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ABSTRACT

This report presents a generalization of some recent results of Cheng and Pearson[1]. It is shown that the conditions obtained under certain assumptions in [1] are valid under more general assumptions. We therefore obtain the solution to a more general linear multivariable regulator problem than that previously solved by Wonham and Pearson [2].

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1. Introduction

In this report, we consider a linear time invariant system of the form

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_3 x_2(t) + B_1 u(t) \\ \dot{x}_2(t) &= A_2 x_2(t) \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) \\ z(t) &= D_1 x_1(t) + D_2 x_2(t) + D_3 u(t) \end{aligned} \quad (1)$$

which may be interpreted as a plant with state vector $x_1(t)$ coupled to an exogenous system with state vector $x_2(t)$. The control input is $u(t)$; $y(t)$ is the measured output and $z(t)$ is the regulated output. The control problem is to find a feedback controller with input $y(t)$ and output $u(t)$ such that for every initial state of the closed-loop system we obtain $\lim_{t \rightarrow \infty} z(t) = 0$.

A less general version of this problem (i.e., $D_3 = 0$) was formulated and solved in [2] using the geometric approach. In this paper a frequency domain formulation is used, thus permitting the general case ($D_3 \neq 0$) to be treated directly without the additional complications involved in the geometric approach. A robust version of this more general problem has been treated by Davison and Goldenberg [3], but the robust regulator problem is considerably simpler than the problem posed here.

Laplace transforming (1) and devoting initial conditions by a constant vector w , we may write

$$\begin{aligned} \hat{y}(s) &= -H_1(s)\hat{u}(s) + G_1(s)w \\ \hat{z}(s) &= H_2(s)\hat{u}(s) + G_2(s)w \end{aligned} \quad (2)$$

where $\hat{y}$, $\hat{z}$ and $\hat{u}$ represent the Laplace transforms of y , z and u respectively and

$$\begin{aligned} -H_1(s) &= C_1(s - A_1)^{-1} B_1 \\ H_2(s) &= D_1(s - A_1)^{-1} B_1 + D_3 \end{aligned} \quad (3)$$

We seek a proper controller $C(s)$ of the type

$$\hat{u}(s) = C(s)\hat{y}(s)$$

such that

$$[H_2(s)C(s)(1 + H_1(s)C(s))^{-1}G_1(s) + G_2(s)]$$

is analytic for $\text{Re } s' \geq 0$ and

$$\psi_1(s)\psi_C(s) | 1 + H_1(s)C(s) |$$

is a Hurwitz polynomial, where $\psi_1(s)$ and $\psi_C(s)$ are the characteristic polynomials of $H_1(s)$ and $C(s)$ respectively and $|X(s)|$ represents the determinant of a square matrix $X(s)$. This problem is called RPIS (Regulator Problem with Internal Stability) and when $H_1(s) = F(s)P(s)$ and $H_2(s) = P(s)$, has been solved by Cheng and Pearson [1] under the assumptions that $F(s)$ and $P(s)$ have no hidden unstable modes and that the pair $(F(s), P(s))$ is admissible, i.e.,

$$\psi_F^+(s)\psi_P^+(s) = \psi_{FP}^+(s)$$

where $\psi^+(s)$ represents the unstable factor of the polynomial $\psi(s)$.

In this report we show how the results obtained in [1] apply directly to the problem specified by (1) or (2). Basically this involves replacing the admissibility of $(F(s), P(s))$ by the stabilizability and detectability of (C_1, A_1, B_1) and establishing that this is all that is needed to allow the proof of Theorem 1 [1] to go through as originally stated. In §2 we furnish a new proof, using the results of [1], of the well known fact that a system (C, A, B) can be stabilized by output feedback through a proper controller if and only if (C, A, B) is stabilizable and detectable. In §3 we use this result to establish a Lemma to replace Lemma 1 of [1] and state the main result of the report.

