WILLIAM MARSH RICE UNIVERSITY

AN EVALUATION OF THE FIXED POINT METHOD
OF VIBRATION ANALYSIS FOR A PARTICULAR
SYSTEM WITH INITIAL DAMPING

by

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ABSTRACT

The object of this thesis is to investigate the validity of a popular approximate procedure which is used to select the optimum damping for a given mechanical system. Since this cannot be done in general, a particular system is chosen.

The exact determination of the optimum damping for a very simple mechanical system is performed. This value of optimum damping is then compared with the value obtained by the popular, so-called "fixed point" method.

Since the fixed point solution is obtained independently of the initial damping of the system, the exact solution is worked out for sets of values of initial damping, which are typical of practical mechanical devices, to determine the influence of the initial damping on the optimum damping.

It is found that for this particular system the fixed point method and the exact method give the same value for optimum damping when there is no initial damping in the system. As more and more initial damping is added to the system the value for optimum damping increases significantly, but it becomes much less serious to use a value different from the optimum value.

NOTATION

A^T	denotes the transpose of matrix A
B	the matrix defined by Equation (E7)
C	the square damping matrix
c_o	the damper constant which is to be optimized
c_{ij}	the ij^{th} element of the matrix C, the j^{th} term in the i^{th} column
c_i	a damper constant, lb.sec/in
D	the matrix defined by Equation (E8)
e	Naperian base
$\bar{F}$	the column matrix of force amplitudes
F_j	the j^{th} element of the matrix $\bar{F}$
G	the matrix defined by Equation (25)
i	$\sqrt{-1}$
I	the identity matrix
K	the square spring matrix
k_{ij}	the ij^{th} element of the matrix K, the j^{th} term in the i^{th} column
k_i	a spring constant, lb/in
M	the square mass matrix
m_{ij}	the ij^{th} element of the matrix M
m_i	a mass constant, lb.sec ² /ft
M_{ij}	mobility, absolute value of displacement at Point i over force at Point j
$M_{eff, ij}^F$	effective mass, defined by Equation (A29)
$\bar{p}$	a column matrix of coordinates
p_r	the r^{th} element of the matrix $\bar{p}$
Q	the square undamped modal matrix defined by Equation (A17)
$\bar{q}$	a column matrix of coordinates
q_r	the r^{th} element of the matrix $\bar{q}$

2	a damping factor defined by Equation (50)
t	time - seconds
$\bar{w}$	a column matrix of coordinates
w_r	the r^{th} element of the matrix $\bar{w}$
$\bar{x}$	the column matrix of displacements
x_1	the displacement of the 1^{th} degree of freedom
$\bar{x}_0^r$	the r^{th} undamped eigenvector or mode shape
x_{01}^r	the i^{th} element of the vector $\bar{x}_0^r$
$\dot{x}_1$	the velocity of the 1^{th} degree of freedom
$\ddot{x}_1$	the acceleration of the 1^{th} degree of freedom
$\bar{y}$	a column matrix of coordinates
$\bar{z}$	a column matrix of coordinates
$\bar{z}_0^r$	the r^{th} damped eigenvector or mode shape
β	equals $-1/\lambda$
λ_r	the r^{th} eigenvalue of the matrix G
φ	a phase angle
θ	the square damped modal matrix defined by Equation (33)
ω	frequency - rad/sec
ω_r	the r^{th} natural frequency

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INTRODUCTION

Many times the amplitude of motion of certain members of a vibratory system are of such magnitude as to be undesirable. These amplitudes usually occur at frequencies of vibration which correspond to the natural frequencies of the system.

There are various methods used to reduce these amplitudes of motion over the frequency range of interest. The natural frequencies of the system can be changed such that they no longer occur in the range of frequencies of the exciting forces. To shift the natural frequencies of the system to frequencies higher than the exciting frequency the elements connecting various parts of the structure must be stiffened. Often this can result in an undesirable increase in weight. To shift the natural frequencies of the system to frequencies below the exciting frequency the stiffness of these connecting elements can be reduced. This, on the other hand, can lead to excessive static deflection.

A vibration absorber can be added to the system which will reduce the amplitude of motion of a single element over a narrow frequency range or over a somewhat wider range of frequencies if damping is added to the absorber. Many times vibration absorbers are impractical due to the effects of the added weight.

The addition of damping to the system reduces vibration amplitudes by increasing energy dissipation in the system. Often a single damper is added between two elements of the system and designed such that the maximum amplitude of motion in question (the value at resonance) is made a minimum.

Two methods used to determine the damper constant which will make this maximum amplitude a minimum are the exact method and the so-called

"fixed point" method. The theory for each method is presented in the appendices of this thesis. Both methods are applied to a simple particular mechanical system so that their results can be compared and the accuracy of the approximate fixed point method can be evaluated for the case of the system considered.

Since the fixed point method does not take into account the initial damping of the system, and it is apparent that the optimum damping will depend on the initial damping, it is to be suspected that for very small values of initial damping the two methods will yield approximately the same result, whereas for large values of initial damping this may not be the case. For this reason the solution by the exact method was obtained for various values of initial damping to determine the range of initial damping for which the fixed point method gives a suitable solution. The initial damping in each case is proportional to the elastic constants since this is believed to be a common occurrence.

A STATEMENT OF THE PROBLEM

The theory applicable to an n-degree of freedom damped linear vibratory system is applied to a damped two degree of freedom system.

(See Figure 1.)*

Figure 1

For selected values of c_1 and c_2 , the method given in Appendix E is used to obtain the damped normal coordinates of the system. The response of the system, amplitude of motion divided by force, $|x_2/F_1|$, versus frequency, ω , is then determined over the frequency range of interest. The damper constant, c_0 , is adjusted such that the maximum response, response at resonance, is a minimum for each pair of values of c_1 and c_2 .

The fixed point method (developed in Appendix C) is used in Appendix D to obtain an approximation for the optimum value of c_0 . This solution is then compared with the exact solution for the values of c_1 and c_2 considered.

This comparison should show how closely the fixed point solution approximates the exact solution for various values of initial damping for the particular system considered.

* The particular problem chosen could easily be solved by classical means since it is only a two degree of freedom problem. The procedure used here, however, is valid for any number of degrees of freedom. The principal difference the number of degrees of freedom makes is in the amount of computing time involved.

THE ANALYSIS

The equations of motion for the system shown in Figure 1 are

$$M \ddot{\bar{x}} + C \dot{\bar{x}} + K \bar{x} = R(\bar{f} e^{i\omega t}) \quad (1)$$

where,

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad (2)$$

$$C = \begin{bmatrix} c_1 + c_2 + c_0 & -(c_2 + c_0) \\ -(c_2 + c_0) & c_2 + c_0 \end{bmatrix} \quad (3)$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad (4)$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5)$$

and

$$\bar{f} = \begin{bmatrix} 0 \\ F_1 \end{bmatrix} \quad (6)$$

Take the transformation

$$\bar{x} = Q \bar{q} \quad (7)$$

where Q is the modal matrix given by

$$Q = \begin{bmatrix} x_{o1}^1 & x_{o1}^2 \\ x_{o2}^1 & x_{o2}^2 \end{bmatrix} \quad (8)$$

or as is given by Equation (B 10)

$$Q = \begin{bmatrix} 0.6495 & 0.7602 \\ 2.4045 & -2.0542 \end{bmatrix} \quad (9)$$

Using Equation (9), Equation (1) becomes

$$M \dot{Q} \ddot{\bar{q}} + C \dot{Q} \dot{\bar{q}} + K Q \bar{q} = \bar{f} e^{i\omega t} \quad (10)$$

Premultiplying Equation (10) by Q^T there results

$$Q^T M \dot{Q} \ddot{\bar{q}} + Q^T C \dot{Q} \dot{\bar{q}} + Q^T K Q \bar{q} = Q^T \bar{f} e^{i\omega t} \quad (11)$$

But from Equations (A 18) and (A 19)

$$Q^T M Q = I \quad (12)$$

and

$$Q^T K Q = [\omega_r^2]_o \quad (13)$$

So Equation (11) becomes

$$I \ddot{\bar{q}} + Q^T C \dot{Q} \dot{\bar{q}} + [\omega_r^2]_o \bar{q} = Q^T \bar{f} e^{i\omega t} \quad (14)$$

As in Appendix E the auxiliary equation

$$I \ddot{\bar{q}} - I \dot{\bar{q}} = 0 \quad (15)$$

is needed. Writing Equations (14) and (15) in matrix form there results

$$\begin{bmatrix} 0 & I \\ I & Q^T C Q \end{bmatrix} \begin{bmatrix} \ddot{\bar{q}} \\ \dot{\bar{q}} \end{bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & [\omega_r^2]_o \end{bmatrix} \begin{bmatrix} \dot{\bar{q}} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} 0 \\ Q^T \bar{f} \end{bmatrix} e^{i\omega t} \quad (16)$$

