Design of Space-Time Codes with Optimal Coding Gain

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Abstract—Space time codes have been proposed in the literature as an efficient means for improving the data rates over fading channels with multiple transmit antennas. In particular, the rank and the determinant of code difference matrices have been shown to be important in the design of space time codes for fading channels. In the present work, we investigate the problem of maximizing the coding gain of space-time codes, given by the minimum of the determinants of all the code difference matrices. We rely on equality of the singular values of the code difference matrices as a necessary and sufficient condition for obtaining the optimal coding gain. Finally, we discuss the construction of trellis codes based on the criterion derived in [7] and present simulation results.

I. INTRODUCTION

Space-time codes have been proposed as an efficient means of providing diversity and coding gains for wireless channels. The space time coding technique is meant to provide the system designer with the flexibility of achieving high data rates or improving the error performance at lower data rates. A good introduction to this topic can be found in [2], [1]. The capacity analysis of multiple transmit and receive antenna systems, performed by Telatar [5] and Foschini and Gans [4], has played an important role in the development of the space-time coding concept. Space-time design criteria based on the rank and the determinant of code difference matrices have been given by Tarokh *et al.* [1]. The role of the Euclidean distance in determining the error performance was analyzed by Ionescu in [7], [8], where a criterion based on *equal singular values* of the difference code matrices was introduced. The product distance as a design parameter in fast fading channels (a variant of the criterion for rapid fading seen in [1]) was first introduced in [9]. It was also shown that the Hamming distance plays a bigger role than the Euclidean distance in the design of codes for fast fading channels. A relationship between the criteria relevant in the quasistatic and rapid fading scenarios is discussed in [8]. We discuss the instantaneous performance (per-

formance in a given realization of the channel) of the codes in addition to the average performance which is important for the quasi-static fading channel model assumed in this work. It is to be noted that a good instantaneous performance would necessarily lead to a good average performance but not vice-versa. We exploit the fact that the determinant criterion—linked to the coding gain of space time codes—is strengthened when all the singular values of code difference matrices are made equal. This will be referred to as the *equal singular values* (ESV) criterion in the sequel. In this paper, we further analyze the ESV criterion of [7] and establish the *robust* nature of the codes designed by this criterion. We distinguish between the space time block codes based on orthogonal designs [3] and codes constructed from the ESV criterion. We show that the conditions underlying the orthogonal design of [3] are sufficient conditions for the ESV criterion and, in general, constitute a stronger requirement. We design codes which offer good performance under different channel conditions and compare their performance with similar codes from the literature. The design and the construction of the codes follow a systematic procedure. The paper is organized as follows. The problem setting along with the channel model and notations are given in section II. The ESV criterion is revisited and its optimality in maximizing the diversity gain and the coding gain is discussed in section III. In section IV we show that the ESV criterion leads to robust code performance, i.e. good performance under different channel conditions. We introduce codes based on the ESV criterion in section V and then discuss the construction of an 8-state QPSK EDTCM code in section VI. Simulation results are presented in section VII and conclusions are drawn in section VIII.

II. PROBLEM SETTING

We consider a wireless communication system with L transmit antennas and M receive antennas. The fading

process on any channel is assumed to be frequency non-selective. Data is encoded and sent over the L transmit antennas simultaneously. Let c_t^i denote the symbol transmitted on antenna i at time instant t . Further, let h_t^j be the channel coefficient corresponding to the channel between the i th transmit antenna and the j th receive antenna. Finally, let n_t^j be the proper complex Gaussian noise with variance $1/2$ per dimension. It is assumed that the fading is quasi-static, i.e., the fading coefficients do not change significantly over a frame length. Then, the signal received at the j th receive antenna at time instant t , given by y_t^j , is

$$y_t^j = \sqrt{E_s} \sum_{i=1}^L h_t^j c_t^i + n_t^j \quad (1)$$

where E_s is the transmitted symbol energy. As pointed out in [1], for a system with M receive antennas, the total diversity achieved by the system is M times the diversity of the rest of the system. Hence, we focus our attention on $M = 1$, i.e. the case of one receive antenna only. We drop the super-script on the channel coefficient and the additive noise in the sequel. Denote by l the number of symbol epochs, spanned by a codeword (also called frame length). A codeword is the collection of all the symbols transmitted over the L antennas during the corresponding l consecutive symbol epochs. Hence, a codeword $\mathbf{c}$ can be written out as a matrix of L rows and l columns given by

$$\mathbf{D}_{\mathbf{c}} = \begin{bmatrix} c_1^1 & c_2^1 & \dots & c_l^1 \\ c_1^2 & c_2^2 & \dots & c_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^L & c_2^L & \dots & c_l^L \end{bmatrix} \quad (2)$$

