Numerical pathologies in snap-through simulations

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Abstract

Aircraft structures operating in severe environments may experience snap-through, causing the curvature on part or all of the structure to invert inducing fatigue damage. This paper examines the performance of beam and continuum nonlinear finite element formulations in conjunction with several popular implicit time stepping algorithms to assess the accuracy and stability of numerical simulations of snap-through events. Limitations for the structural elements are identified and we provide examples of interaction between spatial and temporal discretizations that affect the robustness of the overall scheme and impose strict limits on the size of the time step. These limitations need to be addressed in future works in order to develop accurate, robust and efficient simulation methods for response prediction of structures encountering extreme environments.

Keywords: snap-through, finite element, curved beam, solid (continuum)

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1. Introduction

Curved beams or panels can often be found as components of complex structures in civil, mechanical, and aerospace applications. They may experience snap-through causing the curvature on part or all of the structure to invert due to a large inward loading (Figure 1(a)) caused by extreme loading conditions [1]. When snap-through happens, the load-deflection diagram presents a jump, e.g., from point B to point C, as shown in Figure 1(b). In some cases, when the load is reduced, *hysteresis* is observed: the structure will snap-back to its original configuration but on a different path (represented by the jump from point D to point A in the same figure).

Figure 1: Snap-through buckling of shallow structures: (a) initial and post-snap configuration, (b) load-deflection curve.

Snap-through is characterized by large nonlinear deformations, changes in the system stability and large stress reversals, which accelerate fatigue damage. Since no analytical solutions for general systems with snap-through exist, numerical models that can capture these phenomena are needed in

order to predict the fatigue life of the structure. Among the numerical techniques, the Finite Element Method (FEM) *provides the most generality and can be applied to systems with arbitrarily complex geometries*. This paper analyzes the performance of several finite element formulations (2D and 3D) and the stability of the time-stepping schemes in simulating a curved beam undergoing snap-through: we identify the important features that affect the numerical accuracy and robustness and the region where the schemes are stable for such simulations.

The snap-through of curved beams or shallow arcs has been studied analytically for particular cases by numerous authors, including Murphy et al. [2], Virgin [3], Bradford et al. [4], and Plaut and Virgin [5]. Snap-through can also be studied qualitatively using a truss system [6]. Simplifying assumptions are usually needed in finding the analytical estimates. While we are paying the price of the lack of physical intuition by using the complex FEM, we often need to do so in order to use more general analysis techniques that allow for complex effects to be captured. Therefore, we test here the performance of this method when applied to a simple curved beam structure (geometry and properties introduced in section 2). This structure is representative since it exhibits the type of phenomena we wish to study (snap-through) but simple enough to allow for comparisons with solutions obtained with analytical methods as well.

In this paper, the numerical simulations are performed with the Finite Element Analysis Program (FEAP), a research code that includes most commonly used finite elements and solvers and provides a reliable framework for developing and implementing new user formulations [7]. The factors taken

into consideration in this study are (1) the time-stepping algorithm, (2) the element type, and (3) the size of the time step. The element formulations and the time-stepping algorithms discussed are either available from the standard FEAP distribution or implemented as user subroutines. The main criterion used to verify the robustness of the results is the energy conservation throughout the simulation.

For discretization, 2D straight beam elements and 3D solid linear and quadratic elements are used. All formulations account for large deformations. The 2D beam elements utilized are (1) beam elements without shear deformation with *large displacement* and small rotation (2nd order theory) with cubic interpolation and (2) beam elements with shear deformation formulation and *large displacement and large rotation* that can consider inelastic behavior for bending and axial effects but retain linear elastic response in the transverse shear terms [7]. The 3D solid elements used are linear (8-nodes) elements with displacement, mixed (B-bar), and enhanced formulations and quadratic (27-nodes) elements with displacement formulation. Unlike the *structural* elements, e.g., beams, the continuum (3D solid) elements do not include a built in kinematic assumption. *This characteristic makes them suitable for consistent incorporation of other effects, such as thermomechanical coupling in future studies.* Note that in the current study, no coupling, and no material or boundary nonlinearities are included.

Conventional approaches in the stability analysis of structures often make use of static considerations, but in fact, even when the loads are applied statically, buckling and snap-through are inherently dynamic processes and a full description of the structural behavior can be obtained only through

a dynamic analysis [3]. Numerical simulations of such phenomena require access to stable time-stepping schemes and in general to robust simulation environments.

