SENSITIVITY OF ACHIEVABLE RATES FOR HALF-DUPLEX RELAY CHANNEL

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ABSTRACT

This paper investigates the sensitivity of achievable rates for the half-duplex AWGN relay channel to the choice of modulation, receiving time fraction, and source-relay correlation. Our study is motivated by practical considerations that limit the choice of parameters in a real relay network. We show that BPSK modulation suffices to achieve much of the relaying throughput in the SNR range where relaying is most advantageous. We also show that smartly choosing either completely correlated or uncorrelated signals incurs negligible loss in rate compared to optimally correlated signals. Furthermore, the achievable rate is not diminished significantly if we use equal time slots for relay transmission and reception, instead of using the optimum time ratio. Finally, we demonstrate that even when time and correlation are both restricted, relaying with appropriate power control still carries substantial gains over direct and two-hop channels.

1. INTRODUCTION

The goal of this paper is to investigate the sensitivity of half-duplex relaying rates to various system parameters. These parameters include choice of modulation, fraction of time in which the half-duplex relay receives, and source-relay correlation. From an information theoretic standpoint, the achievable rate is maximized for a specific combination of these and other parameters. In reality, however, it is challenging to build a communication system that offers complete flexibility in the choice of all parameter values. We will show that such freedom is in fact not necessary. Even when certain parameters are fixed, the loss in throughput is insignificant.

Our problem is motivated by the challenges of relay communication system design. Any practical digital communication system uses a finite constellation. It is important to evaluate whether the loss due to a small constellation nullifies the benefit of relaying over an AWGN channel. Second, for a given relay channel, there is an optimum

correlation ρ between source and relay signals that maximizes achievable rate. But, it is a challenge to design arbitrarily correlated source and relay codebooks. The design becomes much simpler if we can work with either independent, or completely dependent codebooks. This prompts us to ask: how much do we lose if we use $\rho = 0$ or $\rho = 1$ instead of the optimum ρ ? Third, in a network where there are several relay channels, it is easier to schedule transmissions with equal time slots during which relays receive/transmit, instead of the optimum ratio dictated by information theory, which is different for each relay channel. The rate loss from using equal time slots again needs to be quantified.

Our main results are as follows. First, we show that by using BPSK modulation, we can achieve most of the relay channel throughput at low to medium SNRs. In particular, BPSK performs reasonably well in the SNR range where relaying carries a significant advantage over both direct and two-hop communication. Second, we discover that the achievable rate for relaying is robust to the choice of correlation. In particular, smartly choosing either completely correlated or uncorrelated signals incurs negligible loss in rate compared to signals with information theoretically optimal correlation. In the high SNR regime, we prove this analytically, in addition to demonstrating it numerically. Next, we look into the effect of using equal time slots rather than using the optimum time ratio, and find that achievable rates are robust to this parameter also. Finally, we study the effect of simultaneously equating time slots and limiting correlation to 0 or 1. Substantial relaying gain is observed even with both constraints imposed simultaneously.

The information-theoretic relay channel was first studied by van der Meulen [1]. A few years later, some of the most important capacity results on relaying were published in [2]. After the initial interest, however, the idea of relaying received little attention for nearly two decades. Recent years have seen a renewed interest in relaying in the context of wireless networks. In [3], the authors demonstrated the feasibility of user cooperation in a wireless network by an information theoretic exposition of the gains, and a practical CDMA implementation. New information theoretic results were found in [4, 5, 6, 7]. The study of

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relay protocols and their outage analysis in fading environments was done in [8, 9]. As a practical alternative to conventional decode-and-forward and amplify-and-forward relay protocols, in [10, 11], the authors investigated relaying through coded cooperation. It is worth noting that several of the above research contributions have been based on the premise of half-duplex relaying [6, 7, 10, 9, 8].

This paper is organized as follows. Section 2 describes our notations and assumptions. Section 3 briefly recapitulates known capacity results for direct, two hop, and relay channels. In Section 4 we present our finds and finally, we conclude in Section 5.

(a) General relay channel (b) Half-duplex relay modes

Fig. 1. The relay channel

2. SYSTEM DESCRIPTION

The relay channel is shown in Fig. 1 (a). The source (S) sends data to the destination (D), and in doing so it is aided by the relay (R). In the particular case of a half-duplex relay channel, the relay cannot transmit and receive simultaneously in the same band. Communication takes place over two time slots of time fractions t and $(1 - t)$. In the first slot, S transmits information, which is received by R and D. We will call this the BC (broadcast) mode of communication. In the second slot, both S and R transmit information to D. This, we will refer to as the MAC (multiple access) mode. In the context of a two-hop channel also, we will use the terms BC mode and MAC mode to denote the phases in which the relay receives and transmits respectively. These two modes are depicted in Fig. 1 (b).

