RICE UNIVERSITY

Three Circle Theorems in Partial
Differential Equations and Applications
to Improperly Posed Problems

by

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1. Introduction. For complex valued functions analytic and single valued on an annulus, we know by Hadamard's three circle theorem [16] that the modulus on an intermediate circle must go to zero as we let the bound on the inner circle go to zero, the bound on the outer circle being fixed. Suppose $f(z)$ is analytic and single valued on the annulus $a < |z| < 1$, continuous on the closure, and

$$\begin{aligned} |f(z)| &\leq m = a^\alpha, & |z| &= a, \\ |f(z)| &\leq 1, & |z| &= 1. \end{aligned}$$

Then, Hadamard's result is that, for $a \leq r \leq 1$

$$|f(z)| \leq m^{\frac{\log r}{\log a}} = r^\alpha, \quad |z| = r.$$

Further, this is an optimum bound; for when α , defined by $m = a^\alpha$, is an integer, the power function z^α satisfies the hypotheses and assumes the bound. This theorem establishes the Holder continuous dependence of $f(z)$ on the data for analytic continuation from boundary data on the inner circle, provided $f(z)$ has a prescribed bound on the outer circle.

In this paper analogous three circle theorems are found for harmonic functions on a disc, but with respect to L_2 or even weaker bounds on the inner and outer circles. In fact, it is sufficient that the function correspond to a Schwartz distribution on the outer circle. Analogous three line theorems are found for periodic solutions of the heat equation and for periodic solutions of the Cauchy problem for Laplace's equation. In the process, a very general and flexible method is presented for finding uniform bounds on the solution and its partial derivatives in terms of L_2 bounds on the data. Moreover, these bounds are very precise; in

certain cases they are, except for a factor of at most two, the best possible bounds. The method involves reduction of the problem to a consideration of the Fourier coefficients of the solution and maximization of linear functionals of the coefficients with respect to quadratic constraints.

These results lead to numerical methods for three of the classical improperly posed problems in partial differential equations: harmonic and analytic continuation, backward solution of the heat equation, and the Cauchy problem for Laplace's equation. See F. John [11] for a discussion of the wide class of improperly posed problems for which continuous dependence on data can be restored by restricting attention to those solutions satisfying a prescribed bound. See also pages 226-231 of [3]. If data for harmonic continuation is given on a whole circle, the method involves nothing more than the computation of a certain number of Fourier coefficients. If data is given only at a finite number of interior points, the method involves the solution of a set of linear inequalities. Both methods extend immediately to analytic continuation, in which case the solution is found in Taylor series form. Completely analogous methods are found for backward solution of the heat equation on a rectangle, and the Cauchy problem for Laplace's equation on a rectangle.

For the sake of notational simplicity, all three circle results will be derived on the unit disc and with the prescribed bound on the outer circle normalized. Similarly, for the three line results the period will be 2π and the prescribed bound will be normalized. This is sufficient, since our equations are linear, with constant coefficients.

The following simple result will be used repeatedly.
Maximization lemma. The maximum of the linear functional

$$F \{x_n\} = \sum_{n=0}^K x_n A_n$$

with respect to the quadratic constraint

$$G \{x_n\} = \sum_0^K x_n^2 B_n^2 \leq c^2$$

is

$$(1.1) \quad c \left(\sum_0^K \left(\frac{A_n}{B_n} \right)^2 \right)^{\frac{1}{2}}.$$

Further, the same result is true for $K = \infty$ if (1.1) is convergent. If (1.1) is divergent, then the supremum of such $F \{x_n\}$ is infinity. Proof: If K is finite or (1.1) is convergent, write

$$\sum_0^K x_n A_n = \sum_0^K (x_n B_n) \left(\frac{A_n}{B_n} \right).$$

As

$$x_n^* = c \left(\sum_0^K \left(\frac{A_n}{B_n} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{A_n}{B_n} \right)$$

satisfies the constraint and $F \{x_n^*\}$ assumes the bound (1.1), the result follows from the Schwarz inequality.

If $K = \infty$ and (1.1) is divergent, consider the maximum of F over all sequences $\{x_{n,N}\}$ which terminate after N terms and which satisfy the constraint. This maximum was shown to be

$$c \left(\sum_0^N \left(\frac{A_n}{B_n} \right)^2 \right)^{\frac{1}{2}},$$

which diverges as N increases to infinity. Hence, the lemma is proved.

2. Three circle theorems for harmonic functions. For harmonic functions on the annulus, we see that no theorem similar to Hadamard's is true. In fact, if we demand harmonicity merely on the annulus we do not even have unique dependence upon the data on the inner circle, much less continuous dependence. For example, the data zero has solutions $u = 0$ or $u = ar^{-1} \cos \theta - a^{-1}r \cos \theta$. We therefore consider only functions harmonic on the whole open unit disc. A three circle theorem for the disc follows from Hadamard's theorem. If u is small on the inner disc and bounded on the outer disc, then its harmonic conjugate can be made small on the disc of radius $a - \epsilon$ and bounded on the disc of radius $1 - \epsilon$. Thus we may apply Hadamard's theorem to get bounds on u on intermediate circles. However, these are not the best possible bounds, and they are not clean cut.

Before treating three-circle theorems, let us consider some preliminaries. Harmonic functions on the open unit disc have there the Fourier expansion

$$(2.1) \quad u(r, \theta) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n, \quad r < 1.$$

Denote the uniform norm and the L_2 norm on the circle of radius r as follows:

$$\begin{aligned} |u|_r &= \sup_{\theta} |u(r, \theta)|, \\ (2.2) \quad \|u\|_r &= \left(\frac{1}{2\pi} \int_0^{2\pi} u^2(r, \theta) d\theta \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2} \sum_{n=0}^{\infty} (a_n^2 + b_n^2) r^{2n} \right)^{\frac{1}{2}}, \quad r < 1, \end{aligned}$$

with the convention that $b_0 = 0$. Notice that we have written the

constant term in (2.1) as $\frac{a_0}{\sqrt{2}}$, rather than the usual Fourier series notation $\frac{a_0}{2}$, in order that $\|u\|_r$ may be in the simple form (2.2). Each term of the sum in the L_2 norm is an increasing function of r . Hence, if $\|u\|_r$ is bounded for $r < 1$, the sum converges for $r = 1$, and we may define

$$(2.3) \quad \|u\|_1 = \lim_{r \uparrow 1} \|u\|_r = \sup_{r < 1} \|u\|_r =$$

$$= \left(\frac{1}{2} \sum_0^{\infty} (a_n^2 + b_n^2) \right)^{\frac{1}{2}}.$$

For an integer $N \geq 0$ let the corresponding low and high order parts of u be

$$(2.4) \quad u_N(r, \theta) = \frac{a_0}{\sqrt{2}} + \sum_1^N (a_n \cos n\theta + b_n \sin n\theta) r^n,$$

$$u_N^{\infty}(r, \theta) = \sum_{N+1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n, \quad r < 1.$$

For the real number α , let $[\alpha]$ denote the greatest integer such that $[\alpha] \leq \alpha$.

Let us begin now the consideration of three circle theorems by deriving the optimum L_2 bound for L_2 constraints.

Theorem 1. If u is harmonic for $r < 1$, and if

$$\|u\|_a \leq m = a^{\alpha},$$

$$\|u\|_1 \leq 1,$$

then, for $a \leq r \leq 1$,

$$(2.5) \quad \|u\|_r \leq m^{\frac{\log r}{\log a}} = r^{\alpha}.$$

Further, this is an optimum bound, since $u = \sqrt{2} r^\alpha \cos \alpha \theta$ satisfies the hypotheses and assumes the bound when α is an integer.

Proof: Recall that

$$\|u\|_r^2 = \frac{1}{2} \sum_0^\infty (a_n^2 + b_n^2) r^{2n}.$$

Let

$$A_n = \frac{1}{2}(a_n^2 + b_n^2)$$

and consider the analytic function

$$F(z) = \sum_0^\infty A_n z^{2n}, \quad |z| < 1.$$

Since $A_n \geq 0$, the maximum of F on the complex circle $|z| = r$ is assumed on the real axis; therefore

$$\|u\|_r^2 = F(r) = \max_{|z|=r} |F(z)|.$$

The result follows immediately from Hadamard's theorem.

L_2 bounds are often less interesting for applications than uniform bounds, and we shall turn to the study of uniform bounds. It will not be necessary to prescribe uniform bounds on the constraint circles; somewhat weaker bounds will be sufficient. The result will be based on lemmas bounding the low and high order parts.

Lemma 1. Let $L_N = \{u_N : \|u_N\|_a \leq m\}$, and let

$$L(N, r) = \sup_{u_N \in L_N} \|u_N\|_r.$$

Then,

$$(2.6) \quad L(N, r) = m \left(1 + 2 \sum_1^N \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}.$$

Proof: L_N contains the rotations and the negatives of its member functions. Therefore $L(N, r)$ is the supremum of $u_N(r, 0)$ over L_N . Thus, the problem is to maximize the linear functional

$$u_N(r, 0) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N a_n r^n$$

with respect to the quadratic constraint

$$\|u_N\|_a^2 = \frac{1}{2} \sum_{n=0}^N (a_n^2 + b_n^2) r^{2n} \leq m^2.$$

It is sufficient to maximize $u_N(r, 0)$ over the subclass for which $b_n = 0$. Therefore, by the maximization lemma,

$$\max_{u_N \in L_N} u_N(r, 0) = m\sqrt{2} \left(\frac{1}{2} + \sum_{n=1}^N \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}.$$

Lemma 2. Let $H_N = \left\{ u_N^\infty : \|u_N^\infty\|_1 \leq 1 \right\}$,

and let

$$H(N, r) = \sup_{u_N^\infty \in H_N} |u_N^\infty|_r.$$

Then,

$$(2.7) \quad H(N, r) = \left(2 \sum_{n=N+1}^{\infty} r^{2n} \right)^{\frac{1}{2}}.$$

Proof: Exactly as in the previous lemma, the problem reduces to the maximization of the linear functional

$$u_N^\infty(r, 0) = \sum_{n=N+1}^{\infty} a_n r^n$$

with respect to the quadratic constraint

$$\|u_N^\infty\|_1^2 = \frac{1}{2} \sum_{N+1}^{\infty} a_n^2 \leq 1.$$

Therefore, by the maximization lemma,

$$\max_{u_N^\infty \in H_N} u_N^\infty(r, 0) = \left(2 \sum_{N+1}^{\infty} r^{2n} \right)^{\frac{1}{2}}.$$

Theorem 2. Let Γ be the class of harmonic functions $u(r, \theta)$ on $r < 1$ which satisfy

$$(2.8) \quad \begin{aligned} (a) \quad & \|u\|_a \leq m \leq 1, \\ (b) \quad & \|u\|_1 \leq 1. \end{aligned}$$

For $a \leq r < 1$, let

$$M_r = \sup_{u \in \Gamma} |u|_r.$$

Then,

$$(2.9) \quad \max(L, H) \leq M_r \leq L + H,$$

where

$$(2.10) \quad \begin{aligned} L &= L([\alpha], r) = m \left(1 + 2 \sum_1^{[\alpha]} \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}, \\ H &= H([\alpha], r) = \left(2 \sum_{[\alpha+1]}^{\infty} r^{2n} \right)^{\frac{1}{2}} \end{aligned}$$

and where α is defined by $m = a^\alpha$.

Moreover,

$$(2.11) \quad L + H < m^{\frac{\log r}{\log a}} \left[\left(2 \sum_0^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \left(\frac{2}{1-r^2} \right)^{\frac{1}{2}} \right]$$

Proof: Consider $u \in \Gamma$. Let $N \geq 0$ and write

$$u = u_N + u_N^\infty.$$

We notice that both the low and high order parts of u are also in Γ , since

$$\begin{aligned}
 (aL) \quad \|u_N\|_a^2 &= \frac{1}{2} \sum_0^N (a_n^2 + b_n^2) a^{2n} \leq m = a^2, \\
 (bL) \quad |u_N|_1^2 &= \frac{1}{2} \sum_0^N (a_n^2 + b_n^2) \leq 1; \\
 (2.12) \quad (aH) \quad |u_N^\infty|_a^2 &= \frac{1}{2} \sum_{N+1}^\infty (a_n^2 + b_n^2) a^{2n} \leq m^2 = a^{2\alpha}, \\
 (bH) \quad |u_N^\infty|_1^2 &= \frac{1}{2} \sum_{N+1}^\infty (a_n^2 + b_n^2) \leq 1.
 \end{aligned}$$

By (2.12 aL), u_N is in the class L_N of lemma 1. By (2.12 aH), u_{N+1}^∞ is in the class H_N of lemma 2. We therefore have, for any N ,

$$M_r \leq L(N, r) + H(N, r).$$

In particular this is true for $N = [\alpha]$, so the upper bound of our theorem has been obtained.