Unfortunately, due to the kinematic assumptions incorporated in the structural elements, the use of these elements coupled with the time-stepping integrators is prone to numerical difficulties that affect the accuracy of the results as shown in section 4. Numerical instabilities mask the true physical behavior rendering the structural response prediction inaccurate.

An important component of the simulation environment is the time-stepping scheme. We examine here common choices for structural mechanics simulations: (1) the traditional Newmark integrator [8] and (2) energy-momentum conserving algorithms. The Newmark method is the most widely used time integrator in the area of structural analysis. For certain combinations of parameters, it is unconditionally stable for linear problems. However, its stability is not guaranteed for nonlinear problems. The traditional time-stepping algorithms developed for structural dynamics applications usually perform well for linear problems. In the nonlinear regime, however, numerical instabilities appear due to the energy increase of the discrete system. Hence, energy-momentum schemes were developed to overcome the lack of conservation [9]. These schemes belong to a class of algorithms designed to satisfy various conservation laws by construction. The energy-momentum algorithms used are (1) the algorithm based on the work by Simo and Tarnow [10], Simo et al. [11], and Gonzalez [12] and (2) a composite algorithm based on the trapezoidal rule and the three point backward Euler proposed by Bathe [13] that is stable for large time steps. These algorithms will be re-

ferred as conserving A and conserving B, respectively, in the rest of this paper. Many authors used conserving A to solve various types of problems [14]. In recent work, Garikipati et al. [15] used this algorithm to model growth in biological tissue. Bathe [13] showed that the conserving B algorithm, which is incorporated in the commercial FEM software ADINA 8.7, is able to solve a specific type of problems where the Newmark algorithm is unstable and does not conserve energy and momentum. Although the conserving B scheme allows us to use larger time steps, interactions between the time-stepping schemes and the spatial discretizations might still occur when structural elements are used.

This paper is organized as follows. Section 2 describes the representative structure used throughout the paper. Section 3 compares the results of the static analyses obtained using different element types and formulations. Section 4 analyzes the numerical results of the nonlinear dynamic simulations of a curved beam undergoing snap-through. Various time integrators and element types and formulations are used. A summary of our findings is presented in the concluding section.

2. Representative Structure

The representative system discussed in this paper is a curved beam that undergoes snap-through under a concentrated load. The geometry of the beam is shown in Figure 2. The beam is symmetrical with an angle $\theta = 5.674^\circ$ at the supports. The thicker lines in the figure represent the parts of the beam that are clamped; their horizontal length is L_c . The free portion of the beam has a horizontal projection L_h . The total horizontal projection of the

Figure 2: Geometry of the structure. A curved beam clamped at the supports.

Figure 3: 3D representation of the geometry of the structure (undeformed and post-snap configurations)

beam, including the clamped parts is $L = 362.204$ mm. The beam can be considered as a thin beam and has a rectangular cross section with depth d much larger than the thickness t . Figure 3 shows the 3D representation of the beam.

The dimensions of the beam are listed in Table 1 and Table 2 lists the material properties of the structure. Without loss of generality, the type of load used throughout this paper is a point load applied at the center of the beam.

Table 1: Normalized dimensions of the curved beam

Normalized Dimensions	Values
Bottom radius / L	5.058
Free horizontal length (L_h) / L	0.8415
Clamp horizontal length (L_c) / L	0.07925
Depth (d) / L	0.035
Thickness (t) / L	0.0014
Rise-to-thickness (H/t)	17.67

Table 2: Material properties

Properties	Values
Young's Modulus [N/mm ²]	206843
Poisson's Ratio	0.28
Density [N s ² /mm ⁴]	7.83x10 ⁻⁹

3. Static Analysis

In this section we study the effect of the type of elements and formulations on the accuracy of the results in quasistatic simulations. Based on these preliminary results, the options that are less accurate in the static analysis are eliminated, and as a result a decision regarding the best elements to be used in the dynamic analysis can be made.

All simulations presented later in this paper are performed with meshes that ensure spatial convergence. Figure 4 shows a mesh convergence study for 3D solid quadratic elements (27 nodes). The reference number of elements

Figure 4: Mesh refinement study for 3D solid quadratic elements.

is taken to be 3072 elements along the arch. The coarsest mesh utilized has 6 elements along the arch and gives a relative error of 98.2 % in estimating the displacement after snap-through. Based on this study, we use a mesh of 200 elements along the arc to ensure less than 1 % relative error. A similar study was performed for discretization with beam elements [16].

