RICE UNIVERSITY

Torsional Buckling of an
Extended Twisted Cylindrical
Rod Under One-Sided Constraint

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science

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Houston, Texas

January, 1964

3 1272 00317 8470

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Acknowledgments

The author would like to express his appreciation to Dr. P. R. Paslay for helpful advice and encouragement during the preparation of this thesis, to his wife, Glenda, for painstaking preparation of the manuscript and to the National Aeronautics and Space Administration for financial aid during his graduate work at Rice University.

Abstract

The problem considered is that of the determination of stability criteria for an extended twisted cylindrical rod resting on a rigid plane inclined with respect to a gravity field. It was assumed that the tensile strain of the rod was no larger than that compatible with infinitesimal elasticity theory but that the torsion might be finite. The energy theory of buckling was used.

Prior to consideration of the specific problem, a formulation of the strain energy of an arbitrarily displaced cylindrical rod was made. It was assumed for this purpose that the rod suffers no lateral deformation and that sections of the rod initially plane and perpendicular to the center line remain plane and perpendicular to the tangent of the center line curve.

The potential energy contributed by external loadings (the gravity force, axial tensile force and axial moment) was then computed and added to the strain energy expression above. An equilibrium configuration was then assumed and the total energy variation for perturbations of this configuration considered. The variation of energy for arbitrary displacements was found to be positive to first order effects so that the configuration is stable in the classical sense. Consideration of second order effects revealed that instability may develop for sufficiently large displacements from the equilibrium configuration. The conditions under which such instabilities can occur were formulated and reduced to a simple sequence of calculations for application.

Nomenclature

A	Unity plus a small strain, e .
$\mathcal{A}$	Cross sectional area of the bar.
A_{ij}	Elements of the orthogonal matrix describing the motion.
B	Axial strain per unit length due to gravity loading.
$E_{\alpha\beta}$	Component of the Green - St. Venant strain tensor.
e	Axial strain due to tensile forces and twist.
e_0	Axial strain due only to tensile forces.
F	Displacement magnitude parameter.
F^*	See equation (V.4.).
G	The elastic shear modulus.
$G_{\alpha\beta}$	The material coordinate metric tensor component in the unstrained material.
g	Acceleration of gravity.
I	The second moment of area of the bar cross section.
L	The length of the buckled portion of the bar.
M	The axial moment applied to the bar.
P_0	The axial tension at the top of the bar.
P	The axial tension at the bottom of the buckled bar.
U	Total strain energy of the bar.
V	Strain energy of the bar per unit unstrained state volume.
$\vec{V}_x, \vec{V}_y, \vec{V}_z$	A set of base vectors associated with the material coordinates.
x^i	General term for spacial coordinates, here xyz or $r\theta z$.
X^α	General term for material coordinates, here XYZ or $R\Theta Z$.
x_0, y_0, z_0	Spacial coordinates of the bar center line.

(Continued)

Nomenclature (Continued)

β	The group $\phi' - \delta' \sin \epsilon$.
δ, ϵ, ϕ	Rotation angles describing cross section orientation.
∇^i_{α}	The displacement gradient $\frac{\partial x^i}{\partial X^{\alpha}}$.
ρ	The mass density of the bar.
T	Total potential energy of the bar.
ψ	Twist per unit length of the bar previous to buckling.
$\psi_{\alpha\beta}$	Elements of the material coordinate metric tensor for the strained material.
Ω	Potential energy of the external loads on the bar.
①	The quantity $x_o'^2 + y_o'^2 + z_o'^2$.
②	The quantity $y_o'^2 + z_o'^2$.

I. General Theoretical Considerations

I.1. The Theory of Elastic Stability

The theory of elastic stability is well established: an excellent summary and bibliography is given by Langhaar, [1]¹ or [2], and inclusive treatments are given by Pearson [3], Ziegler [4], Koiter [5] and Thompson [6]. There are various methods used in the solution of buckling problems, the most predominant being the energy method, the adjacent equilibrium method and the dynamic formulation method. The question of the area of applicability of each method has been discussed by Pearson [3], Ziegler [4], and Green and Shield [7]. It will be sufficient here to point out that the energy method (used in this investigation) is generally held to be valid under the condition of conservative forces, assumed satisfied herein. (But see the general objection voiced in [7].)

A general criterion for stability of a configuration of a system is that all kinematically admissible small perturbations away from this configuration spontaneously tend to zero in time. For conservative systems this is equivalent to the requirement that the change in potential energy of the system be positive for all virtual displacements from the configuration, and the problem of stability as treated by the energy method is the determination of the possibility of existence of deformations for which this is not satisfied. The necessary conditions for stability can easily be derived by considering the change in potential

¹Figures in square brackets refer to the reference list appearing in Appendix E.

energy of the system as a polynomial expansion in the displacement parameters U_i , $i = 1, 2, \dots, n$

$$\Delta T = P_1 + P_2 + P_3 + \dots \quad (I.1.)$$

where P_j represents terms of the j th order in U_i . The requirement that ΔT be positive for all kinematically admissible sets U_i , for sufficiently small displacements and holomorphic constraints, leads to the conditions:

$$(1) \quad P_1 = 0 \text{ and } P_2 \geq 0 \quad (I.2.)$$

$$(2) \quad \text{For } P_2 = 0, P_3 = 0 \text{ and } P_4 \geq 0$$

etc.

Koiter [6] goes on to formulate a necessary and sufficient condition for stability, but in most buckling problems examination of the first two terms in (I.1.) suffices.

In the present case it is found that the constraint on one displacement parameter is not holomorphic: the displacement is required to be non-negative. This in turn leads to $P_1 > 0$ which insures stability for small perturbations. There remains the possibility, however, that for somewhat larger displacements the second order term may be negative and become sufficiently large to dominate the first order term. Assuming that the higher order terms remain insignificant, the displacement set at which the rate of increase of the first order term is canceled by the rate of decrease of a negative second order term represents another position of equilibrium. If the second order term continues to remain negative the position is one of unstable equilibrium. The error involved in ignoring the higher order terms in this analysis is not felt to be great in this application because the first order coefficient is seen to be quite small in comparison to the second order coefficients, so that the displacement associated with the secondary equilibrium position is small.

The usual procedure of using arbitrarily chosen approximate variational displacements for the calculation of the energy change is used herein. It has been pointed out that this procedure results in buckling parameters which are upper bounds on the true parameters. There exists no correspondingly simple general procedure for obtaining lower bounds, although Trefftz [8] has suggested a method which has been successfully applied by Boresi [9] and in a somewhat different manner by Budiansky and Hu [10]. No attempt was made in the present case to determine a lower bound solution.

I.2. A Note on Method of Formulation

The method used in formulating the problem is somewhat similar to that of Green and Zerna [11]. At some instant in time the body is described as a continuum in a Euclidean frame and at all other times each point in the body retains these same "material" coordinates. If the deformation is continuous, as is presumed, these coordinates at any time form a curvilinear set, and the motion is assumed to be such that this coordinate set remains at all times Euclidean. A set of inertial Euclidean coordinate axes are also defined: these will be referred to as the spacial set. Denoting the spacial coordinates by x^i ($i = 1, 2, 3$) and the material coordinates by X^α ($\alpha = 1, 2, 3$), the configuration of the body is described at any time by the set of functions

$$x^i = x^i(X^\alpha, t). \quad (\text{I.3.})$$

The metric tensor associated with the material set at any time t is denoted $\Psi_{\alpha\beta}(t)$ and in particular at the time the body can be regarded as being in its relaxed or unstrained state, $G_{\alpha\beta}$. The metric tensor of the spacial set is denoted g_{ij} . Basic principles which will be used

in this work are:

- (1) The Green - St. Venant strain is defined as

$$E_{\alpha\beta} = \frac{1}{2} (\Psi_{\alpha\beta} - G_{\alpha\beta}). \quad (I.4.)$$

- (2) Employing the summation convention on repeated indices in products,

$$\Psi_{\alpha\beta} = \frac{\partial \chi^i}{\partial X^\alpha} \frac{\partial \chi^j}{\partial X^\beta} g_{ij}, \quad (I.5.)$$

presuming the functions (I.3.) are differentiable.

- (3) The classical elastic strain energy per unit volume in the relaxed state is given by

$$V = \frac{\lambda}{2} I_1^2 + G I_2 \quad (I.6.)$$

where I_1 and I_2 are the strain invariants

$$I_1 = G^{\alpha\beta} E_{\beta\alpha} \quad \text{and} \quad I_2 = G^{\alpha\gamma} E_{\gamma\beta} G^{\beta\delta} E_{\delta\alpha} \quad (I.7.)$$

and λ and G the Lamé elastic moduli.

If the relations (I.6.), (I.7.) and (I.4.) are combined, the unit strain energy becomes

$$V = \frac{\lambda}{8} (G^{\alpha\gamma} \Psi_{\gamma\alpha} - 1)^2 + \frac{1}{4} G (G^{\alpha\gamma} \Psi_{\gamma\beta} - \delta_\beta^\alpha) (G^{\beta\delta} \Psi_{\delta\alpha} - \delta_\alpha^\beta) \quad (I.8.)$$

where δ_β^α is the Kronecker Delta symbol, zero if α and β are different, unity if they are the same.

II. Statement of the Problem, Precedents

The buckling of a long thin cylindrical rod, initially extended and twisted and resting upon a rigid plane inclined with respect to a gravity field is to be considered (see Figure 1). It will be assumed that the initial deformations are of the order usually considered in infinitesimal elasticity theory (all related approximations will be noted as they are applied). It is assumed that in the equilibrium position the center line of the bar is straight and the bar rests on the plane along its whole length, but that the center line of the buckled bar assumes a three-dimensional curve. Cross sections of the bar initially perpendicular to the center line are assumed to remain plane and perpendicular to the center line curve. Further, no lateral contraction of the cross sections will be considered. The boundary conditions on the center line curve are that the lateral displacements and their derivatives with respect to the longitudinal coordinate vanish at both ends of the bar. An obvious kinematic constraint is that the displacement of the center line perpendicular to the rigid plane must be positive.

It should be expected that the assumption of no lateral contraction would have an effect on the buckling parameters. The buckling of an extended and twisted rod consists primarily of the exchange of high torsional strain in the equilibrium position for bending strain in the buckled position. The lateral contraction would thus add to the total energy primarily in the buckled configuration, so that the change in energy found herein should be less than that for a contracting bar. Thus the buckling parameters found here would be less than those for the contracting bar. Since the parameters herein found must represent

an upper bound on the actual parameters, if applied to a contracting bar, it is impossible to state exactly the relation of the predicted parameters and the actual ones. (The effect of contraction presumably would not be great.)

The problem of the buckling of a rod due to torsion in conjunction with extension or compression has a large literature. Some investigations of interest are those of Greenhill [12], Trösch [13], Langhaar [14], Goodier [15], and Green and Spencer [16]. All of these except the last deal with the case of compressed rods. Of these, Goodier, using the adjacent equilibrium method of formulation, makes use of a motion formulation similar to the one used here, and his results are similar to those found herein. Green and Spencer present a very elegant treatment of the buckling under finite extension and torsion of a "Neo-Hookian" material (an extension to finite strain of the incompressible elastic material). None of these treatments includes gravity effects.

III. Evaluation of the Potential Energy for General Displacements

The configuration is taken as indicated in Figure 1. The Cartesian spacial coordinates, x, y, z are directed so that the z axis lies along the axis of the undistorted rod, the x axis is parallel to the constraining plane and the y axis is perpendicular to the plane such that the triad xyz is right handed. The material coordinates XYZ are such that they coincide with the spacial axes when the bar is in the unstrained state. It is found convenient to use interchangeably with these the spacial set $r\theta z$ and the material set $R\Theta Z$ which are cylindrical coordinates oriented such that $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$ and such that R, Θ and Z coincide with r, θ and z when the material is in the unstrained state.

The strain energy of the bar subjected to arbitrary displacements will be formulated before introducing the equilibrium and variational displacement representations. The only assumptions made in this formulation are those previously mentioned, (1) that there is no lateral deformation, and (2) that cross sections originally perpendicular to the axis of the bar remain plane and perpendicular to the tangent of the displaced center line.