Letting

$$\bar{z} = \begin{bmatrix} \dot{\bar{q}} \\ \bar{q} \end{bmatrix} \quad (17)$$

Equation (16) becomes

$$\begin{bmatrix} 0 & I \\ I & Q^T C Q \end{bmatrix} \dot{\bar{z}} + \begin{bmatrix} -I & 0 \\ 0 & [\omega_r^2]_o \end{bmatrix} \bar{z} = \begin{bmatrix} 0 \\ Q^T \bar{f} \end{bmatrix} e^{i\omega t} \quad (18)$$

To obtain the homogeneous solution of the system, the equation

$$\dot{\bar{z}} = \bar{z} \cdot e^{\beta t} \quad (19)$$

is substituted into the homogeneous part of Equation (18) to give

$$\beta \begin{bmatrix} 0 & -I \\ I & Q^T C Q \end{bmatrix} \bar{z}_o = - \begin{bmatrix} -I & 0 \\ 0 & [\omega_r^2]_o \end{bmatrix} \bar{z}_o \quad (20)$$

If the change of coordinates

$$\bar{z}_0 = \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \bar{p} \quad (21)$$

is introduced into Equation (20) and the result is premultiplied by

$$\begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix}$$

there results

$$\beta \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & c_{11} & c_{12} \\ 0 & 1 & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \bar{p} \quad (22)$$

$$= - \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & \omega_1^2 & \\ & & & \omega_2^2 \end{bmatrix} \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \bar{p}$$

or

$$\beta \begin{bmatrix} 0 & 0 & \frac{i}{\omega_1} & 0 \\ 0 & 0 & 0 & \frac{i}{\omega_2} \\ \frac{i}{\omega_1} & 0 & \frac{c_{11}}{\omega_1^2} & \frac{c_{12}}{\omega_1 \omega_2} \\ 0 & \frac{i}{\omega_2} & \frac{c_{21}}{\omega_1 \omega_2} & \frac{c_{22}}{\omega_2^2} \end{bmatrix} \bar{p} = - \bar{p} \quad (23)$$

or finally

$$G \bar{p} = \lambda \bar{p} \quad (24)$$

where,

$$G = \begin{bmatrix} 0 & 0 & \frac{i}{\omega_1} & 0 \\ 0 & 0 & 0 & \frac{i}{\omega_2} \\ \frac{i}{\omega_1} & 0 & \frac{c_{11}}{\omega_1^2} & \frac{c_{12}}{\omega_1 \omega_2} \\ 0 & \frac{i}{\omega_2} & \frac{c_{21}}{\omega_1 \omega_2} & \frac{c_{22}}{\omega_2^2} \end{bmatrix} \quad (25)$$

and

$$\lambda = -\frac{1}{\beta}$$

Equation (24) represents a complex eigenvalue problem, the eigenvalues⁽²⁶⁾ which are, in general, complex. The complex roots of the characteristic equation occur as complex conjugate pairs with corresponding complex modal columns.

In Equation (22)

$$Q^T C Q = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad (27)$$

where the elements, c_{ij} , must be obtained from Equations (3) and (9) as follows,

$$Q^T C Q = \begin{bmatrix} 0.6495 & 2.4045 \\ 0.7602 & -2.0542 \end{bmatrix} \begin{bmatrix} c_1 + c_2 + c_0 & -(c_2 + c_0) \\ -(c_1 + c_0) & c_2 + c_0 \end{bmatrix} \begin{bmatrix} 0.6495 & 0.7602 \\ 2.4045 & -2.0542 \end{bmatrix} \quad (28)$$

or

$$Q^T C Q = [c_{ij}] = \begin{bmatrix} 0.42185 c_1 + 3.0800(c_2 + c_0) & 0.49375 c_1 - 4.9393(c_2 + c_0) \\ 0.49375 c_1 - 4.9393(c_2 + c_0) & 0.57790 c_1 + 7.9026(c_2 + c_0) \end{bmatrix} \quad (29)$$

Substituting Equation (29) into Equation (25) G becomes

$$G = \begin{bmatrix} 0 & 0 & \frac{i}{\omega_1} & 0 \\ 0 & 0 & 0 & \frac{i}{\omega_2} \\ \frac{i}{\omega_1} & 0 & \frac{0.42185 c_1 + 3.0800(c_2 + c_0)}{\omega_1^2} & \frac{0.49375 c_1 - 4.9393(c_2 + c_0)}{\omega_1 \omega_2} \\ 0 & \frac{i}{\omega_2} & \frac{0.49375 c_1 - 4.9393(c_2 + c_0)}{\omega_1 \omega_2} & \frac{0.57790 c_1 + 7.9026(c_2 + c_0)}{\omega_2^2} \end{bmatrix} \quad (30)$$

There are various methods used to solve the eigenvalue problem represented by Equation (24). For large matrices (50 x 50) the Jacobi method* has been used with much success and is particularly suitable for the case of repeated roots of the secular equation. Due to the small size of the matrix given by Equation (30) the following method is used here.

The determinant, $|[G] - \lambda[I]|$, is expanded and set equal to zero to obtain the secular equation which is quadratic in λ . Only real coefficients appear in the secular equation so the solution of the eigenvalue problem is reduced to the solution of a quadratic equation with real coefficients.

The eigenvectors can easily be obtained by writing Equation (24) as

$$[G - \lambda I] \bar{p} = 0 \quad (31)$$

or

$$\begin{bmatrix} -\lambda & 0 & \frac{i}{\omega_1} & 0 \\ 0 & -\lambda & 0 & \frac{i}{\omega_2} \\ \frac{i}{\omega_1} & 0 & g_{33} - \lambda & g_{34} \\ 0 & \frac{i}{\omega_2} & g_{43} & g_{44} - \lambda \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = 0 \quad (32)$$

Since the magnitude of the vector $\bar{p}$ is undetermined, p_1 can be chosen arbitrarily and p_2 , p_3 , and p_4 can be determined from three of the equations of Equation (32). The fourth equation can be used to check the solution.

Corresponding to each eigenvalue, λ_r , an eigenvector, $\bar{p}^r$, is obtained. To get the normal coordinates the transformation given by Equation (21)

$$\bar{z}_0 = \begin{bmatrix} i & & & \\ & i & & \\ & & \frac{1}{\omega_1} & \\ & & & \frac{1}{\omega_2} \end{bmatrix} \bar{p} \quad (21)$$

* G. F. Forsythe and P. Henrici, "The Cyclic Jacobi Method for Computing the Principal Values of a Complex Matrix," Transactions of the American Mathematical Society, 1960, Vol. 94, No. 1, pp. 1-23.

must be made.

If a modal matrix is formed by using the damped mode shapes, $\bar{z}_o^r$, as the columns of the matrix, there results as in Equation (E 28)

$$\Theta = \begin{bmatrix} \bar{z}_{o1}^1 & \bar{z}_{o1}^2 & \bar{z}_{o1}^3 & \bar{z}_{o1}^4 \\ \bar{z}_{o2}^1 & \bar{z}_{o2}^2 & \bar{z}_{o2}^3 & \bar{z}_{o2}^4 \\ \bar{z}_{o3}^1 & \bar{z}_{o3}^2 & \bar{z}_{o3}^3 & \bar{z}_{o3}^4 \\ \bar{z}_{o4}^1 & \bar{z}_{o4}^2 & \bar{z}_{o4}^3 & \bar{z}_{o4}^4 \end{bmatrix} \quad (33)$$

If Equation (18) is written as

$$B \ddot{\bar{z}} + D \dot{\bar{z}} = \bar{f}' e^{i\omega t} \quad (34)$$

where

$$B = \begin{bmatrix} 0 & I \\ I & Q^T C Q \end{bmatrix} \quad (35)$$

$$D = \begin{bmatrix} -I & 0 \\ 0 & [\omega_r^2]_o \end{bmatrix} \quad (36)$$

and

$$\bar{f}' = \begin{bmatrix} 0 \\ Q^T \bar{f} \end{bmatrix} \quad (37)$$

and a solution of the form

$$\bar{z} = \bar{y} e^{i\omega t} \quad (38)$$

is assumed, Equation (34) becomes

$$i\omega B \bar{y} + D \bar{y} = \bar{f}' \quad (39)$$

Changing to the damped normal coordinates by the transformation

$$\bar{y} = \Theta \bar{w} \quad (40)$$

and premultiplying the result by Θ^T there results

$$i\omega \Theta^T B \Theta \bar{w} + \Theta^T D \Theta \bar{w} = Q^T \bar{f}' \quad (41)$$

As shown in Appendix E, $\Theta^T B \Theta$ and $\Theta^T D \Theta$ are diagonal so that each ω_r can be obtained individually as

$$\omega_r = \frac{\bar{z}_0^{*T} \bar{f}'}{i\omega [\Theta^T B \Theta]_r + [\Theta^T D \Theta]_r} \quad (42)$$

But from Equation (40)

$$\bar{y} = \Theta \bar{w} \quad (40)$$

and from Equation (17)

$$\bar{y} = \begin{bmatrix} i\omega \bar{z}_0 \\ \bar{z}_0 \end{bmatrix} \quad (17)$$