We consider the lower bound. Let Γ_N be the class of low order harmonic functions satisfying (2.12 aL) and (2.12 bL), and let

Γ_N^∞ be the class of high order harmonic functions satisfying (2.12 aH) and (2.12 bH). Let ℓ_N and h_N^∞ be the supremums of $|u_N|_r$ and $|u_N^\infty|_r$ for all functions in Γ_N and Γ_N^∞ respectively. Γ_N and Γ_N^∞ are subclasses of Γ ; hence,

$$\ell_N \leq M_r ,$$

$$h_N^\infty \leq M_r ,$$

or,

$$(2.13) \quad \max (\ell_N, h_N^\infty) \leq M_r .$$

Evaluation of ℓ_N or h_N^∞ would involve the maximization of a linear functional of the coefficients with respect to two quadratic constraints. Fortunately, for the proper choice of N one of the constraints implies the other, so we need maximize only with respect to one constraint.

In (2.12 aL) and (2.12 aH) we bring the $a^{2\alpha}$ to the left and have

$$(2.14) \quad \begin{aligned} \text{(aL)} \quad & \frac{1}{2} \sum_0^N (a_n^2 + b_n^2) a^{2(n-\alpha)} \leq 1 , \\ \text{(aH)} \quad & \frac{1}{2} \sum_{N+1}^{\infty} (a_n^2 + b_n^2) a^{2(n-\alpha)} \leq 1 . \end{aligned}$$

For $N \leq \alpha$, all the terms $a^{2(n-\alpha)}$ in (aL) are greater than one; hence, the inner constraint (aL) implies the outer constraint (bL), and Γ_N coincides with L_N of lemma 1. Therefore,

$$(2.15) \quad \ell_N = L(N, r), \quad N \leq \alpha .$$

For $N+1 \geq \alpha$, all the terms $a^{2(n-\alpha)}$ in (aH) are less than one; hence, the outer constraint (bH) implies the inner constraint (aH). Therefore,

$$(2.16) \quad h_N^\infty = H(N, r), \quad N+1 \geq \alpha .$$

Since $[\alpha] \leq \alpha < [\alpha+1]$, the lower bound of our theorem follows from (2.13), (2.15), and (2.16) for the case $N = [\alpha]$.

Finally, we manipulate the bounds (2.10) into more convenient form (2.11).

Recall that $m^{\frac{\log r}{\log a}} = r^\alpha$. From (2.10),

$$\begin{aligned} L &< m \left(2 \sum_0^{[\alpha]} \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}, \\ &= r^\alpha \left(\frac{a}{r} \right)^\alpha \left(\frac{r}{a} \right)^{[\alpha]} \left(2 \sum_0^{[\alpha]} \left(\frac{r}{a} \right)^{2(n-[\alpha])} \right)^{\frac{1}{2}} \\ &< r^\alpha \left(2 \sum_0^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}}. \end{aligned}$$

Also,

$$\begin{aligned} H &= r^{[\alpha+1]} \left(2 \sum_0^\infty r^{2n} \right)^{\frac{1}{2}} \\ &< r^\alpha \left(\frac{2}{1-r^2} \right)^{\frac{1}{2}}. \end{aligned}$$

The bound (2.11) follows, and the proof is completed.

It follows from (2.9) that

$$(2.17) \quad \frac{1}{2} (L + H) \leq M_r \leq (L + H);$$

i.e., $L + H$ is, except for a factor of at most two, the best possible bound.

Suppose we had assumed uniform rather than L_2 constraints. That is, suppose (2.12 a) replaced by $|u|_a \leq m = a^\alpha$ and (2.12 b) replaced by $|u|_1 \leq 1$. Let M_r' be the supremum of $|u|_r$ over the corresponding class. If α is an integer, then $u = r^\alpha \cos \alpha \theta$ satisfies the constraints; hence, M_r' must be at least as great as $r^\alpha = m^{\frac{\log r}{\log a}}$.

Therefore the ratio M_r/M_r' is no greater than the value of the bracket

$$\left[\left(2 \sum_0^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \left(\frac{2}{1-r^2} \right)^{\frac{1}{2}} \right]$$

from (2.11). We see that this bracket tends essentially to

$$(2.18a) \quad (2\alpha + 2)^{\frac{1}{2}} = (2(\log m / \log a) + 2)^{\frac{1}{2}}$$

as r decreases to a , is bounded by

$$(2.18b) \quad \left(\frac{2}{1 - (a/r)^2} \right)^{\frac{1}{2}} + \left(\frac{2}{1 - r^2} \right)^{\frac{1}{2}}$$

independently of m for $a < r < 1$, and tends to infinity like

$$(2.18c) \quad \left(\frac{1}{1-r} \right)^{\frac{1}{2}}$$

as r tends to 1. Therefore, we conclude roughly that the bound for L_2 constraints differs greatly from the bound for uniform constraints only near the boundary $r = 1$.

We may obtain three circle theorems for more general constraints on the outer circle. In fact, it is sufficient that the boundary values of u correspond to a Schwartz distribution. See pages 795 and 797 of [3]. Notice that $\|u\|_1 \leq 1$ implies

$$|a_n|, |b_n|, \leq \sqrt{2}, \quad n \geq 1.$$

Thus, Γ is contained in C_0 , where the general class C_k is defined in the following theorem. Notice that C_0 also includes harmonic functions with delta functions for boundary values, for example, the kernel of Poisson's integral on the unit disc.

Theorem 3. For an integer k , let C_k be the class of harmonic functions on $r < 1$ which satisfy

$$(2.19) \quad \begin{aligned} & (a) \|u\|_a \leq m = a^\alpha \leq 1, \\ & (b) |a_n|, |b_n| \leq \sqrt{2} n^k, \quad n \geq 1. \end{aligned}$$

Then, for $u \in C_k$ and $a \leq r < 1$,

$$(2.20) \quad |u|_r < m^{\frac{\log r}{\log a}} \left[\left(2 \sum_{n=0}^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + 2 \sum_{n=0}^{\infty} (n+[\alpha+1])^k r^n \right].$$

Moreover, for $k > 0$,

$$(2.21) \quad |u|_r < m^{\frac{\log r}{\log a}} (\alpha+1)^k B_k(r),$$

where

$$B_k(r) = 4 \sum_{n=0}^{\infty} (1+n)^k r^n < \infty, \quad a \leq r < 1.$$

For $k = 0$ or $k = -1$,

$$(2.22) \quad |u|_r < m^{\frac{\log r}{\log a}} \left[\left(2 \sum_{n=0}^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \frac{2}{1-r} \right].$$

For $k \leq -2$,

$$(2.23) \quad |u|_r < m^{\frac{\log r}{\log a}} \left[\left(2 \sum_{n=0}^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \frac{\pi^2}{3} \right].$$

Proof: We write

$$u = u_{[\alpha]} + u_{[\alpha]}^{\infty}.$$

It follows from (2.19 a) and lemma 1, as in theorem 2, that $|u_{[\alpha]}|_r$ is bounded by the first term in (2.20). From (2.19 b),

$$\begin{aligned}
(2.24) \quad |u_{[\alpha]}^{\infty}(r, \theta)| &= \left| \sum_{[\alpha+1]}^{\infty} (a_n \cos n \theta + b_n \sin n \theta) r^n \right| \\
&\leq \sum_{[\alpha+1]}^{\infty} (a_n^2 + b_n^2)^{\frac{1}{2}} r^n \leq \sum_{[\alpha+1]}^{\infty} 2n^k r^n \\
&\leq r^{[\alpha+1]} 2 \sum_{[\alpha+1]}^{\infty} n^k r^{n-[\alpha+1]} \\
&< r^{\alpha} 2 \sum_0^{\infty} (n + [\alpha+1])^k r^n.
\end{aligned}$$

Therefore, (2.20) is proved.

Then (2.21), (2.22), and (2.23) follow from (2.20). For $k > 0$,

$$\begin{aligned}
&\left[\left(2 \sum_0^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + 2 \sum_0^{\infty} (n + [\alpha+1])^k r^n \right] \\
&\leq (2[\alpha+1])^{\frac{1}{2}} + 2[\alpha+1]^k \sum_0^{\infty} \left(1 + \frac{n}{[\alpha+1]} \right)^k r^n \\
&< 4(\alpha+1)^k \sum_0^{\infty} (1+n)^k r^n;
\end{aligned}$$

hence, (2.21) follows. In like manner, (2.22) follows. Finally, (2.23) is an immediate result of (2.20) and the identity

$$\sum_1^{\infty} n^{-2} = \frac{\pi^2}{6},$$

and the theorem is proved.

Not only u , but also all of its partial derivatives, tend to zero on intermediate circles as we let the inner constraint approach zero.

Lemma 3. If u is in the class Γ of theorem 2, then

$$(2.25) \quad \left| \frac{\partial^{i+j} u}{\partial r^i \partial \theta^j} \right|_r < \frac{\log r}{m \log a} (\alpha+1)^{i+j+\frac{1}{2}} B_{i,j}(r),$$

where

$$B_{i,j}(r) = \frac{2}{r^i} \left(2 \sum_0^{\infty} (1+n)^{2(i+j)} r^{2n} \right)^{\frac{1}{2}} < \infty,$$

$$a \leq r < 1.$$

Proof: Again, it is sufficient to find a bound at the point $(r, 0)$. Differentiating the series (2.1) for u , we have,

$$\begin{aligned} \frac{\partial^{i+j} u}{\partial r^i \partial \theta^j} (r, 0) &= \\ &= \sum_1^{\infty} A_n n^j (n)(n-1) \dots (n-i+1) r^{n-i}, \end{aligned}$$

where A_n is either a_n , b_n , $-a_n$, or $-b_n$, as $j \pmod{4}$ is either 0, 1, 2, or 3. Suppose, for example, that $j \pmod{4}$ is 0. Then $A_n \equiv a_n$ and

$$(2.26) \quad \left| \frac{\partial^{i+j} u}{\partial r^i \partial \theta^j} (r, 0) \right| < \frac{1}{r^i} \sum_1^{[\alpha]} |a_n| n^{i+j} r^n$$

$$+ \frac{1}{r^i} \sum_{[\alpha+1]}^{\infty} |a_n| n^{i+j} r^n.$$

Consider the constraints (2.8) for α in the form (2.12). The maximum of the first term of (2.26) with respect to

$$(2.27) \quad (aL) \quad \sum_1^{[\alpha]} a_n^2 a^{2n} \leq 2 m^2$$

is

$$(2.28) \quad m \left(2 \sum_1^{[\alpha]} n^{2(i+j)} \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}} \\ < m r^{[\alpha]} [\alpha]^{i+j} \frac{1}{r^i} \left(2 \sum_1^{[\alpha]} \left(\frac{a}{r} \right)^{2([\alpha]-n)} \right)^{\frac{1}{2}} \\ < m \frac{\log r}{\log a} [\alpha]^{i+j+\frac{1}{2}} \frac{\sqrt{2}}{r^i}.$$

The maximum of the second term of (2.26) with respect to

$$(2.29) \quad (bH) \quad \sum_{[\alpha+1]}^{\infty} a_n^2 \leq 2$$

is

$$(2.30) \quad \frac{1}{r^i} \left(2 \sum_{[\alpha+1]}^{\infty} n^{2(i+j)} r^{2n} \right)^{\frac{1}{2}} \\ < \frac{r^{[\alpha+1]}}{r^i} \left(2 \sum_0^{\infty} (n+[\alpha+1])^{2(i+j)} r^{2n} \right)^{\frac{1}{2}} \\ m \frac{\log r}{\log a} [\alpha+1]^{i+j} \frac{1}{r^i} 2 \sum_0^{\infty} (1+n)^{2(i+j)} r^{2n} \quad \frac{1}{2}$$

Finally, (2.25) follows easily from (2.28) and (2.30).

The unbounded character of $|u|_r$ in theorem 2 as r approaches one, as shown by (2.18 c), points out the fact that L_2 bounds imply good uniform bounds only where the function is smooth. We can impose this smoothness by prescribing a bound on the L_2 norm of either the tangential or the radial derivative of u on the outer circle. Notice that

$$(2.31) \quad \|u_r\|_1 = \|u_\theta\|_1 = \left(\frac{1}{2} \sum_1^\infty n^2 (a_n^2 + b_n^2) \right)^{\frac{1}{2}}.$$

Lemma 4. If u is harmonic for $r < 1$, and if

$$(2.32) \quad \begin{aligned} (a) \quad & \|u\|_a \leq m = a^\alpha \leq 1, \\ (b) \quad & \|u_r\|_1 = \|u_\theta\|_1 \leq 1, \end{aligned}$$

then, for $a \leq r < 1$,

$$(2.33) \quad |u|_r \leq m^{\frac{\log r}{\log a}} \left[\left(2 \sum_0^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \frac{\pi^2}{6} \right].$$

Proof: The bound for $|u_{[\alpha]}|_r$ follows from (2.32 a) as in theorem 2. We find a bound for $|u_{[\alpha]}|_r$ with respect to (2.32 b), using the formula (2.31). The desired bound,

$$|u_{[\alpha]}|_r \leq m^{\frac{\log r}{\log a}} \left(\sum_1^\infty \frac{1}{n^2} \right) = m^{\frac{\log r}{\log a}} \frac{\pi^2}{6},$$

follows immediately, and the lemma is proved.

