FREQUENCY DOMAIN SYNTHESIS OF
MULTIVARIABLE LINEAR REGULATORS

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ABSTRACT

This paper considers a general class of multivariable linear regulators and determines conditions under which there exist proper controllers such that output regulation is achieved with internal stability.

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1. Introduction

The Regulator Problem with Internal Stability (RPIS) has been treated in considerable generality in a state space framework in [1]. Recent work by Bengtsson [2] and Francis [3] involves solution, in the frequency domain, of a restricted class of regulators in which the regulated outputs are also the measured outputs. This paper is concerned with a somewhat more general problem in which the regulated outputs are not necessarily the measured outputs and necessary and sufficient conditions are obtained which guarantee output regulation with internal stability.

The primary motivation of this paper is to obtain a frequency domain characterization of internally stable regulators in order to determine structural requirements that all such systems must satisfy. Since internal stability and some type of regulation is involved in the design of many control systems, it is of interest to understand when and how these basic objectives can be achieved.

In § 2 we formulate RPIS in the frequency domain and state our main results. In § 3 we show how our results reduce in the special case considered by Bengtsson [2] to his results and in § 4 we prove the existence of proper controllers which solve the problem. We defer the discussion of an algorithm for the computation of proper controllers to a future publication. Preliminary results on this problem and a nontrivial example can be found in [4].

Notation

$|G|$ is the determinant of the square matrix G . If $A(s)$ is a square polynomial matrix, $\sigma(A)$ is the collection of roots of the polynomial $|A|$. C^+ and C^- denote the closed right half and open left half of the complex plane respectively. If A is rectangular $|A|_i$ represents the monic greatest common divisor (g.c.d.) of all the i th order minors of A . $a|b$ denotes a divides b . I_p is the $p \times p$ identity matrix. The order of a rational matrix $P(s)$, denoted by $o(P)$ is such that

$$\lim_{s \rightarrow \infty} s^{-o(P)} P(s)$$

is real and nonzero. A polynomial is Hurwitz if its roots are in C^- .

$\{P\}_+$ represents that part of a partial fraction expansion of $P(s)$ containing poles in C^+ . $i \in \underline{r}$ means $i \in (1, 2, 3, \dots, r)$.

2. Problem Formulation and Solution

The model under consideration in this paper is described as follows:

$$\begin{aligned} y(s) &= -F(s)P(s) u(s) + D_1(s)w \\ z(s) &= P(s) u(s) + D_2(s)w \end{aligned} \quad (1)$$

where $y(s)$ is a vector corresponding to the measured outputs and $z(s)$ a vector corresponding to the regulated outputs. $F(s)$, $P(s)$, $D_1(s)$ and $D_2(s)$ are rational matrices in the indeterminate s of dimensions $p \times \ell$, $\ell \times n$, $p \times m$ and $\ell \times m$ respectively denoting transfer function matrices. $u(s)$ is a vector representing the control inputs and w is a constant vector. For the sake of simplicity the indeterminate s will be dropped as long as this causes no confusion.

This model is motivated by the model used in [5] in that if P is the plant, then the regulated variables involve plant outputs and the measured variables involve plant outputs measured by means of sensors with transfer function matrix F . Actually the model (1) is more general than this in that F does not necessarily represent the transfer function matrix of a physical system. On the other hand a more general model would involve replacement of FP by H and this will be the subject of further study.

With F , P , D_1 and D_2 given, our objective is to determine a suitable controller described by an $n \times p$ transfer function matrix

$$u = Cy \quad (2)$$

such that the resulting system behaves in some desirable manner. In particular, regulation of z requires that the rational matrix

$$[PC(I_p + FPC)^{-1} D_1 + D_2] \quad (3)$$

be analytic in $\text{Re } s \geq 0$. The requirement of internal stability means that the cascade connection of systems P , F and C having minimal state space realizations of dimensions n_p , n_F and n_C should result in $n_p + n_F + n_C$ stable roots of the characteristic polynomial of the system

$$\phi(s) = \psi_F(s)\psi_P(s)\psi_C(s) | I_p + F(s)P(s)C(s) | \quad (4)$$

where ψ_F , ψ_P and ψ_C are the characteristic polynomials of F , P and C respectively. We therefore state the problem of interest as follows.

Regulator Problem with Internal Stability (RPIS)

Given F , P , D_1 and D_2 . Under what conditions does there exist a proper C such that (3) is analytic for $\text{Re } s \geq 0$ and (4) is a Hurwitz polynomial.

The most effective way to proceed is to first drop the constraint that C be proper and then replace C by a more convenient function. In this, we follow Youla, et al [5] and say that the pair (F, P) is admissible if

$$(\psi_{FP})_+ = (\psi_F)_+(\psi_P)_+$$

where $(\psi)_+$ represents the unstable factor of the polynomial ψ . This condition prevents unstable cancellation in the matrix product FP and has been shown to be necessary and sufficient for the existence of a controller C (not necessarily proper) that makes (4) Hurwitz [5 - Lemma 2].

Let

$$FP = A^{-1}B = B_1A_1^{-1} \quad (5)$$

be minimal representations where A and B are relatively left prime (r.l.p) and A_1 and B_1 are relatively right prime (r.r.p.) [6]. Consider the polynomial matrices X and Y satisfying

$$AX + BY = I_p \quad (6)$$

and let K be any rational matrix satisfying

$$C(I_p + FPC)^{-1} = (Y + A_1K)A \quad (7)$$

This matrix K was first introduced in [5] and derives its significance from the following

Lemma [5 - Lemma 3]

Let (F,P) be admissible. There exists a controller C such that the closed loop system is internally stable if and only if there exists a K analytic in $\text{Re } s \geq 0$ and satisfying (7) and

$$|X - B_1 K| \neq 0 \quad (8)$$

The stabilizing controller associated with a particular K is given by

$$C = (Y + A_1 K)(X - B_1 K)^{-1} \quad (9)$$

The apparent simplicity of this result belies its significance in dealing with internal stability while attempting to satisfy simultaneously other objectives in control system synthesis. This is a fundamental characterization of all stabilizing controllers and appears to be one of the most useful results obtained so far in frequency domain characterizations of linear systems. Its significance in the Wiener-Hopf problem was demonstrated in [5] and here we demonstrate its significance in RPIS.

In terms of K it is possible to rewrite (3) as follows

$$\begin{aligned} PC(I_p + FPC)^{-1} D_1 + D_2 &= P(Y + A_1 K)AD_1 + D_2 \\ &= PA_1 KAD_1 + (PYAD_1 + D_2) \end{aligned}$$

and we will concentrate on the following problem which we continue to refer to as RPIS.

(RPIS)

Assume (F,P) is admissible. Under what conditions does there exist a rational function K , satisfying

$$\{PA_1 KAD_1 + (PYAD_1 + D_2)\}_+ = 0 \quad (10)$$

and

$$\{K\}_+ = 0 \quad (11)$$

In this statement of RPIS, output regulation is equivalent to (10) and internal stability to (11).

Notice that the two problem statements do not appear to be equivalent. The existence of a proper C has been dropped from the second. Our main result, Theorem 4, will show that there exists a proper C solving RPIS if and only if there exists a K satisfying (10) and (11). It is for this reason that we have called both problems RPIS.

We proceed to solve RPIS in the form just stated. Our main result of this section is

Theorem 1

Assume (F,P) is admissible. The following are equivalent.

(i) RPIS is solvable, i.e. there exists K such that

$$\begin{aligned} \{PA_1 KAD_1 + (PYAD_1 + D_2)\}_+ &= 0 \\ \{K\}_+ &= 0 \end{aligned}$$

(ii) There exist polynomial matrices N, A_2, B_2, W_1 and a rational matrix K such that

$$Q_d^{-1} Q_{d1} = N \quad (12)$$

$$A_2 P_{d_2} N \Pi_1 = -B_2 + W_1 \tilde{Q}_{d_1} \quad (13)$$

$$A_2^{-1} B_2 = B_3 A_3^{-1} K \quad (14)$$

with $|A_2|$ Hurwitz and $\{K\}_+ = 0$

(iii) There exist polynomial matrices N, V and W such that

$$Q_{d_2}^{-1} Q_{d_1} = N$$

and

$$P_{d_2} N \Pi_1 = B_3 V + W \tilde{Q}_{d_1} \quad (15)$$

where $\{A D_1\}_+ = P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} P_{d_1} \quad (16)$

$$\{P Y A D_1 + D_2\}_+ = P_{d_2} Q_{d_2}^{-1} \quad (17)$$

$$P A_1 = B_3 A_3^{-1} \quad (18)$$

are minimal representations, and

$$\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m \quad (19)$$

The proof of Theorem 1 will be developed by means of the following lemmas.

