A NOTE ON THE CHARACTERIZATION OF
BIVARIATE DENSITIES BY CONDITIONAL DENSITIES

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ABSTRACT

The compatibility of pairs of conditional densities and the uniqueness of the resulting bivariate densities are discussed. Examples are given involving beta, gamma, and Gaussian conditional densities.

INTRODUCTION

Given two marginal densities, there always exists a bivariate density with the given marginals, because of the existence of the product density, and the bivariate density is not uniquely determined by the marginals. The question of the compatibility of marginal densities, not necessarily univariate, in the multivariate case is more profound, and a survey of this area is given by Dall'Aglia (1972). Seshadri and Patil (1964) discuss the compatibility of a univariate marginal and a conditional density of the

same random variable, as well as the uniqueness of the bivariate density so determined, in a number of special cases. Roux (1971) presents a multivariate generalization of some of Seshadri and Patil's results. Brucker (1979) and Fraser and Streit (1980) note recently that given two conditional univariate Gaussian densities, the bivariate density is uniquely determined to be bivariate Gaussian.

We discuss here some questions of compatibility and uniqueness when a pair of conditional densities is given. This work generalizes Brucker's example considerably and yields a number of interesting examples of characterizations of bivariate densities in terms of conditional densities of the components. There remain, however, many similar questions of interest for investigation, particularly with respect to multivariate extensions.

RESULT

Let $p_1(x|y)$ and $p_2(y|x)$ be given conditional density functions. If the given conditional density functions are compatible, that is, if there is some bivariate density, $a(x,y)$ with the given conditional densities, then

$$\int a(x,y)dx \cdot p_1(x|y) = \int a(x,y)dy \cdot p_2(y|x)$$

or

$$\frac{p_1(x|y)}{p_2(y|x)} = \frac{\int a(x,y)dy}{\int a(x,y)dx} ,$$

assuming the denominators are nonzero. Thus a necessary condition for compatibility of a pair of conditional densities is that their ratio be of the form $g(x)/h(y)$ where g and h are nonnegative integrable functions such that $\int g(x)dx = \int h(y)dy$. That this is also sufficient for compatibility of the conditional densities as well as uniqueness of the resulting bivariate density follows from the identification of $g(x)/\int g(x)dx$ with $f_1(x)$ and $h(y)/\int h(y)dy$ with $f_2(y)$ and the reconstruction of the bivariate density as $f_2(y)p_1(x|y)$ or $f_1(x)p_2(y|x)$.

Some examples follow. In these examples, both conditional densities are chosen to be of the same continuous type. It is

clear that the same ideas apply when the conditionals are of different types or when the distributions are discrete.

EXAMPLES

Ex. 1 Beta Conditional Densities

Let the conditional densities be given by

$$p_1(x|y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{(x-a(y))^{p-1} (b(y)-x)^{q-1}}{(b(y)-a(y))^{p+q-1}}$$

and

$$p_2(y|x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{(y-c(x))^{r-1} (d(x)-y)^{s-1}}{(d(x)-c(x))^{r+s-1}}.$$

Then their ratio becomes

$$\frac{p_1(x|y)}{p_2(y|x)} = \frac{\Gamma(p+q)\Gamma(r)\Gamma(s)}{\Gamma(r+s)\Gamma(p)\Gamma(q)} \frac{(d(x)-c(x))^{r+s-1} (x-a(y))^{p-1} (b(y)-x)^{q-1}}{(b(y)-a(y))^{p+q-1} (y-c(x))^{r-1} (d(x)-y)^{s-1}}.$$

If $a(y)=0$, $c(x)=0$, $b(y)=b-y$, $d(x)=b-x$, and $q=s$, the bivariate density becomes the bivariate Dirichlet density, Johnson and Kotz (1972), p. 231, Lee (1971), Mardia (1970), p. 87.