Associated with the material coordinate set is a set of base vectors, denoted $\vec{V}_x, \vec{V}_y, \vec{V}_z$. Considering those associated with the center of each cross section ($X = Y = 0$), it is possible to state (1) that they form at all times an orthogonal set, (2) that $\vec{V}_x$ and $\vec{V}_y$ are of unit magnitude throughout, and (3) that $\vec{V}_z$ is tangent to the curve of the center line of the bar. At any cross section (i.e. a fixed value of Z), a Cartesian coordinate system is defined such that $X^* = X$, $Y^* = Y$ and

Z^* is directed along $\vec{V}_Z$. The position of this set relative to the set xyz can be expressed by a rigid translation and rotation. The rotation can be described by three successive rotations (see Goldstein, pp. 107-109 [17]): here the three are a rotation of δ (counterclockwise) about $\vec{V}_X$, followed by a rotation of ϵ (clockwise) about $\vec{V}_Y$ and then ζ (counterclockwise) about $\vec{V}_Z$. Note that these are not the usual Eulerian angles. If x_0, y_0, z_0 denote the center line coordinates of the cross section considered, in matrix notation

$$\begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix} = [\alpha] \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \quad (\text{III.1.})$$

where

$$[\alpha] = \begin{bmatrix} \cos \zeta & \sin \zeta & 0 \\ -\sin \zeta & \cos \zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \epsilon & 0 & \sin \epsilon \\ 0 & 1 & 0 \\ -\sin \epsilon & 0 & \cos \epsilon \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \delta & \sin \delta \\ 0 & -\sin \delta & \cos \delta \end{bmatrix} \quad (\text{III.2.})$$

is an orthogonal transformation. This relation is easily inverted to yield

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A] \begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad (\text{III.3.})$$

where $[A] = [\alpha]^T =$

$$\begin{bmatrix} \cos \zeta \cos \epsilon & -\sin \zeta \cos \epsilon & -\sin \epsilon \\ \sin \zeta \cos \delta - \cos \zeta \sin \epsilon \sin \delta & \cos \zeta \cos \delta + \sin \zeta \sin \epsilon \sin \delta & -\sin \delta \cos \epsilon \\ \sin \zeta \sin \delta + \cos \zeta \sin \epsilon \cos \delta & \cos \zeta \sin \delta - \sin \zeta \cos \epsilon \sin \delta & \cos \epsilon \cos \delta \end{bmatrix} \quad (\text{III.4.})$$

The components A_{ij} of the rotation matrix and the center line displacements x_0 , y_0 and z_0 must be functions of Z only. The spacial position of points in a given cross section of the bar is given then by (III.3.) with $X = X^*$, $Y = Y^*$, and $Z^* = 0$, or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0(Z) \\ y_0(Z) \\ z_0(Z) \end{bmatrix} + [A(Z)] \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix}. \quad (\text{III.5.})$$

A further relation may be found by noting that for the relation (III.1.) a unit vector in the Z^* direction can be expressed

$$\frac{\vec{\nabla}_Z}{|\vec{\nabla}_Z|} = \alpha_{31} \hat{i} + \alpha_{32} \hat{j} + \alpha_{33} \hat{k}$$

where $|\vec{\nabla}_Z|$ denotes the magnitude of $\vec{\nabla}_Z$ and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the x , y and z directions respectively. Noting relation (III.4.), this can be written in terms of the elements A_{ij} of the matrix $[A]$:

$$\frac{\vec{\nabla}_Z}{|\vec{\nabla}_Z|} = A_{13} \hat{i} + A_{23} \hat{j} + A_{33} \hat{k}. \quad (\text{III.6.})$$

From differential geometry the unit tangent to a curve expressed parametrically as $x_0(Z)$, $y_0(Z)$ and $z_0(Z)$ is

$$T = \frac{x'_0}{\sqrt{x_0'^2 + y_0'^2 + z_0'^2}} \hat{i} + \frac{y'_0}{\sqrt{x_0'^2 + y_0'^2 + z_0'^2}} \hat{j} + \frac{z'_0}{\sqrt{x_0'^2 + y_0'^2 + z_0'^2}} \hat{k}. \quad (\text{III.7.})$$

Equating (III.6.) and (III.7.), and denoting the group $x_0'^2 + y_0'^2 + z_0'^2$ by ① results in

$$\begin{aligned} A_{13} &= -\sin \epsilon = \frac{x'_0}{\sqrt{\textcircled{1}}} \\ A_{23} &= -\sin \delta \cos \epsilon = \frac{y'_0}{\sqrt{\textcircled{1}}} \\ A_{33} &= \cos \delta \cos \epsilon = \frac{z'_0}{\sqrt{\textcircled{1}}}. \end{aligned}$$

From these relations, noting that the cosines of the angles must be positive since δ , ϵ and ϕ are assumed less than 90° in absolute value, it is possible to obtain by use of trigonometric identities

$$\begin{aligned}\sin \epsilon &= \frac{-x'_0}{\sqrt{①}} & \cos \epsilon &= \frac{\sqrt{②}}{\sqrt{①}} \\ \sin \delta &= \frac{-y'_0}{\sqrt{②}} & \cos \delta &= \frac{z'_0}{\sqrt{②}}\end{aligned}\tag{III.8.}$$

where $① = x_0'^2 + y_0'^2 + z_0'^2$ and $② = y_0'^2 + z_0'^2$.

Thus the parameters necessary to describe the motion can be taken as $x_0(Z)$, $y_0(Z)$, $z_0(Z)$ and $\phi(Z)$.

To evaluate the strain energy $\Psi_{\alpha\beta}$ must be obtained, using the relation (III.5.) in the equation (I.5.). In this case, since the spacial set is Cartesian $g_{ij} = \delta_{ij}$, if a deformation gradient matrix is defined as

$$[\nabla^i_\alpha] = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix}\tag{III.9.}$$

the relation (I.5.) becomes

$$[\Psi_{\alpha\beta}] = [\nabla^i_\alpha]^T [\nabla^i_\alpha].\tag{III.10.}$$

From expression (III.5.), since the elements A_{ij} of $[A]$ are functions of Z only,

$$[\nabla^i_\alpha] = \begin{bmatrix} A_{11} & A_{12} & x'_0 + A'_{11}X + A'_{12}Y \\ A_{21} & A_{22} & y'_0 + A'_{21}X + A'_{22}Y \\ A_{31} & A_{32} & z'_0 + A'_{31}X + A'_{32}Y \end{bmatrix}\tag{III.11.}$$

where the primes denote differentiation with respect to Z . The details of the application of (III.11.) to (III.10.) making use of relations (III.4.) are given in Appendix A. The results of those calculations are

$$\begin{aligned}
\psi_{xx} = \psi_{yy} &= 1 & \psi_{xy} = \psi_{yx} &= 0 \\
\psi_{xz} = \psi_{zx} &= -Y(\phi' - \delta' \sin \epsilon) \\
\psi_{yz} = \psi_{zy} &= X(\phi' - \delta' \sin \epsilon) \\
\psi_{zz} &= \textcircled{1} + 2X(\sqrt{\textcircled{1}} \epsilon' \cos \phi + \sqrt{\textcircled{2}} \delta' \sin \phi) \\
&\quad + 2Y(-\sqrt{\textcircled{1}} \epsilon' \sin \phi + \sqrt{\textcircled{2}} \delta' \cos \phi) \\
&\quad + R^2(\phi' - \delta' \sin \epsilon)^2 \\
&\quad + \{X(\epsilon' \cos \phi + \delta' \cos \epsilon \sin \phi) - Y(\epsilon' \sin \phi - \delta' \cos \epsilon \cos \phi)\}^2.
\end{aligned} \tag{III.12.}$$

All of the terms involving δ and ϵ can be expressed in terms of the center line displacements x_0 , y_0 and z_0 but it is found more convenient to delay this.

Now the relations (III.12.) can be applied to equation (I.8.) to evaluate the unit strain energy of the bar. Use of the first two of expressions (III.12.) and the fact that for this choice of axes $G_{\alpha\beta} = \delta_{\alpha\beta}$, the unit strain energy becomes

$$V = \frac{\lambda}{8} (\psi_{zz} - 1)^2 + \frac{G}{4} \{ (\psi_{zz} - 1)^2 + 2\psi_{xz}^2 + 2\psi_{yz}^2 \}$$

but since for a material which exhibits no lateral contraction the first Lamé constant is zero, the relation becomes

$$V = \frac{G}{4} \{ (\psi_{zz} - 1)^2 + 2\psi_{xz}^2 + 2\psi_{yz}^2 \}. \tag{III.13.}$$

Substitution of the remaining expressions (III.12.) into this equation yields an expression for the strain energy in terms of δ , ϵ , ϕ , $\textcircled{1}$ and $\textcircled{2}$ and thus in terms of x_0 , y_0 , z_0 and ϕ , the chosen parameters of displacement. The expansion of (III.13.) is found in Appendix B; it should be noted here that the subscript "eff." used hereafter denotes the fact

that certain terms which arise in the expansion but which will not contribute to the quantity after integration over the whole bar, due to its axial symmetry, are deleted for convenience in manipulation.

Thus

$$V_{\text{eff}} = \frac{G}{4} \left\{ (\textcircled{1}-1)^2 + (R^2\beta^2 + 2\textcircled{1})R^2\beta^2 + 2(R^2\beta^2 + 3\textcircled{1}-1)(X^2\eta^2 + Y^2\chi^2) + X^4\eta^4 + 6X^2Y^2\eta^2\chi^2 + Y^4\chi^4 \right\} \quad (\text{III.14.})$$

where

$$\begin{aligned} \beta &= \phi' - \delta' \sin \epsilon \\ \eta &= \epsilon' \cos \phi + \delta' \sin \phi \cos \epsilon \\ \chi &= \epsilon' \sin \phi - \delta' \cos \phi \cos \epsilon. \end{aligned}$$

Noting that the groups β , η , χ and $\textcircled{1}$ are functions of Z only, and converting to cylindrical coordinates, the strain energy of the entire

bar can be expressed as

$$U = \iiint V R dR d\Theta dZ = \int_{Z=0}^L \int_{R=a}^b \int_{\Theta=0}^{2\pi} \frac{G}{4} \left\{ (\textcircled{1}-1)^2 R + (R^5\beta^2 + 2\textcircled{1}R^3)\beta^2 + 2(R^5\beta^2 + 3\textcircled{1}R^3 - R^3)(\eta^2 \cos^2 \Theta + \chi^2 \sin^2 \Theta) + R^5(\eta^4 \cos^4 \Theta + 6\eta^2\chi^2 \cos^2 \Theta \sin^2 \Theta + \chi^4 \sin^4 \Theta) \right\} d\Theta dR dZ$$

and this can be integrated with respect to Θ immediately to yield

$$U = \int_0^L \int_a^b \frac{G}{4} \left\{ 2\pi(\textcircled{1}-1)^2 R + 2\pi(R^5\beta^2 + 2\textcircled{1}R^3)\beta^2 + 2\pi(R^5\beta^2 + 3\textcircled{1}R^3 - R^3)(\eta^2 + \chi^2) + \frac{3}{4}\pi R^5(\eta^4 + \chi^4) + \frac{3}{4}\pi R^5\eta^2\chi^2 \right\} dR dZ.$$

It is easily seen that $\eta^2 + \chi^2 = \epsilon'^2 + \delta'^2 \cos^2 \epsilon$ so that the expression

becomes

$$U = \int_0^L \int_a^b \frac{G}{4} \left\{ 2\pi(\textcircled{1}-1)^2 R + 2\pi(R^5\beta^2 + 2\textcircled{1}R^3)\beta^2 + 2\pi(R^5\beta^2 + 3\textcircled{1}R^3 - R^3)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon) + \frac{3}{4}\pi R^5(\epsilon'^2 + \delta'^2 \cos^2 \epsilon)^2 \right\} dR dZ.$$

Integration over R is easily performed to yield

$$U = \int_0^L \frac{G}{4} \left\{ \mathcal{A}(\Theta - 1)^2 + 2I(h\beta^2 + 2\Theta)\beta^2 + 2I(h\beta^2 + 3\Theta - 1)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon) + \frac{3}{4}Ih(\epsilon'^2 + \delta'^2 \cos^2 \epsilon)^2 \right\} dZ \quad (\text{III.15.})$$

where

$$\begin{aligned} \mathcal{A} &= \pi(b^2 - a^2), \\ I &= \frac{\pi}{4}(b^4 - a^4) \\ \text{and } h &= \frac{2}{3} \frac{(b^6 - a^6)}{(b^4 - a^4)}. \end{aligned} \quad (\text{III.16.})$$

Obviously $\mathcal{A}$ is the cross sectional area of the bar, I the second moment of the cross sectional area and hI a third moment of the cross sectional area. The expression $\epsilon'^2 + \delta'^2 \cos^2 \epsilon$ can be evaluated easily using relations (C.1) and (C.2) of Appendix C.

The second contribution to the potential energy of the bar is that of the external loads and the body forces. The gravity force will be regarded as an external load distributed uniformly along the axis of the bar and acting through the axis. The magnitude of this load is $\rho \mathcal{A} g \cos \alpha$ in the positive z direction and $\rho \mathcal{A} g \sin \alpha$ in the negative y direction, where ρ is the mass density of the bar, $\mathcal{A}$ its cross sectional area, g the acceleration of gravity and α the inclination of the constraining plane with respect to the gravity field. It is assumed that the bar is subjected to axial tensile forces and moments at each end. The force and moment at the end $Z = 0$ are denoted P_0 and M_0 respectively and those at the end $Z = L$, P and M respectively. Recognizing that under the boundary conditions (the vanishing of the lateral displacement gradients at each end of the bar) $\zeta(0)$ and $\zeta(L)$ represent the angles of axial rotation of the bar at these points, the potential energy of the external loadings is easily seen to be expressible as

$$\Omega = -P z_o(L) + P_o z_o(0) - M \phi(L) + M_o \phi(0) + \int_0^L \rho A g (y_o \sin \alpha - z_o \cos \alpha) dZ. \quad (\text{III.17.})$$

The total potential energy of the deflected bar is the sum of (III.17.)

and (III.15.) or

$$\begin{aligned} \Upsilon = U + \Omega = \int_0^L \frac{G}{4} \left[A (\chi_o'^2 + y_o'^2 + z_o'^2 - 1)^2 \right. \\ + 2I [h (\phi' - \delta' \sin \epsilon)^2 + 2(\chi_o'^2 + y_o'^2 + z_o'^2)] (\phi' - \delta' \sin \epsilon)^2 \\ + 2I [h (\phi' - \delta' \sin \epsilon)^2 + 3(\chi_o'^2 + y_o'^2 + z_o'^2) - 1] (\epsilon'^2 + \delta'^2 \cos^2 \epsilon) \\ \left. + \frac{3}{4} I h (\epsilon'^2 + \delta'^2 \cos^2 \epsilon)^2 \right] dZ \\ + \int_0^L \rho A g (y_o \sin \alpha - z_o \cos \alpha) dZ \\ - P z_o(L) + P_o z_o(0) - M \phi(L) + M_o \phi(0), \end{aligned} \quad (\text{III.18.})$$

where

$$\epsilon' = \frac{\chi_o' (y_o' y_o'' + z_o' z_o'') - \chi_o'' (y_o'^2 + z_o'^2)}{(\chi_o'^2 + y_o'^2 + z_o'^2) \sqrt{y_o'^2 + z_o'^2}}, \quad (\text{C.1})$$

$$\delta' = \frac{y_o' z_o'' - y_o'' z_o'}{y_o'^2 + z_o'^2}, \quad (\text{C.2})$$

$$\sin \epsilon = \frac{-\chi_o'}{\sqrt{\chi_o'^2 + y_o'^2 + z_o'^2}}, \quad \cos \epsilon = \frac{\sqrt{y_o'^2 + z_o'^2}}{\sqrt{\chi_o'^2 + y_o'^2 + z_o'^2}}, \quad (\text{III.8.})$$

$$A = \pi(b^2 - a^2), \quad I = \frac{\pi}{4}(b^4 - a^4) \quad \text{and} \quad h = \frac{2(b^6 - a^6)}{3(b^4 - a^4)}. \quad (\text{III.16.})$$

The expression (III.18.) is a completely general expression for the potential energy of a bar of cylindrical cross section for any set of center line displacements x_o , y_o and z_o and twist ϕ , subject only to the assumptions of (1) no lateral contraction of the cross sections; and (2) the cross sections originally normal to the center line remaining plane and normal to the distorted center line curve. This expression cannot be extended to bars of other cross sections, even if warping and lateral effects are ignored, without re-examination of the calculations in Appendix B and in the last paragraphs.