Lemma 1

Assume (F, P) is admissible.

$$\{P A_1\}_+ = 0$$

Proof: Let $F = B_F A_F^{-1}$, $P = B_P A_P^{-1}$ and $A_F^{-1} B_P = \bar{B}_P \bar{A}_F^{-1}$ be minimal representations. Then $FP = B_F A_F^{-1} B_P A_P^{-1} = (B_F \bar{B}_P) (A_P \bar{A}_F)^{-1}$. Since $FP = B_1 A_1^{-1}$ is minimal $A_P \bar{A}_F = A_1 S$ for some polynomial matrix S . Therefore

$$|S| \psi_{FP} = \psi_P |\bar{A}_F|$$

and by admissibility

$$|S|_+ (\psi_F)_+ = |\bar{A}_F|_+$$

However $|\bar{A}_F|_+$ divides $|A_F|_+ = (\psi_F)_+$ so $|S|$ is Hurwitz and

$$P A_1 = B_P \bar{A}_F S^{-1}$$

Q.E.D.

Lemma 2 (Smith Form of a Matrix)

For any $m \times n$ polynomial matrix $A(s)$ there exist unimodular matrices $M(s)$ and $N(s)$ such that

$$S = MAN$$

and

$$S = \begin{cases} (Q, 0) & m < n \\ Q & m = n \\ \begin{pmatrix} Q \\ 0 \end{pmatrix} & m > n \end{cases}$$

$$Q = \text{diag}(e_1, \dots, e_r, 0 \dots 0)$$

where r is the rank of $A(s)$,

$$e_i \text{ divides } e_{i+1}, \quad i = 1, 2, \dots, r-1$$

and

$$e_i = |A|_i / |A|_{i-1}, \text{ with } |A|_0 = 1$$

for $i = 1, 2, \dots, r$.

Proof: See Rosenbrock [7].

Lemma 3

Let B_1 and A_1 be polynomial matrices of dimension $m \times n$ and $n \times n$ respectively. The following are equivalent

1. B_1 and A_1 are r.r.p.
2. $\begin{vmatrix} A_1 \\ B_1 \end{vmatrix}_n = 1$
3. The Smith form of $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ is $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$
4. There exist r.l.p. polynomial matrices X_1 and Y_1 such that

$$X_1 A_1 + Y_1 B_1 = I_n$$
5. Rank of $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = n$ for all s .

Proof: See Rosenbrock [7].

Lemma 4

If $B_1 A_1^{-1}$ is a minimal representation and A is a square matrix with
 $\sigma(A) \cap \sigma(A_1) = \emptyset$, then $A B_1 A_1^{-1}$ is a minimal representation.

Proof: This follows easily from part 5 of Lemma 3. If s_1 is a value of

s for which

$$\text{rank} \begin{pmatrix} A_1(s_1) \\ A(s_1)B_1(s_1) \end{pmatrix} < n$$

then $|A_1(s_1)| = |A(s_1)| = 0$ and

so $\sigma(A) \cap \sigma(A_1) \neq \emptyset$

Q.E.D.

Lemma 5

Let B and A be r.r.p. polynomial matrices with A invertible.

Polynomial matrices (V,X,Y) satisfy the equation

$$V = XB + YA$$

if and only if there exists a polynomial matrix W such that

$$\begin{aligned} X &= VX_1 + W\tilde{A} \\ Y &= VY_1 - W\tilde{B} \end{aligned}$$

where

$$X_1B + Y_1A = I$$

and

$$BA^{-1} = \tilde{A}^{-1}\tilde{B} \text{ is minimal.}$$

Proof: Sufficiency is obvious. To prove necessity, write

$$V = [X \ Y] \begin{bmatrix} B \\ A \end{bmatrix}$$

Since $[X_1 \ Y_1]$ is a left inverse for $\begin{bmatrix} B \\ A \end{bmatrix}$ and $[\tilde{A} - \tilde{B}]$ is a minimal basis [8] for its left kernel, it follows that

$$[X \ Y] = V[X_1 \ Y_1] + W[\tilde{A} - \tilde{B}]$$

for some polynomial matrix W.

Q.E.D.

In the following we will be concerned with the solvability of the equation $BV + WD = P$, for V and W, given B,D, and P. This problem has previously been studied by Roth [9], but because of its close relation to RPIS, we study it from a different standpoint and bring to light another set of conditions.

Assume that B is $p \times n$, D and P are $p \times p$, $\text{rank } B = r$ and $\text{rank } D = p$.

According to Lemma 2, unimodular matrices M_b, N_b, M_d and N_d exist such that

$$M_b B N_b = B_d = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} \quad (20)$$

where $B_0 = \text{diag}(b_1, b_2, \dots, b_r)$

$$M_d D N_d = \text{diag}(d_1, d_2, \dots, d_p) \quad (21)$$

where $d_1 d_2 \dots d_p \neq 0$.

Let

$$P' = M_b P N_d \quad (22)$$

Pre-multiplying by M_b and post-multiplying by N_d upon both sides of $BV + WD = P$, we obtain

$$B_d V' + W' D_d = P' \quad (23)$$

where $V' = N_b^{-1} V N_d$ and $W' = M_b W M_d^{-1}$.

Now we are ready to present the following.

Lemma 6

Assume B, D and P are polynomial matrices of dimension $p \times n$, $p \times p$ and $p \times p$ respectively with rank $B = r$ and rank $D = p$. The equation

$$BV + WD = P \quad (24)$$

is solvable for polynomial matrices V and W if and only if

1) $g_{ij} | p'_{ij}$ for all $i \in \underline{r}$ and $j \in \underline{p}$

and 2) $d_j | p'_{ij}$ if $p > r$, for $i = r+1, \dots, p$ and $j \in \underline{p}$

where p'_{ij} is the (i, j) element of P' which is defined in (22) and g_{ij} is the monic g.c.d. of b_i and d_j , which are defined in (20) and (21) respectively.

Proof:

(only if) Assume (24) is solvable, so is (23).

Thus $b_i v'_{ij} + d_j w'_{ij} = p'_{ij}$ holds for some v'_{ij} and w'_{ij} for $i \in \underline{r}$

and $j \in \underline{p}$. Since $b_i = g_{ij} \bar{b}_{ij}$ and $d_j = g_{ij} \bar{d}_{ij}$ for some polynomial $\bar{b}_{ij}$ and $\bar{d}_{ij}$, it follows that

$$g_{ij} \bar{b}_{ij} v_{ij}^! + g_{ij} \bar{d}_{ij} w_{ij}^! = p_{ij}^!.$$

By assumption $d_j \neq 0$ for all j , so $g_{ij} \neq 0$ for all $i \in \underline{r}$, $j \in \underline{p}$. Hence the above equation implies that $g_{ij} | p_{ij}^!$ for all $i \in \underline{r}$, $j \in \underline{p}$.

Furthermore, solvability of (23) also implies that

$$d_j w_{ij}^! = p_{ij}^! \quad \text{for } i = r+1, \dots, p \text{ and } j \in \underline{p}.$$

therefore $d_j | p_{ij}^!$ for $i = r+1, \dots, p$, $j \in \underline{p}$ as asserted.

(if)

First let $b_i = g_{ij} \bar{b}_{ij}$ and $d_j = g_{ij} \bar{d}_{ij}$ for some $i \in \underline{r}$ and $j \in \underline{p}$, where $\bar{b}_{ij}$ and $\bar{d}_{ij}$ are relatively prime.

Since $g_{ij} \neq 0$ and $g_{ij} | p_{ij}^!$ is hypothesized, $p_{ij}^! / g_{ij}$ is a polynomial. Further, the relative primeness of $\bar{b}_{ij}$ and $\bar{d}_{ij}$ guarantees the existence of two polynomials v_{ij} and w_{ij} such that

$$\bar{b}_{ij} v_{ij} + \bar{d}_{ij} w_{ij} = 1$$

$$\text{or equivalently } g_{ij} \bar{b}_{ij} v_{ij} \frac{p_{ij}^!}{g_{ij}} + g_{ij} \bar{d}_{ij} w_{ij} \frac{p_{ij}^!}{g_{ij}} = p_{ij}^!.$$

Hence

$$b_i v_{ij}^! + d_j w_{ij}^! = p_{ij}^! \tag{25}$$

for some polynomials $v_{ij}^!$ and $w_{ij}^!$. Note that the above arguments hold for all $i \in \underline{r}$ and $j \in \underline{p}$.

Additionally, by hypothesis polynomials $w_{ij}^!$ exist such that

$$d_j w_{ij}^! = p_{ij}^! \tag{26}$$

for $i = r+1, \dots, p$ and $j \in \underline{p}$.

In light of (25) and (26) we conclude that $B_d V^! + W^! D_d = P^!$, or equivalently $BV + WD = P$ is solvable.

Q.E.D.