3. Three line theorems for periodic solutions of the heat equation. We consider solutions of the heat equation on the upper half plane which are periodic in x with period 2π . That is, consider functions $u(x, t)$ satisfying

$$(3.1) \quad \begin{aligned} (a) \quad & u(x, t) \in C^2, \quad t > 0, \\ (b) \quad & u_{xx} = u_t, \quad t > 0, \\ (c) \quad & u(x, t) \text{ is periodic in } x \text{ with period } 2\pi. \end{aligned}$$

The functions of this class have the representation

$$(3.2 \text{ a}) \quad u(x, t) = \frac{a_0}{\sqrt{2}} + \sum_1^{\infty} (a_n \cos n x + b_n \sin n x) e^{-n^2 t}, \quad t > 0,$$

where $\{a_n, b_n\}$ is any sequence such that

$$(3.2 \text{ b}) \quad \sum_1^{\infty} (a_n^2 + b_n^2) e^{-2n^2 t} < \infty, \quad t > 0.$$

The proof of this representation is as follows. Given $\{a_n, b_n\}$ satisfying (3.2 b), then the sum (3.2 a) is a solution of (3.1), since it converges subuniformly on $t > 0$ and its individual terms are solutions of (3.1).

Conversely, given a solution $u(x, t)$ of (3.1), let $a_{n, \epsilon} e^{-n^2 \epsilon}$ and $b_{n, \epsilon} e^{-n^2 \epsilon}$ denote the Fourier coefficients of $u(x, \epsilon)$ for some $\epsilon > 0$. Since $u(x, \epsilon)$ is in C^2 , its Fourier series converges uniformly; hence,

$$u_{\epsilon}(x, t) = \frac{a_{0, \epsilon}}{\sqrt{2}} + \sum_1^{\infty} (a_{n, \epsilon} \cos n x + b_{n, \epsilon} \sin n x) e^{-n^2 t},$$

$t \geq \epsilon,$

is a solution of the initial value problem on $t \geq \epsilon$ for the data $u(x, \epsilon)$. By the uniqueness for this problem,

$$u_{\epsilon}(x, t) = u(x, t), \quad t \geq \epsilon.$$

Clearly, the coefficients

$$a_n = a_{n,\varepsilon}, \quad b_n = b_{n,\varepsilon},$$

are independent of ε ; for if $0 < \varepsilon' < \varepsilon$, then

$$u_{\varepsilon'}(x, \varepsilon) = u(x, \varepsilon) = u_{\varepsilon}(x, \varepsilon) .$$

Thus (3.2 a) is proved. Finally, define the L_2 norm on the horizontal line at time t

$$\begin{aligned} (3.3) \quad \|u\|_t &= \left(\frac{1}{2\pi} \int_0^{2\pi} u^2(x, t) dx \right)^{\frac{1}{2}}, \\ &= \left(\frac{1}{2} \sum_0^{\infty} (a_n^2 + b_n^2) e^{-2n^2 t} \right)^{\frac{1}{2}}, \quad t > 0. \end{aligned}$$

Since u is continuous, this integral is finite for $t > 0$; hence (3.2 b) is established and the representation is proved.

For such periodic solutions of the heat equation we will derive three line theorems which correspond exactly to the three circle theorems of section 2. Only the series representation has been changed ($e^{-n^2 t}$ of this section corresponds to r^n of section 2), and in most cases we will merely point out the similarity to the corresponding proof of section 2.

Let $|u|_t$ denote the uniform norm on the horizontal line at time t . We may again define

$$(3.4) \quad \|u\|_0 = \lim_{t \downarrow 0} \|u\|_t = \left(\frac{1}{2} \sum_0^{\infty} (a_n^2 + b_n^2) \right)^{\frac{1}{2}} .$$

We again write $u = u_N + u_N^{\infty}$, where these low and high order parts are defined analogously to (2.4).

We have first the theorem corresponding to theorem 1.

Theorem 4. If $u(x, t)$ is a solution of the heat equation for $t > 0$, periodic in x with period 2π , and if

$$\begin{aligned}\|u\|_a &\leq m = e^{-\alpha^2 a} \leq 1, \\ \|u\|_0 &\leq 1,\end{aligned}$$

then for $0 \leq t \leq a$,

$$(3.5) \quad \|u\|_t \leq m^{\frac{t}{a}} = e^{-\alpha^2 t}.$$

Further, this is an optimum bound, since $u = \sqrt{2} e^{-\alpha^2 t} \cos \alpha x$ satisfies the hypotheses and assumes the bound when α is an integer.

The proof follows exactly as the proof of theorem 1, except that we let $|z| = e^{-t}$ instead of $|z| = r$.

Lemma 5. Let $L_N = \{u_N : \|u_N\|_a \leq m\}$, and let

$$L(N, r) = \sup_{u_N \in L_N} |u_N|_r.$$

Then,

$$(3.6) \quad L(N, r) = m \left(1 + 2 \sum_{n=1}^N e^{2n^2(a-t)} \right)^{\frac{1}{2}}.$$

The proof is completely analogous to the proof of lemma 1.

Lemma 6. Let $H_N = \{u_N^\infty : \|u_N^\infty\|_0 \leq 1\}$, and let

$$H(N, r) = \sup_{u_N^\infty \in H_N} |u_N^\infty|_r.$$

Then,

$$(3.7) \quad H(N, r) = \left(2 \sum_{n=N+1}^{\infty} e^{-2n^2 t} \right)^{\frac{1}{2}}.$$

The proof is completely analogous to the proof of lemma 2.

Theorem 5. Let Γ be the class of solutions $u(x, t)$ of the heat equation on $t > 0$ which are periodic in x with period 2π , and which satisfy

$$(3.8) \quad \begin{aligned} (a) \quad \|u\|_a &\leq m \leq 1, \\ (b) \quad \|u\|_0 &\leq 1. \end{aligned}$$

For $0 < t \leq a$, let

$$M_t = \sup_{u \in \Gamma} |u|_r.$$

Then,

$$(3.9) \quad \max(L, H) \leq M_t \leq L + H,$$

where

$$(3.10) \quad \begin{aligned} L &= L([\alpha], t) = m \left(1 + 2 \sum_{n=1}^{[\alpha]} e^{2n^2(a-t)} \right)^{\frac{1}{2}}, \\ H &= H([\alpha], t) = \left(2 \sum_{n=[\alpha]+1}^{\infty} e^{-2n^2 t} \right)^{\frac{1}{2}}, \end{aligned}$$

and where α is defined by $m = e^{-\alpha^2 a}$.

Moreover,

$$(3.11) \quad \begin{aligned} L + H &< m^{\frac{t}{a}} \left[\left(2 \sum_{n=0}^{[\alpha]} e^{-2n^2(a-t)} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(2 \sum_{n=0}^{\infty} e^{-2n^2 t} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

The proof for (3.9) and (3.10) is completely analogous to the proof for (2.9) and (2.10) of theorem 2. Again, for $N = [\alpha]$, the low order inner constraint (aL) implies the outer constraint (bL), and the high order outer constraint (bH) implies the inner constraint (aH). Thus, in each case, we need maximize with respect to only one quadratic constraint. Finally, (3.11) follows easily from (3.9) and (3.10).

Theorem 6 . For an integer k , let C_k be the class of solutions $u(x, t)$ of the heat equation on $t > 0$ which are periodic in x with period 2π , and which satisfy

$$\begin{aligned} ((3.12) \quad (a) \quad \|u\|_a &\leq m = e^{-\alpha^2 a} \leq 1 \\ (b) \quad |a_n|, |b_n| &\leq \sqrt{2} n^k, \quad n \geq 1. \end{aligned}$$

Then, for $u \in C_k$ and $0 < t \leq a$,

$$\begin{aligned} (3.13) \quad |u|_t &< m^{\frac{t}{a}} \left[\left(2 \sum_{n=0}^{[\alpha]} e^{-2n^2(a-t)} \right)^{\frac{1}{2}} + \right. \\ &\quad \left. + 2 \sum_{n=0}^{\infty} (n + [\alpha+1])^k e^{-n^2 t} \right]. \end{aligned}$$

Moreover, for $k > 0$,

$$(3.14) \quad |u|_t < m^{\frac{t}{a}} (\alpha+1)^k B_k(t),$$

where

$$B_k(t) = 4 \sum_{n=0}^{\infty} (1+n)^k e^{-n^2 t} < \infty, \quad 0 < t \leq a.$$

For $k = 0$ or $k = -1$,

$$(3.15) \quad |u|_t < m^{\frac{t}{a}} \left[\left(2 \sum_0^{[\alpha]} e^{-2n^2(a-t)} \right)^{\frac{1}{2}} + \sum_0^{\infty} e^{-n^2 t} \right].$$

For $k \leq -2$,

$$(3.16) \quad |u|_t < m^{\frac{t}{a}} \left[\left(2 \sum_0^{[\alpha]} e^{-2n^2(a-t)} \right)^{\frac{1}{2}} + \frac{\pi^2}{3} \right].$$

The proof of this theorem is completely analogous to the proof of theorem 3 in section 2.

One may also easily obtain the results analogous to lemma 3 and lemma 4.

4. Three line theorems for periodic solutions of the Cauchy problem for Laplace's equation. We consider functions $u(x, y)$ harmonic on an infinite strip $0 < y < a$ and periodic in x with period 2π . The functions of this class have the representation

$$(4.1 \text{ a}) \quad u(x, y) = \frac{a_0 + c_0 y}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cosh n y + c_n \sinh n y) \cos n x + \sum_{n=1}^{\infty} (b_n \cosh n y + d_n \sinh n y) \sin n x ,$$

where $\{a_n, b_n, c_n, d_n\}$ is any sequence such that

$$(4.1 \text{ b}) \quad \sum_{n=1}^{\infty} (A_{n,y}^2 + B_{n,y}^2) < \infty, \quad 0 < y < a ,$$

and where $A_{n,y}$ and $B_{n,y}$ denote the sine and cosine coefficients

$$(4.1 \text{ c}) \quad \begin{aligned} A_{n,y} &= a_n \cosh n y + c_n \sinh n y, \quad n \geq 1, \\ B_{n,y} &= b_n \cosh n y + d_n \sinh n y, \quad n \geq 1. \end{aligned}$$

The proof of this representation follows in like manner to the proof of the representation in section 3, although being slightly more complicated. In the proof we use the inequality

$$(4.2) \quad |A_{n,y}| \leq |A_{n,y_1}| e^{-n(y-y_1)} + |A_{n,y_2}| e^{-n(y_2-y)}, \quad y_1 < y < y_2 ,$$

which follows from the identity

$$(4.3) \quad A_{n,y} = A_{n,y_1} \frac{\sinh n(y_2 - y)}{\sinh n(y_2 - y_1)} + A_{n,y_2} \frac{\sinh n(y - y_1)}{\sinh n(y_2 - y_1)} .$$

Notice that if u satisfies a certain L_2 constraint, then so do $(a_0 + c_0 y)/2$ and $u = (a_0 + c_0 y)/2$. Therefore, without loss of generality, we may assume that the mean value term $(a_0 + c_0 y)$ is zero. That is, we consider only harmonic functions of the form

$$(4.4. a) \quad u(x,y) = \sum_{n=1}^{\infty} (a_n \cosh n y + c_n \sinh n y) \cos n x + \sum_{n=1}^{\infty} (b_n \cosh n y + d_n \sinh n y) \sin n x, \quad 0 < y < a;$$

$$(4.4 b) \quad \begin{aligned} A_{n,y} &= a_n \cosh n y + c_n \sinh n y, \\ B_{n,y} &= b_n \cosh n y + d_n \sinh n y; \end{aligned}$$

$$(4.4 c) \quad \|u\|_y = \left(\frac{1}{2} \sum_{n=1}^{\infty} (A_{n,y}^2 + B_{n,y}^2) \right)^{\frac{1}{2}} < \infty, \quad 0 < y < a .$$

We see that $A_{n,y}^2 + B_{n,y}^2$ is a positive convex function, which therefore takes on its maximum at the end points 0 or a . If $\|u\|_y$ is bounded on $0 < y < a$, the sum converges uniformly to a continuous convex function on $0 \leq y \leq a$. Thus, we may define

$$\begin{aligned}
 \|u\|_0 &= \lim_{y \downarrow 0} \|u\|_y = \left(\frac{1}{2} \sum_1^{\infty} (A_{n,0}^2 + B_{n,0}^2) \right)^{\frac{1}{2}} \\
 (4.5) \quad \|u\|_a &= \lim_{y \uparrow a} \|u\|_y = \left(\frac{1}{2} \sum_1^{\infty} (A_{n,a}^2 + B_{n,a}^2) \right)^{\frac{1}{2}}.
 \end{aligned}$$

Notice that, formally

$$\begin{aligned}
 u(x,0) &= \sum_1^{\infty} (a_n \cos n x + b_n \sin n x), \\
 u_y(x,0) &= \sum_1^{\infty} (n c_n \cos n x + n d_n \sin n x), \\
 u_x(x,0) &= \sum_1^{\infty} (-n a_n \sin n x + n b_n \cos n x).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|u\|_0 &= \left(\frac{1}{2} \sum_1^{\infty} (a_n^2 + b_n^2) \right)^{\frac{1}{2}}, \\
 (4.6) \quad \|u_y\|_0 &= \left(\frac{1}{2} \sum_1^{\infty} (n^2 c_n^2 + n^2 d_n^2) \right)^{\frac{1}{2}}, \\
 \|u_x\|_0 &= \left(\frac{1}{2} \sum_1^{\infty} (n^2 a_n^2 + n^2 b_n^2) \right)^{\frac{1}{2}}
 \end{aligned}$$

It turns out to be better, for the sake of greater generality and symmetry, to hypothesize constraints on

$$\sum_1^{\infty} (c_n^2 + d_n^2) \text{ instead of on } \sum_1^{\infty} (n^2 c_n^2 + n^2 d_n^2).$$

Now, if u is of the form (4.4), let $D^{-1} u_y$ denote the termwise x integral of u_y ,

$$\begin{aligned}
 (4.7 \text{ a}) \quad D^{-1} u_y(x, y) = & \sum_{n=1}^{\infty} (a_n \sinh n y + c_n \cosh n y) \sin n x \\
 & - \sum_{n=1}^{\infty} (b_n \sinh n y + d_n \cosh n y) \cos n x, \quad 0 < y < a
 \end{aligned}$$

which we immediately recognize as the harmonic conjugate of $-u$.