IV. Variation of the Potential Energy, Buckling Modes

As noted in section I it is now necessary to hypothesize an equilibrium configuration and consider the energy variation created by small perturbations of the configuration. It is possible to assume a completely general equilibrium configuration and use the vanishing of a portion of the first variation of the energy to evaluate the corresponding displacement coordinates, but in view of the elementary nature of the equilibrium problem it is more convenient to assume a more restricted equilibrium configuration. The assumed equilibrium configuration is expressed in cylindrical coordinates as

$$r = R, \quad \theta = \Theta + \Psi z, \quad \text{and} \quad z = (A + BZ)Z. \quad (\text{IV.1.})$$

The motivation for this is obvious: it is assumed that there is no lateral contraction, Ψ represents a twist per unit final length of the bar, A represents unity plus a small strain due to the tensile loading, and B represents the linearly varying strain due to the tensile effect of the gravity force. Any deformation due to contact with the rigid plane on which the bar lies is ignored: this omission would be expected to lead to the configuration (IV.1.) being only approximately in equilibrium with the given external effects. It will be assumed in all that follows that the quantities $e = A-1$ and BL are sufficiently small that their squares may be neglected with respect to the quantities themselves. In terms of center line displacements and the angle ϕ , the assumed equilibrium configuration is given by

$$x_{oe} = y_{oe} = 0, \quad z_{oe} = (A + BZ)Z, \quad \text{and} \quad \phi_e = \Psi(A + BZ)Z. \quad (\text{IV.2.})$$

A momentarily unspecified variation in each of the four displacement parameters can now be added to these base values and the

total displacements used in (III.18.) to calculate the potential energy in the perturbed configuration; subtracting the value obtained by use of the equilibrium displacements produces the variation of energy due to the perturbation. The actual calculation of these quantities is relegated to Appendix D. It need only be noted here that terms of order greater than the second in the variations of the displacements are neglected. Using the notation of the calculus of variations (which is not completely appropriate here, since it is anticipated that the variations in the displacement coordinates will later be taken to be small but not infinitesimal), the total variation of the potential energy is

$$\begin{aligned}
\Delta \mathcal{T} = & -P \delta z_0(L) + P_0 \delta z_0(0) - M \delta \phi(L) + M_0 \delta \phi(0) \\
& + \int_0^L \left\{ \frac{G}{4} \left[8[A(e+2BZ) + I\Psi^2] \delta z'_0 + 8I\Psi(1+3e+6BZ) \delta \phi' \right] \right. \\
& \quad \left. + \rho A g (\delta y_0 \sin \alpha - \delta z_0 \cos \alpha) \right\} dZ \\
& + \int_0^L \frac{G}{4} \left\{ 4A(1+3e+6BZ) \delta z_0'^2 + 16I\Psi(1+2e+4BZ) \delta z'_0 \delta \phi' \right. \\
& \quad + 4I(1+2e+4BZ) \delta \phi'^2 + 4[A(e+2BZ) + I\Psi^2] (\delta x_0'^2 + \delta y_0'^2) \\
& \quad + 16BI\Psi \delta x_0' \delta y_0' - 8I\Psi(1+e+2BZ) \delta x_0' \delta y_0'' \\
& \quad \left. - 16BI(\delta x_0' \delta x_0'' + \delta y_0' \delta y_0'') + 4I(1+e+2BZ)(\delta x_0''^2 + \delta y_0''^2) \right\} dZ.
\end{aligned} \tag{D.6}$$

Here $e = A-1$, and terms of the order of e^2 and B^2L^2 are neglected with respect to terms of the order of e and BL . It is assumed that $\Psi^2 b^2$, Ψbe and terms of similar order are negligible with respect to unity but no other restrictions are placed on the magnitude of Ψ .

The first necessary condition for equilibrium of the body in the assumed equilibrium configuration, as previously stated, is $\delta \mathcal{T} \geq 0$, where $\delta \mathcal{T}$ denotes the first order terms in (D.6). The expression for $\delta \mathcal{T}$ can be simplified by integrating partially the terms involving $\delta z'_0$ and

$\delta\zeta'$. The result is

$$\begin{aligned}\delta T = & \left\{ -P + 2G[\mathcal{A}(e + 2BL) + I\psi^2] \right\} \delta z_0(L) \\ & + \left\{ P_0 - 2G(\mathcal{A}e + I\psi^2) \right\} \delta z_0(0) \\ & + \left\{ -M + 2GI\psi(1 + 3e + 6BL) \right\} \delta\zeta(L) \\ & + \left\{ M_0 - 2GI\psi(1 + 3e) \right\} \delta\zeta(0) \\ & + \int_0^L \left\{ \mathcal{S}\mathcal{A}g \sin\alpha \delta y_0 - (4G\mathcal{A}B + \mathcal{S}\mathcal{A}g \cos\alpha) \delta z_0 \right. \\ & \quad \left. - 12GIB\psi \delta\zeta \right\} dZ.\end{aligned}$$

If it is required that $\delta T \geq 0$ hold, since δz_0 and $\delta\zeta$ are not restricted but $\delta y_0 \geq 0$ then

$$\begin{aligned}-P + 2G[\mathcal{A}(e + 2BL) + I\psi^2] &= 0, \\ P_0 - 2G(\mathcal{A}e + I\psi^2) &= 0, \\ -M + 2GI\psi(1 + 3e + 6BL) &= 0, \\ M_0 - 2GI\psi(1 + 3e) &= 0, \\ -4G\mathcal{A}B - \mathcal{S}\mathcal{A}g \cos\alpha &= 0 \\ \text{and } -12GIB\psi &= 0.\end{aligned}$$

These relations are satisfied subject to the approximation $3e + 6BL \ll 1$, by

$$\begin{aligned}(1) \quad M &= M_0 = 2GI\psi \quad \text{or} \quad \psi = \frac{M}{2GI}, \\ (2) \quad e &= \frac{P_0}{2G\mathcal{A}} - \frac{I}{\mathcal{A}}\psi^2 = e_0 - \frac{I}{\mathcal{A}}\psi^2, \\ (3) \quad B &= -\frac{\mathcal{S}g \cos\alpha}{4G},\end{aligned}\tag{IV.3.}$$

$$\text{and (4) } P = P_0 + \mathcal{S}\mathcal{A}gL \cos\alpha,$$

excepting, of course, the last, which is only approximately satisfied for $12GIB\psi$ very small. This approximation is not extremely serious since it is to be expected that B will usually be rather small and ψ usually significantly less than one. Presumably the neglect of contact deformation is responsible for this anomaly. Note the division of e into e_0 and $-\frac{I}{\mathcal{A}}\psi^2$: the determination of buckling parameters requires the separation of the effects due to axial tension from those due to torsion.

Assuming that the contribution to the energy of the term $12GIB\psi$ is negligible, the only contribution of the first variation of the energy to the total change is

$$\delta T = \int_0^L \mathcal{H} g \sin \alpha \delta y_0 dZ \geq 0. \quad (\text{IV.4.})$$

Thus the position of equilibrium will be stable for all small displacements therefrom which involve a lifting of the bar from the constraining plane. For displacements in which there is no such lifting the integral (IV.4.) is zero but, as will be shown below, for such motions the second variation of the energy is positive definite so that the position is without question stable in the classical sense.

Now it is necessary, as previously stated, to examine the possibility of the second variation of the energy being sufficiently negative to overcome the positive first variation for somewhat larger perturbations. From (D.6) and (IV.4.)

$$\begin{aligned} \Delta T = & \mathcal{H} g \sin \alpha \int_0^L \delta y_0 dZ + \int_0^L \frac{G}{4} \left\{ 4\mathcal{H}(1+3e+6BZ) \delta z_0'^2 \right. \\ & + 16I\psi(1+2e+4BZ) \delta z_0' \delta \zeta_0' + 4I(1+2e+4BZ) \delta \zeta_0'^2 \\ & + 4[\mathcal{H}(e+2BZ) + I\psi^2](\delta x_0'^2 + \delta y_0'^2) + 16BI\psi \delta x_0' \delta y_0' \\ & - 8I\psi(1+e+2BZ) \delta x_0' \delta y_0'' - 16BI(\delta x_0' \delta x_0'' + \delta y_0' \delta y_0'') \\ & \left. + 4I(1+e+2BZ)(\delta x_0''^2 + \delta y_0''^2) \right\} dZ. \end{aligned} \quad (\text{IV.5.})$$

The only terms in this expression which are not positive definite it is assumed $e+2BZ > 0$ (that the bar is under tension throughout its length) are

$$\begin{aligned} (1) & 16 I \psi (1+2e+4BZ) \delta z_0' \delta \zeta_0' \\ (2) & 16 I \psi B \delta x_0' \delta y_0' \\ (3) & 8 I \psi (1+e+2BZ) \delta x_0' \delta y_0'' \\ \text{and (4)} & 16 B I (\delta x_0' \delta x_0'' + \delta y_0' \delta y_0''). \end{aligned} \quad (\text{IV.6.})$$

Attention to the derivation in Appendix D reveals that the last term of the four arises from expansion of squared terms and thus cannot decrease the positive nature of the integral. The quantity ψ_B has already been assumed quite small and thus the second term would not be expected to contribute significantly. It is to be expected from the nature of the problem that the quantity δz_0 will be significantly smaller than δx_0 or δy_0 ; this, together with the fact that the coefficient of (IV.6., 1) is proportional to ψ while all other terms in (IV.5.) involving the variables δz_0 and $\delta \zeta$ have coefficients proportional to unity, rules out any effective contribution from the first term. Thus (IV.6., 3) must be the significant term in reducing the positive nature of the second variation. Because of this, if the y displacement is identically zero $\delta^2 T$ is positive definite, as was previously stated.

As pointed out above, it is to be expected that the contribution of the terms δz_0 and $\delta \zeta$ to the energy variation will be essentially positive. Since buckling will occur at the lowest energy level it must occur in such a way that

$$\delta z_0 = 0 \quad \text{and} \quad \delta \zeta = 0. \quad (\text{IV.7.})$$

Thus it is presumed that the buckling mode will include no extensional or torsional effects other than those arising from the lateral deflection of the bar: this is quite reasonable physically since with the ends of the bar restrained it would not be expected to spontaneously elongate or untwist.

The requirement that $\delta \zeta$ be zero does not of course rule out any untwisting of the bar. The term $\beta = \zeta' - \delta' \sin \epsilon$ encountered in the expressions for ψ_{xz} and ψ_{yz} is a measure of the "absolute" torsion of the bar: it can easily be demonstrated that β represents the component

in the direction of the disturbed center line of the rate of change of angular position of the cross sections with respect to Z.

The latter contribution to β , $\delta' \sin \epsilon$, represents the twisting of the bar due to the lateral deformation and in fact gives rise to (IV.6., 3), the key term in the energy expression. This is not unexpected, but the fact that this contribution to the energy is not symmetric in δx_0 and δy_0 is at first glance surprising since neither displacement was repressed in the algebraic manipulations leading to the formulation of the energy variation. The cause of this lack of symmetry is the choice of the non-symmetric parameter ϕ to express the twisting motion of the bar. In one of his various formulations of the problem of compression and torsion buckling Goodier [13] expresses the cross-section location in a manner similar to that used in section III but chooses the opposite order of rotations about $\bar{V}_x$ and $\bar{V}_y$. The final rotation angle is used as the twisting parameter and assuming that it does not vary in the buckling, Goodier derives an energy expression in which the term analogous to (IV.6., 3) involves $-\delta y'_0 \delta x''_0$ rather than $\delta x'_0 \delta y''_0$ as here. Another formulation of the problem in that paper involves the use of a rotation parameter which is the average of these two non-symmetric parameters, and the energy term corresponding to (IV.6., 3) then involves $(\delta x''_0 \delta y'_0 - \delta x'_0 \delta y''_0)$. (This parameter is the usual infinitesimal rotation angle; see also Pearson [3], p. 137 and Love [18], pp. 383-386.) This ambivalence of rotation parameters raises some theoretical questions regarding the calculation of the second variation of the energy, but it will most probably be of no significance in practical applications since the quantities involved would not be expected to differ significantly. More critical cases might warrant investigation of the theoretical implications.