Consider polynomial matrices B_3 and $\tilde{Q}_{d_1}$. B_3 is $p \times n$ and of rank r and $\tilde{Q}_{d_1}$ is $p \times p$ and of rank p . From Lemma 2 unimodular matrices M_1 , N_1 , M_2 and N_2 exist such that

$$M_1 B_3 N_1 = \begin{bmatrix} b_1 & & & & 0 \\ & \ddots & & & \\ & & b_r & & \\ & & & 0 & \\ 0 & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \quad (27)$$

$$M_2 \tilde{Q}_{d_1} N_2 = \text{diag}(d_1, \dots, d_p) = D_d \quad (28)$$

Let $b_i = b_i^+ \cdot b_i^-$ where $\sigma(b_i^+) \subset \mathbb{C}^+$ and $\sigma(b_i^-) \subset \mathbb{C}^-$, for $i \in \underline{r}$,

$$B_0^+ = \begin{bmatrix} b_1^+ & & & & 0 \\ & \ddots & & & \\ & & b_r^+ & & \\ & & & d & \\ 0 & & & & \ddots & \\ & & & & & d \end{bmatrix} \quad (29)$$

$$B_0^- = \begin{bmatrix} b_1^- & & & & 0 \\ & \ddots & & & \\ & & b_r^- & & \\ & & & 0 & \\ 0 & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

where $d = d_1 d_2 \dots d_p$, B_0^+ is $p \times p$ and B_0^- is $p \times n$.

Then with

$$B_3^+ = M_1^{-1} B_0^+,$$

$$B_3^- = B_0^- N_1^{-1},$$

(30)

we note that

$$B_3 = B_3^+ B_3^-.$$

With the aid of Lemma 6, we are now able to prove the following important result.

Lemma 7.

Consider B_3 , $\tilde{Q}_{d_1}$, and P , of dimension $p \times n$, $p \times p$ and $p \times p$ respectively
with $\sigma(\tilde{Q}_{d_1}) \subset \mathbb{C}^+$ and rank $(\tilde{Q}_{d_1}) = p$.

Let $B_3 = B_3^+ B_3^-$ according to (27) - (30).

Then $B_3^+ V + \tilde{W} \tilde{Q}_{d_1} = P$ is solvable for V and W if and only if
 $B_3 \bar{V} + \tilde{W} \tilde{Q}_{d_1} = P$ is solvable for $\bar{V}$ and $\bar{W}$.

Proof.

(only if) Assume $B_3^+ V + \tilde{W} \tilde{Q}_{d_1} = P$ is solvable. Pre-multiplying by M_1 and post-multiplying by N_2 upon both sides of $B_3^+ V + \tilde{W} \tilde{Q}_{d_1} = P$, it follows

$$M_1 B_3^+ V N_2 + M_1 W M_2^{-1} \tilde{Q}_{d_1} N_2 = M_1 P N_2 = P', \text{ or}$$

$$\text{diag}(b_1^+, \dots, b_r^+, d, \dots, d) V N_2 + M_1 W M_2^{-1} \cdot \text{diag}(d_1, \dots, d_p) = M_1 P N_2.$$

The solvability of the above equation implies, according to Lemma 6, that

$$1) \text{ g.c.d.}(b_i^+, d_j) \text{ divides } p_{ij}^! \text{ for } i \in \underline{r}, j \in \underline{p}. \quad (31a)$$

$$2) \text{ g.c.d.}(d, d_j) = d_j \text{ divides } p_{ij}^!, \text{ for } i = r+1, \dots, p, j \in \underline{p} \quad (31b)$$

Note that for b_i^- , $\sigma(b_i^-) \subset \mathbb{C}^-$, $i \in \underline{r}$, we have

$$\text{g.c.d.}(b_i^+ b_i^-, d_j) = \text{g.c.d.}(b_i^+, d_j) \text{ for } i \in \underline{r}, j \in \underline{p}. \quad (32)$$

where the fact $\sigma(d_j) \subset \mathbb{C}^+$ is utilized.

(31a) and (32) imply

$\text{g.c.d.}(b_i^+ b_i^-, d_j)$ divides $p_{ij}^!$ for $i \in \underline{r}, j \in \underline{p}$ which, together with (31), according to Lemma 6 ensures the solvability of

$$\text{diag}(b_1^+, \dots, b_r^+, d, \dots, d) \begin{bmatrix} b_1^- & & & 0 \\ & \ddots & & \\ & & b_r^- & \\ 0 & & & 0 \end{bmatrix} V' +$$

$$W' \cdot \text{diag}(d_1, \dots, d_p)$$

$$= P'$$

or equivalently that of

$$B_0^+ B_0^- V' + W' D_d = M_1 P N_2.$$

Hence $B_3^+ B_3^- \bar{V} + \bar{W} Q_{d_1} = P$ is solvable for $\bar{V}$ and $\bar{W}$.

(if) By hypothesis, $B_3^+ B_3^- \bar{V} + \bar{W} Q_{d_1} = P$ holds for some $\bar{V}$ and $\bar{W}$. Then obviously $B_3^- \bar{V}$ and $\bar{W}$ is a solution to $B_3^+ V + \tilde{W} Q_{d_1} = P$.

Q.E.D. ✓

Proof of Theorem 1

We will show that (i) implies (ii), (ii) implies (iii), and finally (iii) implies (i).

(i) $\rightarrow$ (ii)

Assume (10) and (11) hold, and let

$$P A_1 K = A_2^{-1} B_2 \quad (33)$$

Since $\{P A_1\}_+ = 0$ (from Lemma 1) and $\{K\}_+ = 0$, it follows that A_2 is Hurwitz. Hence define Φ as

$$\begin{aligned} \Phi &= \{P A_1 K A D_1 + (P Y A D_1 + D_2)\}_+ \\ &= \{P A_1 K \{A D_1\}_+ + \{P Y A D_1 + D_2\}_+\}_+ \\ &= \{P A_1 K P_{d_1} Q_{d_1}^{-1} + P_{d_2} Q_{d_2}^{-1}\}_+ \end{aligned}$$

where equations (16) and (17) are used

Using (33), we have

$$\begin{aligned}\Phi &= \{A_2^{-1}B_2P_{d_1}Q_{d_1}^{-1} + P_{d_2}Q_{d_2}^{-1}\}_+ \\ &= \{A_2^{-1}(B_2P_{d_1}Q_{d_1}^{-1} + A_2P_{d_2}Q_{d_2}^{-1})\}_+.\end{aligned}$$

We claim that for $\Phi = 0$, it is necessary that

$$B_2P_{d_1}Q_{d_1}^{-1} + A_2P_{d_2}Q_{d_2}^{-1} = M \quad (34)$$

a $\ell \times m$ polynomial matrix. To see this, let $M = P_0Q_0^{-1}$ and $A_2^{-1}P_0 = P_{01}A_{21}^{-1}$ be minimal. Therefore $X_3P_0 + Y_3Q_0 = I_m$ and $X_4P_{01} + Y_4A_{21} = I_m$ for some polynomial matrices X_3, Y_3, X_4 and Y_4 according to Lemma 3. Note that $Y_4(X_3P_0 + Y_3Q_0)A_{21} = Y_4A_{21} = I_m - X_4P_{01}$ and so $(Y_4X_3A_2 + X_4)P_{01} + (Y_4Y_3)Q_0A_{21} = I_m$, or $P_{01}(Q_0A_{21})^{-1}$ is a minimal representation. Suppose Q_0 is not unimodular, then according to (34), $\sigma(Q_0) \subset \mathbb{C}^+$. Then since $A_2^{-1}P_0Q_0^{-1} = P_{01}(Q_0A_{21})^{-1}$ is minimal, $\Phi = \{P_{01}(Q_0A_{21})^{-1}\}_+ \neq 0$.

Hence (34) is true with M a polynomial matrix and thus

$$A_2P_{d_2}Q_{d_2}^{-1}Q_{d_1} = -B_2P_{d_1} + MQ_{d_1} \quad (35)$$

Recall that $\sigma(A_2) \subset \mathbb{C}^-$, $\sigma(Q_{d_2}) \subset \mathbb{C}^+$ and $P_{d_2}Q_{d_2}^{-1}$ is minimal. Lemma 4 tells us that $A_2P_{d_2}Q_{d_2}^{-1}$ is minimal. The fact that $A_2P_{d_2}Q_{d_2}^{-1}$ is minimal and $A_2P_{d_2}Q_{d_2}^{-1}Q_{d_1}$ is a polynomial matrix implies that $Q_{d_2}^{-1}Q_{d_1}$ is a polynomial matrix. Hence

$$Q_{d_2}^{-1}Q_{d_1} = N$$

for some polynomial matrix N .

With N thus defined, (35) becomes

$$A_2P_{d_2}N = -B_2P_{d_1} + MQ_{d_1}. \quad (36)$$

Since $P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1}$ is minimal, there exist polynomial matrices Π_1 and Π_2 such that

$$\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m$$

by virtue of Lemma 3. Note also that

$$(\tilde{Q}_{d_1} - \tilde{P}_{d_1}) \begin{pmatrix} P_{d_1} \\ Q_{d_1} \end{pmatrix} = 0$$

According to Lemma 5 there exists some polynomial matrix W such that

$$-B_2 = A_2 P_{d_2} N \cdot \Pi_1 + W \tilde{Q}_{d_1}$$

$$M = A_2 P_{d_2} N \cdot \Pi_2 - W \tilde{P}_{d_1}$$

Finally, note that (14) follows immediately from (33). According to Lemma 1 $\{PA_1\}_+ = 0$ and so $\sigma(A_3) \subset \mathbb{C}^-$.