We compare half-duplex relaying with direct as well as two-hop communication. For fair comparison, the middle node is assumed to operate in a half-duplex fashion in the two-hop channel. We normalize transmitted power with noise power, which is assumed to be the same at each receiver. An average transmission power constraint is imposed on the nodes, denoted by the symbol Θ .

$$\Theta : tP_{SBC} + (1 - t)(P_{SMAC} + P_{RMAC}) \leq P \quad (1)$$

where P represents total power ¹, P_{SBC} is source transmission power in BC mode, P_{SMAC} is source transmission

power in MAC mode, and P_{RMAC} is relay transmission power in MAC mode. In the two-hop case, the same power constraint holds with $P_{SMAC} = 0$. For the direct link, the above power constraint is true with $t = 1$.

All channels are assumed to be time-invariant 1-D AWGN channels ². The distance between S and D is normalized to unity, and R is assumed to lie on the straight line joining S and D. The relay position, denoted d , represents its distance from the source. The SD channel gain is given by $\gamma_{SD} = 1$, the SR gain is $\gamma_{SR} = \frac{1}{d^\lambda}$, and the RD gain is $\gamma_{RD} = \frac{1}{(1-d)^\lambda}$, where λ is the channel attenuation exponent. We will use $\lambda = 2$ in this paper.

3. REVIEW OF CAPACITY RESULTS

We review known capacity results for the direct, two hop and half-duplex relay channels [12]. For convenience, we define the following functions

$$C(x) = \frac{1}{2} \log(1 + x) \quad (2)$$

$$H(X) = \int_{-\infty}^{\infty} -f_X(x) \log(f_X(x)) dx \quad (3)$$

and use the following notations for received power

$$\begin{aligned} P_{SR} &= P_{SBC} \gamma_{SR} & P_{SD1} &= P_{SBC} \gamma_{SD} \\ P_{RD} &= P_{RMAC} \gamma_{RD} & P_{SD2} &= P_{SMAC} \gamma_{SD} \end{aligned} \quad (4)$$

The capacity of the AWGN channel is known to be $C(P\gamma_{SD})$. The AWGN channel with BPSK modulation has capacity

$$C_{BAWGN} = H(Y) - H(Z) \quad (5)$$

$$f_Y(y) = \frac{1}{2} \sum_{b=1, -1} f_Z(y - b\sqrt{P\gamma_{SD}}) \quad (6)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right) \quad (7)$$

The capacity of the two-hop AWGN channel is given by

$$C_{2HOP} = \sup_{\Theta, 0 \leq t \leq 1} \min\{tC(P_{SR}), (1 - t)C(P_{RD})\} \quad (8)$$

Its BPSK counterpart has a capacity

$$C_{B2HOP} = \sup_{\Theta, 0 \leq t \leq 1} \min\{tC_{SR}, (1 - t)C_{RD}\} \quad (9)$$

$$C_{SR} = H(Y_1) - H(Z) \quad (10)$$

$$C_{RD} = H(Y_2) - H(Z) \quad (11)$$

$$f_{Y_1}(y_1) = \frac{1}{2} \sum_{b=1, -1} f_Z(y_1 - b\sqrt{P_{SR}}) \quad (12)$$

$$f_{Y_2}(y_2) = \frac{1}{2} \sum_{b=1, -1} f_Z(y_2 - b\sqrt{P_{RD}}) \quad (13)$$

²We treat only the case of 1-D AWGN, since an extension to circularly symmetric AWGN is straightforward and brings no additional insight.

¹ P is also the equivalent relay channel SNR.

with f_Z defined in (7).

An upper bound on the capacity of the half-duplex relay channel is obtained from an extension of the max-flow-min-cut theorem to networks with multiple states [12].

$$R \leq \sup_{0 \leq t \leq 1} \min\{tI(X_1; Y_1, V_1) + (1-t)I(X_2; Y_2|W_2), \\ tI(X_1; Y_1) + (1-t)I(X_2, W_2; Y_2)\} \quad (14)$$

where X, Y, V, W are transmitted and received signals shown in Fig. 1, the subscript 1 denotes BC mode, and 2 denotes MAC mode. An achievable lower bound, which is in fact the capacity for the degraded half-duplex relay channel is given by

$$R \geq \sup_{0 \leq t \leq 1} \min\{tI(X_1; V_1) + (1-t)I(X_2; Y_2|W_2), \\ tI(X_1; Y_1) + (1-t)I(X_2, W_2; Y_2)\} \quad (15)$$

The above achievable rate corresponds to *decode and forward* relaying. There are other achievable rates corresponding to *amplify and forward* and *estimate and forward* (also called *compress and forward*) relaying, but we will not consider them in this paper. Also, we will only be concerned with the achievable rate for the relay, and not the upper bound since the upper bound may not be achievable.