To further establish the nature of the probable buckling mode it can be noted that the center line of a twisted unconstrained thin rod assumes a helical shape. It would be expected that even under the constraint and gravity loading encountered here the center line curve would approximate a helix. It is at any rate an established practical rule that the buckling parameters are not critically dependent upon the exact nature of the buckling displacement assumed as long as no significant modes are repressed, e.g. here to assume a motion for which $\int_0^L \delta x_0' \delta y_0'' dZ = 0$. With this in mind the deformations

$$\begin{aligned} \delta x_0 &= F \left(\sin \frac{2\pi}{L} Z - \frac{1}{2} \sin \frac{4\pi}{L} Z \right) \\ \text{and } \delta y_0 &= F \left(1 - \cos \frac{2\pi}{L} Z \right) \end{aligned} \quad (\text{IV.8.})$$

are assumed. These, with the first of (IV.7.) approximate a helix and satisfy the boundary conditions of vanishing displacements and slopes at the ends of the buckled bar (see Figure 3 for graphs of these functions). A single wave of displacement is considered but the length, L , of the buckled bar will be carried as an undetermined constant, as will the displacement magnitude parameter F .

V. Determination of Buckling Parameters

With the displacements (IV.7.) and (IV.8.) the energy variation equation (IV.5.) becomes

$$\begin{aligned}\Delta T = & \mathcal{S}A g \sin \alpha \int_0^L F(1 - \cos \frac{2\pi}{L} Z) dZ \\ & + \int_0^L \frac{G}{4} [4A(e_0 + 2BZ)(4\frac{\pi^2}{L^2} F^2)(\cos^2 \frac{4\pi}{L} Z - 2\cos \frac{4\pi}{L} Z + 1) \\ & - 16BI\psi(4\frac{\pi^2}{L^2} F^2)(\cos \frac{4\pi}{L} Z \sin \frac{2\pi}{L} Z - \cos \frac{2\pi}{L} Z \sin \frac{4\pi}{L} Z) \\ & + 8I\psi(1+e_0+2BZ)(8\frac{\pi^3}{L^3} F^2)(\cos \frac{4\pi}{L} Z \cos \frac{2\pi}{L} Z - \cos^2 \frac{2\pi}{L} Z) \\ & - 16BI(8\frac{\pi^3}{L^3} F^2)(\cos \frac{4\pi}{L} Z \sin \frac{2\pi}{L} Z + \cos \frac{2\pi}{L} Z \sin \frac{4\pi}{L} Z \\ & - \cos \frac{4\pi}{L} Z \sin \frac{4\pi}{L} Z) \\ & + 4I(1+e_0+2BZ)(16\frac{\pi^4}{L^4} F^2)(1 - 2\sin \frac{2\pi}{L} Z \sin \frac{4\pi}{L} Z + \sin^2 \frac{4\pi}{L} Z)] dZ\end{aligned}$$

using the relation $e = e_0 - \frac{I}{A} \psi^2$ and neglecting $\frac{I}{A} \psi^2$ with respect to unity. This expression is easily integrated to yield

$$\begin{aligned}\Delta T = & \mathcal{S}A g \sin \alpha FL + GAe_0(4\frac{\pi^2}{L^2} F^2)(\frac{3}{2}L) \\ & + GA(2B)(4\frac{\pi^2}{L^2} F^2)(\frac{3}{4}L^2) + 2GI\psi(1+e_0)(8\frac{\pi^3}{L^3} F^2)(-\frac{L}{2}) \\ & + 2GI\psi(2B)(8\frac{\pi^3}{L^3} F^2)(-\frac{L^2}{4}) + GI(1+e_0)(16\frac{\pi^4}{L^4} F^2)(\frac{3L}{2}) \\ & + GI(2B)(16\frac{\pi^4}{L^4} F^2)(\frac{3L^2}{4})\end{aligned}$$

or, rearranging terms,

$$\begin{aligned}\Delta T = & (\mathcal{S}A g L \sin \alpha) F \\ & + \left[\frac{6GA\pi^2}{L}(e_0 + BL) + \frac{24GI\pi^4}{L^3}(1+e_0+BL) - \frac{8GI\psi\pi^3}{L^2}(1+e_0+BL) \right] F^2.\end{aligned}\tag{V.1.}$$

The change of energy is thus made to depend on the single displacement magnitude parameter F . From (IV.8.) $\delta y_0 \geq 0$ implies $F \geq 0$ and the position $F = 0$ represents the stable equilibrium configuration. If there

exists a point F^* which also represents a position of equilibrium it must satisfy the relation

$$\left. \frac{\partial \Delta \mathcal{I}}{\partial F} \right|_{F=F^*} = 0, \quad (V.2.)$$

Further, if this is to be a stable equilibrium F^* must satisfy

$$\left. \frac{\partial^2 \Delta \mathcal{I}}{\partial F^2} \right|_{F=F^*} > 0. \quad (V.3.)$$

These expressions are the one-dimensional equivalents of those noted in section I for multi-dimensional system stability (with holomorphic constraints). Applying (V.2.) to (V.1.) results in

$$\rho A g L \sin \alpha + 2F^* [] = 0$$

where the brackets denote the terms enclosed by square brackets in (V.1.). Solving for F^* ,

$$F^* = - \frac{\rho A g L \sin \alpha}{2 []}. \quad (V.4.)$$

Considering the requirement (V.3.) it can be seen that the configuration is stable, of indeterminate stability or unstable, according to whether the term $[] =$

$$\left[\frac{6G A \pi^2}{L} (e_0 + B L) + \frac{24G I \pi^4}{L^3} (1 + e_0 + B L) - \frac{8G I \pi^3}{L^2} (1 + e_0 + B L) \psi \right] \quad (V.5.)$$

is positive, zero, or negative, respectively. For any fixed values of all of the other quantities there exists a ψ sufficiently large to make this quantity negative and thus lead to instability of the system in the displaced equilibrium configuration. (Note that under compression rather than tension the first term in (V.5.) also contributes to the negativeness.) Moreover, since $F \geq 0$, from the constraint, the quantity F^* given by (V.4.) exists only for negative values of (V.5.); thus if

such an equilibrium configuration exists it is unstable. For (V.5.) small but negative F^* becomes very large, and to the restricted validity of the calculation an infinite displacement is required to reach a position of equilibrium when (V.5.) vanishes.

The quantity (V.5.) multiplied by F is the second variation of the potential energy, and for the case of no gravity loading ($\rho g \sin \alpha = 0$) its vanishing determines the lowest buckling twist. Calling the value of ψ for which (V.5.) vanishes ψ_{CR} it can be seen that with all other parameters fixed the system possesses a second, unstable, equilibrium position at $F^*(\psi)$ if $\psi > \psi_{CR}$; it cannot buckle for $\psi < \psi_{CR}$ and at $\psi = \psi_{CR}$ a very large lateral displacement is necessary to reach a position of equilibrium. The critical value of ψ is found by setting (V.5.) equal to zero:

$$\psi_{CR} = \frac{3\pi}{L} + \frac{3}{4} \frac{A(e_0 + BL)}{I(1 + e_0 + BL)} \frac{L}{\pi}. \quad (V.6.)$$

As a partial check on this expression it can be noted that for no extension of the bar and no gravity field this yields a (classical) buckling twist of

$$\psi_{CR} = \frac{3\pi}{L}$$

which compares favorably with that quoted in the literature for the same conditions:

$$\text{Ziegler ([4], p. 389): } \psi_{CR} = 2.861 \frac{\pi}{L};$$

$$\text{Greenhill ([12], p. 205): } \psi_{CR} = 4 \frac{\pi}{L}.$$

The question of the influence of the wave length of the displacement must now be considered. First, ψ_{CR} possesses a relative minimum with respect to L if

$$\frac{\partial \psi_{CR}}{\partial L} = 0 \quad \text{and} \quad \frac{\partial^2 \psi_{CR}}{\partial L^2} > 0.$$

From (V.6.)

$$\frac{\partial \psi_{CR}}{\partial L} = - \frac{3\pi}{L^2} + \frac{3A(e_0 + BL)}{4I(1 + e_0 + BL)\pi} + \frac{3ABL}{4I(1 + e_0 + BL)^2\pi}.$$

This is simplified if BL is assumed much smaller than e_0 , to

$$\frac{\partial \psi_{CR}}{\partial L} = -\frac{3\pi}{L^2} + \frac{3Ae_0}{4I(1+e_0)\pi}$$

and further,

$$\frac{\partial^2 \psi_{CR}}{\partial L^2} = \frac{6\pi}{L^3} > 0$$

so that ψ_{CR} always possesses a minimum with respect to L. This minimum occurs at

$$\frac{\pi^2}{L^2} = \frac{Ae_0}{4I(1+e_0)} \quad \text{or} \quad L^* = 2\pi \sqrt{\frac{I(1+e_0)}{Ae_0}} \quad (V.7.)$$

and here

$$\psi_{CRMIN} = \frac{3}{2} \frac{Ae_0}{I(1+e_0)} + \frac{3Ae_0}{4I(1+e_0)} \sqrt{\frac{4I(1+e_0)}{Ae_0}}$$

or

$$\psi_{CRMIN} = 3 \sqrt{\frac{Ae_0}{I(1+e_0)}} \quad (V.8.)$$

As e_0 approaches zero ψ_{CRMIN} approaches zero and L^* becomes infinite.

The lowest value of ψ_{CR} , then, would in most cases for very small tensions be the value given by (V.6.) with the total available length of the bar substituted for L.

The nature of the relationship between ψ and L is illustrated in Figure 4. The upper graph contains curves of ψ_{CR} versus L for various values of the extension e_0 for a typical case. The lower graph isolates one of these curves. It is seen that for ψ greater than ψ_{CRMIN} buckling is possible with wave length between L_1 and L_2 (which are the two roots of equation (V.6.) for $\psi_{CR} = \psi$):

$$L_1 = \pi \frac{\psi - \sqrt{\psi^2 - 9 \frac{Ae_0}{I(1+e_0)}}}{\frac{3}{2} \frac{Ae_0}{I(1+e_0)}} = 6\pi \frac{\psi - \sqrt{\psi^2 - \psi_{CRMIN}^2}}{\psi_{CRMIN}^2} \quad (V.9.)$$

and

$$L_2 = \pi \frac{\psi + \sqrt{\psi^2 - 9 \frac{Ae_0}{I(1+e_0)}}}{\frac{3}{2} \frac{Ae_0}{I(1+e_0)}} = 6\pi \frac{\psi + \sqrt{\psi^2 - \psi_{CRMIN}^2}}{\psi_{CRMIN}^2}$$

To determine exactly the wave length at which buckling may be expected, consider the expression for F^* (again neglecting BL with respect to e_0):

$$F^* = \frac{5A g \sin \alpha}{4 \left[\frac{4GI\pi^3}{L^3} (1+e_0) \psi - \frac{3GA\pi^2}{L^2} e_0 - \frac{12GI\pi^4}{L^4} (1+e_0) \right]}. \quad (V.10.)$$

F^* becomes infinite at L equal to L_1 and L_2 and is positive in the interval between them so it must possess a relative minimum in that region, being continuous. (A typical plot of F^* versus L is shown in Figure 5.) Maximizing the denominator of (V.10.) yields the two values

$$\frac{\pi}{L_B} = \frac{\psi \pm \sqrt{\psi^2 - 8 \frac{Ae_0}{I(1+e_0)}}}{8} = \frac{\psi \pm \sqrt{\psi^2 - \frac{8}{9} \psi_{CRMIN}^2}}{8}. \quad (V.11.)$$

This always yields solutions for $\psi > \psi_{CRMIN}$. It can be shown that the positive sign is the correct choice here since the negative sign yields a value of L_B greater than L_2 .

If the total length of the bar available is less than the value (V.11.) the buckling would be expected to occur over the full length of the bar.

VI. Results

The calculations indicated previously can be systematically applied to determine the possibility of instabilities in a given system using the following procedure:

- (1) Determine the quantities

$$e_o = \frac{P_o}{2GA}$$

$$B = - \frac{\rho g \cos \alpha}{4G}$$

$$\psi = \frac{M}{2GI}$$

where P_o is the axial tensile force applied to the bar at the uppermost point, M the torque applied, ρg the weight density, A the cross-sectional area, I the second moment of area of the cross section, G the shear modulus for the bar, and α the inclination of the bar with respect to the direction of the gravity field (Figure 1).

- (2) Compare the total length of the bar, l , to the quantity

$$L^* = 2\pi \sqrt{\frac{I(1+e_o)}{Ae_o}},$$

- (a) If $l \geq L^*$ the bar cannot buckle for

$$\psi < 3 \sqrt{\frac{Ae_o}{I(1+e_o)}}.$$

If ψ is greater than this quantity buckling is possible if (3) and (4) are fulfilled.

- (b) If $l < L^*$ the bar cannot buckle for

$$\psi < \frac{3\pi}{l} + \frac{3A(e_o + B l)}{4I(1+e_o + B l)} \frac{1}{\pi}.$$

If ψ is greater than this quantity buckling is possible if (3) and (4) are fulfilled.