The structure is symmetrical, but since nonsymmetrical configurations after bifurcation exist under certain conditions, all simulations were performed on the full beam.

3.1. Incremental loading using the Newton Raphson scheme

The first static analysis is performed with a load control algorithm and uses the Newton-Raphson iterative method. An increasing point load is applied at the midspan of the beam.

Figure 5: Load-displacement curve from static analysis with Newton Raphson.

The load-deflection curves for the static analyses show that the 2D beam elements with and without shear deformation, the 3D solid linear elements with enhanced formulation, and the 3D solid quadratic elements estimate similar snap-through loads (Figure 5), $P_{crit} = 1.51$ N. However, the 3D solid linear elements with displacement and mixed (B-bar) formulations experience locking; enhanced strain formulations can be used instead to avoid this problem while keeping the elements linear, but the use of higher order elements provides the same algorithmic improvement without the added computational burden (the enhanced elements require a static condensation step and inversion in every element) and without the increased storage requirements (additional local variables inside every element). Therefore, the 2D beam elements and 3D solid quadratic (27 nodes) elements will be the elements used in the transient analysis. Note that there is no warping observed in the

simulation that uses 3D solid quadratic elements. Therefore the use of beam elements is acceptable even though they cannot capture warping. Also note that no torsional deformation is observed when the discretization uses 3D solid quadratic elements.

We compared the values of the buckling load and the displacement after buckling obtained from the static analysis with the analytical approximation proposed in Bradford et al. [4]. They have shown that these values provide reasonable approximations for the symmetric buckling of fixed arches. From the comparison, we obtained a relative difference of 4.93% for the buckling load and a relative difference of 5.47% for the displacement after buckling.

Simpler models (link systems) were also used for qualitative comparisons [17] and an extensive investigation of the post snap-through behavior of the curved beam including identification of solutions with non-symmetric deformation response and transient responses under a variety of forcing patterns is presented in [18].

3.2. Hysteresis analysis

The hysteresis analysis illustrates the path-dependence behavior. The structure is loaded with a concentrated force (at the middle of the beam) increased up to a value slightly larger than the snap-through value and then it is unloaded with the same rate. The results under a load controlled scenario confirm that the structure experiences hysteresis causing the equilibrium path to be dependent on the history (Figure 6).

Figure 6: Load-displacement curve from hysteresis analysis.

3.3. Path following analysis using an arclength method

In order to obtain the whole equilibrium path, including the unstable configurations, an arclength algorithm based on Schweizerhof and Wriggers [19] is used. The method controls neither the load nor the displacement but traces the equilibrium configurations, both stable and unstable (Figure 7), with increments in the length of an arc on this curve. Even though unstable paths are hard to capture experimentally, having access to the whole solution space provides significant insight into the system behavior. If incremental loading is used, the system advances to point 1 then snaps through to point 2. If the load is increased, it will travel to point 3. When the unloading starts, the system will travel back from point 3 to point 2 then point 4 before snapping back to point 5 and then returning to the original (undeformed)

Figure 7: Load-displacement curve from static analysis with arclength method.

position.

Static analyses provide some information, such as the load level at which snap-through and snap-back occur for specific material properties and geometry. However, a dynamic analysis is required to capture the transient behavior after the snap-through occurs.

4. Transient Analysis

This section analyzes the performance of time integrators when applied to transient problems involving snap-through, i.e., the dynamic jump of the system from a “quasi static” configuration to oscillations about a remote equilibrium configuration.

The concentrated force applied at the midspan of the curved beam ramps-

up to some value above the snap-through load and then remains constant (Figure 8).

Figure 8: Load for transient analysis.

In the following simulations, the choice of finite elements used to discretize the system is based on the preliminary results of the previous section: 2D beam elements with and without shear deformation and 3D solid quadratic (27 nodes) elements ¹.

Due to the snap-through phenomenon, the structure undergoes changes in the stability behavior. The study of the accuracy and robustness of the numerical methods for transient simulations of structures that are likely to traverse unstable regimes raises the following issues: (1) the existence of a critical time step for the stability of the numerical integrator, (2) the introduction of unwanted artificial numerical damping, and (3) the lack of exact energy conservation and other numerical pathologies. These issues are discussed in detail in the rest of this section.