For the AWGN channel, the achievable rate is

$$C_{\text{relay}} \geq \sup_{\Theta, 0 \leq t, \rho \leq 1} \min\{tC(P_{SR}) \\ + (1-t)C((1-\rho^2)P_{SD2}), tC(P_{SD1}) \\ + (1-t)C(P_{SD2} + P_{RD} + 2\rho\sqrt{P_{SD2}P_{RD}})\} \quad (16)$$

For the BPSK AWGN channel, the relay achievable rate is given by (15) and the following mutual information terms

$$\begin{aligned} I(X_1; V_1) &= H(V_1) - H(Z) \\ I(X_1, Y_1) &= H(Y_1) - H(Z) \\ I(X_2; Y_2|W_2) &= \frac{1}{2} \left\{ \sum_{b_2=-1,1} H(Y_2|W_2 = b_2) \right\} - H(Z) \\ I(X_2, W_2; Y_2) &= H(Y_2) - H(Z) \end{aligned} \quad (17)$$

where

$$f_{V_1}(v_1) = \sum_{b_1=1,-1} f_{X_1}(b_1) f_Z(v_1 - b_1 \sqrt{P_{SR}}) \quad (18)$$

$$f_{Y_1}(y_1) = \sum_{b_1=1,-1} f_{X_1}(b_1) f_Z(y_1 - b_1 \sqrt{P_{SD1}}) \quad (19)$$

$$f_{Y_2}(y_2) = \sum_{b_1=1,-1} \sum_{b_2=1,-1} f_{X_2, W_2}(b_1, b_2) \\ f_Z(y_2 - b_1 \sqrt{P_{SD2}} - b_2 \sqrt{P_{RD}}) \quad (20)$$

$$f_{Y_2|W_2}(y_2|b_2) = \sum_{b_1=1,-1} f_{X_2|W_2}(b_1|b_2) \\ f_Z(y_2 - b_1 \sqrt{P_{SD2}} - b_2 \sqrt{P_{RD}}) \quad (21)$$

with f_Z defined in (7). In the above, f_{X_1} is uniform and the optimal input distribution $f_{X_2, W_2}(b_1, b_2)$ is

$$\begin{aligned} f_{X_2, W_2}(1, 1) &= f_{X_2, W_2}(-1, -1) = \frac{1+\rho}{4} \\ f_{X_2, W_2}(1, -1) &= f_{X_2, W_2}(-1, 1) = \frac{1-\rho}{4} \end{aligned} \quad (22)$$

Fig. 2. Rate vs. SNR for direct, two-hop and relay channels.

The capacities of the direct link and two-hop channels are compared with the achievable rate of the half-duplex relay channel in Fig. 2. Rates are plotted for gaussian as well as BPSK signaling. The relay position is $d = 0.5$. At low SNR, the performance of relaying is close to that of two-hop communication, whereas at high SNR, relaying is closer to direct communication. In the intermediate SNR range, half-duplex relaying affords sizeable gains. A maximal gain of nearly 40% over both direct and two-hop links is obtained at 3dB SNR with gaussian signaling. With BPSK, a maximum 35% gain is obtained at -1dB.³

4. SENSITIVITY OF ACHIEVABLE RATE

The achievable rate for half-duplex relaying depends on several factors. We will focus only on modulation, ρ and t .

4.1. Role of modulation

As seen in Fig. 2, relaying yields a sizeable gain in throughput at medium SNRs. In this range, antipodal modulation per channel dimension, or BPSK⁴, performs fairly well. In

³Note that the rates in Fig. 2 correspond to normalized distances not exceeding unity. Translated to a practical network, the same rates would correspond to higher and more realistic SNRs.

⁴We will call the modulation scheme BPSK since we assume that the channel (and also the noise) is 1-dimensional. In a real channel where each frequency has both I and Q components, our results correspond to QPSK.

particular, BPSK relaying outperforms both direct and two-hop AWGN channels (for any modulation) at SNRs up to 4dB. The conclusion that binary channel codes can achieve most of the relaying gain at low to intermediate SNRs is of value to coding theorists.