According to Equation (7)

$$\bar{x} = Q \bar{y} \quad (7)$$

so

$$\bar{x} = Q \Theta \bar{w}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{01} & x_{02} \\ x_{03} & x_{04} \end{bmatrix} \begin{bmatrix} z_{03} & z_{04} & z_{05} & z_{06} \\ z_{07} & z_{08} & z_{09} & z_{10} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} \quad (43)$$

From Equation (37)

$$\bar{f}' = \begin{bmatrix} 0 \\ \rho^T \bar{f} \end{bmatrix} \quad (44)$$

or

$$\bar{f}' = \begin{bmatrix} 0 \\ 0 \\ .6495 & 2.4045 \\ .7602 & -2.0542 \end{bmatrix} \begin{bmatrix} F_1 e^{i\omega t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .6495 \\ .7602 \end{bmatrix} F_1 e^{i\omega t} \quad (45)$$

Using Equation (45) in Equation (42), w_r becomes

$$w_r = \frac{(.6495 \beta_{03}^r + .7602 \beta_{04}^r) F_1}{i\omega [\Theta^T B \Theta]_0^r + [\Theta^T D \Theta]_0^r} \quad (46)$$

So Equation (43) becomes

$$\begin{bmatrix} \frac{\chi_1}{F_1} \\ \frac{\chi_2}{F_1} \end{bmatrix} = \begin{bmatrix} \frac{(.6495 \beta_{03}^{i_1} + .7602 \beta_{04}^{i_1})(.6495 \beta_{03}^{i_1} + .7602 \beta_{04}^{i_1})}{i\omega [\Theta^T B \Theta]_0^{i_1} + [\Theta^T D \Theta]_0^{i_1}} \\ \frac{(2.4045 \beta_{03}^{i_2} - 2.0542 \beta_{04}^{i_2})(.6495 \beta_{03}^{i_2} + .7602 \beta_{04}^{i_2})}{i\omega [\Theta^T B \Theta]_0^{i_1} + [\Theta^T D \Theta]_0^{i_1}} \end{bmatrix} \quad (47)$$

$$\begin{aligned} &+ \frac{(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})}{i\omega [\Theta^T B \Theta]_0^{i_2} + [\Theta^T D \Theta]_0^{i_2}} + \frac{(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})}{i\omega [\Theta^T B \Theta]_0^{i_3} + [\Theta^T D \Theta]_0^{i_3}} \\ &+ \frac{(2.4045 \beta_{03}^{i_3} - 2.0542 \beta_{04}^{i_3})(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})}{i\omega [\Theta^T B \Theta]_0^{i_2} + [\Theta^T D \Theta]_0^{i_2}} + \frac{(2.4045 \beta_{03}^{i_3} - 2.0542 \beta_{04}^{i_3})(.6495 \beta_{03}^{i_3} + .7602 \beta_{04}^{i_3})}{i\omega [\Theta^T B \Theta]_0^{i_3} + [\Theta^T D \Theta]_0^{i_3}} \\ &+ \left[\frac{(.6495 \beta_{03}^{i_4} + .7602 \beta_{04}^{i_4})(.6495 \beta_{03}^{i_4} + .7602 \beta_{04}^{i_4})}{i\omega [\Theta^T B \Theta]_0^{i_4} + [\Theta^T D \Theta]_0^{i_4}} \right. \\ &\left. + \frac{(2.4045 \beta_{03}^{i_4} - 2.0542 \beta_{04}^{i_4})(.6495 \beta_{03}^{i_4} + .7602 \beta_{04}^{i_4})}{i\omega [\Theta^T B \Theta]_0^{i_4} + [\Theta^T D \Theta]_0^{i_4}} \right] \end{aligned}$$

Defining

$$\begin{aligned} A_{34}^r &= .6495 \beta_{03}^r + .7602 \beta_{04}^r \\ B_{34}^r &= 2.4045 \beta_{03}^r - 2.0542 \beta_{04}^r \\ g^r(\omega) &= i\omega [\Theta^T B \Theta]_0^r + [\Theta^T D \Theta]_0^r \end{aligned} \quad (48)$$

Equation (47) becomes

$$\begin{bmatrix} \frac{\chi_1}{F_1} \\ \frac{\chi_2}{F_1} \end{bmatrix} = \begin{bmatrix} \frac{(A_{34}^{i_1})^2}{g^{i_1}(\omega)} + \frac{(A_{34}^{i_2})^2}{g^{i_2}(\omega)} + \frac{(A_{34}^{i_3})^2}{g^{i_3}(\omega)} + \frac{(A_{34}^{i_4})^2}{g^{i_4}(\omega)} \\ \frac{(A_{34}^{i_1})(B_{34}^{i_1})}{g^{i_1}(\omega)} + \frac{(A_{34}^{i_2})(B_{34}^{i_2})}{g^{i_2}(\omega)} + \frac{(A_{34}^{i_3})(B_{34}^{i_3})}{g^{i_3}(\omega)} + \frac{(A_{34}^{i_4})(B_{34}^{i_4})}{g^{i_4}(\omega)} \end{bmatrix} \quad (49)$$

The procedure for determining the optimum value for c_0 for a set of initial damper constants, c_1 and c_2 , is as follows:

1. Choose c_1
2. Since the value of c_2 is always selected so that $c_1/c_2 = k_1/k_2$, the value of c_2 may be found as $c_2 = c_1/10$.
3. Choose a value for c_0 .
4. Calculate $[G]$ from Equation (30).
5. Expand $|[G] - \lambda [I]|$ and set the result equal to zero to obtain the secular equation.
6. Solve the quadratic secular equation for λ to obtain the eigenvalues.
7. Solve Equation (32) to obtain the eigenvectors, $\bar{p}^r$.
8. Obtain the damped normal coordinates by the transformation given in Equation (21). Form the modal matrix given by Equation (33).
9. Form the matrices B and D from Equations (35) and (36). $Q^T C Q$ is given by Equation (29).
10. Make the matrix multiplications $\Theta^T B \Theta$ and $\Theta^T D \Theta$.
11. Calculate A^r_{34} and B^r_{34} from Equation (48). Choose a value for ω and calculate $g^r(\omega)$ from Equation (48).
12. Calculate the response, $|x_2/F_1|$, from Equation (49). $|x_1/F_1|$ can also be computed by Equation (49).
13. Select enough values of ω to define the response curve in the neighborhood of its stationary values.
14. Choose another value for c_0 and go back to step 3. Sufficient values of c_0 must be selected to define the "maximum response versus c_0 " curve in the range of its minimum.

To represent various values of initial damping in the system a parameter is defined as

$$Z = \frac{\sqrt{k_1 m_1}}{c_1} \quad (50)$$

so in terms of $\mathcal{Z}$, c_1 becomes

$$c_1 = \frac{\sqrt{k_1 m_1}}{\mathcal{Z}} = \frac{100}{\mathcal{Z}} \quad (51)$$

This $\mathcal{Z}$ would correspond to the damping ratio of a single degree of freedom system made up of k_1 , m_1 , and c_1 . The above calculations were made for values of $\mathcal{Z}$ ranging from $\mathcal{Z} = 100$ to $\mathcal{Z} = 6$. For many mechanical systems $\mathcal{Z} = 100$ represents an almost undamped case and $\mathcal{Z} = 6$ represents very large damping.

Figure 2 shows the response, $|x_2/F_1|$, versus frequency, ω , for $\mathcal{Z} = 100$ and selected values of c_0 between one and ten. The maximum value of $|x_2/F_1|$ for each curve shown in Figure 2 is shown in Figure 3 as a function of c_0 . This shows that the optimum value for c_0 is about $c_0 = 6.5$. This value should be compared with the value $c_0 = 6.4$ which is obtained in Appendix D using the fixed point method. The calculations for the case of $\mathcal{Z} = 100$ and $c_0 = 6$ are presented in numerical detail in Appendix F.

Figure 3 contains the curves for all values of $\mathcal{Z}$ considered from $\mathcal{Z} = 100$ to $\mathcal{Z} = 6$. The reader should note that the optimum value of c_0 increases from $c_0 = 6.5$ for the initially undamped case to $c_0 = 20$ for the most damped case. It should also be noted that the curves in Figure 3 are much flatter for lower values of $\mathcal{Z}$ so that choosing a value for c_0 different from the optimum value would not be as serious in the case of low values of $\mathcal{Z}$ as it would be for high values of $\mathcal{Z}$.

Figure 4 shows the values obtained for the optimum maximum response $|x_2/F_1|$, for values of $\mathcal{Z}$ ranging from $\mathcal{Z} = 100$ to $\mathcal{Z} = 6$ by the exact method and the fixed point method. Usually the fixed point method is used to determine only the optimum value for c_0 . Then using this value of c_0 the response, $|x_2/F_1|$, is obtained by the exact method.