(ii) $\rightarrow$ (iii)

First of all, consider unimodular matrices such that

$$M_1 B_3 N_1 = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } B_0 = \text{diag}(b_1, b_2, \dots, b_r),$$

according to Lemma 2. Let $b_i = b_i^+ b_i^-$, $i = 1, 2, \dots, r$, such that $\sigma(b_i^+) \subset \mathbb{C}^+$ and $\sigma(b_i^-) \subset \mathbb{C}^-$.

Then if we define

$$B_0^+ = \text{diag}(b_1^+, b_2^+, \dots, b_r^+, d, \dots, d)$$

$$\text{and } B_0^- = \begin{bmatrix} B_{01} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } B_{01} = \text{diag}(b_1^-, b_2^-, \dots, b_r^-)$$

and $d = d_1 \cdot d_2 \cdot \dots \cdot d_p$, it follows that $M_1 B_3 N_1 = B_0^+ B_0^-$, where $\sigma(B_0^+) \subset \mathbb{C}^+$, B_0^+ is $\ell \times \ell$, and B_0^- is $\ell \times n$.

$$\text{Let } B_3^+ = M_1^{-1} B_0^+ \quad \text{and} \quad B_3^- = B_0^- N_1^{-1}. \quad \text{Then } B_3 = B_3^+ B_3^-.$$

This process is analogous to that described by equations (27) - (30).

(14) then becomes

$$A_2 B_3^+ B_3^- (A_3)^{-1} K = B_2 \quad (37)$$

Note that since A_2 and B_3^+ are nonsingular and of dimension $\ell \times \ell$, so $|A_2 B_3^+|_\ell = |A_2|_\ell \cdot |B_3^+|_\ell$, and thus there exist two square polynomial matrices $\tilde{G}^+$ and $\tilde{A}_2$ such that

$$A_2 B_3^+ = \tilde{G}^+ \tilde{A}_2 \quad (38)$$

where $|\tilde{G}^+|_\ell = |B_3^+|_\ell$ and $|\tilde{A}_2|_\ell = |A_2|_\ell$. This follows easily using Lemma 2.

Using (37), (38) becomes

$$\tilde{G}^+ \tilde{A}_2 B_3^- (A_3)^{-1} K = B_2$$

Let $\tilde{A}_2 B_3^- (A_3)^{-1} K = R^{-1} S$ be a minimal representation. Since $|A_3|$ is Hurwitz, and $\{K\}_+ = 0$, so $|R|$ is Hurwitz.

Now we have $\tilde{G}^+ R^{-1} S = B_2$.

The fact that $\sigma(\tilde{G}^+) \subset \mathbb{C}^+$ and $\sigma(R) \subset \mathbb{C}^-$ implies that

$$\left| \begin{array}{c} R \\ \tilde{G}^+ \end{array} \right|_\ell = 1. \text{ By virtue of Lemma 3, } \tilde{G}^+ \text{ and } R \text{ are r.r.p. and so } \tilde{G}^+ R^{-1}$$

is minimal. The minimality of $\tilde{G}^+ R^{-1}$ and $\tilde{G}^+ R^{-1} S = B_2$ imply, that $R^{-1} S = -V_1$ is a polynomial matrix. Therefore,

$$B_2 = -\tilde{G}^+ V_1 \quad (39)$$

for some polynomial matrix V_1 .

Equations (39) and (13) then combine to yield

$$A_2 (P_{d_2} N \Pi_1) + \tilde{G}^+ (-V_1) = W Q_{d_1} \quad (40)$$

Note that from (38), we have

$$A_2^{-1} \tilde{G}^+ = B_3^+ \tilde{A}_2^{-1} \quad (41)$$

Since $\sigma(B_3^+) \subset \mathbb{C}^+$ and $\sigma(\tilde{A}_2) \subset \mathbb{C}^-$, it follows that

$$\left| \begin{array}{c} B_3^+ \\ \tilde{A}_2 \end{array} \right|_\ell = 1 \text{ and so } B_3^+ \tilde{A}_2^{-1} \text{ is minimal by Lemma 3.}$$

Similarly, A_2 and $\tilde{G}^+$ are r.l.p., which ensures the existence of polynomial matrices T_1 and T_2 such that

$$A_2(T_1) + \tilde{G}^+(T_2) = I_\ell \quad (42)$$

A dual result of Lemma 5 can be obtained that if A and B are r.l.p. with A being invertible, then $V = AX + BY$ if and only if $X = X_1V + \tilde{B}W$ and $Y = Y_1V - \tilde{A}W$ for some polynomial W , where $AX_1 + BY_1 = I_p$ and $A^{-1}B = \tilde{B}\tilde{A}^{-1}$ is minimal. Application of this result and equations (41) and (42) upon (40) then yields

$$P_{d_2} N \Pi_1 = T_1(\tilde{W} \tilde{Q}_{d_1}) + B_3^+ Z$$

for some polynomial matrix Z .

Denoting $T_1 W$ by W' and Z by V' , it follows that

$$P_{d_2} N \Pi_1 = B_3^+ V' + W' \tilde{Q}_{d_1}$$

for some polynomial matrices V' and W' .

Finally, Lemma 7 is invoked to prove that

$$P_{d_2} N \Pi_1 = B_3 V + W \tilde{Q}_{d_1}$$

holds for some polynomial matrices V and W .

(iii) $\rightarrow$ (i)

Assume (12) and (15) hold, i.e.,

$$Q_{d_2}^{-1} Q_{d_1} = N$$

$$P_{d_2} N \Pi_1 = B_3 V + W \tilde{Q}_{d_1}$$

for some polynomial matrices N , V , and W .

Premultiplying (15) by some arbitrary Hurwitz polynomial matrix A_2 , we have

$$A_2 P_{d_2} N \Pi_1 = A_2 B_3 V + (A_2 W) \tilde{Q}_{d_1} \quad (43)$$

Let

$$B_2 = -A_2 B_3 V \quad (44)$$

$$K = -A_3(V + N_B L) \quad (45)$$

where N_B is a basis for the kernel of B_3 and L is any arbitrary rational matrix such that $\{L\}_+ = 0$.

Equations (43), (44) and (45) combine to yield

$$A_2 P_{d_2} N \Pi_1 = -B_2 + (A_2 W) \tilde{Q}_{d_1} \quad (46)$$

$$A_2^{-1} B_2 = B_3 A_3^{-1} K \quad (47)$$

Note here that $\{K\}_+ = 0$ from (45).

By post-multiplying (46) by P_{d_1} and using

$$P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1} \quad \text{or} \quad \tilde{Q}_{d_1} P_{d_1} = \tilde{P}_{d_1} Q_{d_1}$$

and

$$\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m$$

we have

$$A_2 P_{d_2} N (I_m - \Pi_2 Q_{d_1}) = -B_2 P_{d_1} + (A_2 W) \tilde{P}_{d_1} Q_{d_1}, \text{ or}$$

$$A_2 P_{d_2} N = -B_2 P_{d_1} + (A_2 P_{d_2} N \Pi_2 + A_2 W \tilde{P}_{d_1}) Q_{d_1}.$$

Hence

$$A_2 P_{d_2} Q_{d_2}^{-1} Q_{d_1} = -B_2 P_{d_1} + (A_2 P_{d_2} N \Pi_2 + A_2 W \tilde{P}_{d_1}) Q_{d_1}, \text{ or}$$

$$\begin{aligned} P_{d_2} Q_{d_2}^{-1} &= -A_2^{-1} B_2 P_{d_1} Q_{d_1}^{-1} + A_2^{-1} (A_2 P_{d_2} N \Pi_2 + A_2 W \tilde{P}_{d_1}) \\ &= -A_2^{-1} B_2 P_{d_1} Q_{d_1}^{-1} + (P_{d_2} N \Pi_2 + W \tilde{P}_{d_1}). \end{aligned}$$

Using (47), the above leads to

$$B_3 A_3^{-1} K P_{d_1} Q_{d_1}^{-1} + P_{d_2} Q_{d_2}^{-1} = (P_{d_2} N \Pi_2 + W \tilde{P}_{d_1}), \text{ or}$$

$$P A_1 K \{AD_1\}_+ + \{PYAD_1 + D_2\}_+ = (P_{d_2} N \Pi_2 + W \tilde{P}_{d_1}).$$

Therefore we obtain

$$\{P A_1 K \{AD_1\}_+ + \{PYAD_1 + D_2\}_+\}_+ = \{P_{d_1} N \Pi_2 + W \tilde{P}_{d_1}\}_+ = 0$$

or

$$\{PA_1 KAD_1 + (PYAD_1 + D_2)\}_+ = 0$$

where the fact that $\{PA_1\}_+ = \{K\}_+ = 0$ is utilized.

Q.E.D.

In the next section, we examine a simplified version of RPIS which has been treated by G. Bengtsson [2].