To quantify the robustness of the numerical method, one of the measures

¹All elements account for large deformation

used is the total energy. The values of the algorithmic parameters utilized for Newmark and the energy momentum schemes ensure energy conservation for linear problems. We will use the deviation from this norm as a measure of the loss of numerical robustness and of the increase in the likelihood of the numerical nature of the instability (if instability is encountered).

4.1. Critical time step

We first present the numerical studies performed in order to identify the critical time steps for both the Newmark method and the conserving A algorithm for different finite element formulations (2D beams and 3D solids). The critical time steps are identified by systematic numerical experiments. We also presents simulation results obtained using the conserving B algorithm, where we will show that there is no critical time step (the simulation does not “blow up”) but accurate results can be obtained only for small time steps.

Simulations with 2D beam elements without shear may converge even for Δt larger than the critical time step. However, for each Δt , a different structural behavior is identified. This observation clearly suggests that even though the nonlinear solver converges, *this is not necessarily a solution of the physical system but rather a numerical artifact*. An example of such unstable solution ($\Delta t = 10^{-2}$ s) suggests that the structure *undergoes intermittent snap-through before it settles into periodic oscillations around an intermediate point that is neither the original configuration nor the snap-through configuration* (Figure 9(a)). The *total energy plot* for this simulation shows that the energy is not conserved; it increases significantly when the response settles into oscillations about this intermediate configuration (Figure 9(b)).

Figure 9: 2D beam without shear deformation; Newmark method; $\Delta t = 10^{-2}$ s (a) Displacement, (b) Total energy.

The simulations using the 2D beam elements without shear deformations are repeated with the conserving A algorithm (Figure 10(a)). The energy plot for an unstable time step also shows that the energy increases rapidly at a certain point in the simulation that corresponds to the time increment where the response becomes unphysical (Figure 10(b)). The total energy obtained in simulations that converge to nonphysical solutions is much larger when the

Figure 10: 2D beam without shear deformation; conserving A scheme; $\Delta t = 10^{-2}$ s (a) Displacement, (b) Total energy.

conserving A scheme is used than when using the Newmark method. This makes *the identification of the unphysical solutions much easier*.

The result of a stable simulation shows that the structure experiences periodic oscillations, which begin right after snap-through occurs (Figure 11(a)). Even though a stable numerical behavior is exhibited, we observe that the numerical analysis provides a damped response. This response is obtained

Figure 11: Displacement vs. time; Newmark method (a) 2D beam with no shear ($\Delta t = 5.10^{-5}$ s), (b) 3D solid quadratic ($\Delta t = 10^{-4}$ s).

with the Newmark method with a combination of parameters that should have precluded numerical dissipation, so clearly this damping is a numerical artifact, as will be discussed in more detail in the next subsection.

The transient simulation performed with 3D solid quadratic elements and a stable time step shows periodic oscillations in the post-snap response of the structure; the amplitude of oscillations stays constant in the region of constant load (Figure 11(b)), not showing the dissipation that appears when the 2D beam elements without shear deformation are used in the simulation.

The critical time step obtained from systematic numerical experimentations are summarized in Table 3. The preceding study shows that the critical time step depends on the type of elements used to discretize the structure. Studies performed using other loading patterns show that these bounds also depend on the load pattern used (results are not shown here).

Simulations performed using the conserving B scheme [20] are numerically stable at large time steps. However, this scheme introduces numerical damp-

Table 3: Bounds of the critical time step for different finite elements

Element type and formulation	Critical time step
2D beam element with shear deformation	$\Delta t_{cr} \approx 10^{-4}\text{s}$
2D beam element without shear deformation	$\Delta t_{cr} \approx 5.10^{-5}\text{s}$
3D solid quadratic (27 nodes) element	$\Delta t_{cr} \approx 10^{-4}\text{s}$

ing, which affects the accuracy of the solutions greatly when a large time step is used (Figure 12(a)). When small time steps are used (Figure 12(b)), the solution matches the results obtained using the Newmark method and the conserving A algorithm.

Figure 12: Displacement vs. time; 3D solid quadratic; conserving B method (a) $\Delta t=10^{-3}\text{s}$, (b) $\Delta t=10^{-4}\text{s}$.