$$a(x,y) = \frac{\Gamma(p+q+r)}{\Gamma(p)\Gamma(q)\Gamma(r)} x^{p-1} y^{r-1} (b-x-y)^{q-1}.$$

where $x \geq 0$, $y \geq 0$, $x+y \leq b$, with beta marginals

$$f_1(x) = \frac{\Gamma(p+q+r)}{\Gamma(p)\Gamma(q+r)} x^{p-1} (b-x)^{q+r-1}$$

where $0 \leq x \leq b$, and

$$f_2(y) = \frac{\Gamma(p+q+r)}{\Gamma(p+r)\Gamma(r)} y^{r-1} (b-y)^{p+q-1}$$

where $0 \leq y \leq b$. If instead $a(y)=a-y$, $c(x)=a-x$, $b(y)=b-y$, $d(x)=b-x$, $p=r$, and $q=s$, the bivariate density has uniform marginal densities on $[a,b]$ and is given explicitly by

$$a(x,y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{(x+y-a)^{p-1} (b-x-y)^{q-1}}{(b-a)^{p+q}}$$

where $x \geq a$, $y \geq a$, $x+y \leq b$. These are the only types of bivariate densities with the given marginals, so that all other marginals except beta and uniform are incompatible with beta conditionals with parameters incorporating the conditioning variables as indicated.

Ex. 2 Gamma Conditional Densities

Let the conditional densities be given by

$$p_1(x|y) = \frac{x^{a(y)-1} e^{-x}}{\Gamma(a(y))}$$

and

$$p_2(y|x) = \frac{y^{b(x)-1} e^{-y}}{\Gamma(b(x))}.$$

Then $p_1(x|y)/p_2(y|x)$ is never of the form $g(x)/h(y)$ so that gamma conditional densities of the given form are never compatible unless the marginals are independent.

Rather than the more general gamma conditionals

$$p_1(x|y) = \frac{x^{a(y)-1} e^{-c(y)x} (c(y))^{a(y)}}{\Gamma(a(y))}$$

and

$$p_2(y|x) = \frac{y^{b(x)-1} e^{-d(x)y} (d(x))^{b(x)}}{\Gamma(b(x))}$$

for which the resulting $g(x)$ and $h(y)$ are rather complicated, it is reasonable to consider the exponential conditional case for which $a(y)=1$ and $b(x)=1$. Then the conditional densities are given by

$$p_1(x|y) = e^{-c(y)x} c(y)$$

and

$$p_2(y|x) = e^{-d(x)y} d(x).$$

Now if $c(y)=ay+c$ and $d(x)=ax+d$, the ratio of conditional densities becomes

$$\frac{p_1(x|y)}{p_2(y|x)} = \frac{e^{-cx} (ay+c)}{e^{-dy} (ax+d)}$$

so that

$$f_1(x) = \frac{ke^{-cx}}{ax+d}$$

and

$$f_2(y) = \frac{ke^{-dy}}{ay+c}$$

where

$$k^{-1} = \frac{1}{a} \{-e^{cd/a} \text{Ei}(-cd/a)\}$$

(see Gradshteyn and Ryzhik (1965) p. 311 and p. 925). Then the bivariate density is given explicitly by

$$a(x,y) = ke^{-\{cx+axy+dy\}}$$

and this is the only bivariate density with exponential conditionals of the given form.

Ex. 3 Gaussian Conditional Densities

Let the conditional densities be given by

$$p_1(x|y) = \frac{1}{\sqrt{2\pi}b(y)} \exp \left\{ -\frac{1}{2} \left(\frac{x-a(y)}{b(y)} \right)^2 \right\}$$

and

$$p_2(y|x) = \frac{1}{\sqrt{2\pi}d(x)} \exp \left\{ -\frac{1}{2} \left(\frac{y-c(x)}{d(x)} \right)^2 \right\}.$$

Then their ratio becomes

$$\frac{p_1(x|y)}{p_2(y|x)} = \frac{d(x)}{b(y)} \frac{\exp \left\{ -\frac{1}{2} \left(\frac{x^2}{b^2(y)} - \frac{2xa(y)}{b^2(y)} + \frac{a^2(y)}{b^2(y)} \right) \right\}}{\exp \left\{ -\frac{1}{2} \left(\frac{y^2}{d^2(x)} - \frac{2yc(x)}{d^2(x)} + \frac{c^2(x)}{d^2(x)} \right) \right\}}.$$