3. Special Case: $D_1 = -D_2$, $F = I_p$

In this section, we confine ourselves to a simplified version of RPIS, with $D_1 = -D_2$ and $F = I_p$. Let $D_2 = D$ and note that the model now becomes

$$-y = z = Pu + Dw \quad (48)$$

with controller

$$u = Cy \quad (49)$$

What we want to discuss is then the following restricted RPIS, or RRPIS.
(RRPIS) Under what conditions does there exist a proper controller C

such that

$$\{(I_p + PC)^{-1}D\}_+ = 0$$

and

$$\psi_P \psi_C |I_p + PC| \text{ is Hurwitz.}$$

Again we replace C by K and thus solve a simpler problem using the conditions of Theorem 1. First notice that under the assumptions

$D_1 = -D_2$, $F = I_p$, we have

$$P = A^{-1}B = B_1 A_1^{-1}$$

and

$$AX + BY = I_p$$

Furthermore, let

$$A\{D_1\}_+ = P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1} \quad (50)$$

where $P_{d_1} Q_{d_1}^{-1}$ and $\tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1}$ are minimal.

From $AX + BY = I_p$, we easily obtain

$$\begin{aligned} I_p &= A^{-1}(AX + BY)A = XA + A^{-1}BYA \\ &= XA + PYA \end{aligned} \quad (51)$$

Hence PYA is a polynomial matrix. Now note that

$$\begin{aligned} \{(I_p + PC)^{-1}D\}_+ &= \{-PC(I_p + PC)^{-1}D + D\}_+ \\ &= \{-PA_1KAD + (-PYAD + D)\}_+ \\ &= \{-PA_1KA\{D\}_+ + (XA)\{D\}_+\}_+ \end{aligned}$$

where use has been made of Lemma 1 and (51).

Let

$$XA\{D\}_+ = P_{d_2} Q_{d_2}^{-1}$$

be minimal. Observe that

$$P_{d_2} Q_{d_2}^{-1} = XA\{D\}_+ = -XA\{D_1\}_+ = -XP_{d_1} Q_{d_1}^{-1}$$

where (50) is used. Since $P_{d_2} Q_{d_2}^{-1}$ is minimal, it follows that

$$Q_{d_2} N = Q_{d_1} \quad (52)$$

for some polynomial matrix N .

Such being the case, we note that $P_{d_2} Q_{d_2}^{-1} = -XP_{d_1} Q_{d_1}^{-1}$ implies that

$$P_{d_2} N = -XP_{d_1} \quad (53)$$

Now we are ready to present the solution to RRPIS as a Corollary to Theorem 1.

Corollary 1. RRPIS is solvable if and only if there exist polynomial matrices

V and W such that

$$-XP_{d_1} \Pi_1 = B_1 V + W \tilde{Q}_{d_1} \quad (54)$$

where Π_1 is such that

$$\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m$$

Proof. It suffices to show that the two conditions in Theorem 1, i.e., (12) and (15) reduce to (54).

First note that (52) holds and so (12), i.e.,

$$Q_{d_2}^{-1} Q_{d_1} = N$$

follows automatically. Furthermore, using (53), we have $P_{d_2} N \Pi_1 = -X P_{d_1} \Pi_1$. Finally, since $P = B_1 A_1^{-1}$, it is obvious that $PA_1 = B_1 = B_3$. Equation (54) follows from (15) accordingly.

Q.E.D.

Condition (54) can be further reduced by virtue of the next Lemmas.

Lemma 8. For $P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1}$ and $\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m$, the following identity holds.

$$I_p = P_{d_1} \Pi_1 + ((I_p - P_{d_1} \Pi_1)T + P_{d_1} \Pi_2 U) \tilde{Q}_{d_1} \quad (55)$$

where U and T are such that

$$\tilde{P}_{d_1} U + \tilde{Q}_{d_1} T = I_p \quad (56)$$

Proof. First notice that the existence of U and T are guaranteed by the relatively left primeness of $\tilde{P}_{d_1}$ and $\tilde{Q}_{d_1}$.

From $\Pi_1 P_{d_1} + \Pi_2 Q_{d_1} = I_m$, we observe that

$$Q_{d_1} (\Pi_1 P_{d_1} + \Pi_2 Q_{d_1}) Q_{d_1}^{-1} = I_m, \text{ or}$$

$$Q_{d_1} \Pi_1 P_{d_1} Q_{d_1}^{-1} + Q_{d_1} \Pi_2 = I_m, \text{ or}$$

$$Q_{d_1} \Pi_1 \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1} = I_m - Q_{d_1} \Pi_2. \quad (57)$$

Since $\tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1}$ is minimal, and $Q_{d_1} \Pi_1 \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1}$ is a polynomial matrix, $Q_{d_1} \Pi_1 \tilde{Q}_{d_1}^{-1} = \bar{U}$ is a polynomial matrix.

Therefore, we have from the above equation and (57):

$$Q_{d_1} \Pi_1 = \bar{U} \tilde{Q}_{d_1} \quad (58)$$

$$\widetilde{UP}_{d_1} + Q_{d_1} \Pi_2 = I_m \quad (59)$$

Now note the following operations:

$$\begin{aligned} I_p - \widetilde{P}_{d_1} \bar{U} &= \widetilde{Q}_{d_1} T + \widetilde{P}_{d_1} U - \widetilde{P}_{d_1} \bar{U} (\widetilde{Q}_{d_1} T + \widetilde{P}_{d_1} U) \\ &\quad ((56) \text{ used}) \\ &= (\widetilde{Q}_{d_1} - \widetilde{P}_{d_1} \bar{U} \widetilde{Q}_{d_1}) T + \widetilde{P}_{d_1} (I_m - \bar{U} \widetilde{P}_{d_1}) U \\ &= (\widetilde{Q}_{d_1} - \widetilde{P}_{d_1} Q_{d_1} \Pi_1) T + \widetilde{P}_{d_1} Q_{d_1} \Pi_2 U \\ &\quad ((58) \text{ and } (59) \text{ used}) \\ &= (\widetilde{Q}_{d_1} - \widetilde{Q}_{d_1} P_{d_1} \Pi_1) T + \widetilde{Q}_{d_1} P_{d_1} \Pi_2 U \\ &\quad ((50) \text{ used}) \\ &= \widetilde{Q}_{d_1} ((I_p - P_{d_1} \Pi_1) T + P_{d_1} \Pi_2 U) \end{aligned}$$

Finally, note that

$$\begin{aligned} ((I_p - P_{d_1} \Pi_1) T + P_{d_1} \Pi_2 U) \widetilde{Q}_{d_1} &= \widetilde{Q}_{d_1}^{-1} (I_p - \widetilde{P}_{d_1} \bar{U}) \widetilde{Q}_{d_1} \\ &= I_p - \widetilde{Q}_{d_1}^{-1} \widetilde{P}_{d_1} \bar{U} \widetilde{Q}_{d_1} \\ &= I_p - P_{d_1} Q_{d_1}^{-1} Q_{d_1} \Pi_1 \\ &\quad ((58) \text{ used}) \\ &= I_p - P_{d_1} \Pi_1 . \end{aligned}$$

Q.E.D.

As a dual to this Lemma, we present the next

Lemma 9. For $A^{-1}B = B_1 A_1^{-1}$ and $AX + BY = I_p$ the following holds

$$I_n = YB + A_1(X_1(I_n - YB) + Y_1XB) \quad (60)$$

where X_1 and Y_1 are such that

$$Y_1 B_1 + X_1 A_1 = I_n .$$

Proof. Along the same lines as in Lemma 8, we offer a brief proof. First,

note that $XA + A^{-1}BYA = I_p$, or

$$XA + B_1 A_1^{-1} Y A = I_p .$$

Hence we have

$$YA = A_1 \bar{Y}_1 \quad (61)$$

and

$$XA + B_1 \bar{Y}_1 = I_p \quad (62)$$

for some polynomial matrix $\bar{Y}_1$. Observe that

$$\begin{aligned} I_n - \bar{Y}_1 B_1 &= (Y_1 B_1 + X_1 A_1) - (Y_1 B_1 + X_1 A_1) \bar{Y}_1 B_1 \\ &= Y_1 (I_p - B_1 \bar{Y}_1) B_1 + X_1 (A_1 - A_1 \bar{Y}_1 B_1) \\ &= Y_1 X A B_1 + X_1 (I_n - Y B) A_1 \\ &= (Y_1 X B + X_1 (I_n - Y B)) A_1. \end{aligned}$$

$$\text{Hence } I_n - Y B = A_1 (I_n - \bar{Y}_1 B_1) A_1^{-1} = A_1 (X_1 (I_n - Y B) + Y_1 X B) .$$

Q.E.D.

Now it is easy to prove the following

Lemma 10. For A, B, A_1, B_1, X and Y as defined in (5) and (6), the following holds.

$$XB = B_1 (X_1 (I_n - Y B) + Y_1 X B) \quad (63)$$

Proof. Recall that $AX + BY = I_p$.

$$\text{Hence } A^{-1}(AX + BY)B = A^{-1} \cdot I_p \cdot B = A^{-1}B, \text{ or}$$

$$XB + A^{-1}BYB = A^{-1}B .$$

$$\text{Accordingly, } XB = A^{-1}B(I_n - YB)$$

$$= A^{-1}B \cdot A_1 (X_1 (I_n - YB) + Y_1 X B)$$

where Lemma 9 is utilized.