FIGURE 2. RESPONSE VERSUS FREQUENCY

FIGURE 3. MAXIMUM RESPONSE VERSUS c_0

FIGURE 4. OPTIMUM MAXIMUM RESPONSE
VERSUS THE DAMPING FACTOR, Z

CONCLUSIONS

The value obtained for the optimum damper constant, c_o , by the exact method and by the fixed point method coincide for the initially undamped case. As initial damping is added to the system the value for the optimum damper, c_o , increases significantly, although the error incurred in using the value given by the fixed point method does not increase proportionately due to the flatness of the curves in Figure 3 for lower values of $\mathcal{L}$.

The maximum value of the response, $|x_2/F_1|$, obtained by the fixed point method differs appreciably from those obtained by the exact method for values of $\mathcal{L}$ less than 50. Therefore the fixed point method should be used only as a means to determine the optimum value of c_o . The exact method should then be used with the c_o obtained from the fixed point method to determine the response curve.

APPENDIX A - THE UNDAMPED CASE - THEORY.

The equations of motion for an n-degree of freedom spring-mass system with no gyroscopic coupling and with external forces which vary sinusoidally with time, all of which have the same frequency and are either in phase or 180 degrees out of phase with one another, can be written as the real part of

$$M \ddot{\bar{x}} + K \bar{x} = \bar{f} e^{i\omega t} \quad (A1)$$

where,

M is the symmetric mass matrix. The i^{th} term in the j^{th} row, m_{ij} , being the component of the external force system which does work on the j^{th} degree of freedom when this external force system and the configuration are such that the i^{th} degree of freedom has unit acceleration, zero velocity, and zero displacement while the remaining degrees of freedom have zero displacement, velocity, and acceleration.

K is the symmetric spring matrix. The i^{th} term in the j^{th} row, k_{ij} , being the component of the external force system which does work on the j^{th} degree of freedom when this external force system and configuration are such that the i^{th} degree of freedom has unit displacement, zero velocity, and zero acceleration while the remaining degrees of freedom have zero displacement, velocity, and acceleration.

$\bar{f}$ is a column matrix of force amplitudes where the i^{th} element is the component of the amplitude of the harmonic exciting external force system which does work on the i^{th} degree of freedom.

$\bar{x}$ is a column matrix of displacements.* The i^{th} term, x_i , gives the position of the i^{th} degree of freedom relative to its static equilibrium position. $\dot{x}_2$ is called the velocity and $\ddot{x}_2$ the acceleration of the i^{th} degree of freedom.

* These displacements can be angular as well as linear.

To solve Equation (A 1) consider the case of free vibrations given by

$$M \ddot{\bar{x}} + K \bar{x} = 0 \quad (A2)$$

The solution of Equation (A 2) is of the form

$$\bar{x} = \bar{x}_0 e^{i(\omega t + \phi)} \quad (A3)$$

so that

$$-\omega^2 M \bar{x}_0 + K \bar{x}_0 = 0 \quad (A4)$$

or

$$[K - \omega^2 M] \bar{x}_0 = 0 \quad (A5)$$

The solution of Equation (A 5) is an eigenvalue problem and non-trivial solutions are possible only if the determinant

$$|K - \omega^2 M| = 0 \quad (A6)$$

Expanding Equation (A 6) results in an n^{th} degree algebraic equation in ω^2 known as the secular equation. When the roots of the secular equation are substituted into Equation (A 5) it is possible to solve for the mode shapes (eigenvectors) by solving $n-1$ linearly independent equations for $n-1$ of the elements in terms of one arbitrarily chosen element. Corresponding to each value ω_r^2 there exists a vector solution $\bar{x}_0^r$. Since K and M are real symmetric matrices ω_r^2 and $\bar{x}_0^r$ are real.

An important property of the mode shapes is orthogonality, which is established as follows. Let ω_p^2 and ω_q^2 be two eigenvalues and denote the corresponding eigenvectors by $\bar{x}_0^p$ and $\bar{x}_0^q$, respectively, so that the equations

$$K \bar{x}_0^p = \omega_p^2 M \bar{x}_0^p \quad (A7)$$

$$K \bar{x}_0^q = \omega_q^2 M \bar{x}_0^q \quad (A8)$$

are satisfied. Transposing Equation (A 7) and postmultiplying the result by $\bar{x}_0^q$ leads to (recalling that $[AB]^T = B^T A^T$ and that $A^T = A$ for symmetric matrices)

$$\bar{x}_0^{pT} K \bar{x}_0^q = \omega_p^2 \bar{x}_0^{pT} M \bar{x}_0^q \quad (A9)$$

Premultiplying Equation (A8) by $\bar{x}_o^{PT}$ there results

$$\bar{x}_o^{PT} K \bar{x}_o^s = \omega_o^2 \bar{x}_o^{PT} M \bar{x}_o^s \quad (A10)$$

If Equation (A10) is subtracted from Equation (A9) there results

$$(\omega_p^2 - \omega_o^2) \bar{x}_o^{PT} M \bar{x}_o^s = 0 \quad (A11)$$

If $\omega_p^2 \neq \omega_o^2$ *

$$\bar{x}_o^{PT} M \bar{x}_o^s = 0 \quad (A12)$$

and if Equation (A12) is substituted into Equation (A10) there results

$$\bar{x}_o^{PT} K \bar{x}_o^s = 0 \quad (A13)$$

Since the amplitude of the vectors, $\bar{x}_o^r$, are not fixed, they may be adjusted so that

$$\bar{x}_o^{rT} M \bar{x}_o^r = 1 \quad (A14)$$

These $\bar{x}_o^r$'s are called the normalized mode shapes. Premultiplying Equation (A7) by $\bar{x}_o^{PT}$ it follows that

$$\bar{x}_o^{rT} K \bar{x}_o^r = \omega_r^2 \quad (A15)$$

Equations (A12) and (A13) define the orthogonality relations between the mode shapes with respect to the mass matrix and spring matrix respectively.

To solve the undamped forced vibrations case Equation (A3) is used in Equation (A1) resulting in

$$-\omega^2 M \bar{y}_o + K \bar{y}_o = \bar{f} \quad (A16)$$

The undamped mode shapes are used to form a modal matrix, the r^{th} column of which is the mode shape associated with the eigenvalue ω_r^2 . The modal matrix can be written as

* If two of the roots are equal, i.e., $(\omega^2 - \omega_r^2)^2$ is a factor of the secular equation, it can be shown that two linearly independent solutions exist when $\omega^2 = \omega_r^2$, hence two orthogonal eigenvectors corresponding to ω_r^2 can be determined. See F. B. Hildebrand, "Methods of Applied Mathematics," Prentice-Hall, 1960, p. 59.

$$Q = \begin{bmatrix} \chi_{o1}^1 & \chi_{o1}^2 & \dots & \chi_{o1}^n \\ \chi_{o2}^1 & \chi_{o2}^2 & \dots & \chi_{o2}^n \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{on}^1 & \chi_{on}^2 & \dots & \chi_{on}^n \end{bmatrix} \quad (A17)$$

Using the normalized mode shapes it follows from Equation (A14) that

$$Q^T M Q = I \quad (A18)$$

and

$$Q^T K Q = [\omega_r^2]_D \quad (A19)$$

where the subscript "D" indicates a diagonal matrix. Equation (A16) is multiplied by Q^T to obtain

$$-\omega^2 Q^T M \bar{\chi}_o + Q^T K \bar{\chi}_o = Q^T \bar{f} \quad (A20)$$

and the coordinates are transformed to the normal coordinates by

$$\bar{\chi}_o = Q \bar{q} \quad (A21)$$

to give

$$-\omega^2 Q^T M Q \bar{q} + Q^T K Q \bar{q} = Q^T \bar{f} \quad (A22)$$

or, due to Equation (A18) and (A19)

$$[-\omega^2 + \omega_r^2]_D \bar{q} = Q^T \bar{f} \quad (A23)$$

The left hand side of Equation (A23) is now diagonalized making it possible to solve for each q_r as

$$q_r = \frac{\bar{\chi}_o^r{}^T \bar{f}}{\omega_r^2 - \omega^2} \quad (A24)$$

Using Equation (A21) to transform back to the original coordinates there results

$$\chi_{on} = \sum_{r=1}^n \chi_{on}^r q_r \quad (A25)$$

Substituting Equation (A24) into Equation (A25) there results

$$\chi_{on} = \sum_{r=1}^n \frac{(\bar{\chi}_o^r{}^T \bar{f}) \bar{\chi}_{on}^r}{\omega_r^2 - \omega^2} \quad (A26)$$

or in matrix form

$$\bar{\chi}_o = \sum_{r=1}^n \frac{(\bar{\chi}_o^T \bar{f}) \bar{\chi}_o^r}{\omega_r^2 - \omega^2} \quad (\text{A27})$$

In the case where the force system does work only on the j^{th} degree of freedom (only non-zero $\bar{f}$ term in F_j) Equation (A26) becomes

$$\frac{\chi_{ok}}{F_j} = \sum_{r=1}^n \frac{\chi_{oj}^r \chi_{ok}^r}{\omega_r^2 - \omega^2} e^{i\omega t} \quad (\text{A28})$$

Defining

$$\frac{1}{\chi_{oj}^r \chi_{ok}^r} = M_{\text{eff}jk}^r$$

where $M_{\text{eff}jk}^r$ is called the effective mass, Equation (A28) becomes

$$\frac{\chi_{ok}}{F_j} = \sum_{r=1}^n \frac{e^{i\omega t}}{M_{\text{eff}jk}^r (\omega_r^2 - \omega^2)} \quad (\text{A29})$$

APPENDIX B - APPLICATION OF APPENDIX A TO THE UNDAMPED CASE.