4.2. Unwanted artificial numerical damping

The discussion presented in the previous section shows that the 2D beam element without shear deformation introduces artificial numerical dissipation in the transient analysis of structures undergoing changes in stability for time steps smaller than the $\Delta t_{critical}$. This behavior is observed even though the algorithm used is supposed to conserve energy, which suggests that there exists

some correlation between the kinematic assumptions built into a “structural” finite element and the numerical stability and conservation properties in the transient simulation when the problem is highly nonlinear. (Note that the energy conservation property is proven to hold for linear problems, but does not hold for nonlinear problems).

Figure 13: Decrease in amplitude in the simulations using 2D beam elements without shear deformation.

The transient analyses using 2D beam elements without shear deformation exhibit more numerical damping for smaller time steps for both the Newmark method and the conserving A algorithm (Figure 13). The responses obtained with the Newmark method show more dissipation than those obtained with the conserving A scheme. Therefore, in addition to the time step size, the amount of dissipation also depends on the time integrator used in the analysis. The relationship between dissipation and $\log \Delta t$ is ap-

proximately linear (Figure 13 with log scale representation on the horizontal axis). The dissipation might be due to the fact that the use of no-shear beam formulations for dynamic problems leads to a discretized system that is parabolic and consequently inadequate for wave propagation studies [21]. It is therefore expected that the Newmark method that is designed for hyperbolic and hyperbolic-parabolic problems may encounter difficulties in the parabolic case.

To isolate the possible cause of the artificial numerical damping in analyses with 2D beam elements with shear deformation, simulation of one other transient problem is performed; a curved cantilever beam, essentially representing half of the representative structure described in Figure 2 that has the same properties and is loaded at the free end with a concentrated force increased in the first 2 seconds and then decreased to zero rapidly (Figure 14). Notice that snap-through is not exhibited here, reducing the problem to non-linear deformation / vibration only. In this problem, the beam is discretized with 2D beam elements (with and without shear deformation).

Figure 14: Load for transient analysis of the curved cantilever beam.

When the beam is discretized with 2D beam elements without shear deformation, damping is also present in the case of the curved cantilever beam. However, the damping is only noticeable when the simulation time is long. Therefore, we can conclude that snap-through aggravates the amount of the damping.

Figure 15: Curved cantilever beam; Newmark; $\Delta t = 10^{-4}$ s; Various 2D beam elements (a) Displacement vs. time, (b) Zoom in.

Comparison of the responses for $\Delta t = 10^{-4}$ s obtained with 2D beam elements with and without shear deformation shows that the beam element with small rotation assumption and no shear is clearly more rigid than the beam element with large rotation and shear, both in the initial quasi-static loading phase and in the amplitudes of the oscillations in the transient regime (Figure 15). Since the beam is very thin and flexible, the inclusion of the shear effects should not bring significant changes to the results and we suspect that the cause of this more rigid behavior is the limiting assumption of small rotation.

Figure 16: Total energy vs. time; Newmark method (a) 2D beam with no shear ($\Delta t = 5.10^{-5} \text{ s}$), (b) 3D solid quadratic ($\Delta t = 10^{-4} \text{ s}$).

4.3. Algorithmic conservation of energy

In addition to the existence of a critical time step and the introduction of artificial numerical damping, the transient analysis of a curved beam experiencing snap-through also raises several issues concerning the conservation of energy of the system.

The analysis using 2D beam elements shows that the energy conservation is not satisfied throughout the simulation. The total energy plot for 2D beam elements without shear deformation shows that the total energy is significantly larger (Figure 16(a)) than the total energy in the analysis with 3D solid quadratic elements (Figure 16(b)). The total energy includes the kinetic energy, strain energy, and the work applied. Under the energy conservation measure, the conserving A algorithm performs better than the Newmark method. Simulations performed using the conserving B algorithm clearly do not satisfy the conservation of energy; artificial numerical damping is introduced into the system [20]: the larger the time step, the larger the

Figure 17: Total energy vs. time; 3D solid quadratic; conserving B method (a) $\Delta t=10^{-3}$ s, (b) $\Delta t=10^{-4}$ s.

decrease in energy (Figure 17).