So,

$$XB = B_1 A_1^{-1} A_1 (X_1 (I_n - YB) + Y_1 X B)$$

$$= B_1 (X_1 (I_n - YB) + Y_1 X B)$$

Q.E.D.

We are now ready to conclude with the next

Theorem 2. RRPIS is solvable if and only if there exist polynomial matrices
 $\bar{V}$ and $\bar{W}$ such that

$$I_p = B\bar{V} + \bar{W}\tilde{Q}_{d_1}. \quad (64)$$

Proof.

(only if) Assume RRPIS is solvable. According to Corollary 1, polynomial matrices V and W exist such that

$$-XP_{d_1}\Pi_1 = B_1V + W\tilde{Q}_{d_1}.$$

Premultiplying by A and noting that $AX = I_p - BY$, and $AB_1 = BA_1$, we obtain

$$P_{d_1}\Pi_1 = B(YP_{d_1}\Pi_1 - A_1V) + (-AW)\tilde{Q}_{d_1}$$

Using Lemma 8, i.e.,

$$I_p - P_{d_1}\Pi_1 = ((I_p - P_{d_1}\Pi_1)T + P_{d_1}\Pi_2U)\tilde{Q}_{d_1}$$

it follows that

$$I_p = B(YP_{d_1}\Pi_1 - A_1V) + ((I_p - P_{d_1}\Pi_1)T + P_{d_1}\Pi_2U - AW) \cdot \tilde{Q}_{d_1}.$$

(if) Assume that $\bar{V}$ and $\bar{W}$ exist such that

$$I_p = B\bar{V} + \bar{W}\tilde{Q}_{d_1}$$

Again using Lemma 8, we have

$$P_{d_1}\Pi_1 = B\bar{V} + (\bar{W} - (I_p - P_{d_1}\Pi_1)T - P_{d_1}\Pi_2U)\tilde{Q}_{d_1}$$

Hence

$$XP_{d_1}\Pi_1 = XB\bar{V} + X(\bar{W} - (I_p - P_{d_1}\Pi_1)T - P_{d_1}\Pi_2U)\tilde{Q}_{d_1}$$

$$= B_1(X_1(I_n - YB) + Y_1XB)\bar{V} +$$

$$X(\bar{W} - (I_p - P_{d_1}\Pi_1)T - P_{d_1}\Pi_2U)\tilde{Q}_{d_1}$$

(Lemma 10
used)

Then according to Corollary 1, RRPIS is solvable.

Q.E.D.

Remark 1:

Clearly we have not shown the existence of a proper C that solves RRPIS. Theorem 4 will show that the existence of $\bar{V}$ and $\bar{W}$ satisfying (64) does indeed guarantee the existence of a proper C . This is the main result obtained by Bengtsson [2].

Remark 2:

An interesting comparison can now be made concerning equations (15) and (64). The zeros of $|\tilde{Q}_{d1}|$ in (64) are unstable exogenous poles that are not also plant poles. The plant zeros are the values of s that reduce the rank of B . Thus from Lemma 6, (64) is solvable when the plant zeros are disjoint from the zeros of $|\tilde{Q}_{d1}|$. In this case the zeros of $|\tilde{Q}_{d1}|$ must be introduced as additional poles into the loop so that regulation can take place. This procedure of introducing an "internal model" and then stabilizing the loop is the subject of [2], is implicit in [5], [10] and [11] and for single input, single output, unity feedback systems is generally well known by control engineers. In the more general case represented by (15), it is clear that this coprimeness is no longer necessary and in some cases regulation takes place only because some plant zeros are the same as some unstable exogenous poles. The concept of "internal model" is less clear in this more general case.

4. Existence of Proper Controllers

Recall that it is the matrix K rather than the controller C itself that has been utilized in order to solve RPIS. The K as constructed in the proof of Theorem 1 is given by $K = -A_3(V + N_B L)$, where N_B is a basis for $\text{Ker } B_3$ and L is an arbitrary rational matrix. Indeed if L is selected with some desired poles, then as shown in [5], the resultant system will have these poles as long as $N_B \neq 0$. However, if P , and hence B_3 , are of full column rank, $N_B = 0$ and $K = -A_3 V$ is free of poles, which implies a non-dynamic system.

In this section we first present conditions in terms of order relationships which will be used to prove the existence of proper controllers.

Define

$$S = (I_p + FPC)^{-1} \quad (65)$$

and

$$R = CS \quad (66)$$

Then

$$C = R(I_p - FPR)^{-1} \quad (67)$$

The next lemma provides sufficient conditions for C to be proper.

Lemma 11. If $\sigma(R) \leq \min(0, -1 - \sigma(FP))$, then

$$\lim_{s \rightarrow \infty} (I_p + FPC)^{-1} = I_p \text{ and } C \text{ is proper.}$$

Proof. Since $S = (I_p + FPC)^{-1}$, it follows that

$$S = I_p - FPR$$

where use is made of (66).

By assumption $\sigma(R) \leq -1 - \sigma(FP)$. Hence $\sigma(FPR) \leq \sigma(FP) + \sigma(R) \leq -1$.*

* Here we make use of the easily proved inequality $\sigma(AB) \leq \sigma(A) + \sigma(B)$.

Let $\text{adj } S$ denote the adjoint of $S(s)$ and note that $\lim_{s \rightarrow \infty} S =$

$\lim_{s \rightarrow \infty} (I_p - FPR) = I_p$ and $o(S) = 0$, which implies that $o(\text{adj } S) \leq 0$, or

$\lim_{s \rightarrow \infty} (\text{adj } S)$ is another constant. Coupled with the fact that $\lim_{s \rightarrow \infty} |S| = 1$,

the above suggests that $\lim_{s \rightarrow \infty} S^{-1}$ exists as a constant matrix and so

$$o(S^{-1}) = 0.$$

Hence $o(C) \leq o(R) + o(S^{-1}) = o(R) \leq 0$. where (66) is used.

Q.E.D.

In § 2, it was established that RPIS can be solved if and only if there exist polynomial matrices B_2 , W , and A_2 , with $|A_2|$ Hurwitz and a stable rational matrix K such that

$$A_2 P_{d_2} N \Pi_1 = -B_2 + W \tilde{Q}_{d_1}$$

$$A_2^{-1} B_2 = B_3 A_3^{-1} K$$

where P_{d_2} , N , Π_1 , $\tilde{Q}_{d_1}$, B_3 and A_3 are all given.

Let

$$R = C(I_p + FPC)^{-1} = (Y + A_1 K)A \quad (68)$$

K is said to be admissible if it is stable and satisfies (13) and (14). R is said to be admissible if it satisfies (68) for admissible K .

According to Theorem 1, solvability of RPIS ensures the existence of polynomial matrices V and W such that

$$P_{d_2} N \Pi_1 = B_3 V + W \tilde{Q}_{d_1}$$

Assume as before that $\text{rank } P = r$. Since $PA_1 = B_3 A_3^{-1}$, it follows that $\text{rank } B_3 = r$. Then according to Lemma 2, unimodular matrices M_1 , N_1 , M_2 and N_2 exist such that

$$M_1 B_3 N_1 = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \equiv B_d \quad (69)$$

and

$$M_2 \tilde{Q}_{d_1} N_2 = \text{diag}(d_1, d_2, \dots, d_p) \equiv D_d \quad (70)$$

where $B_0 = \text{diag}(b_1, b_2, \dots, b_r)$.

Let

$$M_1 P_{d_2} N_{\Pi_1} N_2 = \begin{bmatrix} P_{21} \\ P_{22} \end{bmatrix} \quad (71)$$

where P_{21} is $r \times p$ and P_{22} is $(\ell-r) \times p$.

The following theorem concerns the structure of K .

Theorem 3. If RPIS is solvable, then an admissible

$$K \text{ is } K = -A_3 N_1 \begin{bmatrix} A_{21}^{-1} & \bar{V}_1 \\ & L_2 \end{bmatrix} N_2^{-1} \quad (72)$$

where $A_{21} = \text{diag}(a_1, \dots, a_r)$ with $a_1, a_2, \dots, a_r$ being arbitrary Hurwitz polynomials,

$$A_{21} P_{21} = B_0 \bar{V}_1 + \bar{W}_1 D_d \quad (73)$$

for some polynomial matrices $\bar{V}_1$ and $\bar{W}_1$ and L_2 is an arbitrary stable rational matrix of dimension $(n-r) \times p$.

Proof: Assume RPIS is solvable.