In this appendix the undamped problem is solved to obtain the modal matrix which can be used to make the transformation to the undamped normal coordinates. The system for no damping is shown below.

Figure B-1

The equations of motion for the system shown in Figure B-1 are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (\text{B1})$$

Substituting

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} e^{i\omega t} \quad (\text{B2})$$

into Equation (B1) there results

$$\begin{bmatrix} -m_1 \omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2 \omega^2 + k_2 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = 0 \quad (\text{B3})$$

To find the natural frequencies the determinant of the coefficient matrix is set equal to zero

$$\begin{vmatrix} -m_1 \omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2 \omega^2 + k_2 \end{vmatrix} = 0 \quad (\text{B4})$$

which gives

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2) \omega^2 + k_1 k_2 = 0 \quad (\text{B5})$$

Substituting the values given in Figure B-1 for k_1 , k_2 , m_1 and m_2 into Equation (B 5) there results

$$\omega^4 - 2.1 \times 10^4 \omega^2 + 10^8 = 0 \quad (B6)$$

the roots of which are $\omega_1^2 = 7,298 \text{ (rad/sec)}^2$ and $\omega_2^2 = 13,702 \text{ (rad/sec)}^2$ or $\omega_1 = 85.4 \text{ rad/sec}$ and $\omega_2 = 117 \text{ rad/sec}$.

Solving Equation (B3) gives the amplitude ratio, x_{o2}/x_{o1} , for a particular natural frequency as

$$\left. \frac{x_{o2}}{x_{o1}} \right|_{\omega = \omega_n} = \left(\frac{-m_1 \omega_n^2 + k_1 + k_2}{k_2} \right) \quad (B7)$$

when the eigenvectors are normalized according to the condition that

$m_1 x_{o1}^2 + m_2 x_{o2}^2 = 1$, there results

$$\text{for } \omega_n = \omega_1 = 85.4 \text{ rad/sec, } \begin{bmatrix} x_{o1}^1 \\ x_{o2}^2 \end{bmatrix} = \begin{bmatrix} 0.6495 \\ 2.4045 \end{bmatrix} \quad (B8)$$

$$\text{for } \omega_n = \omega_2 = 117 \text{ rad/sec, } \begin{bmatrix} x_{o1}^2 \\ x_{o2}^2 \end{bmatrix} = \begin{bmatrix} 0.7602 \\ -2.0542 \end{bmatrix}$$

From Equation (A17) the modal matrix, Q , is therefore

$$Q = \begin{bmatrix} 0.6495 & 0.7602 \\ 2.4045 & -2.0542 \end{bmatrix} \quad (B9)$$

In Appendix A it is shown that the mass matrix, M , and the spring matrix, K , are simultaneously diagonalized by the transformations $Q^T M Q$ and $Q^T K Q$. This fact is used in the analysis section of this thesis to relate the $\bar{x}$ coordinates to the normal coordinates, $\bar{q}$.

APPENDIX C - THE FIXED POINT METHOD - THEORY*

The Fixed Point Method is an empirical procedure to help select an approximate value for optimum damping. The salient point underlying this method is the following theorem.

Theorem: For a specified force excitation which varies sinusoidally in time applied to an undamped linear system there are certain frequencies at which any prescribed steady state motion resulting from the specified force system is not altered by the addition of a single damper to the original undamped system.

In other words, if the amplitude of the specified steady state motion is plotted as a function of frequency with the constant for the added damper as a parameter, there are certain points, called "fixed points", common to all curves.** Since this is true the fixed point in the frequency range of interest which represents maximum motion, $F P_{\max}$, is a lower bound on the minimum maximum value of the specified motion.

By the Fixed Point Method the stationary value of the response, $|x/F|$, versus frequency, ω , curve is forced to occur at the $F P_{\max}$. In the remainder of this section the theory associated with the Fixed Point Method is given.

A general representation of the system is shown in Figure C-1, where the figure is understood to represent any linear undamped n-degree of freedom vibratory system with no gyroscopic coupling which has the external force applied at any Point 1 and the motion is observed at any Point 2. A single damper is connected between any two Points 3 and 4.

* The results presented here were first published by R. Plunkett, "The Calculation of Optimum Concentrated Damping for Continuous Systems", Journal of Applied Mechanics, 1958, Vol. 25, No. 2, pp. 219-224.

** A semi-proof of this is presented later in this Appendix.

Figure C-1

The expression to be minimized is the mobility, M_{1j} , the displacement at 1 due to force excitation at j . It is to be minimized with respect to the exciting frequency, ω , and the damper constant, c_0 .

First, the expression for M_{1j} , that is

$$M_{ij} = x_i / F_j = f(\omega, c_0) \quad (C1)$$

must be found. This is found by first solving the undamped case and then adding the damper and solving again in terms of the undamped mobilities. The displacement to be minimized is at Point 2, the force system acts at Point 1, and the damper is placed between two Points 3 and 4. (See Figure C-1). The expression for displacements in terms of forces are as follows:

$$x_2 = M_{21} F_1 + M_{23} F - M_{24} F \quad (C2)$$

$$x_3 = M_{31} F_1 + M_{33} F - M_{34} F \quad (C3)$$

$$x_4 = M_{41} F_1 + M_{43} F - M_{44} F \quad (C4)$$

where the M_{1j} 's are the mobilities of the undamped system. F can be expressed in terms of c_0 as

$$c_0 (\dot{x}_3 - \dot{x}_4) = -F \quad (C5)$$

but

$$\dot{\chi} = i \omega \chi \quad (C6)$$

so

$$i \omega c_o (\chi_3 - \chi_4) = -F \quad (C8)$$

Using Equation (C8) to eliminate F from Equations (C2), (C3) and (C4) there results

$$\chi_2 = M_{21} F_1 - i \omega c_o (M_{23} - M_{24}) (\chi_3 - \chi_4) \quad (C9)$$

$$\chi_3 = M_{31} F_1 - i \omega c_o (M_{33} - M_{34}) (\chi_3 - \chi_4) \quad (C10)$$

$$\chi_4 = M_{41} F_1 - i \omega c_o (M_{43} - M_{44}) (\chi_3 - \chi_4) \quad (C11)$$

Subtracting Equation (C11) from Equation (C10), solving the result for $(\chi_3 - \chi_4)$ and noting that the right hand side of Equation (A28) is symmetric and j and k so that $M_{ij} = M_{ji}$, there results

$$(\chi_3 - \chi_4) = \frac{(M_{31} - M_{41}) F_1}{1 + i \omega c_o (M_{33} - 2 M_{34} + M_{44})} \quad (C12)$$

Substituting Equation (C12) into Equation (C9) gives

$$\frac{\chi_2}{F_1} = \frac{M_{21} - i \omega c_o [-M_{21} (M_{33} - 2 M_{34} + M_{44}) + (M_{23} - M_{24}) (M_{31} - M_{41})]}{1 + i \omega c_o (M_{33} - 2 M_{34} + M_{44})} \quad (C13)$$

Since the M_{ij} 's are for an undamped case they are pure real and Equation (C13) can be written as*

$$\frac{\chi_2}{F_1} = b_1 \cdot \frac{1 + i \omega c_o b_2}{1 + i \omega c_o b_3} \quad (C14)$$

where

$$b_1 = M_{21}$$

$$b_2 = (M_{33} - 2 M_{34} + M_{44}) - \frac{(M_{23} - M_{24})(M_{31} - M_{41})}{M_{21}} \quad (C15)$$

$$b_3 = (M_{33} - 2 M_{34} + M_{44})$$

* Clearly, if damping were present in the initial system an expression of this form would not be possible. Consequently the Fixed Point Method does not take into account damping in the initial system.

Since $|a + ib/c + id| = (a^2 + b^2/c^2 + d^2)^{1/2}$, $|x_2/F_1|^2$ can be written as

$$\left| \frac{x_2}{F_1} \right|^2 = b_1^2 \cdot \frac{1 + \omega^2 c_0^2 b_2^2}{1 + \omega^2 c_0^2 b_3^2} \quad (C16)$$

If the modulus of $|x_2/F_1|$ is to be independent of c_0 then, as can be seen from Equation (C16), it must be when

$$b_2^2 = b_3^2 \quad (C17)$$

This requires that $b_2 = \pm b_3$. For $b_2 = +b_3$ there must exist frequencies at which either $M_{32} = M_{42}$ or $M_{13} = M_{14}$ or both and M_{21} is not equal to zero. $M_{32} - M_{42}$ means $(x_3 - x_4)/F_2 = 0$ which means there must be a frequency at which the relative motion between Points 3 and 4 is zero for a force applied at Point 2. $M_{13} - M_{14} = 0$ means $(x_3 - x_4)/F_1 = 0$ which means there must be a frequency at which the relative motion between Points 3 and 4 is zero for a force applied at Point 1. It is a matter of experience that such points do in fact exist.