Pathologies in the variation of energy also appear when 2D beam elements with shear deformation are used for the spatial discretization of the system. The results obtained from analyses performed with the Newmark method ($\Delta t = 10^{-4}$ s) for a system discretized with 2D beam elements with shear deformation show that the refinement of the mesh increases the energy of the system (Figure 18). Note that the energy does not change when a smaller time step is used, showing that the increase in energy depends only on the mesh refinement.

Further examination shows that as the mesh is refined, it is the kinetic energy that changes significantly in magnitude while the strain energy remains the same. This phenomenon is observed only when the 2D beam elements with shear deformation capabilities are used in the analysis. The analyses using 2D beam elements without shear deformation and 3D solid quadratic elements show no dependence of the kinetic energy on the level of

Figure 18: Arch fixed at both ends; 2D beam with shear; Newmark method; $\Delta t=10^{-4}$ s; Various meshes (a) Total energy vs. time, (b) Zoom in.

mesh refinement.

Several attempts were made to understand this behavior. A closed-form solution based on the assumptions of negligible displacement in the X direction and small rotation shows that the kinetic energy for one element with length h and two elements with length $\frac{h}{2}$ in one time step is the same, but once again, such results no longer hold for large deformation [16].

Simulations performed on the curved cantilever beam used in the previous subsection examined whether the increase in energy appears in other transient problems solved with 2D beam elements with shear deformation capabilities. Recall that this formulation also accounts for large deformation and large rotation. The geometry represents half of the representative structure described in Figure 2 and the loading is shown Figure 14. The kinetic energy plots show that there is an increase in kinetic energy when the mesh is refined (Figure 19) for this case as well. Therefore, we can conclude that the dependency of the kinetic energy on the mesh refinement is a feature of

Figure 19: Curved cantilever beam; Newmark; 2D with shear; $\Delta t=10^{-4}$ s; Various meshes (a) Kinetic energy vs. time, (b) Zoom in.

the 2D beam with shear elements and not necessarily induced by the type of the problem (e.g., snap-through).

5. Conclusions

This paper analyzes the performance of several finite element formulations (2D and 3D) and the stability of the time-stepping schemes in simulating a curved beam undergoing snap-through by identifying the important features that affect the numerical accuracy and robustness and the region where the schemes are stable for such simulations. We examine the interaction between the two most important components of the finite element analysis applied to structural dynamics problem: (1) the time-stepping schemes and (2) the finite element formulations used to spatially discretize the structure. The integrators studied are (1) Newmark method, (2) a conserving integrator (referred as conserving A) [10, 11, 12], and (3) a composite integrator (referred as conserving B) proposed by Bathe [13]. The finite element formu-

lations used are the 2D beam elements with and without shear deformation capabilities and the 3D solid elements.

The study shows that the Newmark method and the conserving A scheme have a restrictive bound on the size of the time step to ensure numerical stability. The critical time step depends on the finite element formulations used to discretize the structure and on the loading pattern. Simulations performed using the conserving B algorithm do not have this restriction. However the numerical damping introduced by this algorithm increases with the increase in the time step and greatly affects the accuracy of the solution when large time steps are used.

The study also shows that the spatial and temporal discretizations may interact and such interactions may induce unwanted numerical effects such as artificial damping, lack of energy conservation, and most importantly, misleading numerical results that seem to indicate a chaotic response when in fact the simulation simply converged to non-physical solutions. These issues are very severe when structural elements are used to discretize the beam while the use of 3D solid elements has less severe effects. Therefore, we recommend the use of 3D solid elements in solving snap-through problems.

Due to the nonlinearity of the problem, the conservation of energy is sometimes lost when *structural* elements (e.g., beams) introduce kinematic assumptions in the discretization of the structure. The total energy is approximately conserved when 3D solid quadratic elements are used with the Newmark method and conserving A algorithm. When the conserving B algorithm is used, the conservation of energy is not satisfied because numerical damping is introduced to the system. The inspection of energy plots for the

analyses performed with beam formulation with shear deformation also show another numerical artifact: the kinetic energy is mesh dependent (increases as the number of elements increase; reducing the time step does not eliminate this unwanted behavior).

In conclusion, we have shown that several of the currently available finite element formulations are not robust or accurate enough to simulate snap-through. In particular, assumptions built into the formulation of structural elements (e.g., beam elements) lead to unwanted numerical behavior that is often times amplified by the presence of the snap-through phenomenon in the system.

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