From (69), (70), (71) and (15), it follows

$$M_1 \cdot P_{d_2} N_{\Pi_1} \cdot N_2 = M_1 B_0 N_1 V^1 + W^1 M_2 \tilde{Q}_{d_1} N_2$$

or

$$P^1 = B_d V^1 + W^1 \cdot D_d \quad (74)$$

where

$$V^1 = N_1^{-1} V N_2$$

$$W^1 = M_1 W M_2^{-1}$$

and

$$P^1 = M_1 P_{d_2} N_{\Pi_1} N_2$$

The solvability of (15) and hence (74) implies that p_{ij}^1 is divisible by the g.c.d. of b_i and d_j for all $i \in \underline{p}$ and $j \in \underline{p}$ according to Lemma 6. Thus for any a_i , $i \in \underline{p}$, $a_i p_{ij}^1$ is also divisible by the g.c.d.

of b_i and d_j for all $i, j \in p$, which according to Lemma 6, in turn implies the solvability of

$$\bar{A}_2 P^1 = B_d \bar{V} + \bar{W} D_d \quad (75)$$

where

$$\bar{A}_2 = \begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$A_{21} = \text{diag}(a_1, \dots, a_r) \text{ and } A_{22} = \text{diag}(a_{r+1}, \dots, a_p),$$

or equivalently

$$M_1^{-1} \bar{A}_2 M_1 P_{d_2} N_{\Pi_1} = B_3 \tilde{V} + \tilde{W} Q_{d_1} \quad (76)$$

where

$$\tilde{V} = N_1 \bar{V} N_2^{-1} \text{ and } \tilde{W} = M_1^{-1} \bar{W} M_2$$

For the reason that will be clear later, let $\sigma(\bar{A}_2) \subset \mathbb{C}^-$.

Since by hypothesis, RPIS is solvable, there exist a Hurwitz A_2 and B_2 and W_1 such that

$$A_2 P_{d_2} N_{\Pi_1} = -B_2 + W_1 \tilde{Q}_{d_1}$$

which, in light of (76), is satisfied by

$$A_2 = M_1^{-1} \bar{A}_2 M_1$$

$$B_2 = -B_3 \tilde{V}$$

$$W_1 = M_1^{-1} \bar{W} M_2$$

Theorem 1 then says the stable K should satisfy

$$A_2^{-1} B_2 = B_3 A_3^{-1} K$$

Note that for this particular A_2 , we have

$$\begin{aligned} A_2^{-1} B_2 &= -(M_1^{-1} \bar{A}_2 M_1)^{-1} B_3 \tilde{V} \\ &= -M_1^{-1} \bar{A}_2^{-1} M_1 B_3 \cdot N_1 \cdot \bar{V} N_2^{-1} \\ &= -M_1^{-1} \bar{A}_2^{-1} B_d \bar{V} N_2^{-1} \end{aligned}$$

Since $\bar{A}_2$ and B_d are diagonal, it is obvious that

$$\begin{aligned}
\bar{A}_2^{-1} B_d &= \begin{bmatrix} A_{21}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A_{21}^{-1} B_0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{21}^{-1} & 0 \\ A_4 & A_5 \end{bmatrix}
\end{aligned}$$

where A_4 , and A_5 , are arbitrary rational matrices of dimensions $(n-r) \times r$ and $(n-r) \times (n-r)$ respectively.

Consequently,

$$A_2^{-1} B_2 = -M_1^{-1} \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{21}^{-1} & 0 \\ A_4 & A_5 \end{bmatrix} \cdot \bar{V}N_2^{-1}$$

and thus (14) becomes

$$-B_d \begin{bmatrix} A_{21}^{-1} & 0 \\ A_4 & A_5 \end{bmatrix} \bar{V}N_2^{-1} = M_1 B_3 N_1 (A_3 N_1)^{-1} K$$

and so

$$K = -A_3 N_1 \begin{bmatrix} A_{21}^{-1} & 0 \\ A_4 & A_5 \end{bmatrix} \bar{V}N_2^{-1} + \begin{bmatrix} 0 \\ L_1 \end{bmatrix} \quad (77)$$

where L_1 is an arbitrary rational matrix.

(75) can be partitioned as follows.

$$\begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} P_{21} \\ P_{22} \end{bmatrix} = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} + \begin{bmatrix} \bar{W}_1 \\ \bar{W}_2 \end{bmatrix} \cdot D_d$$

or equivalently

$$\begin{aligned}
A_{21} P_{21} &= B_0 \bar{V}_1 + \bar{W}_1 D_d \\
A_{22} P_{22} &= \bar{W}_2 D_d
\end{aligned} \quad (78)$$

Let us note in passing that the solvability of (78) is implied by that of (15), and that $\bar{V}_2$ is arbitrary.

With the above partitions, (77) takes the following form:

$$K = -A_3 N_1 \begin{bmatrix} A_{21}^{-1} \bar{V}_1 N_2^{-1} \\ (A_4 \bar{V}_1 + A_5 \bar{V}_2) N_2^{-1} + L_1 \end{bmatrix}$$

$$= -A_3 N_1 \begin{bmatrix} A_{21}^{-1} \bar{V}_1 \\ L_2 \end{bmatrix} \cdot N_2^{-1}$$

where L_2 is an arbitrary rational matrix of dimension $(n-r) \times p$, which is analytic for $\text{Re } s \geq 0$.

Q.E.D.

For the K constructed in the above theorem, define the rational matrix G as follows.

$$\begin{aligned} R &= (Y + A_1 K) A \\ &= Y A - A_1 A_3 N_1 \cdot \begin{bmatrix} A_{21}^{-1} \bar{V}_1 \\ L_2 \end{bmatrix} \cdot N_2^{-1} A \\ &= A_1 A_3 N_1 G N_2^{-1} A \end{aligned} \quad (79)$$

where

$$G = \begin{bmatrix} H \\ H^* \end{bmatrix} - \begin{bmatrix} A_{21}^{-1} \bar{V}_1 \\ L_2 \end{bmatrix} \quad (80)$$

and

$$\begin{bmatrix} H \\ H^* \end{bmatrix} = (A_1 A_3 N_1)^{-1} Y N_2 \quad (81)$$

For the construction of proper controllers, the following lemma proves to be useful.

Lemma 12. For any nonnegative integer β and admissible K , there exists G as defined in (79) such that

$$o(G) = -\beta \quad (82)$$

Proof. Note that $o(G) = -\beta$ can be accomplished by setting

$$(i) \quad o(H - A_{21}^{-1} \bar{V}_1) = -\beta \quad \text{and} \quad (ii) \quad o(H^* - L_2) = -\beta.$$

Case (i):

Let $p_{ij} = (P_{21})_{ij}$, $v_{ij} = (\bar{V}_1)_{ij}$, $w_{ij} = (\bar{W}_1)_{ij}$, and c_{ij} = monic g.c.d. of b_i and d_j .

Equation (73) implies that

$$a_i p_{ij} = b_i v_{ij} + w_{ij} d_j \quad \text{for all } i \in \underline{r}, j \in \underline{p}$$

or,

$$a_i \frac{p_{ij}}{c_{ij}} = \frac{b_i}{c_{ij}} v_{ij} + w_{ij} \frac{d_j}{c_{ij}}$$

Note that solvability of RPIS and hence (15) implies that p_{ij}/c_{ij} is a polynomial. Furthermore, b_i/c_{ij} and d_j/c_{ij} are relatively prime, and so x_{ij} and y_{ij} can be found such that

$$\begin{bmatrix} \frac{b_i}{c_{ij}} & \frac{d_j}{c_{ij}} \end{bmatrix} \cdot \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} = 1 \quad (83)$$

which together with the observation

$$\begin{bmatrix} \frac{b_i}{c_{ij}} & \frac{d_j}{c_{ij}} \end{bmatrix} \cdot \begin{bmatrix} \frac{d_j}{c_{ij}} & -\frac{b_i}{c_{ij}} \end{bmatrix}^T = 0$$

enables us to write v_{ij} as

$$\begin{aligned} v_{ij} &= a_i \cdot \frac{p_{ij}}{c_{ij}} \cdot x_{ij} + \ell_{ij} \cdot \frac{d_j}{c_{ij}} \\ &= a_i u_{ij} + \ell_{ij} v_{ij} \end{aligned} \quad (84)$$

where

$$u_{ij} = (p_{ij} x_{ij})/c_{ij} \quad (85)$$

$$v_{ij} = d_j/c_{ij} \quad (86)$$

and ℓ_{ij} is some arbitrary polynomial.

Now return to (80) and let $g^1_{ij} = (G^1)_{ij}$, where $G^1 \equiv H - A_{21}^{-1} \bar{V}_1$. Therefore, with $z_{ij}/f_{ij} \equiv H_{ij}$, it follows that $g^1_{ij} = z_{ij}/f_{ij} - v_{ij}/a_i$

$$= \frac{(z_{ij} - f_{ij} u_{ij}) a_i - \ell_{ij} (f_{ij} v_{ij})}{f_{ij} a_i} \quad (87)$$

where use is made of (84).

$o(H - A_{21}^{-1} \bar{V}_1) = o(G^1) = -\beta$ can be achieved by letting

$$o(g_{ij}^1) = -\beta, \text{ or}$$

$$o\left(\frac{(z_{ij} - f_{ij} u_{ij})a_i - l_{ij}(f_{ij} v_{ij})}{f_{ij} a_i}\right) = -\beta \quad (88)$$

for all $i \in \underline{r}$ and $j \in \underline{p}$.