If

$$b^2(y) = (b_2y^2 + b_1y + b_0)^{-1},$$

$$d^2(x) = (d_2x^2 + d_1x + d_0)^{-1},$$

$$a(y) = b^2(y) \cdot (a_2y^2 + a_1y + a_0),$$

and

$$c(x) = d^2(x) \cdot (c_2x^2 + c_1x + c_0),$$

where $b_2=d_2$, $b_1=-2c_2$, $d_1=-2a_2$, and $a_1=c_1$, then the ratio of conditional densities becomes

$$\frac{p_1(x|y)}{p_2(y|x)} = \frac{d(x)}{b(y)} \frac{\exp \left\{ -\frac{1}{2} \left(b_0x^2 - 2a_0x - c^2(x)/d^2(x) \right) \right\}}{\exp \left\{ -\frac{1}{2} \left(d_0y^2 - 2c_0y - a^2(y)/b^2(y) \right) \right\}}.$$

Although $p_1(x|y)/p_2(y|x)$ is of the form $g(x)/h(y)$ in this case, it is not always true that the given conditionals define a valid bivariate density. If for example $c_0=c_1=0$, $d_0=d_1=0$, and $c_2^2/d_2^2 > b_0$, then

$$g(x) = \frac{1}{\sqrt{d_2}x} \exp\{-\frac{1}{2}(b_0 - c_2^2/d_2^2)x^2 - 2a_0x\}$$

which is not integrable. Thus, the condition that $p_1(x|y)/p_2(y|x)$ is of the form $g(x)/h(y)$ is an easily verifiable necessary condition on the existence of a valid bivariate density, but it is not always sufficient without the restriction that $g(x)$ and $h(y)$ are nonnegative integrable functions with equal integrals. In the particular case of Gaussian conditionals when $b_0=1$, $b_1=b_2=0$, $d_0=1$, $d_1=d_2=0$, the sufficient conditions are satisfied, and the bivariate density is the bivariate Gaussian density of Brucker's example.

REMARKS

Although the necessary condition given here is rather obvious, it does not appear to be well-known. This approach provides a method for the explicit construction of bivariate densities and a way of characterizing bivariate densities in terms of their conditionals.

A question arises as to when the conditional and marginal densities can be of the same type. Also, although a particular multivariate extension of the necessary condition is straightforward, there remain many similar questions of compatibility in the multivariate case; for example, it is apparent that given $p_j(x_j|x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and $p_k(x_k|x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$, it is necessary that their ratio be of the form

$$\frac{p_j}{p_k} = \frac{g}{h}$$

where g is not a function of x_k and h is not a function of x_j .

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BIBLIOGRAPHY

- Brucker, J. (1979). A note on the bivariate normal distribution. Commun. Statist.--Theor. Meth., A8, 175-177.
- Dall'Aglio, G. (1972). Frechet classes and compatibility of distribution functions. Symposia Mathematica, Vol. IX, New York: Academic Press, 131-150.
- Fraser, D.A.S. and Streit F. (1980). A further note on the bivariate normal distribution. Commun. Statist.--Theor. Meth., A9, 1097-1099.
- Gradshteyn, I.S. and Ryzhik, I.W. (1965). Table of Integrals, Series, and Products. New York: Academic Press.
- Johnson, N.L. and Kotz S. (1972). Distributions in Statistics: Continuous Multivariate Distributions. New York: Wiley.
- Lee, P.A. (1971). A diagonal expansion for the 2-variate Dirichlet probability density function. SIAM J. Appl. Math., 21, 155-165.
- Mardia, K.V. (1970). Families of Bivariate Distributions. London: Griffin.
- Roux, J.J.J. (1971). A characterization of a multivariate distribution. S. Afr. Statist J., 5, 27-36.
- Seshadri, V. and Patil, G.P. (1964). A characterization of a bivariate distribution by the marginal and conditional distributions of the same component. Inst. Statist. Math. Ann., 15, 215-221.