Thus $D^{-1} u_y$ is also harmonic on the strip $0 < y < a$, and as in (4.5) we may write

$$(4.7 \text{ b}) \quad \|D^{-1} u_y\|_0 = \lim_{\epsilon \downarrow 0} \|D^{-1} u_y\|_{\epsilon} = \left(\frac{1}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2) \right)^{\frac{1}{2}}.$$

If $\|D^{-1} u_y\|_0$ is finite, we may say that the initial values of the harmonic conjugate of u corresponds to an L_2 function, or (more meaningfully for the Cauchy problem) that the initial values of u_y correspond to a Schwartz distribution which is the formal x derivative of an L_2 function. See [3], pages 795 and 797.

Now, the norm $\|D^{-1} u_y\|_0$ for u_y is in the more general and symmetric form desired, but its representation (4.7) merely in terms of the Fourier coefficients c_n and d_n is not always useful. However, suppose $u_y(x, 0) = f(x)$, where $f(x)$ is at least an L_1 function. Then $f(x)$ has a Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} (A_n \cos n x + B_n \sin n x)$$

and by Fatou's theorem [10] its termwise integral

$$D^{-1} f(x) = \sum_{n=1}^{\infty} \frac{A_n \sin n x - B_n \cos n x}{n}$$

converges uniformly to the continuous function

$$(4.8 \text{ a}) \quad D^{-1}f(x) = \int_0^x f(t) dt - \frac{1}{2\pi} \int_0^{2\pi} \int_0^x f(t) dt dx ,$$

in other words, to that indefinite integral of $f(x)$ which itself has mean value zero. Since indefinite integrals differ at most by a constant and this particular one has a zero constant term in its series, it has a smaller L_2 norm than any other indefinite integral. Therefore

$$(4.8 \text{ b}) \quad \|D^{-1}f\| \leq \left\| \int_c^x f(t) dt \right\|$$

for an arbitrary constant c . Also, $\|D^{-1}f\|$ may be bounded by a constant multiple of the L_1 norm of f . Let

$$\|f\| = \frac{1}{2\pi} \int_0^{2\pi} |f(x)| dx .$$

Then

$$|A_n| = \left| \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \right| \leq 2\|f\| ,$$

$$|B_n| = \left| \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \right| \leq 2\|f\| .$$

Hence

$$\begin{aligned} \|D^{-1}f\| &= \left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{A_n^2 + B_n^2}{n^2} \right)^{\frac{1}{2}} \\ &\leq 2\|f\| \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} = 2\|f\| \left(\frac{\pi^2}{6} \right)^{\frac{1}{2}} . \end{aligned}$$

Now, let us find a uniform bound for the Cauchy problem for L_2 constraints on the data.

Theorem 7. If $u(x,y)$ is of the form (4.4), and if

$$(4.9) \quad \begin{aligned} (a) \quad & \|u\|_0 \leq m = e^{-\alpha a} \leq 1, \\ (b) \quad & \|D^{-1}u_y\|_0 \leq m, \\ (c) \quad & \|u\|_a \leq 1, \end{aligned}$$

then, for $0 < y < a$,

$$(4.10) \quad |u|_y < m^{\frac{1-y}{a}} \left[\left(\frac{5}{2} \sum_0^{[\alpha-1]} e^{-2ny} \right)^{\frac{1}{2}} + \right. \\ \left. + m^{\frac{2y}{a}} \left(\frac{2}{1 - e^{-2y}} \right)^{\frac{1}{2}} + \left(\frac{2}{1 - e^{-2(a-y)}} \right)^{\frac{1}{2}} \right].$$

Proof: Since the translations and negatives of such functions also satisfy the constraints, the problem reduces to finding a bound for the linear functional

$$u(0,y) = \sum_1^{\infty} A_{n,y} = \sum_1^{\infty} (a_n \cosh n y + c_n \sinh n y)$$

with respect to the quadratic constraints

$$(4.11) \quad \begin{aligned} (a) \quad & \sum_1^{\infty} A_{n,0}^2 = \sum_1^{\infty} a_n^2 \leq 2m^2, \\ (b) \quad & \sum_1^{\infty} c_n^2 \leq 2m^2, \\ (c) \quad & \sum_1^{\infty} A_{n,a}^2 \leq 2. \end{aligned}$$

Using (4.2), we write

$$(4.12) \quad u(0, y) = \sum_1^{[\alpha]} a_n \cosh n y + \sum_1^{[\alpha]} c_n \sinh n y \\ + \sum_{[\alpha+1]}^{\infty} a_n \frac{\sinh n(a-y)}{\sinh n a} + \sum_{[\alpha+1]}^{\infty} A_{n,a} \frac{\sinh n y}{\sinh n a}.$$

Then we apply the maximization lemma. The maximum of the first term with respect to (4.11 a) is

$$\begin{aligned} & m \left(2 \sum_1^{[\alpha]} \cosh^2 n y \right)^{\frac{1}{2}} \\ & < m e^{[\alpha]y} \left(2 \sum_0^{[\alpha-1]} e^{-2ny} \right)^{\frac{1}{2}} \\ & \leq m^{1-\frac{y}{a}} \left(2 \sum_0^{[\alpha-1]} e^{-2ny} \right)^{\frac{1}{2}}. \end{aligned}$$

The maximum of the second term with respect to (4.11 b) is

$$\begin{aligned} & m \left(2 \sum_1^{[\alpha]} \sinh^2 n y \right)^{\frac{1}{2}} \\ & < m^{1-\frac{y}{a}} \left(\frac{1}{2} \sum_1^{[\alpha]} e^{-2ny} \right)^{\frac{1}{2}}. \end{aligned}$$

The maximum of the third term with respect to (4.11 a) is

$$\begin{aligned}
& m \left(2 \sum_{[+1]}^{\infty} \left(\frac{\sinh n(a-y)}{\sinh n a} \right)^2 \right)^{\frac{1}{2}} \\
& < m e^{-[\alpha+1]} \left(\sum_0^{\infty} e^{-2ny} \right)^{\frac{1}{2}} \\
& m^{1-\frac{y}{a}} m^{\frac{2y}{a}} \left(\frac{2}{1 - e^{-2y}} \right)^{\frac{1}{2}} .
\end{aligned}$$

The maximum of the fourth term with respect to (4.11 c) is

$$\begin{aligned}
& \left(2 \sum_{[+1]}^{\infty} \left(\frac{\sinh n y}{\sinh n a} \right)^2 \right)^{\frac{1}{2}} \\
& < e^{-[\alpha+1](a-y)} \left(2 \sum_0^{\infty} e^{-2n(a-y)} \right)^{\frac{1}{2}} \\
& \leq m^{1-\frac{y}{a}} \left(\frac{2}{1 - e^{-2(a-y)}} \right)^{\frac{1}{2}} .
\end{aligned}$$

The conclusion then follows by addition.

The L_2 constraints (4.9) imply the termwise constraints

$$\begin{aligned}
(4.13) \quad & (a) \quad |a_n|, |b_n| \leq \sqrt{2} m, \\
& (b) \quad |c_n|, |d_n| \leq \sqrt{2} m, \\
& (c) \quad |A_{n,a}|, |B_{n,a}| \leq \sqrt{2}
\end{aligned}$$

Theorem 8. If $u(x,y)$ is of the form (4.4), and if its coefficients satisfy (4.13), then, for $0 < y < a$,

$$(4.14) \quad |u|_y < m^{\frac{1-y}{a}} \left[\left(3 \sum_0^{[\alpha-1]} e^{-ny} \right) + \right. \\ \left. + m^{\frac{2y}{d}} \left(\frac{2}{1 - e^{-y}} \right) + \left(\frac{2}{1 - e^{-(a-y)}} \right) \right].$$

Proof: We divide $u(x,y)$ into four parts as we divided $u(0,y)$ in (4.12). The results then follow as in the previous proof.

Notice that the second terms in (4.10) and (4.14) tend slowly to infinity as we approach the data line $y = 0$. Again, this emphasizes the fact that L_2 bounds imply good uniform bounds only where the function is smooth. In theorem 9 we will impose this smoothness at the data line by requiring that the tangential derivative u_x , as well as the normal derivative u_y , be small. Payne [13] has imposed similar conditions. In terms of the Cauchy problem, this requires that we approximate u_x closely, as well as the usual u and u_y . In theorem 10 we will obtain this smoothness more naturally by letting the data curve be embedded within the solution domain, as was the case in theorem 2.

Theorem 9. If $u(x,y)$ is of the form (4.4), and if

$$(4.15) \quad \begin{aligned} (a) \quad & \| |\text{grad } u| \|_0 \leq m = e^{-\alpha a} \leq 1, \\ (b) \quad & \|u\|_a \leq 1, \end{aligned}$$

then, for $0 < y < a$,

$$(4.16) \quad |u|_y < m^{\frac{1-y}{a}} \left[\frac{2\pi}{\sqrt{6}} + \left(\frac{2}{1 - e^{-2(a-y)}} \right)^{\frac{1}{2}} \right]$$

Proof: By (4.6),

$$\begin{aligned} \| |\text{grad } u| \|_0 &= \left(\|u_x\|_0^2 + \|u_y\|_0^2 \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2} \sum_1^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2) \right)^{\frac{1}{2}}. \end{aligned}$$

Then the problem reduces to finding a bound for the linear functional $u(0,y)$ with respect to the quadratic constraints

$$(4.17) \quad (a) \quad \sum_{n=1}^{\infty} (n^2 a_n^2 + n^2 c_n^2) \leq 2m^2,$$

$$(b) \quad \sum_{n=1}^{\infty} A_{n,a}^2 \leq 2.$$

We write $u(0,y)$ in the form (4.12). Then the maximum of the first two terms of (4.12) with respect to (4.17 a) is

$$\begin{aligned} & m \left(2 \sum_{n=1}^{[\alpha]} \frac{\sinh^2 n y + \cosh^2 n y}{n^2} \right)^{\frac{1}{2}} \\ & < m \left(2 \sum_{n=1}^{[\alpha]} \frac{e^{2ny}}{n^2} \right)^{\frac{1}{2}} \\ & < m^{1-\frac{y}{a}} \left(2 \sum_{n=1}^{[\alpha]} \frac{1}{n^2} \right)^{\frac{1}{2}}. \end{aligned}$$

The maximum of the third term of (4.12) with respect to (4.17 a) is

$$\begin{aligned} & m \left(2 \sum_{[\alpha+1]}^{\infty} \frac{1}{n^2} \left(\frac{\sinh n(a-y)}{\sinh n a} \right)^2 \right)^{\frac{1}{2}} \\ & < m \left(2 \sum_{[\alpha+1]}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, the sum of the first three terms is less than

$$m^{1-\frac{y}{a}} \left[\left(2 \sum_{n=1}^{[\alpha]} \frac{1}{n^2} \right)^{\frac{1}{2}} + \left(2 \sum_{n=[\alpha]+1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \right] \\ < m^{1-\frac{y}{a}} 2 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} = m^{1-\frac{y}{a}} \frac{2\pi}{\sqrt{6}}.$$

Finally, the bound for the fourth term of (4.12) follows from (4.17 b) exactly as in theorem 7, and (4.16) as proved by addition.

We now derive results analogous to lemma 1, lemma 2, and theorem 2 of section 2. We will now be considering functions u which are harmonic on the whole symmetric strip, $-a < y < a$.