To proceed further, the following lemma is needed.

Lemma 13. Given polynomials $z(s)$, $a(s) \neq 0$, $f(s) \neq 0$, $v(s) \neq 0$ and an integer $\gamma \geq 0$. There exists a polynomial $\ell(s)$ such that

$$o\left(\frac{z \cdot a + \ell \cdot v}{f \cdot a}\right) = -\gamma$$

if

$$o(a) \geq o(v) - o(f) + \gamma - 1.$$

Proof. Define the following polynomials.

$$z(s) = \sum_{i=0}^{\xi} z_i s^i, \text{ with } z_{\xi} \neq 0$$

$$v(s) = \sum_{i=0}^{\nu} v_i s^i, \text{ with } v_{\nu} \neq 0$$

$$a(s) = \sum_{i=0}^{\alpha} a_i s^i, \text{ with } a_{\alpha} \neq 0$$

$$\ell(s) = \sum_{i=0}^{\lambda} \ell_i s^i, \text{ with } \ell_{\lambda} \neq 0$$

and

$$f(s) = \sum_{i=0}^{\phi} f_i s^i, \text{ with } f_{\phi} \neq 0$$

where $\lambda = \xi + \alpha - \nu$.

Notice that

$$\begin{aligned} \ell \cdot v &= (\ell_{\lambda} v_{\nu}) s^{\lambda+\nu} + (\ell_{\lambda} v_{\nu-1} + \ell_{\lambda-1} v_{\nu}) s^{\lambda+\nu-1} + \dots + \\ &\quad (\ell_1 v_0 + \ell_0 v_1) s + \ell_0 v_0 \end{aligned}$$

$$\begin{aligned} z \cdot a &= (z_{\xi} a_{\alpha}) s^{\lambda+\nu} + (z_{\xi} a_{\alpha-1} + z_{\xi-1} a_{\alpha}) s^{\lambda+\nu-1} + \dots + \\ &\quad (z_1 a_0 + z_0 a_1) s + z_0 a_0 \end{aligned}$$

To make $o\left(\frac{z \cdot a + l \cdot v}{f \cdot a}\right) = -\gamma$, the leading $\delta \equiv (\lambda + v - \alpha - \phi) + \gamma$ coefficients of $z \cdot a + l \cdot v$ should be set to zero, which yields the following equations.

$$v_v l_\lambda = -(z_\xi a_\alpha) = -z_1^*$$

$$v_v l_{\lambda-1} + v_{v-1} l_\lambda = -(z_\xi a_{\alpha-1} + z_{\xi-1} a_\alpha) = -z_2^*$$

$$v_v l_{\lambda-2} + v_{v-1} l_{\lambda-1} + v_{v-2} l_\lambda = -(z_\xi a_{\alpha-2} + z_{\xi-1} a_{\alpha-1} + z_{\xi-2} a_\alpha) = -z_3^*$$

⋮

Since by assumption $\delta \leq \lambda + 1$, it follows that

$$\begin{bmatrix} v_v & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ v_{v-1} & v_v & 0 & & & & & \\ v_{v-2} & v_{v-1} & v_v & & & & & \\ \vdots & & & & & & & \\ v_0 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & v_0 & & & v_v & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} l_\lambda \\ l_{\lambda-1} \\ \\ \\ l_1 \\ l_0 \end{bmatrix} = - \begin{bmatrix} z_1^* \\ z_2^* \\ \\ \\ z_\delta^* \end{bmatrix} \quad (89)$$

or

$$V_1 \underline{l} = (V_0, 0) \underline{l} = -\underline{z}^*$$

where V_0 is $(\lambda + v - \alpha - \phi + \gamma) \times (\lambda + v - \alpha - \phi + \gamma)$.

Since $v_v \neq 0$, V_1 has full row rank, and (89) is solvable for $\underline{l} =$

$$(l_\lambda, l_{\lambda-1}, \dots, l_1, l_0)^T.$$

Q.E.D.

When $H_{ij} = 0$, f_{ij} is taken to be 1, and hence $f_{ij} \neq 0$ for all $i \in \underline{r}$ and $j \in \underline{p}$. Choose Hurwitz $a_i(s)$ with degree

$$\begin{aligned} o(a_i) &= \max_{j \in \underline{p}} (o(f_{ij} v_{ij}) - o(f_{ij})) + \beta - 1 \\ &= \max_{j \in \underline{p}} o(v_{ij}) + \beta - 1 \end{aligned} \quad (90)$$

Then by virtue of Lemma 13, polynomials l_{ij} can be found so that (88), or $o(g_{ij}^1) = -\beta$ holds for all $j \in \underline{p}$.

Note that the above is true for all $i \in \underline{r}$, and hence (88) holds for all $i \in \underline{r}$ and $j \in \underline{p}$.

Now consider $o(H^* - L_2) = -\beta$, namely

Case (ii):

Let

$$L_2 = A_{22}^{-1} V^* \quad (91)$$

for some polynomial matrices A_{22} and V^* , where A_{22} is diagonal, $|A_{22}|$ Hurwitz, but otherwise arbitrary, and V^* is completely arbitrary.

Let $g_{ij}^2 = (G^2)_{ij}$, where

$$\begin{aligned} G^2 &\equiv H^* - L_2 \\ &= H^* - A_{22}^{-1} V^* \end{aligned}$$

Then

$$\begin{aligned} g_{ij}^2 &= \frac{z_{ij}^*}{f_{ij}^*} - \frac{v_{ij}^*}{a_i^*} \\ &= \frac{z_{ij}^* a_i^* - v_{ij}^* f_{ij}^*}{f_{ij}^* a_i^*} \end{aligned} \quad (92)$$

for $i \in \underline{n-r}$ and $j \in \underline{p}$, where

$$(H^*)_{ij} = v_{ij}^* / f_{ij}^* \quad (93)$$

$$A_{22} = \text{diag}(a_1^*, \dots, a_{n-r}^*) \quad (94)$$

and

$$(V^*)_{ij} = v_{ij}^* \quad (95)$$

Choose a_i^* with degree (see Lemma 13)

$$\begin{aligned} o(a_i^*) &= \max_{j \in p} (o(f_{ij}^*) - o(f_{ij}^*)) + \beta - 1 \\ &= \beta - 1. \end{aligned} \quad (96)$$

Then by means of Lemma 13, polynomials v_{ij}^* can be determined so that $o(g_{ij}^2) = -\beta$ for all $j \in p$.

Consequently, by choosing a_i^* with degree $\beta - 1$ for all $i \in \underline{n-r}$, v_{ij}^* exists such that

$$o(g_{ij}^2) = -\beta \text{ for all } i \in \underline{n-r}, j \in p \quad (97)$$

To recapitulate, by choosing a_i and a_i^* with degrees $\max_{j \in p} o(v_{ij}) + \beta - 1$ and $\beta - 1$ respectively, l_{ij} and v_{ij}^* can be calculated so that (88) and (87) hold, i.e.

$$o(g_{ij}^1) = -\beta \text{ for all } i \in \underline{r}, j \in p$$

and

$$o(g_{ij}^2) = -\beta \text{ for all } i \in \underline{n-r}, j \in p$$

which are equivalent to

$$o(G^1) = -\beta$$

and

$$o(G^2) = -\beta$$

Hence $o(G) = -\beta$ as desired.

Q.E.D.

It is now easy to conclude with the following

Theorem 4.

RPIS is solvable with a proper controller if and only if (12) and (15) hold for polynomial matrices N , V and W .

Proof. The necessity of (12) and (15) is apparent.

For sufficiency, take

$$\beta = \max(0, 1 + o(FP)) + o(A_1 A_{\pi 1} N_1) + o(N_2^{-1} A) \quad (98)$$

In light of Lemma 12, corresponding to admissible K , G as defined in (80) can be constructed such that $\sigma(G) = -\beta$. Hence the order of the resultant admissible R , obtained via (68) is bounded as follows.

$$\begin{aligned}\sigma(R) &= \sigma(A_1 A_{\pi} N_1 G N_2^{-1} A) \leq \sigma(A_1 A_{\pi} N_1) + \sigma(G) + \sigma(N_2^{-1} A) \\ &= -\max(0, 1 + \sigma(FP)),\end{aligned}$$

which, according to Lemma 11, implies that $|S| \neq 0$ and since

$$(X - B_1 K) = S A^{-1}$$

it follows that $|X - B_1 K| \neq 0$ so C exists and by Lemma 11 is proper.

Q.E.D.

5. Conclusion

In this paper we have formulated and solved a general regulator problem in the frequency domain. This seems to be the most natural setting in which to formulate linear, time-invariant control system problems. It is not clear at this stage how best to compute controllers that solve RPIS, when they exist. The computational problems encountered in dealing with polynomial matrices appear to be formidable and a considerable amount of work is called for to develop efficient algorithms for first checking the existence of solutions to RPIS and then properly incorporating this information into actual design algorithms.

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