberhettinger 1954) it follows that

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} < \left[ \frac{\pi}{\nu \sin(\theta)} \right]^{1/2} \sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}}, \quad \eta \neq 1. \tag{AS.18}
\]

The series on the right hand side can be evaluated asymptotically (\( N \to \infty \)) by substituting for \( f_{\mu}^{(\nu)} \) from AS.12 and applying V.27. We find \( (y = \cos \theta \neq 1), \nu \to \infty \),

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} = O \left[ \left( \frac{\gamma + 1/2}{\gamma + 1} \right)^{2n+1/2} \right], \tag{AS.19}
\]

Equation AS.17, in conjunction with the leading term of the asymptotic Legendre expansion,

\[
P_{\nu}[\cos(\theta)] = \left[ \frac{\pi}{\nu \sin(\theta)} \right]^{1/2} \cos \left( (\nu + 1/2) \theta - \frac{\pi}{2} \right)
\]

can be used to show that \( (\nu \to \infty) \)

\[
(\nu \to \infty) \Delta P_{\nu}^{(\mu)}(\nu) = O \left\{ \left( \frac{1}{(2n+1)\Gamma(2n+\frac{1}{2})} \right)^2 \frac{(\gamma + 1/2)^{2n+1/2}}{(\gamma + 1)^{2n+1/2}} \right\}, \tag{AS.20}
\]

and \( (\nu \to \infty) \)

\[
(\nu \to \infty) \Delta P_{\nu}^{(\mu)}(\nu) = O \left[ \left( \frac{\gamma + 1/2}{\gamma + 1} \right)^{2n+1/2} \right]. \tag{AS.21}
\]

Substituting AS.19 - AS.21 into AS.16, we obtain the results of AS.15 directly.
Using the generating function for the Legendre polynomials

\[ (1 - 2zq + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(q), \]

we find that a number of infinite Legendre series can be summed exactly. Such expansions have been exploited previously (Nicholaenko and Rabinowicz 1974) but are more fully detailed here. If \( m \) is a positive integer and \( \xi = \sum_{n=0}^{\infty} z^n P_n(q) \), then \( s \) can be obtained as a linear combination of the first \( m \) derivatives in \( z \) of \( (1 - 2zq + z^2)^{-1/2} \).

For instance,

\[
\sum_{n=0}^{\infty} z^n P_n(q) = \sum_{n=0}^{\infty} \left[ 2^n \frac{d^n}{dz^n} (1 - 2zq + z^2)^{-1/2} \right] P_n(q) + \sum_{n=0}^{\infty} \frac{d^n}{dz^n} \sum_{n=0}^{\infty} z^n P_n(q)
\]

\[
= (2^{n+1} + 32^{1/2} + 2^{1/2}) (1 - 2zq + z^2)^{-1/2}.
\]

(Note that, while the original series is divergent for \( z = 1 \), the algebraic solution will have a value at \( z = 1 \) that is continuous with the solution for \( z < 1 \) and \( y \neq 1 \).

We next seek exact solutions when inverse powers of \((n + m)\), \( m \) an integer, are involved. First, consider the series \( \sum_{n=0}^{\infty} \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q) \), \(-1 \leq u < 1\), for \( m > -1 \).

We find,

\[
\sum_{n=0}^{\infty} \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q) = \int_0^1 \frac{d^m}{dz^m} \left( (1 - 2zq + z^2)^{-1/2} \right) dz\, P_n(q),
\]

\[
= \frac{1}{2} \int_0^1 \frac{d^m}{dz^m} \left[ (1 - 2zq + z^2)^{-1/2} \right] dz\, P_n(q),
\]

\[
= \frac{1}{2} \int_0^1 \frac{d^m}{dz^m} \left[ (1 - 2zq + z^2)^{-1/2} \right] dz\, P_n(q).
\]

When \( m \) is a nonnegative integer the integral has an exact algebraic solution. In particular,

\[
\int_0^1 (1 - 2zq + z^2)^{-1/2} dz = -\ln \left[ \frac{(1 - 2zq + z^2)^{1/2} + 1}{(1 - 2zq + z^2)^{1/2} - 1} \right] + \ln \left( \frac{z}{q} \right)
\]

\[
= \ln \left( \frac{(1 - 2zq + z^2)^{1/2} + 1}{(1 - 2zq + z^2)^{1/2} - 1} \right)
\]

and

\[
\int_0^1 (1 - 2zq + z^2)^{-1/2} dz = \int_0^1 \left( \frac{(1 - 2zq + z^2)^{1/2} + 1}{(1 - 2zq + z^2)^{1/2} - 1} \right)
\]

lending to the solutions

\[
\sum_{n=0}^{\infty} \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q) = \ln \left( \frac{1}{(1 - 2zq + z^2)^{1/2} + 1} \right),
\]

\[
= \frac{1}{2} \int_0^1 \frac{d^m}{dz^m} \left[ (1 - 2zq + z^2)^{-1/2} \right] dz\, P_n(q),
\]

\[
= \frac{1}{2} \int_0^1 \frac{d^m}{dz^m} \left[ (1 - 2zq + z^2)^{-1/2} \right] dz\, P_n(q)
\]

\[
= \frac{1}{2} \int_0^1 \frac{d^m}{dz^m} \left[ (1 - 2zq + z^2)^{-1/2} \right] dz\, P_n(q).
\]

When \( m \) is not a nonnegative integer, the integral representation of the series is not known to have an algebraic solution. Higher orders of \( n \) in the denominator will, in general, result in more complicated integral representations. However, we can construct exact solutions for certain "higher-order" series using the results just obtained. For example,

\[
\sum_{n=0}^{\infty} \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q) = \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q) - \sum_{n=0}^{\infty} \frac{\xi^{(m)}}{\xi^{(m)}} P_n(q).
\]
Similarly, we could obtain exact results for series of the form
\[
\sum_{n=0}^{\infty} \left( \frac{z^n}{n!(m_1!)(m_2!)} \right) R_1(q) = \sum_{n=0}^{\infty} \left( \frac{z^n}{n!(m_1!)(m_2!)} \right) R_2(q)
\]
for m and q nonnegative integers.

Finally, we derive certain results used in section V.C.

In particular,
\[
\sum_{n=1}^{\infty} \frac{x^n}{(n)(m_1!)(m_2!)} R_2(q) = \frac{1 + x}{1 - x} R_2(q)
\]
then (using A6.1, A6.3, A6.4, and A6.6) we have
\[
A(z) = A(1) + \frac{2}{2(z-1)^{1/2}} - 1 + 2 \ln \left[ \frac{2}{2(z-1)^{1/2} + (1-q)} \right]
\]
Finally, we define
\[
A(z) = A(1) - \frac{z^2}{2(z-1)^{1/2} + (1-q)}
\]
and
\[
\frac{1}{2(z-1)^{1/2}} \left[ (1-z)^2 + 2q(1+z)^2 \right] \left( 1 - 2zq + 1 - 2zq + (1)^2 \right)
\]
Taking the limits as z \rightarrow 1 of A6.9, we find the Abelian sum
\[
A(1) = -\sum_{n=1}^{\infty} \frac{z^n}{(n)(m_1!)(m_2!)} R_2(q) = \frac{1}{2(z-1)^{1/2} + (1-q)}
\]
If we now define
\[
A(z) = A(1) - \frac{z^2}{2(z-1)^{1/2} + (1-q)} R_2(q)
\]
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