- (3) The buckling will occur with a wave length given by the larger of 1 and

$$\frac{8\pi}{\psi + \sqrt{\psi^2 - 8 \frac{Ae_0}{I(1+e_0)}}}.$$

- (4) The buckling will occur only if the bar is displaced a distance equal to or greater than

$$F^* = \frac{5Ag \sin \alpha}{4 \left[\frac{4GI\pi^3}{L_w^3} (1+e_0+BL_w) \psi - \frac{3GA\pi^2}{L_w^2} (e_0+BL_w) - \frac{12GI\pi^4}{L_w^4} (1+e_0+BL_w) \right]}$$

where L_w is the wave length found in step (3).

It should be repeated here that the buckling parameters found represent lower limits on the actual parameters and that they are subject to the approximations that

- (i) cross sections of the bar do not deform laterally;
- (ii) cross sections remain plane and normal to the deformed center line of the bar;
- (iii) e_0 and Bl are of the order of the usual small deformation gradient elasticity strains and Bl is much smaller than e_0 ; and
- (iv) $\psi^2 b^2$ where b is the largest radius of the cross section is negligible with respect to unity.

Appendix A

Calculation of the Distorted Metric Elements

From (III.10.) the distorted material coordinate metric tensor can be calculated using (III.11.) for the displacement gradients and (III.4.) to evaluate the unspecified coefficients in (III.11.). First,

$$[\psi_{\alpha\beta}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ x'_0 + A'_{11}X + A'_{12}Y & y'_0 + A'_{21}X + A'_{22}Y & z'_0 + A'_{31}X + A'_{32}Y \end{bmatrix} \cdot$$

$$\begin{bmatrix} A_{11} & A_{12} & x'_0 + A'_{11}X + A'_{12}Y \\ A_{21} & A_{22} & y'_0 + A'_{21}X + A'_{22}Y \\ A_{31} & A_{32} & z'_0 + A'_{31}X + A'_{32}Y \end{bmatrix}$$

or

$$[\psi_{\alpha\beta}] = \begin{bmatrix} (A_{11}^2 + A_{21}^2 + A_{31}^2) & (A_{11}A_{12} + A_{21}A_{22} + A_{31}A_{32}) & A_{11}(x'_0 + A'_{11}X + A'_{12}Y) + A_{21}(y'_0 + A'_{21}X + A'_{22}Y) + A_{31}(z'_0 + A'_{31}X + A'_{32}Y) \\ \text{---} & \text{---} & \text{---} \\ (A_{12}^2 + A_{22}^2 + A_{32}^2) & & A_{12}(x'_0 + A'_{11}X + A'_{12}Y) + A_{22}(y'_0 + A'_{21}X + A'_{22}Y) + A_{32}(z'_0 + A'_{31}X + A'_{32}Y) \\ \text{---} & \text{---} & \text{---} \\ & & (x'_0 + A'_{11}X + A'_{12}Y)^2 + (y'_0 + A'_{21}X + A'_{22}Y)^2 + (z'_0 + A'_{31}X + A'_{32}Y)^2 \end{bmatrix} \quad (A.1)$$

where the other terms are omitted since $[\psi_{\alpha\beta}]$ is symmetric.

From the fact that $[A]$ represents an orthogonal transformation,

$$\sum_{j=1}^3 A_{ji} A_{jk} = \delta_{ik}. \quad (\text{A.2})$$

Applying rule (A.2) to (A.1) results in

$$\Psi_{XX} = \Psi_{YY} = 1, \quad \Psi_{XY} = 0. \quad (\text{A.3})$$

Differentiation of (A.2) results in

$$\sum_{j=1}^3 (A'_{ji} A_{jk} + A_{ji} A'_{jk}) = 0. \quad (\text{A.4})$$

This result will be useful later.

To evaluate Ψ_{XZ} it is first expanded to

$$\begin{aligned} \Psi_{XZ} = & A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 \\ & + X(A_{11} A'_{11} + A_{21} A'_{21} + A_{31} A'_{31}) \\ & + Y(A_{11} A'_{12} + A_{21} A'_{22} + A_{31} A'_{32}). \end{aligned} \quad (\text{A.5})$$

From (A.4), with $i = k$, the coefficient of X is zero. Expansion of

Ψ_{YZ} yields

$$\begin{aligned} \Psi_{YZ} = & A_{12} x'_0 + A_{22} y'_0 + A_{32} z'_0 \\ & + X(A_{12} A'_{11} + A_{22} A'_{21} + A_{32} A'_{31}) \\ & + Y(A_{12} A'_{12} + A_{22} A'_{22} + A_{32} A'_{32}). \end{aligned} \quad (\text{A.6})$$

Again, (A.4), with $i = k$, reveals that the coefficient of Y is zero.

Further, from (A.4), with $i = 1$ and $k = 2$,

$$A_{12} A'_{11} + A_{22} A'_{21} + A_{32} A'_{31} = -(A_{11} A'_{12} + A_{21} A'_{22} + A_{31} A'_{32}) \quad (\text{A.7})$$

so that the Y coefficient of Ψ_{XZ} and the X coefficient of Ψ_{YZ} are equal

but opposite in sign. Now Ψ_{XZ} can be expanded, evaluating the matrix

elements A_{ij} and abbreviating cosine by c and sine by s to yield

$$\begin{aligned} \Psi_{XZ} = & A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 \\ & + Y \left[(c\phi c\epsilon)(-s'\phi c\epsilon - s\phi c'\epsilon) \right. \\ & \quad + (s\phi c\delta - c\phi s\epsilon s\delta)(c'\phi c\delta + c\phi c'\delta + s'\phi s\epsilon s\delta + s\phi s'\epsilon s\delta \\ & \quad \left. + (s\phi s\delta + c\phi s\epsilon c\delta)(c'\phi s\delta + c\phi s'\delta - s'\phi s\epsilon c\delta - s\phi s'\epsilon c'\delta) \right] \end{aligned}$$

where the derivatives of the trigonometric functions are not evaluated.

Expanding this product and making liberal use of trigonometric identities produces

$$\begin{aligned}\Psi_{xz} = & A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 \\ & + Y \left[s' \phi (-c \phi c^2 \epsilon - c \phi s^2 \epsilon) + c' \phi s \phi \right. \\ & + s' \epsilon (-c \phi s \phi s \epsilon) + c' \epsilon (-c \phi s \phi c \epsilon) \\ & + s' \delta (c \delta s \epsilon + s \phi c \phi s \delta s^2 \epsilon) \\ & \left. + c' \delta (-s \delta s \epsilon + s \phi c \phi c \delta s^2 \epsilon) \right].\end{aligned}$$

Now, combining terms and recognizing that, in general,

$$c \theta s' \theta - s \theta c' \theta = \theta'$$

$$s \theta s' \theta + c \theta c' \theta = 0,$$

$$\begin{aligned}\Psi_{xz} = & A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 \\ & + Y (-\phi' + \delta' s \epsilon)\end{aligned}$$

and thus from (A.5), (A.6) and (A.7)

$$\begin{aligned}\Psi_{yz} = & A_{12} x'_0 + A_{22} y'_0 + A_{32} z'_0 \\ & + X (\phi' - \delta' s \epsilon).\end{aligned}$$

The leading terms in these two expressions can be expanded, using (III.4.) and (III.8.) as follows:

$$\begin{aligned}A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 = & \cos \phi \left(\frac{\sqrt{2}}{\sqrt{3}} \right) x'_0 + \sin \phi \left(\frac{z'_0}{\sqrt{2}} \right) y'_0 \\ & - \cos \phi \left(\frac{-x'_0}{\sqrt{2}} \right) \left(\frac{-y'_0}{\sqrt{2}} \right) y'_0 + \sin \phi \left(\frac{-y'_0}{\sqrt{2}} \right) z'_0 \\ & + \cos \phi \left(\frac{-x'_0}{\sqrt{3}} \right) \left(\frac{z'_0}{\sqrt{2}} \right) z'_0\end{aligned}$$

or

$$\begin{aligned}A_{11} x'_0 + A_{21} y'_0 + A_{31} z'_0 = & \sin \phi \left(\frac{y'_0 z'_0}{\sqrt{2}} - \frac{z'_0 y'_0}{\sqrt{2}} \right) \\ & + \cos \phi \left(\frac{x'_0 \sqrt{2}}{\sqrt{3}} - \frac{x'_0 y'^2_0}{\sqrt{3} \sqrt{2}} - \frac{x'_0 z'^2_0}{\sqrt{3} \sqrt{2}} \right) = 0\end{aligned}\tag{A.8}$$

and

$$\begin{aligned}
A_{12} x'_0 + A_{22} y'_0 + A_{32} z'_0 &= -\sin \phi \left(\frac{\sqrt{\mathcal{Q}}}{\sqrt{\mathcal{Q}}} \right) x'_0 + \cos \phi \left(\frac{z'_0}{\sqrt{\mathcal{Q}}} \right) y'_0 \\
&+ \sin \phi \left(\frac{-x'_0}{\sqrt{\mathcal{Q}}} \right) \left(\frac{-y'_0}{\sqrt{\mathcal{Q}}} \right) y'_0 + \cos \phi \left(\frac{-y'_0}{\sqrt{\mathcal{Q}}} \right) z'_0 \\
&- \sin \phi \left(\frac{z'_0}{\sqrt{\mathcal{Q}}} \right) \left(\frac{x'_0}{\sqrt{\mathcal{Q}}} \right) z'_0 = 0
\end{aligned} \tag{A.9}$$

recalling that $\mathcal{Q} = y_0'^2 + z_0'^2$. Thus, final expressions for these two metric terms are

$$\begin{aligned}
\psi_{xz} &= -Y(\phi' - \delta' \sin \epsilon) \\
\psi_{yz} &= X(\phi' - \delta' \sin \epsilon).
\end{aligned} \tag{A.10}$$

The expression for the zz element of the distorted metric (A.1)

expands to

$$\begin{aligned}
\psi_{zz} &= x_0'^2 + 2A'_{11} x'_0 X + 2A'_{12} x'_0 Y + A_{11}'^2 X^2 + 2A'_{11} A'_{12} XY + A_{12}'^2 Y^2 \\
&+ y_0'^2 + 2A'_{21} y'_0 X + 2A'_{22} y'_0 Y + A_{21}'^2 X^2 + 2A'_{21} A'_{22} XY + A_{22}'^2 Y^2 \\
&+ z_0'^2 + 2A'_{31} z'_0 X + 2A'_{32} z'_0 Y + A_{31}'^2 X^2 + 2A'_{31} A'_{32} XY + A_{32}'^2 Y^2 \\
&= (x_0'^2 + y_0'^2 + z_0'^2) + 2X[A'_{11} x'_0 + A'_{21} y'_0 + A'_{31} z'_0] \\
&\quad + 2Y[A'_{12} x'_0 + A'_{22} y'_0 + A'_{32} z'_0] \\
&\quad + X^2[A_{11}'^2 + A_{21}'^2 + A_{31}'^2] \\
&\quad + Y^2[A_{12}'^2 + A_{22}'^2 + A_{32}'^2] \\
&\quad + 2XY[A'_{11} A'_{12} + A'_{21} A'_{22} + A'_{31} A'_{32}].
\end{aligned}$$

First it can be noted that differentiation of (A.4) yields

$$\sum_{j=1}^3 (A_{ji}'' A_{jk} + 2A_{ji}' A_{jk}' + A_{ji} A_{jk}'') = 0$$

or

$$\sum_{j=1}^3 A_{ji}' A_{jk}' = -\frac{1}{2} \sum_{j=1}^3 (A_{ji}'' A_{jk} + A_{ji} A_{jk}'').$$

Here then

$$\begin{aligned}
A_{11}'^2 + A_{21}'^2 + A_{31}'^2 &= - (A_{11} A_{11}'' + A_{21} A_{21}'' + A_{31} A_{31}'') \\
A_{12}'^2 + A_{22}'^2 + A_{32}'^2 &= - (A_{12} A_{12}'' + A_{22} A_{22}'' + A_{32} A_{32}'').
\end{aligned}$$

And similarly differentiation of (A.8) and (A.9) yields

$$A_{11} x_0'' + A_{21} y_0'' + A_{31} z_0'' = - (A_{11}' x_0' + A_{21}' y_0' + A_{31}' z_0')$$

$$A_{12} x_0'' + A_{22} y_0'' + A_{32} z_0'' = - (A_{12}' x_0' + A_{22}' y_0' + A_{32}' z_0').$$

The expression for ψ_{zz} can then be expanded most economically in the form:

$$\begin{aligned} \psi_{zz} = \textcircled{1} & - 2X [A_{11} x_0'' + A_{21} y_0'' + A_{31} z_0''] \\ & - 2Y [A_{12} x_0'' + A_{22} y_0'' + A_{32} z_0''] \\ & - X^2 [A_{11} A_{11}'' + A_{21} A_{21}'' + A_{31} A_{31}''] \\ & - Y^2 [A_{12} A_{12}'' + A_{22} A_{22}'' + A_{32} A_{32}''] \\ & + 2XY [A_{11}' A_{12}' + A_{21}' A_{22}' + A_{31}' A_{32}'] \end{aligned}$$

or $\psi_{zz} = \textcircled{1} - 2X[\alpha] - 2Y[\beta] - X^2[\gamma] - Y^2[\lambda] + 2XY[\mu].$