For u of the form (4.4) on $-a < y < a$ we write $u = u_N + u_N^{\infty}$, where these low and high order parts are defined analogously to (2.4)

Lemma 7. Let L_N denote the class of u_N which satisfy

$$(4.18) \quad \begin{aligned} (a) \quad & \|u_N\|_0 \leq m, \\ (b) \quad & \|D^{-1}(u_N)_y\|_0 \leq m. \end{aligned}$$

and let

$$L(N, y) = \sup_{u_N \in L_N} |u_N|_y.$$

Then,

$$(4.19) \quad L(N, y) = m \left[\left(2 \sum_{n=1}^N \sinh^2 n y \right)^{\frac{1}{2}} + \left(2 \sum_{n=1}^N \cosh^2 n y \right)^{\frac{1}{2}} \right]$$

Proof: The problem reduces to the maximization of

$$u_N(0, y) = \sum_{n=1}^N (a_n \cosh n y + c_n \sinh n y)$$

with respect to

$$(4.20) \quad \begin{aligned} (a) \quad & \sum_{n=1}^N a_n^2 \leq 2m^2, \\ (b) \quad & \sum_{n=1}^N c_n^2 \leq 2m^2, \end{aligned}$$

Then (4.19) follows by the maximization lemma.

Lemma 8. Let H_N be the class of u_N^∞ which satisfy

$$(4.21) \quad \|u_N^\infty\|_a^2 + \|u_N^\infty\|_{-a}^2 \leq 2,$$

and let

$$H(N, y) = \sup_{u_N^\infty \in H_N} |u_N^\infty|_y, \quad -a < y < a.$$

Then

$$(4.22) \quad H(N, y) = \left(2 \sum_{N+1}^{\infty} \left(\frac{\cosh^2 n y}{\cosh^2 n a} + \frac{\sinh^2 n y}{\sinh^2 n a} \right) \right)^{\frac{1}{2}}.$$

Proof: The problem reduces to the maximization of

$$u_N^\infty(0, y) = \sum_{N+1}^{\infty} (a_n \cosh n y + c_n \sinh n y)$$

with respect to

$$\sum_{N+1}^{\infty} (a_n^2 \cosh^2 n a + c_n^2 \sinh^2 n a) \leq 2.$$

Then (4.22) follows by the maximization lemma.

Theorem 10. If $u(x,y)$ is of the form (4.4) on $-a < y < a$, and

$$(4.23) \quad \begin{aligned} (a) \quad & \|u\|_0 \leq m = e^{-\alpha a} \leq 1, \\ (b) \quad & \|D^{-1}u_y\|_0 \leq m, \\ (c) \quad & \|u\|_a^2 + \|u\|_{-a}^2 \leq 2, \end{aligned}$$

then, for $-a < y < a$,

$$(4.24) \quad |u|_y < L + H,$$

where

$$L = L([\alpha], y),$$

$$H = H([\alpha], y),$$

are defined by (4.19) and (4.22). Moreover,

$$(4.25) \quad L + H < m^{\frac{1-|y|}{a}} \left[\left(\frac{a}{2} \sum_0^{[\alpha-1]} e^{-2n|y|} \right)^{\frac{1}{2}} + \left(\frac{10}{1-e^{-2(a-|y|)}} \right)^{\frac{1}{2}} \right]$$

Proof: It follows immediately by lemma 7 and lemma 8 that

$$|u|_y \leq L(N, y) + H(N, y)$$

for every N . Then (4.25) follows by simplification of (4.24).

5. Harmonic continuation, data given on a whole circle.

Suppose there exists at least one function $u(r, \theta)$ having the properties

- (a) $u(r, \theta)$ is harmonic for $r < 1$,
 (b) $\|u - g\|_a \leq m = a^\alpha$, where the L_2 data function
 (5.1) $g(\theta)$ is given,
 (c) $\|u\|_1 \leq 1$.

Problem 1. Find a function approximating every u satisfying (5.1).

Let $g(\theta)$ have the expansion

$$(5.2) \quad g(\theta) = \frac{A_0}{\sqrt{2}} + \sum_1^{\infty} (A_n \cos n \theta + B_n \sin n \theta) a^n ,$$

and set

$$(5.3) \quad g_N(r, \theta) = \frac{A_0}{\sqrt{2}} + \sum_1^N (A_n \cos n \theta + B_n \sin n \theta) r^n ,$$

$r < \infty$.

We let $g[\alpha]$ be our approximation function and derive an a-priori error bound.

Lemma 8. If u satisfies (5.1), then

$$\|u - g[\alpha]\|_r < L + H ,$$

where L and H are defined by (2.10) of theorem 2.

Proof: We write

$$\begin{aligned} (u - g[\alpha]) &= (u - g[\alpha])[\alpha] + (u - g[\alpha])[\alpha]^\infty \\ &= (u - g)[\alpha] + u[\alpha]^\infty . \end{aligned}$$

By (5.1 b), lemma 1, and (2.10),

$$|(u - g)_{[\alpha]}|_r \leq L([\alpha], r) = L.$$

By (5.1 c), lemma 2, and (2.10),

$$|u_{[\alpha]}|_r^\infty \leq H([\alpha], r) = H.$$

Therefore, the lemma is proved.

Recall from (2.17) that $L + H$ is, except for a factor of at most two, the best possible bound. Therefore, $[\]$ is, in a sense, the best possible truncation order. Notice that (5.1 b) implies

$$|a_n a^n - A_n a^n|, |b_n a^n - B_n a^n| \leq \sqrt{2} a^\alpha;$$

while (5.1 c) implies

$$|a_n|, |b_n| \leq \sqrt{2},$$

or

$$|a_n a^n - 0|, |b_n a^n - 0| \leq \sqrt{2} a^n.$$

In other words, we are given two sets of data for the individual coefficients $a_n a^n$ and $b_n a^n$ of u on the data circle $r = a$. The first set claims that $a_n a^n$ and $b_n a^n$ are within $\sqrt{2} a^\alpha$ of the coefficients $A_n a^n$ and $B_n a^n$ of the data function g , while the second set claims that they are within $\sqrt{2} a^n$ of being zero. Therefore by setting the coefficients of our approximation function equal zero for n greater than α , we have used for each coefficient the data with the best claimed accuracy.

The data function g clearly cannot be arbitrary if there is to exist a function satisfying (5.1). We derive conditions on g which can be checked after the Fourier coefficients of g_N have been calculated. Notice that

$$\|g_N - g\|_a = \left(\|g\|_a^2 - \|g_N\|_a^2 \right)^{\frac{1}{2}}$$

is quite easily computed at each step.

Lemma 9. If there exist u satisfying (5.1), then $g[\alpha]$ must satisfy these a-posteriori compatability conditions:

$$(5.4) \quad \begin{aligned} (a) \quad & \|g[\alpha] - g\|_a \leq 2m, \\ (b) \quad & \|g[\alpha]\|_1 \leq 2. \end{aligned}$$

Proof: Given u satisfying (5.1), then

$$(5.5) \quad \begin{aligned} (aL) \quad & \|u_N - g_N\|_a \leq m, \\ (aH) \quad & \|u_N^\infty - g_N^\infty\| \leq m, \\ (bL) \quad & \|u_N\|_1 \leq 1, \\ (bH) \quad & \|u_N^\infty\|_1 \leq 1. \end{aligned}$$

Since $u_N - g_N$ contains no orders higher than N , (5.5 aL) implies

$$(5.5 \text{ aL}') \quad \|u_N - g_N\|_1 \leq m \left(\frac{1}{a}\right)^N.$$

Since u_N^∞ contains no orders lower than $N + 1$, (5.5 bH) implies

$$(5.5. \text{ bh}') \quad \|u_N^\infty\|_a \leq 1 a^{N+1}.$$

Then, using (5.5 aH), (5.5 bH'), (5.5 aL'), and (5.5 bL) we have

$$\begin{aligned} \|g_N - g\|_a &= \|g_N^\infty\|_a \leq \|g_N^\infty - u_N^\infty\|_a + \|u_N^\infty\|_a \\ &\leq m + 1 a^{N+1}, \\ \|g_N\|_1 &\leq \|g_N - u_N\|_1 + \|u_N\|_1 \\ &\leq m \left(\frac{1}{a}\right)^N + 1. \end{aligned}$$

Finally, (5.4) follows from (5.6) for $N = [\alpha]$.

This lemma implies that if there exist u satisfying (5.1), then $g_{[\alpha]}$ itself must almost satisfy (5.1).

The preceding formulation has the disadvantage that the inner and outer constraints, m and l , are ad-hoc assumptions whose compatibility is checked only after all the coefficients of $g_{[\alpha]}$ are computed. However, we see by the following lemma that the computed g_N disclose, step by step, a great deal of information about what constraints are compatible for the given g .

Lemma 10. Let m_N and M_N be defined by

$$(5.7) \quad \begin{aligned} (a) \quad & \|g_N - g\|_a = 2m_N, \\ (b) \quad & \|g_N\|_1 = 2M_N. \end{aligned}$$

Then there exists no function u harmonic on $r \leq 1$ satisfying both

$$(5.8) \quad \begin{aligned} (a) \quad & \|u - g\|_a \leq m_N, \\ (b) \quad & \|u\|_1 \leq M_N. \end{aligned}$$

Proof: Suppose u satisfying (5.8) exists. Then (5.6) becomes

$$\begin{aligned} \|g_N\|_a & \leq m_N + M_N a^{N+1}, \\ \|g_N\|_1 & \leq m_N \left(\frac{1}{a}\right)^N + M_N. \end{aligned}$$

Either

$$M_N a^{N+1} < m_N,$$

and therefore

$$\|g_N - g\|_a < 2m_N,$$

which contradicts (5.7 a); or

$$M_N a^{N+1} \geq m_N,$$

and therefore

$$\|g_N\|_1 < 2M_N,$$

which contradicts (5.7 b).

Now, let E denote the set of pairs (m, N) for which there exists a harmonic function u satisfying the constraints

$$\|u - g\|_a \leq m,$$

Also, extend the definition (5.7) to include

$$(m_{-1}, M_{-1}) = (\frac{1}{2} \|g\|_a, 0).$$

Then, we see by lemma 10 that the rectangles

$$m \geq 2m_N,$$

$$M \geq 2M_N, \quad N \geq -1,$$

are in E , while the rectangles

$$m \leq m_N,$$

$$M \leq M_N, \quad N = -1,$$

are in the complement CE . Moreover, one easily shows that E is a convex set; hence, E contains the convex hull of the first collection of rectangles. This information is illustrated in the diagram below.

It will often be sufficient to terminate computations at some order N less than $[\alpha]$. Let m and M be defined by

$$\begin{aligned} \|u - g\|_a &= m, \\ \|u\|_1 &= M. \end{aligned}$$

Then

$$\begin{aligned} \|u - g_N\|_a &\leq m + 2m_N, \\ \|u - g_N\|_a &\leq M + 2M_N. \end{aligned}$$

By theorem 2 we get a bound for $|u - g_N|_r$ whose leading term is

$$(m + 2m_N)^{\frac{\log r}{\log a}} (M + 2M_N)^{1 - \frac{\log r}{\log a}}.$$

It is therefore sufficient, for good approximation of u , to continue until m_N is as small as the expected m , but to stop before M_N grows too large.

6. Some auxiliary three circle bounds. In section 7 we shall have occasion to consider functions u having the properties

$$(6.1) \quad \begin{aligned} (a) \quad u(r, \theta) &= \frac{a_0}{\sqrt{2}} + \sum_{n=1}^K (a_n \cos n\theta + b_n \sin n\theta) r^n, \\ (b) \quad \|u\|_a &\leq m = a^\alpha, \\ (c) \quad |a_n|, |b_n| &\leq 1, \quad n \geq 0, \end{aligned}$$

where K is a fixed positive integer. Let $C_{0,K}$ denote this subclass of the class C_0 of theorem 3. In this section we will derive certain precise bounds for such functions.