2. Stabilizability of a Linear System

Consider the problem of regulating

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t)$$

$$z(t) = x(t)$$

(4)

or

$$\hat{y}(s) = [C(s - A)^{-1}B]\hat{u}(s) + C(s - A)^{-1}x(0)$$

$$\hat{z}(s) = (s - A)^{-1}B \hat{u}(s) + (s - A)^{-1}x(0)$$

Identify

$$-F(s)P(s) = C(s - A)^{-1}B$$

$$P(s) = (s - A)^{-1}B$$

$$G_1(s) = C(s - A)^{-1}$$

$$G_2(s) = (s - A)^{-1}$$

Since $-F(s) = C$, the pair $(F(s), P(s))$ has no unstable hidden modes if and only if (A, B) is stabilizable and is admissible if and only if (C, A, B) is stabilizable and detectable. In order to use the results of [1] define (omit the argument s for simplicity) the minimal fraction representations

$$FP = Q_1^{-1} P_1 = P_2 Q_2^{-1} \\ \{Q_1 G_1\}_+ = R_1 S_1^{-1} = \tilde{S}_1^{-1} \tilde{R}_1 \quad *$$

$$\{P Y Q_1 G_1 + G_2\}_+ = R_2 S_2^{-1} \quad (6)$$

$$P Q_2 = P_3 Q_3^{-1}$$

and polynomial matrices (X, Y) and (X_1, Y_1) by

$$Q_1 X + P_1 Y = I$$

$$X_1 R_1 + Y_1 S_1 = I$$

From [1] we know that if (F, P) is admissible there exists a proper controller stabilizing the system (4) if and only if there exist polynomial matrices N, V and W such that

$$S_2^{-1} S_1 = N \quad (7)$$

and

$$R_2 N X_1 = P_3 V + W \tilde{S}_1 \quad (8)$$

Since (F, P) is admissible if and only if (C, A, B) is stabilizable and detectable, we will prove the following (well-known) result

* $\{ \ }_+$ denotes that part of a partial fraction expansion containing closed right-half-plane poles.

Lemma 1

There exists a proper controller stabilizing the system (4) if and only if (C, A, B) is stabilizable and detectable.

Proof. Necessity is obvious. For sufficiency we show that there exist polynomial matrices N , V and W satisfying (7) and (8).

First replace (C, A, B) by a minimal (i.e. controllable and observable) representation $(\bar{C}, \bar{A}, \bar{B})$. Then since

$$FP = Q_1^{-1} P_1 = \bar{C}(s - \bar{A})^{-1} \bar{B}$$

it follows that

$$Q_1 \bar{C}(s - \bar{A})^{-1} \bar{B}$$

is a polynomial and since

$$(s - \bar{A})^{-1} \bar{B}$$

is minimal,

$$Q_1 \bar{C}(s - \bar{A})^{-1}$$

is a polynomial. Since (C, A) is detectable

$$\{Q_1 \bar{C}(s - \bar{A})^{-1}\}_+ = \{Q_1 C(s - A)^{-1}\}_+ = 0$$

and so we can choose in (5) $R_1 = 0$, $\tilde{R}_1 = 0$, $S_1 = I$, $\tilde{S}_1 = I$.

Clearly

$$FPY P_1 - FP$$

is a polynomial and from

$$\bar{C}[(s - \bar{A})^{-1} \bar{B} Y Q_1 \bar{C} + I](s - \bar{A})^{-1} \bar{B}$$

it follows that since $(s - \bar{A})^{-1} \bar{B}$ is minimal and

$$\bar{C}[(s - \bar{A})^{-1} \bar{B} Y Q_1 \bar{C} + I]$$

is a polynomial, so then is

$$\begin{aligned} & \bar{C}[(s - \bar{A})^{-1} \bar{B} Y Q_1 \bar{C} + I](s - \bar{A})^{-1} \\ &= \bar{C}(s - \bar{A})^{-1} [\bar{B} Y Q_1 \bar{C}(s - \bar{A})^{-1} + I] \end{aligned}$$