The $1 \times L$ channel vector $\mathbf{h}$ is given by (assuming one receive antenna)

$$\mathbf{h} = [h_1 \ h_2 \ h_3 \ \dots \ h_L],$$

The $1 \times l$ received vector $\mathbf{y}$ is given by

$$\mathbf{y} = \sqrt{E_s} \mathbf{h} \mathbf{D}_{\mathbf{c}} + \mathbf{n} \quad (3)$$

where $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_l]$, $\mathbf{n} = [n_1 \ n_2 \ \dots \ n_l]$.

III. REVIEW OF DESIGN CRITERIA

We define the pair-wise error probability between any two codewords $\mathbf{c}$ and $\mathbf{e}$, $\Pr\{\mathbf{c} \rightarrow \mathbf{e}\}$, as the probability that the codeword $\mathbf{c}$ is wrongly decoded as codeword $\mathbf{e}$ and vice versa, at the receiver. From [1], an upper bound on the pair-wise error probability, conditioned on the channel realization, is given by

$$\Pr\{\mathbf{c} \rightarrow \mathbf{e} | \mathbf{h}\} \leq \exp(-d^2(\mathbf{c}, \mathbf{e}) E_s / 4N_0) \quad (4)$$

where

$$d^2(\mathbf{c}, \mathbf{e}) = \sum_{t=1}^l \left| \sum_{i=1}^M h_i (c_t^i - e_t^i) \right|^2. \quad (5)$$

This follows from an approximation to the Q function (Gaussian tail function). The channel coefficients appear in $d^2(\mathbf{c}, \mathbf{e})$. We define the squared Euclidean distance between $\mathbf{c}$ and $\mathbf{e}$, denoted by $d_E^2(\mathbf{c}, \mathbf{e})$, as $\sum_{t=1}^l \sum_{i=1}^M |c_t^i - e_t^i|^2$. It can be shown that the Euclidean distance so defined is the Frobenius norm of the matrix, $\mathbf{D}_{\mathbf{c}} - \mathbf{D}_{\mathbf{e}}$ [7]. In the sequel, the code difference matrix, $\mathbf{D}_{\mathbf{c}} - \mathbf{D}_{\mathbf{e}}$, will be denoted by $\mathbf{D}_{\mathbf{ce}}$. The average pair-wise error probability over a quasi-static fading channel, at high SNR, is bounded by [1]

$$\Pr\{\mathbf{c} \rightarrow \mathbf{e}\} \leq (\prod_{i=1}^r \lambda_i)^{-1} (E_s / 4N_0)^{-r} \quad (6)$$

where λ_i is the i th non-zero eigenvalue and r is the rank of the matrix $\mathbf{A} = \mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H$ (the superscript H stands for conjugated transposition). The sum of the eigenvalues of $\mathbf{A}$ is equal to the Euclidean distance as defined above.

Using (6), Tarokh *et al.* [1] arrived at the following space-time coding criteria.

1. Rank criterion: Maximize the rank, r , of the matrix $\mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H$, for all possible codewords $\mathbf{c}$ and $\mathbf{e}$. Note that $r \leq L$.
2. Determinant criterion: Maximize the minimum of the products of all the non-zero eigenvalues of $\mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H$ taken over all the distinct codewords $\mathbf{c}$ and $\mathbf{e}$.

The diversity gain of the code is determined by the rank criterion. The determinant criterion, which addresses the coding gain of the space-time code, is non-constructive in nature and difficult to implement. Space time codes which are optimal with respect to this criterion have not yet been designed.

We now observe that the determinant criterion is strengthened by the ESV criterion of [7] in the following sense: given a minimum Euclidean distance between two codewords, the coding gain of a space time code is maximized if and only if all the (non-zero) code difference matrices have equal singular values. Since the sum of the squared singular values is constrained by the Euclidean distance, rather than by the total energy, it follows that the coding gain is further maximized by increasing the minimum Euclidean distance of the code.