Lemma 11. Let $u \in C_{0,K}$. Then, for $a^K < m \leq 1$,

$$(6.2) \quad |u|_r < m^{\frac{\log r}{\log a}} \left[\left(2 \sum_{n=0}^{[\alpha]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \left(2 \sum_{n=0}^{K-[\alpha+1]} r^n \right) \right].$$

For $0 \leq m \leq a^K$,

$$(6.3) \quad |u|_r \leq m \left(2 \sum_{n=0}^K \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}.$$

Proof: For $a^K < m \leq 1$, i.e., $0 \leq \alpha \leq K$, we merely substitute

$$\sum_{[\alpha+1]}^K \text{ everywhere that } \sum_{[\alpha+1]}^{\infty} \text{ appears in the proof of (2.20) of}$$

theorem 3, and (6.2) follows. For $0 \leq m \leq a^K$, i.e., $\alpha \geq K$, by lemma 1 and (6.16) we have

$$|u|_r \leq L(K, r) < m \left(2 \sum_{n=0}^K \left(\frac{r}{a} \right)^{2n} \right)^{\frac{1}{2}}.$$

Lemma 12 Let $u \in C_{0,K}$. Let $u_{ss}(r, \theta)$ be its second order directional derivative in a fixed direction. Let $a < b < 1$. Then, for $a^K < m \leq 1$,

$$(6.4) \quad |u_{ss}|_b \leq m \frac{4}{b^2} \left(2 \sum_{n=1}^{[\alpha]} n^4 \left(\frac{b}{a}\right)^{2n} \right)^{\frac{1}{2}} \\ + \frac{8}{b^2} \sum_{n=[\alpha+1]}^K n^2 b^n .$$

For $0 \leq m \leq a^K$,

$$(6.5) \quad |u_{ss}|_b \leq m \frac{4}{b^2} \left(2 \sum_{n=1}^K n^4 \left(\frac{b}{a}\right)^{2n} \right)^{\frac{1}{2}} .$$

Moreover, we have, independently of K ,

$$(6.6) \quad |u_{ss}|_b \leq D(m), \quad 0 \leq m \leq 1 ,$$

where

$$D(m) = m^{\frac{\log b}{\log a}} (\alpha+1)^{5/2} c(b) ,$$

and

$$c(b) = \frac{16}{b^2} \sum_{n=0}^{\infty} (1+n)^2 b^n .$$

Proof: We write

$$u_{ss} = u_{rr}(r_s)^2 + u_r r_{ss} + u_{\theta\theta}(\theta_s)^2 + u_{\theta} \theta_{ss} .$$

One easily shows that $|r_s|_b \leq 1$, $|r_{ss}|_b \leq \frac{1}{b}$, $|\theta_s| \leq \frac{1}{b}$, and

$$|\theta_{ss}|_b \leq \frac{1}{b^2} .$$

Therefore,

$$|u_{ss}|_b \leq |u_{rr}|_b + \frac{1}{b}|u_r|_b + \frac{1}{b^2}|u_{\theta\theta}|_b + \frac{1}{b^2}|u_\theta|_b.$$

Now

$$u_r(b, \theta) = \sum_1^K (a_n \cos n\theta + b_n \sin n\theta) n b^{n-1},$$

$$u_{rr}(b, \theta) = \sum_2^K (a_n \cos n\theta + b_n \sin n\theta) n(n-1) b^{n-2},$$

$$u_\theta(b, \theta) = \sum_1^K (-a_n \sin n\theta + b_n \cos n\theta) n b^n,$$

$$u_{\theta\theta}(b, \theta) = \sum_1^K (-a_n \cos n\theta + b_n \sin n\theta) n^2 b^n.$$

Therefore, we obtain

$$|u_{ss}|_b \leq \frac{4}{b^2} \sum_1^K (a_n^2 + b_n^2)^{\frac{1}{2}} n^2 b^n.$$

Since (6.1 b) implies

$$\sum_1^{[\alpha]} (a_n^2 + b_n^2) a^{2n} \leq 2m^2,$$

we have by the maximization lemma

$$\sum_1^{[\alpha]} (a_n^2 + b_n^2)^{\frac{1}{2}} n^2 b^n \leq m \left(2 \sum_1^{[\alpha]} n^4 \left(\frac{b}{a}\right)^{2n} \right)^{\frac{1}{2}},$$

If $K \leq \alpha$, we get the same result with $[\alpha]$ replaced by K , which proves (6.5). Next, we apply (6.1 c) to the high order terms, obtaining

$$\sum_{[\alpha+1]}^K (a_n^2 + b_n^2)^{\frac{1}{2}} n^2 b^n \leq 2 \sum_{[\alpha+1]}^K n^2 b^n,$$

which completes the proof of (6.4). Finally, we may replace K by ∞ in (6.4). Then,

$$\begin{aligned}
 |u_{ss}|_b &\leq \frac{4}{b^2} m \left(\frac{b}{a}\right)^{[\alpha]} \left(2 \sum_{n=1}^{[\alpha]} n^4 \left(\frac{a}{b}\right)^{2([\alpha]-n)} \right)^{\frac{1}{2}} + \\
 &+ \frac{8}{b^2} b^{[\alpha+1]} \sum_{n=0}^{\infty} (n + [\alpha+1])^2 b^n < \\
 &< \frac{4}{b^2} b^{\alpha} \left[(2[\alpha]^5)^{\frac{1}{2}} + 2[\alpha+1]^2 \sum_{n=0}^{\infty} \left(1 + \frac{n}{[\alpha+1]}\right)^2 b^n \right] \\
 &< b^{\alpha} (\alpha+1)^{\frac{5}{2}} \left[\frac{16}{b^2} \sum_{n=0}^{\infty} (1+n)^2 b^n \right].
 \end{aligned}$$

Therefore, the proof is completed.

Notice that, for $m \leq a^K$, the bound (6.6) is linear in m ; while, for $m > a^K$, the bound (6.5) is linear on intervals of constant $[\alpha]$, but has jumps where $[\alpha]$ changes from one integer to the next. We put the bound in a more useful form for automatic computation. First, define the step function

$$(6.7) \quad S(j, K) = a^j \frac{4}{b^2} \left(2 \sum_{n=1}^j n^4 \left(\frac{b}{a}\right)^{2n} \right)^{\frac{1}{2}}$$

$$+ \frac{8}{b^2} \sum_{j+1}^K n^2 b^n, \quad 0 \leq j \leq K.$$

By (6.4) this is a bound for $|u_{ss}|_b$ when $[\alpha] = j$, since

$m = a^{\alpha} \leq a^{[\alpha]}$. We now force our bound to be non-decreasing. Let

$$(6.8 \text{ a}) \quad B(m, K) = \min_{0 \leq j \leq [\alpha]} S(j, K), \quad a^K < m \leq 1;$$

$$(6.8 \text{ b}) \quad B(m, K) = \min_{0 \leq j < K} \left[S(j, K), m \frac{4}{b^2} \left(2 \sum_{n=1}^K n^4 \left(\frac{b}{a} \right)^{2n} \right)^{\frac{1}{2}} \right],$$

$$0 \leq m \leq a^K;$$

$$(6.8 \text{ c}) \quad B(m, K) = S(0, K), \quad m > 1.$$

Since a bound on $|u_{ss}|_b$ for a given m is a bound for any smaller m , $B(m, K)$, as defined in (6.8 a) and (6.8. b), is a bound for $|u_{ss}|_b$. Since the outer constraint $\|u\|_a \leq 1$ automatically implies $\|u\|_a \leq 1$, we may extend our bound to include $m > 1$ by means of (6.8 c). Notice that we have increased the bounds (6.4) and (6.5) by a factor of at most $(1/a)$ in obtaining $B(m, K)$. Thus, since $D(m)$ of (6.6) is an upper bound for (6.4) and (6.5), $\frac{1}{a} D(m)$ is an upper bound for $B(m, K)$. We have therefore proved the following lemma:

Lemma 13. Let $u \in C_{0, K}$. Then

$$(6.9) \quad |u_{ss}|_b \leq B(m, K),$$

where $B(m, K)$ is defined by (6.8). $B(m, K)$ is a non-decreasing function of m , defined for $m \geq 0$. $B(m, K)$ is initially a positive multiple of m , and thereafter a step function. $B(m, K)$ is constant for $m > a$. Moreover,

$$(6.10) \quad B(m, K) < \frac{1}{a} D(m), \quad 0 \leq m \leq 1,$$

where $D(m)$ is defined by (6.6).

$B(m, K)$ will frequently be a much better bound for $|u_{ss}|_b$ than $D(m)$.

7. Harmonic continuation, data given at a finite set of points. Suppose there exists at least one function $u(r, \theta)$ having the properties

$$(7.1) \quad \begin{aligned} (a) \quad & u(r, \theta) \text{ is harmonic for } r < 1, \\ (b) \quad & |u(r_j, \theta_j) - g_j| \leq \epsilon_0, \\ & \text{where the data values } g_j \text{ are given at interior points} \\ & (r_j, \theta_j), j = 1, \dots, p, \\ (c) \quad & \|u\|_1 \leq 1. \end{aligned}$$

Let P be a simple closed polygon with vertices (r_j, θ_j) , δ the length of its longest side, and a and b the radii of the inscribed and circumscribed circles with centers at the origin, $0 < a < b < 1$. Problem 2. Find a function approximating every u satisfying (7.1).

First, we outline an explicit method which reduces Problem 2 to Problem 1 by means of solving a Dirichlet problem (a well-posed problem) on the polygon.

Let u_{ss} be the second directional derivative of u with respect to arclength along P . By means of the outer constraint (7.1 c) we can bound $|u_{ss}|_P$, the uniform norm of u_{ss} on P .

Let $\hat{g}$ be the linear interpolation of the g_j onto the whole polygon P . Then we have the explicit interpolation bound

$$(7.2) \quad |u - \hat{g}|_P \leq |\hat{u} - \hat{g}|_P + |u - \hat{u}|_P \leq \epsilon_0 + \frac{\delta^2}{4} |u_{ss}|_P.$$

Now, approximate u on P^0 by solving the Dirichlet problem for the Laplace difference equation, modified as in [8] to deal with the irregular boundary, using boundary data $\hat{g}$ on P . Let $g_{\Delta x}$ denote this solution for a grid size Δx , using bilinear interpolation between the grid points. By means of the outer constraint we can bound the fourth order partials of u on P^0 ; hence, the global truncation error on P^0 is $O(\overline{\Delta x}^2)$. Now, our modified difference scheme with bilinear interpolation still obeys the maximum principle; therefore,

$$(7.3) \quad |u - g_{\Delta x}|_{P^0} \leq |u - \hat{g}|_P + O(\overline{\Delta x}^2) ;$$

or, by (7.2),

$$(7.4) \quad \|u - g_{\Delta x}\|_a \leq |u - g_{\Delta x}|_a \leq |u - \hat{g}|_P + O(\overline{\Delta x}^2) = \epsilon_0 + O(\delta^2) + O(\overline{\Delta x}^2) .$$

Finally, apply the methods of section 5 for problem 1, with $g_{\Delta x}(a, \theta)$ as data.

Next, we develop a method of linear inequalities which has two great advantages over the preceding method. First, the work requirement is usually greatly reduced. Second, the interpolation error is usually greatly reduced, thereby yielding far better accuracy from the given data. The method, in brief, is the following: linearize the outer constraint (7.1 c), then find a harmonic function of finite order which closely fits the data and satisfies the constraints.

We will use an approximation of order K , K to be chosen later. Assume that u satisfies (7.1). Then, at the data points u_K satisfies

$$(7.5) \quad |u_K(r_j, \theta_j) - g_j| \leq |u(r_j, \theta_j) - g_j| + |u_K^\omega(r_j, \theta_j)| \leq \epsilon_0 + H(K, b),$$

where we have used the maximum principle and lemma 2 to bound $|u_K^\omega(r_j, \theta_j)|$.

We write (7.5) in terms of the Fourier coefficients, $a_0, a_1, \dots, a_K, b_1, \dots, b_K$, of u_K , and obtain the following $2p$ linear inequalities:

$$(7.6 \text{ a}) \quad \begin{aligned} & g_j - \epsilon_0 - H(K, b) \leq \\ & \leq \frac{a_0}{\sqrt{2}} + \sum_{n=1}^K \left(a_n(r_j^n \cos n\theta_j) + b_n(r_j^n \sin n\theta_j) \right) \leq \\ & \leq g_j + \epsilon_0 + H(K, b), \quad j = 1, \dots, p. \end{aligned}$$

The given quadratic outer constraint,

$$\|u_K\|_1 = \frac{1}{2} \sum_0^K (a_n^2 + b_n^2) \leq 1 ,$$

implies the linear termwise constraints,

$$|a_n|, |b_n| \leq \sqrt{2} ,$$

or the $4K + 2$ linear inequalities,

$$(7.6 \text{ b}) \quad \begin{aligned} -\sqrt{2} &\leq a_n \leq \sqrt{2}, & n = 0, \dots, K, \\ -\sqrt{2} &\leq b_n \leq \sqrt{2}, & n = 1, \dots, K. \end{aligned}$$

Now, there are well-known numerical methods for the solution of linear inequalities. See [1], [4], and [9] for references. Moreover, if there exist no solutions to the inequalities, these methods will tell us so, which gives an a-posteriori check on the hypotheses (7.1). Therefore, find any solution, $A_0, A_1, \dots, A_K, B_1, \dots, B_K$, to the linear inequalities (7.6). Then let these be the Fourier coefficients for an approximation function v_K ; that is, define

$$(7.7) \quad v_K(r, \theta) = \frac{A_0}{\sqrt{2}} + \sum_1^K (A_n \cos n\theta + B_n \sin n\theta) r^n .$$

We begin the derivation of an a-priori error bound. Write the error function in the form

$$(7.8) \quad u - v_K = E_K + u_K ,$$

where

$$E_K = u_K - v_K .$$

By (7.6 a) ,

$$(7.9) \quad \begin{aligned} |E_K(r_j, \theta_j)| &\leq |u_K(r_j, \theta_j) - g_j| + |v_K(r_j, \theta_j) - g_j| \\ &\leq 2\epsilon_0 + 2H(K, b). \end{aligned}$$

Extend this to a bound for $|E_K|_P$, the uniform norm of E_K on P , by means of the linear interpolation $\hat{E}_K$:

$$(7.10) \quad |E_K|_P \leq |\hat{E}_K|_P + |E_K - \hat{E}_K|_P \\ \leq 2\epsilon_0 + 2H(K, b) + \frac{\delta^2}{4} |(E_K)_{ss}|_P.$$