Again since $\bar{C}(s - \bar{A})^{-1}$ is minimal and $(\bar{B} Y Q_1 \bar{C}(s - \bar{A})^{-1} + I)$ is a polynomial it follows that

$$(s - \bar{A})^{-1} (\bar{B}YQ_1 \bar{C}(s - \bar{A})^{-1} + I)$$

is a polynomial. Because of stabilizability and detectability

$$\begin{aligned} & \{(s - \bar{A})^{-1} \bar{B}YQ_1 \bar{C}(s - \bar{A})^{-1} + (s - \bar{A})^{-1}\}_+ \\ &= \{(s - A)^{-1} BYQ_1 C(s - A)^{-1} + (s - A)^{-1}\}_+ \\ &= \{PYQ_1 G_1 + G_2\}_+ = 0 \end{aligned}$$

Therefore from (6) we may choose $R_2 = 0$, $S_2 = I$.

Since $S_2^{-1} S_1 = I$, (7) is true. Since $R_2 = 0$ and $\tilde{S}_1 = I$, let V be any polynomial matrix and define

$$W = -P_3 V$$

Then (8) is true and we have completed the proof.

3. Main Results

We first establish the necessary generalization of Lemma 1 [1]. Let

$$(s - A_1)^{-1} B_1 = SP^{-1}$$

be a minimal fraction representation. From (3)

$$\begin{aligned} H_1 &= -C_1 SP^{-1} = R_1 P_1^{-1} \\ H_2 &= D_1 SP^{-1} + D_2 = R_2 P_2^{-1} \end{aligned} \tag{9}$$

where (R_1, P_1) and (R_2, P_2) are minimal.

Lemma 2

Assume (C_1, A_1, B_1) is stabilizable and detectable. Then

$$\{H_2 P_1\}_+ = 0.$$

Proof: Since (R_1, P_1) and (R_2, P_2) are minimal, it follows from (9) that

$$P = P_1 G_1 = P_2 G_2$$

and $|G_1|$ is stable by assumption. Thus

$$H_2 P_1 = R_2 P_2^{-1} P_1 = R_2 G_2 G_1^{-1}$$

and so

$$\{H_2 P_1\}_+ = 0.$$

Our main result is stated as follows.

Theorem Given the system (2) with $H_1(s)$ and $H_2(s)$ defined by (3).

Assume (C_1, A_1, B_1) is stabilizable and detectable. There exists a proper controller which solves RPIS if and only if there exist polynomial matrices

N, V and W satisfying

$$S_2^{-1} S_1 = N$$

$$R_2 N X_1 = P_3 V + W \tilde{S}_1$$

where $H_1 = Q_1^{-1} P_1 = P_2 Q_2^{-1}$

$$\{Q_1 G_1\}_+ = R_1 S_1^{-1} = \tilde{S}_1^{-1} \tilde{R}_1$$

$$\{H_2 Y Q_1 G_1 + G_2\}_+ = R_2 S_2^{-1}$$

$$H_2 Q_2 = P_3 Q_3^{-1}$$

are minimal representations and X_1 and Y are polynomial matrices defined by

$$Q_1 X + P_1 Y = I$$

$$X_1 R_1 + Y_1 S_1 = I$$

Proof: The proof is identical to the proofs of Theorems 1 and 4 in [1]. The key step is to replace Lemma 1 in [1] by Lemma 2 as stated above.

4. Conclusion

In this report, we have shown how the results of Cheng and Pearson [1] apply to a more general regulation problem than has been previously studied. The results are not completely satisfactory in that hybrid (i.e. part state space, part frequency domain) conditions have been used. This has been necessary because of the key relation that exists between $H_1(s)$ and $H_2(s)$ as stated by Lemma 1 [1] when $H_1 = FP$ and $H_2 = P$ and by Lemma 2 in the present case. In the general case, all we can say at the present time is that if H_1 has no unstable hidden modes and the relation given by Lemma 2 is satisfied by H_1 and H_2 , then the results obtained in this report can be applied.

5. References

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