A necessary condition for the response, $|x_2/F_1|$, to be at its maximum is that the slope of the response versus frequency curve is zero. To insure that $|x_2/F_1|$ is a maximum when Equation (C17) holds, c_0 must be determined such that the frequency, ω , corresponding to damped resonance is also the frequency at which Equation (C17) is true. The slope of the response curve will be zero if

$$\begin{aligned} \frac{\partial (|x_2/F_1|^2)}{\partial \omega} &= \frac{1 + \omega^2 c_0^2 b_2^2}{1 + \omega^2 c_0^2 b_3^2} \cdot \frac{\partial (b_1^2)}{\partial \omega} + \frac{c_0^2 b_1^2}{1 + c_0^2 \omega^2 b_3^2} \cdot \frac{\partial (\omega^2 b_2^2)}{\partial \omega} \\ &\quad - \frac{c_0^2 b_1^2 (1 + c_0^2 \omega^2 b_2^2)}{(1 + c_0^2 \omega^2 b_3^2)^2} \cdot \frac{\partial (\omega^2 b_3^2)}{\partial \omega} = 0 \end{aligned} \quad (C18)$$

But by Equation (C17), $b_2^2 = b_3^2$, so Equation (C18) becomes

$$\frac{\partial (b_1^2)}{\partial \omega} + \frac{c_0^2 b_1^2}{1 + c_0^2 \omega^2 b_3^2} \cdot \frac{\partial (\omega^2 b_2^2)}{\partial \omega} - \frac{c_0^2 b_1^2}{1 + c_0^2 \omega^2 b_3^2} \cdot \frac{\partial (\omega^2 b_3^2)}{\partial \omega} = 0 \quad (C19)$$

which gives

$$\left(\omega^2 b_3^2 \right) \Big|_{\text{at fixed pt.}} \cdot c_o^2 = \frac{-\frac{\partial b_1^2}{\partial \omega} \Big|_{\text{at fixed pt.}}}{\frac{\partial \left(\frac{b_1^2 b_2^2}{b_3^2} \right)}{\partial \omega}} = \frac{-\left(b_1 \frac{\partial b_1}{\partial \omega} \right) \Big|_{\text{at fixed pt.}}}{\frac{b_1 b_2}{b_3} \cdot \frac{\partial \left(\frac{b_1 b_2}{b_3} \right)}{\partial \omega} \Big|_{\text{at fixed pt.}}} \quad (C20)$$

For $c_o = 0$ Equation (C16) becomes

$$\left| \frac{\chi_z}{F_1} \right|^2 = b_1^2 \quad (C21)$$

and for $c_o = \infty$ Equation (C16) becomes

$$\left| \frac{\chi_z}{F_1} \right|^2 = \frac{b_1^2 b_2^2}{b_3^2} \quad (C22)$$

Therefore the numerator of the right hand side of Equation (C20) is just the product of the fixed point mobility and the derivative of the response versus frequency curve for $c_o = 0$. The denominator is the product of the fixed point mobility and the derivative of the response versus frequency curve for $c_o = \infty$. So the numerator can be designated as

$$- \left[\frac{\chi_z}{F_1} \cdot \frac{\partial}{\partial \omega} \left(\frac{\chi_z}{F_1} \right) \right]_{\substack{\text{at fixed pt.} \\ c_o = 0}} \quad (C23)$$

and the denominator can be designated as

$$\left[\frac{\chi_z}{F_1} \cdot \frac{\partial}{\partial \omega} \left(\frac{\chi_z}{F_1} \right) \right]_{\substack{\text{at fixed pt.} \\ c_o = \infty}} \quad (C24)$$

Using Expressions (C23) and (C24) Equation (C20) becomes

$$\left| \omega b_3 \right| \Big|_{\text{at fixed pt.}} \cdot c_o = \sqrt{\frac{- \left[\frac{\chi_z}{F_1} \cdot \frac{\partial}{\partial \omega} \left(\frac{\chi_z}{F_1} \right) \right]_{\substack{\text{at fixed pt.} \\ c_o = 0}}}{\left[\frac{\chi_z}{F_1} \cdot \frac{\partial}{\partial \omega} \left(\frac{\chi_z}{F_1} \right) \right]_{\substack{\text{at fixed pt.} \\ c_o = \infty}}}} \quad (C25)$$

Both of the Expressions (C23) and (C24) are easy to evaluate since $c_o = 0$

represents the undamped case and $c_0 = \infty$ represents an undamped system with one less degree of freedom. To determine c_0 for a system with a large number of degrees of freedom the procedure would be as follows:

1. Determine the undamped natural frequencies and mode shapes.
2. From the results of Step (1) determine, in equation form,

$$b_1 = M_{21}$$

$$b_2 = (M_{33} - 2M_{34} + M_{44}) - \frac{(M_{23} - M_{24})(M_{31} - M_{41})}{M_{21}}$$

$$b_3 = (M_{33} - 2M_{34} + M_{44})$$

This is most easily accomplished by using effective masses as described at the end of Appendix A.

3. Calculate, as a function of frequency, over the range of interest

$$b_1 \quad \text{and} \quad b_1 b_2 / b_3$$

and determine frequencies where $b_1 = b_1 b_2 / b_3$.

4. Select the fixed point through which the slope of the response curve is to be made zero and at this point determine

$$b_1, \quad \frac{\partial (b_1)}{\partial \omega}$$

$$b_1 b_2 / b_3, \quad \frac{\partial (b_1 b_2 / b_3)}{\partial \omega}$$

5. Use Equation (C25) to determine c_0 . From Equation (C25) it is obvious that real values of c_0 will be obtained only for those fixed points at which the slopes of the two intersecting curves are of opposite algebraic sign.

**APPENDIX D - APPLICATION OF THE FIXED POINT METHOD TO THE PROBLEM
UNDER CONSIDERATION**

Consider the system in Appendix B represented by Figure B-1. If the damper, c_o , is inserted between the two masses the system becomes that given in Figure D-1.

Figure D-1

The force, $F_1 e^{i\omega t}$, is applied to the mass m_1 . The procedure outlined at the end of Appendix C is used in this appendix to obtain the value of damping, c_o , which makes the maximum response, $\left| \frac{x_2}{F_1} \right|_{\text{max}}$, versus frequency, ω , a minimum over the frequency range of interest.

1. The undamped natural frequencies and mode shapes for the system given in Figure D-1 were obtained in Appendix B. They are

$$\begin{aligned} \text{natural frequencies; } \omega_1 &= 85.4 \frac{\text{rad}}{\text{sec}}, \quad \omega_2 = 117 \frac{\text{rad}}{\text{sec}} \\ \text{mode shapes; } \omega_n = \omega_1, \quad \begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} &= \begin{bmatrix} 1 \\ 2.4045 \end{bmatrix}; \quad \omega_n = \omega_2, \quad \begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.0542 \end{bmatrix} \end{aligned}$$

The effective mass was found in Appendix A to be

$$M_{\text{eff } jk}^r = \frac{1}{x_{oj}^r x_{ok}^r} \quad (D-1)$$

$$\text{so } M_{\text{eff}12}^1 = \frac{1}{(2.4045)(.6495)} = +0.6403 \frac{\text{lb sec}^2}{\text{in}} \quad (\text{D-2})$$

$$M_{\text{eff}12}^2 = \frac{1}{(-2.0542)(.7602)} = -0.6403 \frac{\text{lb sec}^2}{\text{in}} \quad (\text{D-3})$$

The response, $\frac{x_2}{F_1}$, for $c_0 = 0$ is, from Equation (A-29)

$$\left. \frac{x_2}{F_1} \right|_{c_0=0} = \frac{1}{.6403} \left(\frac{1}{7,298 - \omega^2} - \frac{1}{13,702 - \omega^2} \right) e^{i\omega t} \quad (\text{D-4})$$

For $c_0 = \infty$ the system given in Figure D-1 becomes a one degree of freedom system where

$$\omega_1 = \sqrt{\frac{10^4}{1.1}} = \sqrt{9091} = 95.34 \frac{\text{rad}}{\text{sec}} \quad (\text{D-5})$$

$$\text{and } M_{\text{eff}12} = m_1 + m_2 = 1.1 \frac{\text{lb-sec}^2}{\text{in}} \quad (\text{D-6})$$

The response, $\frac{x_2}{F_1}$, is then

$$\left. \frac{x_2}{F_1} \right|_{c_0=\infty} = \frac{1}{1.1 (9091 - \omega^2)} \quad (\text{D-7})$$

A plot of the response, given by Equations (D-4) and (D-7), for $c_0 = 0$ and $c_0 = \infty$ is shown in Figure D-2. This plot shows that the fixed point corresponding to the highest response is at about 15 cycles per second.