For a fixed direction, $(E_K)_{ss}$ is also a harmonic function; hence, by the maximum principle,

$$(7.11) \quad |(E_K)_{ss}|_P \leq |(E_K)_{ss}|_b.$$

Define m by

$$(7.12) \quad 2m = |E_K|_P$$

Thus,

$$\|E_K\|_a \leq |E_K|_a \leq |E_K|_P = 2m.$$

Let $a'_n = (a_n - A_n)$ and $b'_n = (b_n - B_n)$ denote the Fourier coefficients of E_K . Then,

$$(7.13) \quad (a) \quad E_K(r, \theta) = \frac{a'_0}{\sqrt{2}} + \sum_1^K (a'_n \cos n\theta + b'_n \sin n\theta) r^n, \\ (b) \quad \|E_K\|_a \leq 2m, \\ (c) \quad |a'_n| |b'_n| \leq 2\sqrt{2}.$$

In other words, $\frac{1}{2} E_K$ is in the class $C_{0,K}$ of section 6. Therefore, by lemma 4,

$$(7.14) \quad |(E_K)_{ss}|_b \leq 2 B(m, K).$$

Then, by (7.11) and (7.14), the inequality (7.10) becomes implicit in m :

$$(7.15) \quad m \leq \epsilon_0 + H(K, b) + \frac{\delta^2}{4} B(m, K).$$

In (7.15) ε_0 is the pointwise data error, $H(K,b)$ the truncation error, and $\frac{\delta^2}{4} B(m,K)$ the interpolation error. These errors are interrelated by the implicitness of (4.15); yet it will be convenient for the purpose of analysis to isolate their individual effects. By (7.15),

$$(7.16) \quad m \leq 3 \max (\varepsilon_0, H(K,b), \frac{\delta^2}{4} B(m,K)) .$$

Suppose $\frac{\delta^2}{4} B(m,K)$ is the maximum above; then m is a root of the equation

$$(7.17) \quad t + \beta = \frac{3}{4} \delta^2 B(t, K)$$

for some number $\beta \geq 0$. If this is so, then we see by the diagram below that there exists a largest root, I_K , to the equation

$$(7.18) \quad t = \frac{3}{4} \delta^2 B(t, K),$$

and that all roots of the equation (7.17) must fall below I_K . If there exists no root of (7.18) other than zero, this is merely a sign that it is impossible for $\frac{\delta^2}{4} B(m,K)$ to dominate in (7.16), in which case we define $I_K = 0$.

We have therefore proved the following lemma:

Lemma 13. If E_K is the difference of two harmonic functions u_K and v_K whose Fourier coefficients satisfy (7.6), and if $2m$ is the uniform norm of E_K on P , then

$$(7.19 \text{ a}) \quad m \leq m_K^* = 3 \max(\epsilon_0, H(K, b), I_K),$$

where I_K is the largest root of

$$(7.19 \text{ b}) \quad t = \frac{3}{4} \delta^2 B(t, K), \quad t \geq 0.$$

Corresponding to m_K^* defined by (7.19), we define α^* by $m^* = a^{\alpha^*}$.

Then,

$$a^* = m_K^* \geq 3 H(K, b) > b^{K+1} > a^{K+1}.$$

Thus, it is always true that

$$(7.20) \quad \alpha^* < K+1.$$

Recall from 7.8 that

$$u - u_K = E_K + u_K^{\oplus}.$$

By (7.13) and (7.20) we may use the bound (6.2), rather than (6.3), for $\frac{1}{2} E_K$. By (7.1 c) we may use the bound $H(K, r)$ of lemma 2 for $u_K^{\oplus}$. After simplification we obtain the following a-priori error bound:

Lemma 14. If u satisfies (7.1) and the Fourier coefficients of v_K satisfy (7.6), then

$$(7.21) \quad |u - v_K|_r < m_K^* \frac{\log r}{\log a} \left[\left(8 \sum_{0}^{[\alpha^*]} \left(\frac{a}{r} \right)^{2n} \right)^{\frac{1}{2}} + \right. \\ \left. + \left(4 \sum_{0}^{K-[\alpha^*+1]} r^n \right) + \left(2 \sum_{K-[\alpha^*]}^{\infty} r^{2n} \right)^{\frac{1}{2}} \right].$$

Moreover, on P and its interior P^0 ,

$$(7.22) \quad |u - v_K|_{P^0} \leq 2m_K^* + H(K, b).$$

Considering (7.21), we see that, as far as the sums in the bracket are concerned, it makes little difference where the division point K between the second and third sums falls. Therefore, what really matters is the smallness of m_K^* .

Now ϵ_0 is fixed, $H(K, b)$ is decreasing exponentially like b^K , and I_K is bounded but increasing slowly, since $B(m, K)$ is bounded for all K but increasing slowly. Thus, the minimum m_K^* will occur either at that integer K at which $H(K, b)$ first ceases to dominate both ϵ_0 and I_K , or at the integer just before this.

Choose K to be that integer at which the minimum m_K^* occurs.

Let us compare the explicit method and the method of linear inequalities with respect to work requirements and interpolation error.

For the first method, we found in (7.4) that

$$(7.23 \text{ a}) \quad |u - g_{\Delta x}|_{P^0} = \epsilon_0 + O(\overline{\Delta x}^2) + O(\delta^2),$$

where $O(\overline{\Delta x}^2)$ is the global truncation error for the Laplace difference equation on P^0 and $O(\delta^2)$ is the data interpolation error on P . The method requires the solution of

$$(7.23 \text{ b}) \quad O(\overline{\Delta x}^2)$$

linear equations.

For the second method, the truncation error $H(K, b)$ is $O(b^K)$. The interpolation error I_K is a root of equation (7.1 b); hence, by the upper bound (6.10) for $B(m, K)$, I_K satisfies an inequality of the form

$$(7.24) \quad I_K \leq O(\delta^2) |\log I_K|^{\frac{5}{2}} I_K^{\frac{\log b}{\log a}}.$$

We absorb $|\log I_K|^{\frac{5}{2}}$ into the exponent and obtain

$$I_K \leq C_\epsilon \delta^2 I_K^{\frac{\log b}{\log a} - \epsilon},$$

or,

$$I_K \leq C'_\epsilon \delta^{\frac{2}{1+\epsilon - \log b / \log a}}.$$

We therefore have

$$(7.25 \text{ a}) \quad |u - v_K|_{P^0} \leq \epsilon_0 + O(b^K) + O\left(\delta^{\frac{2}{-1 + \epsilon - \log b / \log a}}\right).$$

The method requires the solution of

$$(7.25 \text{ b}) \quad 4K + O(\delta^{-1})$$

linear inequalities, for fairly evenly spaced data points.

In the limit, where $b = a$ and all the data points lie on a circle, we would use linear interpolation along the circumference to obtain a bound exactly analogous to (6.6) for $|u_{\theta\theta}|_a$. Therefore (7.24) would become

$$(7.25) \quad I_K \leq O(\delta^2) |\log I_K|^{\frac{5}{2}} I_K,$$

or,

$$(7.26) \quad I_K \leq \exp\left(-\frac{1}{O(\delta^{4/5})}\right).$$

Let us suppose, for the purpose of illustration, that $\log b/\log a$ equals $\frac{1}{2} + \epsilon$ and that ϵ_0 is zero. Then let us compare the two methods from two different viewpoints. One viewpoint, corresponding to many experimental situations, assumes that the number of data points may not be increased without great cost. Our accuracy, therefore, is limited by the interpolation error. In the present example, the interpolation error is $O(\delta^4)$ for the second method, as opposed to $O(\delta^2)$ for the first method, a very great increase in accuracy for the given data. Another viewpoint, corresponding to many computational situations, assumes that the major cost is involved in the computation, and that the number of data points may be increased arbitrarily. The accuracy, therefore, is limited only by the number of calculations which we are willing to perform .

Let $N_1(\eta)$ and $N_2(\eta)$ denote, respectively, the number of linear equations required in the first method, and the number of linear inequalities required in the second method, to guarantee an error on P^0 no larger than η . From 7.23 and 7.25 we obtain

$$\begin{aligned} N_1(\eta) &= O(\eta^{-1}) , \\ N_2(\eta) &= O(\eta^{-\frac{1}{4}}) . \end{aligned}$$

Therefore, we have the comparison

$$N_2 = O(N_1^{\frac{1}{4}}) .$$

The linear inequalities method of problem 2 is quite similar to the linear programming approach used by Douglas and Gallie [7] for harmonic continuation on the half plane, by Douglas [5] for analytic continuation, and by Cannon [2] for backward solution of the heat equation. Douglas [6] has also applied the linear programming approach to improper integral equations.

8. Analytic continuation. The problems and methods for harmonic continuation extend immediately to analytic continuation, since the real and imaginary parts of an analytic function are harmonic. Notice that if

$$f = u_1 + i u_2$$

is an analytic function with the Taylor series

$$f(z) = \frac{a_0 + i b_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n - i b_n) z^n$$

then the Fourier series of u_1 and that of its harmonic conjugate u_2 satisfy the relation

$$(8.1) \quad \begin{aligned} u_1(r, \theta) &= \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n, \\ u_2(r, \theta) &= -\frac{b_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (-b_n \cos n\theta + a_n \sin n\theta) r^n. \end{aligned}$$

Suppose there exists at least one function $f(z)$ having the properties

$$(8.2) \quad \begin{aligned} (a) \quad & f(z) = u_1(z) + i u_2(z) \text{ is analytic for } |z| < 1, \\ (b) \quad & \|u_1 - g_1\|_a \leq m = a, \\ & \|u_2 - g_2\|_a \leq m, \text{ where the } L_2 \text{ data functions} \\ & g_1(\theta) \text{ and } g_2(\theta) \text{ are given,} \\ (c) \quad & \|u_1\|_1 \leq 1, \\ & \|u_2\|_1 \leq 1. \end{aligned}$$

Problem 1. Find a function approximating every $f(z)$ satisfying (8.2).

Let g_1 and g_2 have the expansions

$$g_1(a, \theta) = \frac{A_{01}}{\sqrt{2}} + \sum_1^{\infty} (A_{n1} \cos n\theta + B_{n1} \sin n\theta) a^n, \quad (8.3)$$

$$g_2(a, \theta) = -\frac{B_{02}}{\sqrt{2}} + \sum_1^{\infty} (-B_{n2} \cos n\theta + A_{n2} \sin n\theta) a^n.$$

By (8.1), all the coefficients except the constant terms may be obtained from just one of the data functions. Define the conjugate harmonic functions

$$g_{1,N}(r, \theta) = \frac{A_{01}}{\sqrt{2}} + \sum_1^N (A_{n1} \cos n\theta + B_{n1} \sin n\theta) r^n, \quad r < \infty, \quad (8.4 \text{ a})$$

$$g_{2,N}^*(r, \theta) = -\frac{B_{02}}{\sqrt{2}} + \sum_1^N (-B_{n1} \cos n\theta + A_{n1} \sin n\theta) r^n, \quad r < \infty.$$

Then let the analytic function

$$(8.4 \text{ b}) \quad g_N(z) = g_{1,N}(z) + i g_{2,N}^*(z) =$$

$$= \frac{A_{01} - i B_{02}}{\sqrt{2}} + \sum_1^N (A_{n1} - i B_{n1}) z^n$$

be our approximation function, for $N = [\alpha]$.

For checking purposes also calculate the corresponding coefficients of g_2 . We have the following a-posteriori compatibility condition between g_1 and g_2 :

$$\begin{aligned}
 & \left[\frac{1}{2} \sum_1^N ((A_{n1} - A_{n2})^2 + (B_{n1} - B_{n2})^2) a^{2n} \right]^{\frac{1}{2}} \\
 (8.5) \quad & \leq \left[\frac{1}{2} \sum_1^{\infty} ((A_{n1} - a_n)^2 + (B_{n1} - b_n)^2) a^{2n} \right]^{\frac{1}{2}} \\
 & + \left[\frac{1}{2} \sum_1^{\infty} ((A_{n2} - a_n)^2 + (B_{n2} - b_n)^2) a^{2n} \right]^{\frac{1}{2}} \\
 & = \|g_1 - u_1\|_a + \|g_2 - u_2\|_a \leq 2m.
 \end{aligned}$$

To $(u_1 - g_{1,N})$ we may apply directly the a-priori bounds and compatibility checks of lemmas 8-10. For $(u_2 - g_{2,N}^*)$ the only difference is that instead of having

$$\|u_{2,N} - g_{2,N}^*\|_a \leq m,$$

we have only

$$\begin{aligned}
 (8.6) \quad & \|u_{2,N} - g_{2,N}^*\|_a^2 \\
 & \leq \frac{1}{2} (b_0 - B_{02})^2 + \frac{1}{2} \sum_1^N ((a_n - A_{n1})^2 + (b_n - B_{n1})^2) a^{2n}, \\
 & \leq \|u_2 - g\|_a^2 + \|u_1 - g_1\|_a^2 \leq 2m^2.
 \end{aligned}$$

Now, suppose that we are given only data values $(g_{1,j}, g_{2,j})$ at a finite number of points (r_j, θ_j) , as in problem 2 for harmonic functions. If we merely treat an analytic function as a pair (u_1, u_2) of harmonic functions whose Fourier coefficients are related by the form (8.1), then the statement of problem 2 and the results of section 7 extend in obvious fashion to analytic functions.