Evaluating these coefficients in turn,

$$\begin{aligned} \gamma = & (c\phi c\epsilon)(c''\phi c\epsilon + 2c'\phi c'\epsilon + c\phi c''\epsilon) \\ & + (s\phi c\delta - c\phi s\epsilon s\delta)(s''\phi c\delta + 2s'\phi c'\delta + s\phi c''\delta \\ & - c''\phi s\epsilon s\delta - 2c'\phi s'\epsilon s\delta - 2c'\phi s\epsilon s'\delta - c\phi s''\epsilon s\delta \\ & - 2c\phi s'\epsilon s'\delta - c\phi s\epsilon s''\delta) \\ & + (s\phi s\delta + c\phi s\epsilon c\delta)(s''\phi s\delta + 2s'\phi s'\delta + s\phi s''\delta \\ & + c''\phi s\epsilon c\delta + 2c'\phi s'\epsilon c\delta + 2c'\phi s\epsilon c'\delta + c\phi s''\epsilon c\delta \\ & + 2c\phi s'\epsilon c'\delta + c\phi s\epsilon c''\delta). \end{aligned}$$

After much algebraic manipulation, noting that in general $\cos\theta\cos''\theta + \sin\theta\sin''\theta = -\theta'^2$,

$$\begin{aligned} \gamma = & -\phi'^2 + c'\phi(-2\delta's\epsilon s\phi) + s'\phi(2\delta's\epsilon c\phi) - c^2\phi\epsilon'^2 \\ & + s'\epsilon(-2s\phi c\phi\delta') - (s^2\phi + c^2\phi s^2\epsilon)\delta'^2 \end{aligned}$$

or

$$\begin{aligned} \gamma = & -\phi'^2 + 2\phi'\delta's\epsilon - c^2\phi\epsilon'^2 - 2c\phi s\phi\delta'^2 s'\epsilon \\ & - \delta'^2(s^2\phi + c^2\phi s^2\epsilon). \end{aligned}$$

Recognizing that $\sin' \epsilon = \epsilon' \cos \epsilon$, then

$$\gamma = -\phi'^2 + 2\phi' \delta' s \epsilon - c^2 \phi \epsilon'^2 - 2\delta' \epsilon' s \phi c \phi c \epsilon - \delta'^2 (s^2 \phi + c^2 \phi s^2 \epsilon). \quad (\text{A.11})$$

Continuing,

$$\begin{aligned} \lambda = & (-s \phi c \epsilon) (-s'' \phi c \epsilon - 2s' \phi c' \epsilon - s \phi c'' \epsilon) \\ & + (c \delta c \phi + s \phi s \epsilon s \delta) (c'' \delta c \phi + 2c' \delta c' \phi + c \delta c'' \phi \\ & + s'' \phi s \epsilon s \delta + 2s' \phi s' \epsilon s \delta + 2s' \phi s \epsilon s' \delta + s \phi s'' \epsilon s \delta \\ & + 2s \phi s' \epsilon s' \delta + s \phi s \epsilon s'' \delta) \\ & + (c \phi s \delta - s \phi c \delta s \epsilon) (c'' \phi s \delta + 2c' \phi s' \delta + c \phi s'' \delta \\ & - s'' \phi c \delta s \epsilon - 2s' \phi c' \delta s \epsilon - 2s' \phi c \delta s' \epsilon - s \phi c'' \delta s \epsilon \\ & - 2s \phi c' \delta s' \epsilon - s \phi c \delta s'' \epsilon). \end{aligned}$$

Proceeding as before results in

$$\lambda = -\phi'^2 + 2\phi' \delta' s \epsilon - \epsilon'^2 s^2 \phi + 2\delta' s' \epsilon c \phi s \phi - \delta'^2 (s^2 \phi s^2 \epsilon - c^2 \phi)$$

or

$$\lambda = -\phi'^2 + 2\phi' \delta' s \epsilon + 2\delta' \epsilon' c \phi s \phi c \epsilon - \epsilon'^2 s^2 \phi - \delta'^2 (s^2 \phi s^2 \epsilon + c^2 \phi). \quad (\text{A.12})$$

Now,

$$\begin{aligned} \mu = & (c' \phi c \epsilon + c \phi c' \epsilon) (-s' \phi c \epsilon - s \phi c' \epsilon) \\ & + (s' \phi c \delta + s \phi c' \delta - c' \phi s \epsilon s \delta - c \phi s' \epsilon s \delta - c \phi s \epsilon s' \delta) (c' \phi c \delta \\ & + c \phi c' \delta + s' \phi s \epsilon s \delta + s \phi s' \epsilon s \delta + s \phi s \epsilon s' \delta) \\ & + (s' \phi s \delta + s \phi s' \delta + c' \phi s \epsilon c \delta + c \phi s' \epsilon c \delta + c \phi s \epsilon c' \delta) (c' \phi s \delta \\ & + c \phi s' \delta - s' \phi c \delta s \epsilon - s \phi c' \delta s \epsilon - s \phi c \delta s' \epsilon). \end{aligned}$$

Algebraic manipulation and recognition of the identity $(\sin' \theta)^2 + (\cos' \theta)^2 = \theta'^2$ yields

$$\mu = -c \phi s \phi \epsilon'^2 + c \phi s \phi c^2 \epsilon \delta'^2 + \delta' (c^2 \phi - s^2 \phi) s' \epsilon$$

or

$$\mu = (\delta'^2 c^2 \epsilon - \epsilon'^2) c \phi s \phi + (c^2 \phi - s'^2 \phi) \delta' \epsilon' c \epsilon. \quad (\text{A.13})$$

Finally, using the relations (II.8.),

$$\begin{aligned} \alpha &= \cos \phi \left(\frac{x''_0 \sqrt{2}}{\sqrt{1}} \right) + \sin \phi \left(\frac{y''_0 z'_0}{\sqrt{2}} \right) \\ &\quad - \cos \phi \left(\frac{x'_0 y'_0}{\sqrt{1} \sqrt{2}} y''_0 \right) + \sin \phi \left(\frac{-y'_0 z''_0}{\sqrt{2}} \right) \\ &\quad + \cos \phi \left(\frac{-x'_0 z'_0}{\sqrt{1} \sqrt{2}} z''_0 \right) \\ &= \cos \phi \left(\frac{x''_0 (y'^2_0 + z'^2_0) - x'_0 (y'_0 y''_0 + z'_0 z''_0)}{\sqrt{1} \sqrt{2}} \right) + \sin \phi \left(\frac{y''_0 z'_0 - y'_0 z''_0}{\sqrt{2}} \right) \quad (\text{A.14}) \end{aligned}$$

and

$$\begin{aligned} \beta &= -\sin \phi \left(\frac{x''_0 \sqrt{2}}{\sqrt{1}} \right) + \cos \phi \left(\frac{z'_0 y''_0}{\sqrt{2}} \right) + \sin \phi \left(\frac{x'_0 y'_0}{\sqrt{1} \sqrt{2}} y''_0 \right) \\ &\quad + \cos \phi \left(\frac{-y'_0 z''_0}{\sqrt{2}} \right) - \sin \phi \left(\frac{-x'_0 z'_0}{\sqrt{1} \sqrt{2}} z''_0 \right) \\ &= \sin \phi \left(\frac{x'_0 (y'_0 y''_0 + z'_0 z''_0) - x''_0 (y'^2_0 + z'^2_0)}{\sqrt{1} \sqrt{2}} \right) + \cos \phi \left(\frac{y''_0 z'_0 - y'_0 z''_0}{\sqrt{2}} \right). \quad (\text{A.15}) \end{aligned}$$

Thus the total expression for ψ_{zz} is

$$\begin{aligned} \psi_{zz} &= ① + 2X \left\{ \frac{x'_0 (y'_0 y''_0 + z'_0 z''_0) - x''_0 (y'^2_0 + z'^2_0)}{\sqrt{1} \sqrt{2}} \cos \phi \right. \\ &\quad \left. + \frac{y'_0 z''_0 - y''_0 z'_0}{\sqrt{2}} \sin \phi \right\} \\ &\quad + 2Y \left\{ \frac{x''_0 (y'^2_0 + z'^2_0) - x'_0 (y'_0 y''_0 + z'_0 z''_0)}{\sqrt{1} \sqrt{2}} \sin \phi + \frac{y'_0 z''_0 - y''_0 z'_0}{\sqrt{2}} \cos \phi \right\} \end{aligned}$$

(Continued)

$$\begin{aligned}
& + 2XY \{ (\delta'^2 c^2 \epsilon - \epsilon'^2) c \phi s \phi + \delta' \epsilon' c \epsilon (c^2 \phi - s^2 \phi) \} \\
& + Y^2 \{ \phi'^2 - 2 \delta' \phi' s \epsilon + \epsilon'^2 s^2 \phi - 2 \delta' \epsilon' c \epsilon c \phi s \phi \\
& \quad + \delta'^2 (s^2 \phi s^2 \epsilon + c^2 \phi) \} \\
& + X^2 \{ \phi'^2 - 2 \phi' \delta' s \epsilon + \epsilon'^2 c^2 \phi + 2 \delta' \epsilon' s \phi c \phi c \epsilon \\
& \quad + \delta'^2 (s^2 \phi + c^2 \phi s^2 \epsilon) \}.
\end{aligned}$$

This can be written more economically using equations (C.1) and (C.2) for the X and Y coefficients and denoting $X^2 + Y^2$ by R^2 ,

$$\begin{aligned}
\psi_{zz} = & \textcircled{1} + 2X \{ \sqrt{\textcircled{1}} \epsilon' \cos \phi + \sqrt{\textcircled{2}} \delta' \sin \phi \} \\
& + 2Y \{ -\sqrt{\textcircled{1}} \epsilon' \sin \phi + \sqrt{\textcircled{2}} \delta' \cos \phi \} \\
& + R^2 \{ \phi'^2 - 2 \delta' \phi' s \epsilon + \delta'^2 \sin^2 \epsilon \} \\
& + X^2 \{ \delta'^2 (\sin^2 \phi + \cos^2 \phi \sin^2 \epsilon - \sin^2 \epsilon) \\
& \quad + 2 \delta' \epsilon' \sin \phi \cos \phi \cos \epsilon + \epsilon'^2 \cos^2 \phi \} \\
& + 2XY \{ \delta'^2 \cos^2 \epsilon \cos \phi \sin \phi - \epsilon'^2 \cos \phi \sin \phi \\
& \quad + \delta' \epsilon' \cos \epsilon (\cos^2 \phi - \sin^2 \phi) \} \\
& + Y^2 \{ \delta'^2 (\cos^2 \phi + \sin^2 \phi \sin^2 \epsilon - \sin^2 \epsilon) \\
& \quad + \epsilon'^2 \sin^2 \phi - 2 \delta' \epsilon' \cos \epsilon \sin \phi \cos \phi \}.
\end{aligned}$$

This can then be factored to yield

$$\begin{aligned}
\psi_{zz} = & \textcircled{1} + 2X \{ \sqrt{\textcircled{1}} \epsilon' \cos \phi + \sqrt{\textcircled{2}} \delta' \sin \phi \} \\
& + 2Y \{ -\sqrt{\textcircled{1}} \epsilon' \sin \phi + \sqrt{\textcircled{2}} \delta' \cos \phi \} \\
& + R^2 \{ \phi' - \delta' s \epsilon \}^2 \\
& + X^2 \{ \delta' \sin \phi \cos \epsilon + \epsilon' \cos \phi \}^2 \\
& + 2XY \{ \delta' \sin \phi \cos \epsilon + \epsilon' \cos \phi \} \{ \delta' \cos \phi \cos \epsilon - \epsilon' \sin \phi \} \\
& + Y^2 \{ \delta' \cos \phi \cos \epsilon - \epsilon' \sin \phi \}^2
\end{aligned}$$

or finally

$$\begin{aligned}
 \Psi_{zz} = & \textcircled{1} + 2X[\sqrt{\textcircled{1}}\epsilon' \cos \phi + \sqrt{\textcircled{2}}\delta' \sin \phi] \\
 & + 2Y[-\sqrt{\textcircled{1}}\epsilon' \sin \phi + \sqrt{\textcircled{2}}\delta' \cos \phi] \\
 & + R^2[\phi' - \delta' \sin \epsilon]^2 \\
 & + \left\{ X[\delta' \sin \phi \cos \epsilon + \epsilon' \cos \phi] - Y[-\delta' \cos \phi \cos \epsilon + \epsilon' \sin \phi] \right\}^2.
 \end{aligned} \tag{A.16}$$

Appendix B

Calculation of the Unit Strain Energy

The unit strain energy is given by

$$V = \frac{G}{4} [(\psi_{zz} - 1)^2 + 2\psi_{xz}^2 + 2\psi_{yz}^2] \quad (\text{III.13})$$

and the metric components involved are

$$\begin{aligned} \psi_{xz} &= -Y(\phi' - \delta' \sin \epsilon) \\ \psi_{yz} &= X(\phi' - \delta' \sin \epsilon) \\ \psi_{zz} &= ① + 2X[\sqrt{①} \epsilon' \cos \phi + \sqrt{②} \delta' \sin \phi] \\ &\quad + 2Y[-\sqrt{①} \epsilon' \sin \phi + \sqrt{②} \delta' \cos \phi] \\ &\quad + R^2[\phi' - \delta' \sin \epsilon]^2 \\ &\quad + \{X[\epsilon' \cos \phi + \delta' \sin \phi \cos \epsilon] - Y[\epsilon' \sin \phi - \delta' \cos \phi \cos \epsilon]\}^2 \end{aligned} \quad (\text{III.12.})$$