An expanded plot of the neighborhood of this point is shown in Figure D-3.

The point to be adjusted is then at 92.3 radians per second where the absolute value of the response, $\left| \frac{x_2}{F_1} \right|$, for each curve is about 0.00157 in/lb. The slopes of these curves are given as follows:

$$\begin{aligned} \left. \frac{\partial \left(\frac{x_2}{F_1} \right)}{\partial \omega} \right|_{\substack{c_0=0 \\ \omega=92.3 \text{ rad/sec}}} &= \frac{2\omega}{.6403} \left(\frac{1}{(7,298 - \omega^2)^2} - \frac{1}{(13,702 - \omega^2)^2} \right) \bigg|_{\omega=92.3 \frac{\text{rad}}{\text{sec}}} \\ &= 1.83 \times 10^{-4} \frac{\text{in} \cdot \text{sec}}{\text{lb}} \end{aligned}$$

FIGURE D-2. RESPONSE VERSUS FREQUENCY

FIGURE D-3. RESPONSE VERSUS FREQUENCY

FIGURE D-3. RESPONSE VERSUS FREQUENCY

$$\left. \frac{\partial \left(\frac{x_2}{F_1} \right)}{\partial \omega} \right|_{\substack{c_0 = \infty \\ \omega = 92.3 \frac{\text{rad}}{\text{sec}}}} = \frac{2\omega}{1.1} \left(\frac{1}{(9,091 - \omega^2)^2} \right) \bigg|_{\omega = 92.3 \frac{\text{rad}}{\text{sec}}} \\ = 5.15 \times 10^{-4} \frac{\text{in} \cdot \text{sec}}{\text{lb}}$$

It is now necessary to determine $|\omega b_3|$ before Equation (C25) can be applied.

$$\omega b_3 = \omega (M_{33} - 2M_{34} + M_{44})$$

This can be physically interpreted as the relative velocity across the damper connections per unit force across the damper connections, that is, the response $\frac{\omega x_2}{F_1}$, at $\omega = 92.3$ rad/sec for $c_0 = 0$ in the problem shown in Figure D-4.

Figure D-4

From Equation (D-1) the desired effective masses are

$$M_{\text{eff}} \quad \omega = 13.6 \text{ cps} = \frac{1}{(2.4045 - .6495)^2} = .3247 \frac{\text{lb-sec}^2}{\text{in}} \\ M_{\text{eff}} \quad \omega = 18.6 \text{ cps} = \frac{1}{(-2.0542 - .7602)^2} = .1262 \frac{\text{sec}^2}{\text{in}}$$

so that

$$\begin{aligned} \frac{\omega(X_2 - X_1)}{F} \bigg|_{\omega = 92.3 \frac{\text{rad}}{\text{sec}}} &= \frac{\omega}{.3247} \cdot \frac{1}{(7,298 - \omega^2)} + \frac{\omega}{.1262} \cdot \frac{1}{(13,702 - \omega^2)} \bigg|_{\omega = 92.3 \frac{\text{rad}}{\text{sec}}} \\ &= -.0917 \frac{\text{in}}{\text{lb} \cdot \text{sec}} \end{aligned}$$

c_o can be determined from Equation (C25) as

$$(+.0917) c_o = \sqrt{\frac{-(.00157)(1.83)}{(-.00157)(5.15)}}$$

$$c_o = 6.4 \frac{\text{lb} \cdot \text{sec}}{\text{in}}$$

APPENDIX E. THE DAMPED CASE-THEORY

If linear massless viscous dampers are added to the previously considered undamped system the equations of motion are given by

$$M \ddot{\bar{\chi}} + C \dot{\bar{\chi}} + K \bar{\chi} = \bar{f} e^{i\omega t} \quad (E1)$$

where C is the symmetric damping matrix. The i^{th} term in the j^{th} row, c_{ij} , being the component of the external force system which does work on the j^{th} degree of freedom when this external force system and configuration are such that the i^{th} degree of freedom has unit velocity, zero acceleration and zero displacement while the remaining degrees of freedom have zero displacement, velocity, and acceleration.

The steady state solution of Equation (E1) is of the form

$$\bar{\chi} = \bar{\chi}_0 e^{i\omega t} \quad (E2)$$

so that Equation (E1) becomes

$$-\omega^2 M \bar{\chi}_0 + i\omega C \bar{\chi}_0 + K \bar{\chi}_0 = \bar{f} \quad (E3)$$

If Equation (A21)

$$\bar{\chi}_0 = Q \bar{q} \quad (A21)$$

is used the coordinates can be transformed to the undamped normal coordinates to give

$$[\omega_r^2 - \omega^2]_0 \bar{q} + i\omega Q^T C Q \bar{q} = Q^T \bar{f} \quad (E4)$$

While $Q^T M Q$ and $Q^T K Q$ are diagonalized by the same transformation, $Q^T C Q$ is diagonalized only for special cases. In a recent paper* T. K. Caughey points out that "the necessary and sufficient conditions that a damped dynamic system possess classical normal modes is that the damping matrix be diagonalized by the same transformation which uncouples the undamped system." He also shows that a sufficient though not necessary condition

* T. K. Caughey, "Classical Normal Modes in Damped Linear Dynamic Systems," Journal of Applied Mechanics, June 1960, pp 269-271.

for the damping matrix to be diagonalized by the same transformation which diagonalizes the mass and spring matrices is that the damping matrix be a linear combination of the mass matrix and spring matrix.

When the above conditions are not met, a more general method outlined by K. A. Foss* must be employed.

If the auxilliary equation

$$M \ddot{\bar{x}} - M \dot{\bar{x}} = 0 \quad (E5)$$

is written in matrix form with Equation (E1) there results

$$\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}} \\ \dot{\bar{x}} \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{f} e^{i\omega t} \end{bmatrix} \quad (E6)$$

Defining the following quantities

$$B = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \quad (E7)$$

$$D = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \quad (E8)$$

$$\bar{f}' = \begin{bmatrix} 0 \\ \bar{f} \end{bmatrix} \quad (E9)$$

$$\bar{z} = \begin{bmatrix} \ddot{\bar{x}} \\ \dot{\bar{x}} \end{bmatrix} \quad (E10)$$

Equation (E6) becomes

$$B \bar{z} + D \bar{z} = \bar{f}' e^{i\omega t} \quad (E11)$$

To obtain the homogeneous solution to Equation (E11) let

$$\bar{z} = \bar{z}_0 e^{\beta t} \quad (E12)$$

Using Equation (E12) in Equation (E11) there results

$$\beta B \bar{z}_0 + D \bar{z}_0 = 0 \quad (E13)$$

* K. A. Foss, "Co-ordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," Journal of Applied Mechanics, Vol. 25, No. 3, 1958, pp. 361-364.

or

$$[D + \beta B] \bar{z}_0 = 0 \quad (E14)$$

Equation (E14) represents an eigenvalue problem, the secular equation being given by

$$|D + \beta B| = 0 \quad (E15)$$

The solution of Equation (E15) determines the eigenvalues, β_r , which can be used in Equation (E14) to obtain the eigenvectors, $\bar{z}_0^r$. For an n-degree of freedom system there will be 2n real or complex eigenvalues and eigenvectors. The complex roots occur as complex conjugates with corresponding complex conjugate modal columns, such as

$$\beta_r, \quad \bar{z}_0^r = \begin{Bmatrix} \beta_r \bar{\varphi}^r \\ \bar{\varphi}^r \end{Bmatrix} \quad (E16)$$

For any two eigenvalues, β_p and β_q , Equation (E13) can be written as

$$\beta_p B \bar{z}_0^p + D \bar{z}_0^p = 0 \quad (E17)$$

and

$$\beta_q B \bar{z}_0^q + D \bar{z}_0^q = 0 \quad (E18)$$

Transposing Equation (E17) and using the fact that A^T equals A for symmetric matrices (clearly B and D are symmetric since M, C, and K are symmetric) and that $[A.B]^T$ equals $B^T A^T$ there results

$$\beta_p \bar{z}_0^{pT} B + \bar{z}_0^{pT} D = 0 \quad (E19)$$

Postmultiplying Equation (E19) by $\bar{z}_0^q$ and premultiplying Equation (E18) by $\bar{z}_0^{pT}$ there results

$$\beta_p \bar{z}_0^{pT} B \bar{z}_0^q + \bar{z}_0^{pT} D \bar{z}_0^q = 0 \quad (E20)$$

and

$$\beta_q \bar{z}_0^{pT} B \bar{z}_0^q + \bar{z}_0^{pT} D \bar{z}_0^q = 0 \quad (E21)$$