Indeed, from the arithmetic mean - geometric mean inequality, we have

$$\begin{aligned} \text{Det}(\mathbf{A}) &= \prod_{i=1}^r \lambda_i \\ &\leq \left(\frac{\sum_{i=1}^r \lambda_i}{r} \right)^r \\ &= \left(\frac{d_E^2(\mathbf{c}, \mathbf{e})}{r} \right)^r \end{aligned} \quad (7)$$

where the equality is achieved if and only if all the eigenvalues of $\mathbf{A}$ are equal. Recall that $d_E^2(\mathbf{c}, \mathbf{e})$ is the

Euclidean distance between $\mathbf{c}$ and $\mathbf{e}$. Hence, any code with codeword differences that have unequal distribution of the eigenvalues does not obtain the optimum coding gain with a given minimum Euclidean distance between any two codewords. Further, all the eigenvalues are non-zero, thus ensuring that the rank (equal to the number of non-zero eigenvalues) is maximized. Hence, the diversity gain and the coding gain of a space time code are maximized by the ESV criterion.

IV. ANALYSIS OF INSTANTANEOUS ERROR PERFORMANCE

The exponent in the pairwise error event probability, $d^2(\mathbf{c}, \mathbf{e})$, has the following equivalent representation [1]

$$d^2(\mathbf{c}, \mathbf{e}) = \mathbf{h} \mathbf{A} \mathbf{h}^H \quad (8)$$

where $\mathbf{A} = \mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H$. Also, $\mathbf{A}$ is Hermitian so that there exists a unitary matrix $\mathbf{V}$ and a real diagonal matrix $\mathbf{D}$ such that $\mathbf{V} \mathbf{A} \mathbf{V}^H = \mathbf{D}$. The diagonal elements of $\mathbf{D}$, given by λ_i , $i = 1, 2, \dots, M$, are the eigenvalues of $\mathbf{A}$. Define $[\beta_1, \dots, \beta_M] = \mathbf{h} \mathbf{V}^H$ so that $\beta_i = \mathbf{h} \mathbf{v}_i^H$, where the vector $\mathbf{v}_i$ is the i th row of $\mathbf{V}$. Then, we have

$$d^2(\mathbf{c}, \mathbf{e}) = \sum_{i=1}^L \lambda_i |\beta_i|^2 \quad (9)$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\beta_i|^2 &= |\mathbf{h} \mathbf{v}_i^H|^2 \\ &\leq \|\mathbf{h}\|^2 \|\mathbf{v}_i\|^2 \\ &= \|\mathbf{h}\|^2 \end{aligned} \quad (10)$$

Note that $\|\mathbf{v}_i\|^2 = 1$ has been used in the above. Hence,

$$\begin{aligned} d^2(\mathbf{c}, \mathbf{e}) &= \sum_{i=1}^L \lambda_i |\beta_i|^2 \\ &\leq \sum_{i=1}^L \lambda_i \|\mathbf{h}\|^2 \\ &= \|\mathbf{h}\|^2 (\sum_{i=1}^L \lambda_i) \end{aligned} \quad (11)$$

Let $\lambda_{sum} = \sum_{i=1}^L \lambda_i$. Then, the upper bound can be achieved if and only if

- (a) $\mathbf{v}_i = \frac{\mathbf{h}}{\|\mathbf{h}\|^2}$, ($\|\mathbf{h}\| \neq 0$) for some i , along with $\lambda_i = \lambda_{sum}$, and
- (b) $\lambda_j = 0, \forall j \neq i$. This is nothing but beam-forming at the transmitter. The implementation of a strategy like this requires complete channel state information at both the transmitter and the receiver, in addition to adaptive coding and modulation techniques at the transmitter. Also, this is the best performance that can be extracted from the channel, for a given λ_{sum} .

We shall now compare the performance of the ESV criterion with that of the beam-forming described above. With *equal singular values*, we have

$$\begin{aligned} d^2(\mathbf{c}, \mathbf{e}) &= \sum_{i=1}^L \lambda_i |\beta_i|^2 \\ &= \left(\frac{\lambda_{sum}}{L}\right) \sum_{i=1}^L |\beta_i|^2 \\ &= \left(\frac{\lambda_{sum}}{L}\right) \|\mathbf{h}\|^2 \end{aligned} \quad (12)$$

where the last equality follows from $\sum_{i=1}^L |\beta_i|^2 = \|\mathbf{h}\|^2$, which in turn follows directly from the definition of β_i . The implementation of the ESV criterion does not require the channel state information at the transmitter. This result states that the performance of a code which satisfies the ESV criterion is always within a factor of L of the best case performance of the channel, or in other words, no more than $10 \log(L)$ dB less, although the channel state information is not available at the transmitter. This establishes the robust nature of the design criterion to different channel statistics.