Fig. 3. Impact of correlation on rate.

4.2. Role of ρ

We pose the following question: How much do we lose if we have the choice of using $\rho = 0$ or 1 instead of the optimum ρ ? The answer is especially important because it is a challenge to construct codes for intermediate values of ρ . Code construction with $\rho = 1$ is the same as coding for a direct link, whereas coding for $\rho = 0$ is well understood for the MAC channel. In fact, a close look at the information theoretic coding scheme proposed in [6] reveals that for $\rho = 0$, the rates correspond to one of the corner points of the MAC capacity region. It is well known [13] that these corner points can be achieved simply by successive decoding (onion peeling).

Fig. 3 shows relay achievable rates for $\rho = 0, 1$, and the optimum ρ . In general, correlation seems to have little effect on throughput. Specifically, if we can choose between $\rho = 0$ and 1, then the loss in performance from not using the optimum ρ is negligible for all SNRs.

It can be proved analytically that the fractional loss in rate from using $\rho = 0$ or 1 goes to zero at high SNR. For a given SNR, let R_{opt} denote the optimum rate, which is achieved for $\rho = \rho_{opt}$. Now, keeping all other parameters fixed, think of rate as a function of ρ alone. Let R_0 and R_1 denote the rates for $\rho = 0$ and 1 respectively, and let S_0 and S_1 denote the derivatives of the rate with respect to ρ at $\rho = 0$ and 1. We define sensitivity to ρ as follows.

$$\kappa_\rho = \frac{\min(R_{opt} - R_0, R_{opt} - R_1)}{R_{opt}} \quad (23)$$

If $\rho_{opt} = 0$ or 1, then sensitivity is clearly 0. If not, then we note that both the arguments of the min function in (16) are concave functions of ρ , and one is increasing, whereas the other is decreasing. In that case,

$$\kappa_\rho \leq \frac{\min(S_0\rho_{opt}, S_1(\rho_{opt} - 1))}{R_{opt}}$$

The reader may verify that at large SNR, S_1 approaches a constant, whereas R_{opt} in the denominator grows large. Therefore κ_ρ goes to zero.

The optimum ρ depends strongly on both the SNR and relay position. When the relay is close to the receiver, uncorrelated signals are preferable whereas complete correlation is suitable for the reverse scenario. Also, correlated signals perform better as SNR decreases.

Fig. 4. Impact of using $t = 1/2$ for $d = 0.1$.

4.3. Role of t

We are interested to evaluate the loss in throughput when we fix $t = 1/2$ irrespective of relay position. This is important because equal time slots can simplify the design of wireless relay network protocols, and potentially reduce scheduling overhead. Fig. 4 plots rate vs. SNR for $t = 1/2$ for $d = 0.1$. We deliberately choose $d = 0.1$ instead of 0.5 because intuitively $t = 1/2$ would work well for $d = 0.5$. It is clear from the plot that the loss is small for AWGN and extremely small for BPSK AWGN channels. Rate remains superior to both direct and two hop channels at all SNRs.

4.4. Fixing both ρ and t

Our final question is: Do we lose significantly by fixing $t = 1/2$ and $\rho = 0$ or 1 simultaneously? Fig. 5 plots rate vs. SNR for $t = 1/2$ and $\rho = 0, 1$ for $d = 0.1$ and Fig. 6 plots the same for $d = 0.9$. Performance remains superior to direct and two hop channels for both gaussian and BPSK

Fig. 5. Impact of using $t = 1/2$ and $\rho = 0, 1$ for $d = 0.1$.

Fig. 6. Impact of using $t = 1/2$ and $\rho = 0, 1$ for $d = 0.9$.

signaling. For $d = 0.9$ decode and forward relaying yields little benefit over direct communication, as seen in Fig. 6, but the degradation due to t and ρ is nevertheless small. For $d = 0.1$, there is significant relaying gain. The reader will also note that for $d = 0.1$, $\rho = 1$ is preferable, whereas $\rho = 0$ is better for $d = 0.9$, in agreement with our comment at the end of Section 4.2.

5. CONCLUSIONS

We have investigated the effect of modulation size, correlation ρ , and receiving time fraction t for half duplex relays. Our conclusion is that significant relaying gain is achievable even with BPSK modulation and with $t = 0.5$ and $\rho = 0$ or 1. Our observations will simplify relay code design and the design of wireless relay network protocols. From a different point of view, it is also apparent that power control using available channel information, which we have not discussed

in this paper, alone is sufficient to achieve most of the relaying gains, even when other parameters are inflexible.

6. REFERENCES

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