If Equation (E21) is subtracted from Equation (E20) there results

$$(\beta_p - \beta_q) \bar{z}_0^{pT} B \bar{z}_0^q = 0 \quad (E22)$$

Thus unless β_p equals β_q^* , the orthogonality relations are

$$\bar{z}_0^{pT} B \bar{z}_0^q = 0, \text{ when } p \neq q \quad (E23)$$

and

$$\bar{z}_0^{pT} D \bar{z}_0^q = 0, \text{ when } p \neq q \quad (E24)$$

Since only the directions of the modal vectors, $\bar{z}_0^r$, are determined they can be normalized such that

$$\bar{z}_0^{rT} B \bar{z}_0^r = 1 \quad (E25)$$

Substituting Equation (E25) into Equation (E20) there results

$$\bar{z}_0^{rT} D \bar{z}_0^r = \beta_r \quad (E26)$$

It follows as in Appendix A, that the transformation made by the change of coordinates

$$\bar{z} = \theta \bar{w} \quad (E27)$$

where

$$\theta = \begin{bmatrix} \bar{z}_{01}^1 & \bar{z}_{01}^2 & \dots & \bar{z}_{01}^n \\ \bar{z}_{02}^1 & \bar{z}_{02}^2 & \dots & \bar{z}_{02}^n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{z}_{0n}^1 & \bar{z}_{0n}^2 & \dots & \bar{z}_{0n}^n \end{bmatrix} \quad (E28)$$

uncouples the modes of vibration and allows each ω_r to be solved for individually.

If the solution of Equation (E1) is assumed to be of the form

$$\bar{z} = \bar{y} e^{i\omega t} \quad (E29)$$

there results

$$i\omega B \bar{y} + D \bar{y} = \bar{f}' \quad (E30)$$

If Equation (E30) is premultiplied by θ^T and the coordinates are changed to the normal coordinate by the transformation

$$\bar{y} = \theta \bar{w} \quad (E31)$$

* If β_p equals β_q see the footnote on page 17.

the result is

$$i\omega \Theta^T B \Theta \bar{w} + \Theta^T D \Theta \bar{w} = \Theta^T \bar{f}' \quad (\text{E32})$$

or

$$[i\omega + \beta_r]_0 \bar{w} = \Theta^T \bar{f}' \quad (\text{E33})$$

The left hand side of Equation (E33) is diagonal so each w_r can be solved for as

$$w_r = \frac{\bar{z}_0^T \bar{f}'}{i\omega + \beta_r} \quad (\text{E34})$$

or by the same procedure as was used in Appendix A to obtain Equation (A26)

$$\bar{y} = \begin{bmatrix} i\omega \bar{x}_0 \\ \bar{x}_0 \end{bmatrix} = \sum_{r=1}^{2n} \frac{(\bar{z}_0^T \bar{f}') \bar{z}_0^r}{i\omega + \beta_r} \quad (\text{E35})$$

APPENDIX F - EXAMPLE PROBLEM - EXACT METHOD

In this appendix the numerical details of an example problem are presented in which $\mathcal{L}$ equals 100 and c_0 equals $6 \frac{\text{lb-sec}}{\text{in}}$. These values are chosen because they closely approximate the values used and obtained by the Fixed Point Method in Appendix D.

1. From Equation (51)

$$c_1 = \frac{100}{2} = \frac{100}{100} = 1 \text{ lb-sec/in}$$

2. $c_2 = \frac{c_1}{10} = 0.1 \text{ lb-sec/in}$

3. $c_0 = 6.0 \text{ lb-sec/in}$

4. From Equation (30), using $\omega_1 = 85.4$ radians per second and $\omega_2 = 117$ radians per second which were obtained in Appendix B

$$[G] = \begin{bmatrix} 0 & 0 & .0117096i & 0 \\ 0 & 0 & 0 & .0085470i \\ .0117096i & 0 & .0026339 & -.0029660 \\ 0 & .0085470i & -.0029660 & .0035623 \end{bmatrix}$$

5. Expanding $|[G] - \lambda I| = 0$ to obtain the secular equation there results

$$\lambda^4 - .6197667\lambda^3 + 2.107646\lambda^2 - .6810849\lambda + 1.001708 = 0$$

6. The roots to the secular equation are

$$\lambda_1 = .2274249 + i .8876841, \quad \lambda_3 = .2274249 - i .8876841$$

$$\lambda_2 = .08245842 + i 1.089096, \quad \lambda_4 = .08245842 - i 1.089096$$

7. The eigenvectors, from Equation (36) are

$$\bar{p}^1 = \begin{bmatrix} 1.000000 \\ -.8253659 + i1.382559 \\ -.1942209 - i0.7580822 \\ 1.655532 + i0.4893358 \end{bmatrix}, \quad \bar{p}^2 = \begin{bmatrix} 1.000000 \\ .2120293 + i0.4004071 \\ -.0704195 - i0.9300876 \\ .4897597 - i0.3088065 \end{bmatrix}$$

$$\bar{p}^3 = \begin{bmatrix} 1.000000 \\ -.8253659 - i1.382559 \\ .1942209 + i0.7580822 \\ 1.655532 - i0.4893358 \end{bmatrix}, \quad \bar{p}^4 = \begin{bmatrix} 1.000000 \\ .2120193 - i0.4004071 \\ -.0704195 + i0.9300876 \\ .4897597 + i0.3088065 \end{bmatrix}$$

8. The damped normal coordinates are obtained by

$$\bar{z}_0 = \begin{bmatrix} i & & & \\ & i & & \\ & & 0.0117096 & \\ & & & 0.0085470 \end{bmatrix}$$

and the modal matrix is found by

$$\Theta = \begin{bmatrix} 1.0000000 & 1.0000000 \\ -.8253659 - i1.38256 & .2120293 + i0.40040714 \\ -.002274249 - i0.008876841 & -.000824584 - i0.01089096 \\ .001414985 + i0.004182357 & .00418598 - i0.00263937 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000 & 1.0000000 \\ -.9253659 - i1.38256 & .2120293 - i0.40040714 \\ -.00227425 + i0.00887684 & -.000814584 + i0.01089096 \\ .001414985 - i0.00418235 & .00418598 + i0.00263937 \end{bmatrix}$$

9. Using Equations (44 and (45) B and D are obtained as

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 19.20945 & -29.63598 \\ 0 & 1 & -29.63598 & 48.78376 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7,293.2 & 13.10 \\ 0 & 0 & 0 & 13,689 \end{bmatrix}$$

10. Making the multiplication $\Theta^T B \Theta$ and $\Theta^T D \Theta$ there result

$$\Theta^T B \Theta = \begin{bmatrix} \underline{-.0322642 + i0.0240267} & \underline{.00000032 - i0.0000000} \\ .00000032 + i0.00000000 & \underline{-.0322642 - i0.2902672} \\ .00000008 - i0.00000083 & -.00000004 + i0.0000008 \\ -.00000004 - i0.0000008 & -.00000008 + i0.00000083 \end{bmatrix}$$

$$\begin{bmatrix} .00000007 - i0.00000083 & -.00000040 + i0.0000008 \\ -.00000040 + i0.0000008 & .00000008 + i0.00000083 \\ \underline{.00239746 - i0.01770872} & -.00000007 - i0.00000000 \\ -.00000007 + i0.00000000 & \underline{.00239746 + i0.01770872} \end{bmatrix}$$

$$\Theta^T D \Theta = \begin{bmatrix} \underline{2.1946121 + i4.196928} & -.00005280 + i0.00000000 \\ -.0000528 + i0.00000000 & \underline{2.1946121 - i4.196928} \\ -.0008944 - i0.0002166 & -.0001222 - i0.0000117 \\ -.0001222 + i0.0000117 & -.0008944 + i0.0002166 \end{bmatrix}$$

$-.0008943 - i0.0002166$	$-.0001222 + i0.0000117$
$-.0001222 - i0.0000117$	$-.0008944 + i0.0002166$
$\underline{-1.6002316 - i0.3412854}$	$-.0000343 + i0.0000000$
$-.0000343 + i0.0000000$	$\underline{-1.6002316 + i0.34128546}$

The off diagonal elements of the above terms are in general 10^{-4} times the diagonal elements.

Steps 11 and 12 on page 12 were calculated by one program with only the final answer recorded which is plotted in Figure 2.

Figure 2 shows for $\alpha = 100$ and $c_0 = 6 \frac{\text{lb-sec}}{\text{in}}$

$$\left| \frac{x_2}{F_1} \right|_{\max} = 1.6 \times 10^{-3} \frac{\text{in}}{\text{lb}}$$

$$\text{or } \left. \frac{\omega x_2}{F_1} \right|_{\omega=92.4} = .1478 \frac{\text{in}}{\text{lb-sec}}$$

This value is surprisingly close to the value of

$$\left. \frac{\omega x_2}{F_1} \right|_{\text{fixed point}} = .145 \frac{\text{in}}{\text{lb-sec}}$$

obtained by the Fixed Point Method in Appendix D for the initially undamped case with $c_0 = 6.3 \frac{\text{lb-sec}}{\text{in}}$.

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