The computations are simplified by first noting that $\cos \epsilon = \frac{\sqrt{②}}{\sqrt{①}}$ and thus if we define the groups

$$\begin{aligned} \beta &= \phi' - \delta' \sin \epsilon \\ \eta &= \epsilon' \cos \phi + \delta' \sin \phi \cos \epsilon \\ \mathcal{X} &= \epsilon' \sin \phi - \delta' \cos \phi \cos \epsilon \end{aligned}$$

the metric components become

$$\begin{aligned} \psi_{xz} &= -Y\beta & \psi_{yz} &= X\beta \\ \psi_{zz} &= ① + 2\sqrt{①}\eta X - 2\sqrt{①}\mathcal{X} Y \\ &\quad + R^2\beta^2 + (X\eta - Y\mathcal{X})^2 \end{aligned} \quad (\text{B.1})$$

From the first two of the relations (B.1),

$$2\psi_{xz}^2 + 2\psi_{yz}^2 = 2R^2\beta^2,$$

and the third can be applied to calculate the term $(\psi_{zz} - 1)^2$. It will simplify calculations to first note that the expression V_0 will be integrated over the original volume of the bar to obtain the total

strain energy. Since the bar is rotationally symmetric, certain groups of terms will be lost in the integration. This is most easily seen in polar coordinate form: the limits on R and Θ in the integration are independent and such terms as

$$\int_0^{2\pi} X f(R, Z) d\Theta = \int_0^{2\pi} R f(R, Z) \cos \Theta d\Theta$$

will not contribute to the total integral. In the expansion of $(\psi_{zz} - 1)^2$, therefore, all terms involving $f(R, Z)X$, $f(R, Z)Y$, $f(R, Z)(X^2 - Y^2)$, $f(R, Z)X^3$, $f(R, Z)Y^3$, $f(R, Z)X^2Y$, and $f(R, Z)Y^2X$ are not included. (Note that $\textcircled{1}$, β , η , and $\mathcal{X}$ are functions of Z only.) Thus, denoting the deleted quantity by $(\psi_{zz} - 1)^2_{\text{eff.}}$,

$$\begin{aligned} (\psi_{zz} - 1)^2_{\text{eff.}} = & \textcircled{1}^2 + 2\textcircled{1}R^2\beta^2 + 2\textcircled{1}\eta^2X^2 + 2\textcircled{1}\mathcal{X}^2Y^2 - 2\textcircled{1} \\ & + 4\textcircled{1}X^2\eta^2 + 4\textcircled{1}Y^2\mathcal{X}^2 + R^4\beta^4 + 2R^2\beta^2X^2\eta^2 \\ & + 2R^2\beta^2Y^2\mathcal{X}^2 - 2R^2\beta^2 + X^4\eta^4 + 2X^2Y^2\eta^2\mathcal{X}^2 \\ & - 2X^2\eta^2 + 4X^2Y^2\eta^2\mathcal{X}^2 + Y^4\mathcal{X}^4 - 2Y^2\mathcal{X}^2 + 1 \end{aligned}$$

$$\begin{aligned} = & \textcircled{1}^2 - 2\textcircled{1} + 1 + R^2\beta^2(R^2\beta^2 + 2\textcircled{1} - 2) \\ & + X^2(2\textcircled{1} + 2R^2\beta^2 - 2 + 4\textcircled{1})\eta^2 \\ & + Y^2(2\textcircled{1} + 4\textcircled{1} + 2R^2\beta^2 - 2)\mathcal{X}^2 \\ & + X^4\eta^4 + 6X^2Y^2\eta^2\mathcal{X}^2 + Y^4\mathcal{X}^4 \end{aligned}$$

or finally

$$\begin{aligned} (\psi_{zz} - 1)^2_{\text{eff.}} = & (\textcircled{1} - 1)^2 + (R^2\beta^2 + 2\textcircled{1} - 2)R^2\beta^2 \\ & + 2(R^2\beta^2 + 3\textcircled{1} - 1)(X^2\eta^2 + Y^2\mathcal{X}^2) \\ & + X^4\eta^4 + 6X^2Y^2\eta^2\mathcal{X}^2 + Y^4\mathcal{X}^4. \end{aligned}$$

Thus a final expression for the unit strain energy involving these groups is

$$V_{\text{eff.}} = \frac{G}{4} \left[(\textcircled{1}-1)^2 + (R^2\beta^2 + 2\textcircled{1} - 2)R^2\beta^2 \right. \\ \left. + 2(R^2\beta^2 + 3\textcircled{1} - 1)(X^2\eta^2 + Y^2\mathcal{X}^2) \right. \\ \left. + X^4\eta^4 + 6X^2Y^2\eta^2\mathcal{X}^2 + Y^4\mathcal{X}^4 + 2R^2\beta^2 \right]$$

or

$$V_{\text{eff.}} = \frac{G}{4} \left[(\textcircled{1}-1)^2 + (R^2\beta^2 + 2\textcircled{1})R^2\beta^2 \right. \\ \left. + 2(R^2\beta^2 + 3\textcircled{1} - 1)(X^2\eta^2 + Y^2\mathcal{X}^2) \right. \\ \left. + X^4\eta^4 + 6X^2Y^2\eta^2\mathcal{X}^2 + Y^4\mathcal{X}^4 \right] \quad (\text{B.2})$$

where

$$\beta = \phi' - \delta' \sin \epsilon \\ \eta = \epsilon' \cos \phi + \delta' \sin \phi \cos \epsilon \\ \mathcal{X} = \epsilon' \sin \phi - \delta' \cos \phi \cos \epsilon .$$

Appendix C

Calculation of the Gradient of Angular Coordinates

Differentiation of the first of equations (III.8.) yields

$$\begin{aligned}
 \sin' \epsilon = \epsilon' \cos \epsilon &= \frac{\chi'_0 \left(\frac{1}{\sqrt{1}} \right) (2\chi'_0 \chi''_0 + 2y'_0 y''_0 + 2z'_0 z''_0) - \chi''_0 \sqrt{1}}{1} \\
 &= \frac{\chi'_0 (\chi'_0 \chi''_0 + y'_0 y''_0 + z'_0 z''_0) - \chi''_0 (\chi'^2_0 + y'^2_0 + z'^2_0)}{1^{3/2}} \\
 &= \frac{\chi'_0 (y'_0 y''_0 + z'_0 z''_0) - \chi''_0 (y'^2_0 + z'^2_0)}{1^{3/2}} \\
 &= \frac{\chi'_0 (y'_0 y''_0 + z'_0 z''_0) - \chi''_0 (y'^2_0 + z'^2_0)}{1 \sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{1}}
 \end{aligned}$$

or

$$\epsilon' = \frac{\chi'_0 (y'_0 y''_0 + z'_0 z''_0) - \chi''_0 (y'^2_0 + z'^2_0)}{1 \sqrt{2}} \quad (C.1)$$

Differentiation of the second yields

$$\begin{aligned}
 \sin' \delta = \delta' \cos \delta &= \frac{y'_0 \left(\frac{1}{\sqrt{2}} \right) (2y'_0 y''_0 + 2z'_0 z''_0) - y''_0 \sqrt{2}}{2} \\
 &= \frac{y'_0 (y'_0 y''_0 + z'_0 z''_0) - y''_0 (y'^2_0 + z'^2_0)}{2^{3/2}} \\
 &= \frac{y'_0 z'_0 z''_0 - y''_0 z'^2_0}{2^{3/2}} \\
 &= \frac{(y'_0 z''_0 - y''_0 z'_0)}{2} \frac{z'_0}{\sqrt{2}}
 \end{aligned}$$

or

$$\delta' = \frac{y'_0 z''_0 - y''_0 z'_0}{2} \quad (C.2)$$

Differentiation of the remaining two expressions in (III.8.) yields the same results.

Appendix D

Calculation of the Variation of Energy

The potential energy is given by equation (III.18.) as

$$\begin{aligned}
 T = & -P_{z_0}(L) + P_0 z_0(0) - M \dot{\phi}(L) + M_0 \dot{\phi}(0) \\
 & + \int_0^L \left\{ \frac{G}{4} [A(1-1)^2 + 2I(h\beta^2 + 2(1))\beta^2 \right. \\
 & \quad + 2I(h\beta^2 + 3(1) - 1)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon) \\
 & \quad \left. + \frac{3}{4} I h (\epsilon'^2 + \delta'^2 \cos^2 \epsilon)^2 \right] \\
 & \quad \left. + 5A g [y_0 \sin \alpha - z_0 \cos \alpha] \right\} dZ
 \end{aligned} \tag{D.1}$$

where the group $\beta = \dot{\phi}' - \delta' \sin \epsilon$ is retained. Note that in terms of x_0, y_0, z_0 and ϕ ,

$$\beta = \dot{\phi}' - \delta' \sin \epsilon = \dot{\phi}' + \frac{\chi'_0 (y'_0 z''_0 - y''_0 z'_0)}{(2) \sqrt{(1)}}$$

and

$$\begin{aligned}
 \epsilon'^2 + \delta'^2 \cos^2 \epsilon = & \left[\frac{\chi'_0 (y'_0 y''_0 + z'_0 z''_0) - \chi''_0 (y'^2_0 + z'^2_0)}{(1) \sqrt{(2)}} \right]^2 \\
 & + \left[\frac{y'_0 z''_0 - y''_0 z'_0}{\sqrt{(2)} \sqrt{(1)}} \right]^2
 \end{aligned} \tag{D.2}$$

where $(1) = x_0'^2 + y_0'^2 + z_0'^2$ and $(2) = y_0'^2 + z_0'^2$.

Now it is required to calculate the value of the potential energy for the set of displacements

$$\chi_0 = \delta \chi_0, \quad y_0 = \delta y_0, \quad z_0 = z_{0e} + \delta z_0 \quad \text{and} \quad \phi = \phi_e + \delta \phi$$

where $z_{0e} = (A+BZ)Z$ and $\phi_e = \psi(A+BZ)Z$, ψ , A and B being constants.

The potential energy for the set

$$\chi_0 = 0, \quad y_0 = 0, \quad z_0 = z_{0e} \quad \text{and} \quad \phi = \phi_e$$

is to be subtracted from that value. The calculation of the energy for

the perturbed position will be limited to terms of less than third order in the variational displacements. For this purpose it may be observed that β , ① and ② will consist of zero, first and second as well as higher order terms in these displacements, whereas the group $(\epsilon'^2 + \delta'^2 \cos^2 \epsilon)$ consists of only second and higher order terms. For ease in manipulation the following abbreviations will be employed:

$$\begin{aligned}\beta &= \beta_e + \beta_1 + \beta_2 \\ \textcircled{1} &= \textcircled{1}_e + \textcircled{1}_1 + \textcircled{1}_2 \\ \textcircled{2} &= \textcircled{2}_e + \textcircled{2}_1 + \textcircled{2}_2\end{aligned}$$

where the subscripts e, 1 and 2 denote respectively zeroth, first and second order terms in the variational displacements. Using this notation the energy of the perturbed bar is

$$\begin{aligned}T + \Delta T &= -P Z_{oe}(L) - P \delta Z_o(L) + P_o Z_{oe}(0) + P_o \delta Z_o(0) \\ &\quad - M \zeta_e(L) - M \delta \zeta(L) + M_o \zeta_e(0) + M_o \delta \zeta(0) \\ &\quad + \int_0^L \left\{ \frac{G}{4} \left[A(\textcircled{1}_e^2 + 2\textcircled{1}_e \textcircled{1}_1 + 2\textcircled{1}_e \textcircled{1}_2 - 2\textcircled{1}_e \right. \right. \\ &\quad \left. \left. + \textcircled{1}_1^2 - 2\textcircled{1}_1 - 2\textcircled{1}_2 + 1) \right. \right. \\ &\quad \left. + 2I(h\beta_e^2 + 2h\beta_e\beta_1 + 2h\beta_e\beta_2 + h\beta_1^2 \right. \\ &\quad \left. + 2\textcircled{1}_e + 2\textcircled{1}_1 + 2\textcircled{1}_2)(\beta_e^2 + 2\beta_e\beta_1 \right. \\ &\quad \left. + 2\beta_e\beta_2 + \beta_1^2) \right. \\ &\quad \left. + 2I(h\beta_e^2 + 3\textcircled{1}_e - 1)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon) \right] \\ &\quad \left. + 5Ag[\delta y_o \sin \alpha - z_{oe} \cos \alpha - \delta z_o \cos \alpha] \right\} dZ.\end{aligned}$$