We shall now present the construction of a family of codes based on the ESV criterion, then discuss their performance. The codes have been called *Enhanced Dimensionality Trellis Coded Modulation* (EDTCM) codes to highlight that (a) dimensionality of the signaling constellation is increased by joint coding over time and space domains and (b) traditional TCM ideas are used in designing the joint coding-modulation scheme.

V. DESIGN OF EDTCM CODES

In a multiple transmit antenna system, the transmitter has additional degrees of freedom equal to the number of transmit antennas. We explore the consequences of utilizing the dimensions across space and time in a joint fashion. We make use of the ESV criterion, and its connection with the coding gain, in designing an EDTCM code across the space and time domains.

The ESV criterion [7], [8] can be summarized as follows. *Codes designed over l time epochs and L transmit antennas, must satisfy the property that the code difference matrix must have equal singular values for every pair of codewords in the code set.* Mathematically, $\forall \mathbf{c}, \mathbf{e} \in \mathcal{C}$, all the eigenvalues of $\mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H$ are equal. Equivalently, $\mathbf{D}_{\mathbf{ce}} \mathbf{D}_{\mathbf{ce}}^H = \lambda_{ce} \mathbf{I}$, where λ_{ce} is the multiple eigenvalue, specific to $\mathbf{c}$ and $\mathbf{e}$. In practice, it may sometimes be acceptable to insure that the ESV criterion is obeyed for the error event paths that dominate the performance.

We comply with this requirement via the following sufficient conditions. Let $\mathbf{X}$ and $\mathbf{Y}$ be two matrices such that (a) their dimensions are the same and (b) the square matrices $\mathbf{X} \mathbf{X}^H$, $\mathbf{Y} \mathbf{Y}^H$ and $\mathbf{X} \mathbf{Y}^H + \mathbf{Y} \mathbf{X}^H$ have equal eigenvalues. Note that if a square matrix has equal and nonzero eigenvalues then it is diagonal. Returning to $\mathbf{X}$ and $\mathbf{Y}$,

1. the matrix $\mathbf{F} = \mathbf{X} - \mathbf{Y}$ is such that $\mathbf{F} \mathbf{F}^H$ has equal eigenvalues and
2. the matrix $\mathbf{G}$ formed by the concatenation (irrespective of order, e.g. $\mathbf{G} = [\mathbf{X} \mathbf{Y}]$) of the two matrices $\mathbf{X}$ and $\mathbf{Y}$ is such that $\mathbf{G} \mathbf{G}^H$ has equal eigenvalues.

The two sufficient conditions listed above reduce the complexity of space-time codeword design. We simply need to find matrices of lower dimensions which, when concatenated together, satisfy the ESV criterion. It can be easily shown that, if $\mathbf{X}$ and $\mathbf{Y}$ are of the form

$$\begin{bmatrix} C_1 & -C_2^* \\ C_2 & C_1^* \end{bmatrix} \quad (13)$$

(where C_1 and C_2 are any complex numbers in the signal constellation and the superscript $*$ stands for complex conjugation) then they satisfy the sufficient conditions listed above. Hence, if we impose this structure on our codeword matrices, the ESV criterion will be satisfied. The structure in (13) was first exploited by Alamouti in [6]. Similar structures were later developed in [3], using the Radon-Hurwitz transform, under the name of space-time block codes.

We need to make a distinction between the ESV criterion and the requirements placed on space-time block codes. Any space-time block code matrix $\mathbf{D}_c$ verifies $\mathbf{D}_c \mathbf{D}_c^H = \mu \mathbf{I}$, whereas the ESV criterion demands that (all) the differences of code matrices, rather than the code matrices themselves, have an orthogonal structure. In general space-time block codes verify the ESV criterion but not vice versa, which makes the definition of space-time block codes stronger than the requirements of the ESV criterion.

This structure helps to decompose our original goal of maximizing the multiple singular value of the codeword difference matrix into independent tasks by letting us

1. Make the singular values of a codeword difference matrix equal for any given pair of codewords, and
2. Maximize the multiple singular value.