See Douglas [5] for an alternative approach to analytic continuation. Instead of a Taylor's series representation, a Riemann sum approximation to the Cauchy integral is used.

9. Backward solution of the heat equation on a rectangle.

Let us begin with a brief discussion which shows that the problems of backward solution of the heat equation on a rectangle with a prescribed bound may be reduced to the problem of backward continuation of periodic solutions with a prescribed bound. On a rectangle $R = \{(x, t): 0 < x < \pi, 0 < t < a\}$ we consider solutions $u(x, t)$ of the heat equation which are continuous on the closure of R . Let B , T , S_1 and S_2 denote the base, top, and two sides of R .

The backward boundary value problem, for example, is to find u when given its values on the top and two sides. That is, suppose u satisfies

$$(9.1) \quad \begin{aligned} (a) \quad & u_{xx} = u_t \text{ on } R, \\ (b) \quad & u = G \text{ on } T, \\ (c) \quad & u = G_i \text{ on } S_i, \quad |G_i| \leq 1, \quad i = 1, 2. \end{aligned}$$

where G and G_i are known approximately. Moreover, suppose

$$(d) \quad \|u\|_y = \left(\frac{1}{\pi} \int_0^\pi u^2(x, y) dx \right)^{\frac{1}{2}} \leq 1, \quad 0 \leq y \leq a.$$

If $G_1(t) \equiv G_2(t) \equiv 0$, then $u(x, t)$ may be extended antisymmetrically across $x = 0$ and $x = \pi$ by means of its sine expansion to a solution on all $t > 0$, periodic in x with period 2π . If not, let H be any continuous function on B which agrees with G_1 and G_2 at the corners and which satisfies $|H| \leq 1$. Then, let u' be the solution to the forward boundary value problem

$$(9.2) \quad \begin{aligned} (a) \quad & u'_{xx} = u'_t \text{ on } R, \\ (b) \quad & u' = G_i \text{ on } S_i, \\ (c) \quad & u' = H \text{ on } B. \end{aligned}$$

Let G' be defined by

$$(d) \quad u' = G' \text{ on } T.$$

Notice that

$$|u'| \leq 1 \text{ on } R$$

implies

$$\|u\|_y \leq 1, \quad 0 \leq y \leq a.$$

Then $u'' = u - u'$ satisfies

$$(9.3) \quad \begin{aligned} (a) \quad & u''_{xx} = u'_t \text{ on } R, \\ (b) \quad & u'' = G'' = G - G' \text{ on } T, \\ (c) \quad & u'' = 0 \text{ on } S_i, \\ (d) \quad & \|u''\|_y \leq 2, \quad 0 \leq y \leq a. \end{aligned}$$

Therefore, $u = u' + u''$, where the periodic extension of u'' is a solution of the heat equation on $t > 0$, has bounded L_2 norm, and is the backward continuation of G'' . Moreover, since u' is the solution to a well posed problem, small errors in G_i lead to only small errors in u' , and therefore add only small errors to G'' .

We could consider the backward Neumann problem instead, in which u_x , rather than u , is given on the sides. Again, by subtracting the solution to a well posed forward Neumann problem, we can leave $u'' = u - u'$ which has $u''_x = 0$ on the sides, and which can therefore be extended symmetrically across $x = 0$ and $x = \pi$ to a periodic solution on $t > 0$.

We have shown that it is sufficient to consider the following problem for periodic solutions, which corresponds to problem 1 of section 5. Suppose there exists at least one function $u(x, t)$ having the properties

$$(9.4) \quad \begin{aligned} (a) \quad & u(x, t) \text{ is a solution of the heat equation for } t > 0, \text{ periodic in } x \text{ with period } 2\pi, \\ (b) \quad & \|u - g\|_a \leq m = e^{-\alpha^2 a}, \text{ where the } L_2 \text{ function } g(x) \text{ is given,} \\ (c) \quad & \|u\|_0 \leq 1. \end{aligned}$$

Problem 1. Find a function approximating every u satisfying (9.4).

Let

$$(9.5) \quad g(x) = \frac{A_0}{\sqrt{2}} + \sum_1^{\infty} (A_n \cos n x + B_n \sin n x) e^{-n^2 a},$$

and set

$$(9.6) \quad g_N(x, t) = \frac{A_0}{\sqrt{2}} + \sum_1^N (A_n \cos n x + B_n \sin n x) e^{-n^2 t}.$$

This, for $N = [\alpha]$, is our approximation function

We can then easily obtain the a-priori error bound,

$$|u - g_{[\alpha]}|_t < L + H,$$

as in lemma 8 (here L and H are defined in theorem 5), and exact analogies to the a-posteriori compatibility conditions of lemmas 9 and 10.

Again, if the data for u across the top of the rectangle is replaced by data at a finite number of interior points, we may easily show that it is sufficient to consider the following problem for periodic solutions, which corresponds to problem 2 of section 7. Let approximate values g_j for u be given at a finite number of data points (x_j, t_j) in a single period. These points, with their periodic images, either lie on the line $t = a$, with maximum separation δ , or are the vertices of a simple polygonal path P which separates $y = 0$ from $y = a$. Let δ be the maximum segment length of P , and let P itself lie between $y = b$ and $y = a$, $0 \leq b \leq a$. Then the methods of section 7 carry through in obvious analogy.

Cannon [2] has also treated backward solutions of the heat equation. The methods of problem 2 here are similar in many respects to his.

10. The Cauchy problem for Laplace's equation on a rectangle. We develop numerical methods only for the case, corresponding to theorem 10 and corresponding to all the numerical methods of previous sections, that the data curve be embedded in the solution domain. First, let us show that the Cauchy problem on a rectangle with a prescribed bound can be reduced to the Cauchy problem for periodic solutions with a prescribed bound. On a rectangle $R = \{(x,y): 0 < x < \pi, -a < y < a\}$ we consider harmonic functions $u(x,y)$ which are continuous on the closure of R . Let B_1 , C , B_2 , S_1 and S_2 denote the base, horizontal center line, top, and two sides of R . For convenience we also assume that u_y is continuous in a neighborhood of the end points of C .

Consider the Cauchy problem in which u and u_y are given on the center line and u is given on the sides. That is, suppose

$$\begin{aligned}
 (10.1) \quad & (a) \quad u_{xx} + u_{yy} = 0 \text{ on } R, \\
 & (b) \quad u = G \text{ on } C, \\
 & (c) \quad u_y = F \text{ on } C, \\
 & (d) \quad u = G_i \text{ on } S_i, \quad |G_i| \leq 1, \quad i = 1, 2,
 \end{aligned}$$

where G , F , and G_i are known approximately. Moreover, suppose

$$(e) \quad \|u\|_y = \left(\frac{1}{\pi} \int_0^\pi u^2(x,y) dx \right)^{\frac{1}{2}} \leq 1, \quad -a \leq y \leq a.$$

If $G_1(y) \equiv G_2(y) \equiv 0$, then $u(x,y)$ may be extended antisymmetrically across $x = 0$ and $x = \pi$ to a periodic harmonic function of the form (4.4) on $-a < y < a$. If not, let H_1 and H_2 be any continuous functions on B_1 and B_2 which agree with G_1 and G_2 at the corners and which satisfy $|H_1| \leq 1$, $|H_2| \leq 1$. Then, let u' be the solution to the Dirichlet problem

$$\begin{aligned}
 (10.2) \quad & (a) \quad u'_{xx} + u'_{yy} = 0 \text{ on } R, \\
 & (b) \quad u' = G_i \text{ on } S_i, \\
 & (c) \quad u' = H_i \text{ on } B_i,
 \end{aligned}$$

Let G' and F' be defined by

$$\begin{aligned}
 (d) \quad & u' = G' \text{ on } C, \\
 (e) \quad & u'_y = F' \text{ on } C.
 \end{aligned}$$

Then, $u'' = u - u'$ satisfies

$$\begin{aligned}
 (10.3) \quad & (a) \quad u''_{xx} + u''_{yy} = 0 \text{ on } R, \\
 & (b) \quad u'' = G'' = G - G' \text{ on } C, \\
 & (c) \quad u''_y = F'' = F - F' \text{ on } C, \\
 & (d) \quad u'' = 0 \text{ on } S_i, \\
 & (e) \quad \|u''\|_y \leq 2, \quad -a \leq y \leq a.
 \end{aligned}$$

Therefore, $u = u' + u''$, where u' is the solution of a well-posed Dirichlet problem and where the periodic extension of u'' is of the form (4.4) on $-a < y < a$.

Moreover, u' , G' , and even F' (when its $\|D^{-1}F'\|$ norm is considered) depend continuously on G_1 and G_2 . Since u' , G' , and F' depend linearly on G_1 and G_2 , it is sufficient to show they are bounded when G_1 and G_2 are. Now, if

$$(10.4) \quad |G_i| \leq \varepsilon \text{ on } S_i,$$

then by the maximum principle

$$(10.5) \quad |u'| \leq \varepsilon \text{ on } R.$$

But, if u' is harmonic on a disc of radius δ , and $|u'| \leq \varepsilon$ there,

then $|u'_y| \leq \frac{4\varepsilon}{\pi\delta}$ at the center. This is shown by differentiating under the integral in the Poisson integral formula. Thus

$$(10.6) \quad |F'(x)| \leq \frac{4\varepsilon}{\pi\delta},$$

where δ is the distance from $(x,0)$ to the boundary of R . Now, consider the periodic extension of $F'(x)$ antisymmetrically across $x = 0$ and $x = \pi$. Since u'_y is continuous on C even at the end points, this is a piecewise continuous function with zero mean value. Hence, we may apply the bound (4.8 b). Let

$$I(x) = \int_{\pi/2}^x F'(t) dt$$

It follows from (10.6) that

$$|I(x)| < \varepsilon(K_1 + K_2 |\log|\delta|),$$

where the constants K_1 and K_2 depend only on the geometrical ratio $\frac{a}{\pi}$. The L_2 norm of $K_1 + K_2 |\log|\delta|$ is a finite number K , hence,

$$\|I\| < K \varepsilon.$$

Finally, by (4.8 b) we have

$$(10.7) \quad \|D^{-1}F'\| \leq \|I\| < K \varepsilon,$$

and from (10.5) we have

$$\|G'\| \leq \varepsilon.$$

We therefore may conclude that small errors in G_1 and G_2 add only small errors to G'' and F'' .

We have shown that it is sufficient to consider the following problem for periodic harmonic functions, which corresponds to problem 1 of section 5. Suppose there exists at least one function $u(x,y)$ having the properties

$$\begin{aligned}
 & \text{(a) } u(x,y) \text{ is of the form (4.4) on } -a < y < a, \\
 & \text{(b) } \|u - g\|_0 \leq m = e^{-\alpha a}, \\
 & \text{(c) } \|D^{-1}(u - f)\|_0 \leq m, \\
 & \text{(d) } \|u\|_a^2 + \|u\|_{-a}^2 \leq 2,
 \end{aligned}
 \tag{10.9}$$

where $g(x)$ and $f(x)$ are given periodic functions with zero mean value, $g(x)$ in L_2 and $f(x)$ in L_1 .

Problem 1. Find a function approximating every u satisfying (10.9).

$$\begin{aligned}
 & \text{Let} \\
 & g(x) = \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx, \\
 & f(x) \sim \sum_{n=1}^{\infty} n C_n \cos nx + n D_n \sin nx,
 \end{aligned}
 \tag{10.10}$$

and set

$$\begin{aligned}
 g_N(x,y) = & \sum_{n=1}^N (A_n \cos nx + B_n \sin nx) \cosh ny \\
 & + \sum_{n=1}^N (C_n \cos nx + D_n \sin nx) \sinh ny.
 \end{aligned}
 \tag{10.11}$$

This, for $N = [\alpha]$, is our approximation function.

We can then easily obtain the a-priori error bound

$$|u - g_{[\alpha]}|_y < L + H,$$

as in lemma 8 (here L and H are defined in theorem 10), and close analogies to the a-posteriori compatibility conditions in lemmas 9 and 10.

The problem for periodic harmonic functions corresponding to problem 2 of section 7 is the following. We are given approximate values g_j and f_j for u and u_y at a finite number of data points (x_j, y_j) in a single period. These points, with their periodic images, either lie on the x axis, with maximum separation δ , or are the vertices of two simple polygonal paths P_1 and P_2 which enclose the x axis between. Let δ be the maximum segment length of P_1 and P_2 , and let P_1 and P_2 themselves be enclosed by the lines $y = -b$ and $y = +b$, $0 \leq b < a$. Then the methods of section 7 carry through in obvious analogy.

Lavrentiev [12] also treats the approximate solution of the Cauchy problem by Fourier expansion of the data. However, he derives L_2 rather than uniform error bounds and considers only the case in which u is identically zero on the base and two sides of the rectangle. Moreover, his procedure involves a projection or filtering process for the high order coefficients, with no provision for truncation at a finite order.

Pucci [14] has considered a Cauchy problem in which the data is given only at a finite set of points, as in problem 2 here. See also Pucci [15] and Payne [13] for further references on the Cauchy problem for Laplace's equation.

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