Here the higher order terms in the third expression under the integral are dropped, as is the whole term involving $(\epsilon'^2 + \delta'^2 \cos^2 \epsilon)^2$. Expanding the indicated products, keeping only terms of second and lower order and ordering the results yields

$$\begin{aligned}
T + \Delta T = & -P_{z_{oe}}(L) - P \delta z_o(L) + P_o z_{oe}(0) + P_o \delta z_o(0) \\
& - M \zeta_e(L) - M \delta \zeta_e(L) + M_o \zeta_e(0) + M_o \delta \zeta_e(0) \\
& + \int_0^L \left\{ \frac{G}{4} [\mathcal{A}(\mathbb{1}_e - 1)^2 + 2I(h\beta_e^4 + 2\mathbb{1}_e\beta_e^2)] \right. \\
& \quad \left. - 5\mathcal{A}g z_{oe} \cos \alpha \right\} dZ \\
& + \int_0^L \left\{ \frac{G}{4} [\mathcal{A}(2\mathbb{1}_e\mathbb{1}_1 - 2\mathbb{1}_1) \right. \\
& \quad + 2I(4h\beta_e^3\beta_1 + 4\mathbb{1}_e\beta_e\beta_1 + 2\beta_e^2\mathbb{1}_1)] \\
& \quad \left. + 5\mathcal{A}g [\delta y_o \sin \alpha - \delta z_o \cos \alpha] \right\} dZ \\
& + \int_0^L \frac{G}{4} [\mathcal{A}(2\mathbb{1}_e\mathbb{1}_2 + \mathbb{1}_1^2 - 2\mathbb{1}_2) \\
& \quad + 2I(4h\beta_e^3\beta_2 + 6h\beta_e^2\beta_1^2 \\
& \quad + 4\mathbb{1}_e\beta_e\beta_2 + 2\mathbb{1}_e\beta_1^2 \\
& \quad + 4\beta_e\mathbb{1}_1\beta_1 + 2\beta_e^2\mathbb{1}_2) \\
& \quad + 2I(h\beta_e^2 + 3\mathbb{1}_e - 1)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon)] dZ.
\end{aligned} \tag{D.3}$$

The energy in the base state follows from (D.1) and (D.2):

$$\begin{aligned}
T = & -P_{z_{oe}}(L) + P_o z_{oe}(0) - M \zeta_e(L) + M_o \zeta_e(0) \\
& + \int_0^L \left\{ \frac{G}{4} [\mathcal{A}(z_{oe}'^2 - 1)^2 + 2I(h\zeta_e'^2 + 2z_{oe}'^2)\zeta_e'^2] \right. \\
& \quad \left. + 5\mathcal{A}g [-z_{oe} \cos \alpha] \right\} dZ.
\end{aligned} \tag{D.4}$$

From the expressions (D.2) it follows that

$$\begin{aligned}
\beta_e = \zeta_e' \quad , \quad \beta_1 = \delta \zeta_e' \quad , \quad \beta_2 = \frac{1}{2\sqrt{\mathbb{1}}} \left[z_{oe}'' (\delta x_o' \delta y_o') \right. \\
\quad \left. - z_{oe}' (\delta x_o' \delta y_o'') \right], \\
\mathbb{1}_e = z_{oe}'^2 \quad , \quad \mathbb{1}_1 = 2z_{oe}' \delta z_o' \quad \text{and} \quad \mathbb{1}_2 = \delta x_o'^2 + \delta y_o'^2 + \delta z_o'^2.
\end{aligned} \tag{D.5}$$

Noting this, the difference of (D.3) and (D.4) is

$$\begin{aligned}
\Delta T = & -P \delta z_o(L) + P_o \delta z_o(0) - M \delta \zeta(L) + M_o \delta \zeta(0) \\
& + \int_0^L \left\{ \frac{G}{4} \left[4A(z'_{oe}{}^2 - 1)(z'_{oe} \delta z'_o) + 8I(h \zeta_e'^2 + z'_{oe}{}^2) \zeta_e' \delta \zeta' \right. \right. \\
& \quad \left. \left. + 8I \zeta_e'^2 z'_{oe} \delta z'_o \right] \right. \\
& \quad \left. + 5Ag [\delta y_o \sin \alpha - \delta z_o \cos \alpha] \right\} dZ \\
& + \int_0^L \frac{G}{4} \left[2A(z'_{oe}{}^2 - 1)(\delta x_o'^2 + \delta y_o'^2 + \delta z_o'^2) \right. \\
& \quad + 4A z'_{oe}{}^2 \delta z_o'^2 + 8I(h \zeta_e'^2 + z'_{oe}{}^2) \zeta_e' \beta_2 \\
& \quad + 4I(z'_{oe}{}^2 + 3h \zeta_e'^2) \delta \zeta'^2 \\
& \quad + 16I \zeta_e' z'_{oe} \delta z_o' \delta \zeta' \\
& \quad + 4I(\zeta_e'^2)(\delta x_o'^2 + \delta y_o'^2 + \delta z_o'^2) \\
& \quad \left. + 2I(h \zeta_e'^2 + 3z'_{oe}{}^2 - 1)(\epsilon'^2 + \delta'^2 \cos^2 \epsilon) \right] dZ.
\end{aligned}$$

This expression can be more compactly written employing the assumed equilibrium values ζ_e and z_{oe} and combining terms; but first it is necessary to examine the two terms β_2 and $\epsilon'^2 + \delta'^2 \cos^2 \epsilon$. Both of these, as seen from (D.2) and (D.5) contain factors of the form $\frac{1}{\sqrt{1}}$, $\frac{1}{2}$, etc. A Taylor series expansion of such terms can be found. For these power series all terms involving variational displacements must be dropped since in each case the numerator of the fraction involved is already of second and higher order in these variables. A typical expansion might then be

$$\frac{1}{\sqrt{1}} = \frac{1}{1+2BZ} = \frac{1}{1+e+2BZ} = 1 - e - 2BZ + \left\{ \begin{array}{l} \text{higher order} \\ \text{terms in } e \\ \text{and } 2BZ \end{array} \right\}.$$

The higher order terms in e and $2BZ$ will be neglected with respect to e and $2BZ$. (Note that the expansion holds for $|e + 2BZ| < 1$.) Making such expansions it is possible to obtain

$$\begin{aligned}
\epsilon'^2 + \delta'^2 \cos^2 \epsilon \Big|_{\text{Approx.}} &= (1-6e-12BZ) z_{oe}'^2 (z_{oe}'' \delta x_o' - z_{oe}' \delta x_o'')^2 \\
&\quad + (1-4e-8BZ) (z_{oe}'' \delta y_o' - z_{oe}' \delta y_o'')^2 \\
&= (1-4e-8BZ) \left[z_{oe}''^2 (\delta x_o'^2 + \delta y_o'^2) \right. \\
&\quad \left. - 2 z_{oe}'' z_{oe}' (\delta x_o' \delta x_o'' + \delta y_o' \delta y_o'') \right. \\
&\quad \left. + z_{oe}'^2 (\delta x_o''^2 + \delta y_o''^2) \right]
\end{aligned}$$

or finally

$$\begin{aligned}
\epsilon'^2 + \delta'^2 \cos^2 \epsilon \Big|_{\text{Approx.}} &= 4B^2(1-4e-8BZ)(\delta x_o'^2 + \delta y_o'^2) \\
&\quad - 4B(1-3e-6BZ)(\delta x_o' \delta x_o'' + \delta y_o' \delta y_o'') \\
&\quad + (1-2e-4BZ)(\delta x_o''^2 + \delta y_o''^2).
\end{aligned}$$

Then

$$\begin{aligned}
\beta_z \Big|_{\text{Approx.}} &= (1-3e-6BZ) [z_{oe}'' (\delta x_o' \delta y_o') - z_{oe}' (\delta x_o' \delta y_o'')] \\
&= 2B(1-3e-6BZ) \delta x_o' \delta y_o' - (1-2e-4BZ) \delta x_o' \delta y_o''
\end{aligned}$$

so that, substituting these expressions and using the relations for the equilibrium values, again neglecting e^2 and $B^2 Z^2$ for terms of the order of e and BZ ,

$$\begin{aligned}
\Delta T &= -P \delta z_o(L) + P_o \delta z_o(0) - M \delta \zeta(L) + M_o \delta \zeta(0) \\
&\quad + \int_0^L \left\{ \frac{G}{4} [4A(2e+4BZ)(1+e+2BZ) \delta z_o' \right. \\
&\quad \left. + 8I\psi(h\psi^2+1)(1+3e+6BZ) \delta \zeta' \right. \\
&\quad \left. + 8I\psi^2(1+3e+6BZ) \delta z_o' \right] \\
&\quad \left. + 5Ag [\delta y_o \sin \alpha - \delta z_o \cos \alpha] \right\} dZ \\
&\quad + \int_0^L \frac{G}{4} \left\{ 4A(e+2BZ)(\delta x_o'^2 + \delta y_o'^2 + \delta z_o'^2) \right. \\
&\quad \left. + 4A(1+2e+4BZ) \delta z_o'^2 + 8I\psi(h\psi^2+1) [2B \delta x_o' \delta y_o' \right. \\
&\quad \left. - (1+e+2BZ) \delta x_o' \delta y_o''] + 4I(3h\psi^2+1)(1+2e+4BZ) \delta \zeta'^2 \right. \\
&\quad \left. + 16I\psi(1+2e+4BZ) \delta z_o' \delta \zeta' + 4I\psi^2(1+2e+4BZ)(\delta x_o'^2 \right. \\
&\quad \left. + \delta y_o'^2 + \delta z_o'^2) + 2I[(h\psi^2+3)(1+2e+4BZ)-1] [4B^2(\delta x_o'^2 \right. \\
&\quad \left. + \delta y_o'^2) - 4B(\delta x_o' \delta x_o'' + \delta y_o' \delta y_o'') \right. \\
&\quad \left. + (1-2e-4BZ)(\delta x_o''^2 + \delta y_o''^2)] \right\} dZ.
\end{aligned}$$

Now by grouping terms a somewhat more convenient form is obtained:

$$\begin{aligned}
\Delta T = & -P \delta z_0(L) + P_0 \delta z_0(0) - M \delta \phi(L) + M_0 \delta \phi(0) \\
& + \int_0^L \left\{ \frac{G}{4} \left[[8A(e+2BZ) + 8I\Psi^2(1+2e+4BZ)](1+e+2BZ)\delta z'_0 \right. \right. \\
& \quad \left. \left. + 8I\Psi(\Psi^2h+1)(1+3e+6BZ)\delta \phi' \right] \right. \\
& \quad \left. + 5Ag[\delta y_0 \sin \alpha - \delta z_0 \cos \alpha] \right\} dZ \\
& + \int_0^L \frac{G}{4} \left\{ [4A(1+e+2BZ) + 4I\Psi^2](1+2e+4BZ)\delta z'^2_0 \right. \\
& \quad + 16I\Psi(1+2e+4BZ)\delta z'_0 \delta \phi' \\
& \quad + 4I(1+3h\Psi^2)(1+2e+4BZ)\delta \phi'^2 \\
& \quad + [4A(e+2BZ) + 4I\Psi^2(1+2e+4BZ) \\
& \quad \quad \left. + 8IB^2(h\Psi^2+3)[1+2e+4BZ]-1](\delta x'^2_0 + \delta y'^2_0) \right. \\
& \quad + 16I\Psi B(1+h\Psi^2)\delta x'_0 \delta y'_0 \\
& \quad - 8I\Psi(1+h\Psi^2)(1+e+2BZ)\delta x'_0 \delta y''_0 \\
& \quad - 8BI[(h\Psi^2+3)(1+2e+4BZ)-1](\delta x'_0 \delta x''_0 + \delta y'_0 \delta y''_0) \\
& \quad \left. + 2I[(h\Psi^2+3)(1+2e+4BZ)-1](1-2e-4BZ)(\delta x''^2_0 \right. \\
& \quad \left. + \delta y''^2_0) \right\} dZ.
\end{aligned}$$

Since it is to be expected that the twist per unit length, Ψ , will be large if buckling is to occur under the retarding influence of extension and a one-sided constraint, it is not possible to say that Ψ_b is of the order of a classical elastic strain. Therefore, it will be assumed that while terms of the order Ψ_b^2 are negligible with respect to unity, and that terms such as $\Psi_b^2 e$ are negligible with respect to Ψ_b , the Ψ_b^2 terms must be retained in comparison to terms of the order of e and BL . Carrying this out yields the equation

$$\begin{aligned}
\Delta T = & -P \delta z_0(L) + P_0 \delta z_0(0) - M \delta \phi(L) + M_0 \delta \phi(0) \\
& + \int_0^L \left\{ \frac{G}{4} \left[8[A(e+2BZ) + I\Psi^2] \delta z'_0 \right. \right. \\
& \quad \left. \left. + 8I\Psi(1+3e+6BZ) \delta \phi' \right] \right. \\
& \quad \left. + 8Ag [\delta y_0 \sin \alpha - \delta z_0 \cos \alpha] \right\} dZ \\
& + \int_0^L \frac{G}{4} \left\{ 4A(1+3e+6BZ) \delta z_0'^2 \right. \\
& \quad + 16I\Psi(1+2e+4BZ) \delta z'_0 \delta \phi' \\
& \quad + 4I(1+2e+4BZ) \delta \phi'^2 \\
& \quad + 4[A(e+2BZ) + I\Psi^2] (\delta x_0'^2 + \delta y_0'^2) \\
& \quad + 16I\Psi B \delta x'_0 \delta y'_0 \\
& \quad - 8I\Psi(1+e+2BZ) \delta x'_0 \delta y_0'' \\
& \quad - 8IB (\delta x'_0 \delta x_0'' + \delta y'_0 \delta y_0'') \\
& \quad \left. + 4I(1+e+2BZ) (\delta x_0''^2 + \delta y_0''^2) \right\} dZ
\end{aligned} \tag{D.6}$$

and this is the expression used in the rest of the calculations. Note that it is subject to the restrictions: (1) that all terms of greater than second order in the displacements δx_0 , δy_0 , δz_0 and $\delta \phi$ are neglected; and (2) that e and BL are assumed infinitesimal while Ψb is of higher order.

Appendix E

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FIGURE 1: ORIGINAL CONFIGURATION

FIGURE 2: LOCATION OF A CROSS
SECTION

FIGURE 3: ASSUMED BUCKLING MODE

FIGURE 4: Ψ_{CR} vs L CHARACTERISTICS

FIGURE 5: F^* vs L CHARACTERISTICS