Note that the sum of the eigenvalues of $\mathbf{D}_{ce} \mathbf{D}_{ce}^H$ is the squared Euclidean distance between the codewords $\mathbf{c}$ and $\mathbf{e}$. Under the condition that the singular values of the code difference matrix are all equal, maximizing the multiple singular value is equivalent to maximizing the Euclidean distance between the two codewords.

VI. CONSTRUCTION OF EDTCM CODE

We present the construction of an EDTCM code for two transmit antennas—each using QPSK modulation—via a joint coding and modulation approach over the two antennas. This is an extension of the Ungerboeck TCM scheme. We start with the Cartesian product of the two QPSK constellations, then associate with each 2-tuple of the Cartesian product a Hurwitz-Radon matrix of the form (13). This results in a 16 point super-constellation in 4 complex dimensions, where each constellation point is a matrix. The only disadvantage is

that, due to the limitation on the number of Hurwitz-Radon matrices of size 2×2 shown in (13)—only 16 exist—the rate of the EDTCM code is less than 1; i.e., for QPSK and with $L = 2$ transmit antennas, the EDTCM code constructed below sends less than 2 bits/sec/Hz.

We carry out partitioning of this super-constellation of 16 points into subsets with fewer points. It is ensured that the minimum Euclidean distance between any two points in a set formed after partitioning a parent set is greater than that of the parent set. Finally, the EDTCM code is constructed based on the partitioned subsets, in accordance with the following traditional TCM design rules.

1. The distance between any two branches leaving a particular state should be maximum,
2. The distance between any two branches culminating in a particular state should be maximum.
3. All the points in the constellation should occur with equal probability.

An 8 state QPSK EDTCM code was constructed, as an example. The rate of the code is 1.5 bits/sec/Hz. Note that the squared singular values (equal, by construction) of all of the 2×2 Hurwitz-Radon sub-blocks (see (13)) of $\mathbf{D}_{ce}$ add together—for this code, along any error event path—to yield the eigenvalues of $\mathbf{D}_{ce} \mathbf{D}_{ce}^H$. The same holds for the squared Euclidean distances between the sub-blocks, since the squared Euclidean distance is proportional to the eigenvalue (in this equal-eigenvalue case). It is important to note that in the case of the 1.5 bits/sec/Hz EDTCM code, all product matrices $\mathbf{D}_{ce} \mathbf{D}_{ce}^H$ have equal eigenvalues.

VII. SIMULATION RESULTS

In Figures 1 to 3 we provide simulation results with 2 transmit antennas and 1 receive antenna. As already mentioned, the rate of the EDTCM-1.5 code is 1.5 bits/sec/Hz while that of the ATTC code and the Alamouti schemes is 2 bits/sec/Hz. This difference in rate is accounted for, in our simulation, by shrinking the QPSK constellation by a corresponding factor for the EDTCM-1.5 code. A frame length of 130 symbols was used in the simulations. In order to quantify the coding gain of the joint coded modulation scheme used, we have simulated the performance of the Alamouti scheme (13) separately. It has been pointed out in [6] that this scheme is equivalent to diversity of order 2. Simulations have been carried out in a quasistatic fading channel as well as in fading channels with different maximum Doppler shifts.

In the case of quasi-static fading, shown in Figure (1), the outage probability from [5] has been plotted at 1.5

Fig. 1. Performance of 8 state QPSK EDTCM and ATT codes in quasi-static fading. EDTCM-1.5 transmits 1.5 bits/sec/Hz; the outage probability is given for reference.

Fig. 3. Performance of the codes in a fading channel with Doppler 176Hz

Fig. 2. Performance of the codes in a fading channel with Doppler of 88 Hz

bits/sec/Hz and 2 bits/sec/Hz for comparison. It can be seen that the EDTCM code is within 2dB of the outage probability. The ATT code performs better than the Alamouti scheme which has diversity 2 but no coding gain. The simulations at 88 Hz Doppler given in Figure 2 and 176 Hz Doppler given in Figure 3 also show that the EDTCM code preserves its diversity of two even in non-quasistatic fading.

VIII. CONCLUSION

We have shown that the ESV criterion introduced in [7] leads to a systematic design of space time codes having sub-unitary rate and maximum coding gain with a given minimum Euclidean distance for the code. We have briefly discussed the construction of EDTCM codes and presented simulation results in both quasi-static and fast, flat, Rayleigh fading channels.

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