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THE RICE INSTITUTE

ON INVARIANT SUBSPACES OF A NORMAL OPERATOR

by

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A THESIS

SUBMITTED TO THE FACULTY

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

A handwritten signature in cursive script, reading "Arlen Brown".

Houston, Texas
May 1957

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PREFACE

Our principal interest, in the work that follows, is the discovery of conditions on the spectrum of a normal operator which imply that the operator has invariant non-reducing subspaces. We were led to this study by the work of Halmos [5,6]¹, Wermer [14,15], and Bram [1].

In 1950 Halmos published a paper which initiated the study of what he called subnormal operators. An operator B is subnormal on the Hilbert space K provided there is a normal operator A on a Hilbert space H with $K \subset H$ and $A|_K = B$. Notice that the operator B is assumed given in this definition. We consider as given the operator A . One of our purposes is to find criteria for deciding when A is the extension of a subnormal operator. Particularly, we ask the question, "When is A the extension of a subnormal operator B such that B is not normal?" Since $A|_K = B$, K is an invariant subspace of H . If K is also reducing, then $A^*|_K = B^*$. This would imply that B is normal. Thus our question reduces to "When does A have an invariant non-reducing subspace?".

Wermer has studied the problem from a different

¹ The numbers in brackets refer to references listed on pages 73 and 74.

standpoint [14]. He has considered a normal operator A and what he calls property (P). A normal operator A has property (P) if every invariant subspace of A reduces A . The proofs of his Theorems 2 and 3 suggested the use of Borel series for the investigation of our question, at least in the case where the spectrum of A is purely point spectrum. The series and expansions of functions using the method of Runge [13, p.10] turned out to be fruitful in the study of the spectrum of a normal operator.

Suppose we know that A on H is a normal operator which does not have property (P). Let K be an invariant non-reducing subspace for A . If the spectrum of A has a "hole" D in it, Bram has shown that either no point of D belongs to the spectrum of $A|_K$ or D is contained in the spectrum of $A|_K$ [1, p.80]. Some of the results on Borel series (published by A. Denjoy) [2] indicated that we might be able to use Borel series to determine some condition on the spectrum of A in order to be sure that the spectrum of $A|_K$ would fill a given hole in the spectrum of A . Results of this sort are given in the work that follows.

The thesis has been broken up into four chapters.

Chapter I includes introductory material and definitions.

Chapter II deals with questions concerning measures. Two properties in particular are investigated. They are called property (P) and property (R). A compact measure μ has property (P) if and only if for every complex measure ν with $\nu \ll \mu$, $\int z^n d\nu = 0$, $n \geq 0$, implies $\nu = 0$. A compact measure has property (R) if and only if for every complex measure ν with $\nu \ll \mu$, $\int z^n d\nu = 0$, $n \geq 0$, implies $\int (z - \lambda)^n d\nu(z) = 0$ for $n < 0$ and any λ in the complement of the support of μ . Our principal interest lies in the failure of these two properties. We show that if the support of μ is sufficiently dense, say the interior of the support of μ is non-void, then property (P) fails for μ . If the support of μ separates the plane and is sufficiently dense, say the support of μ contains an annulus, then property (R) fails for μ . An approximation theorem related to these properties is added as an appendix to this chapter. The theorem was given to us by G. R. MacLane.

Chapter III is largely the application of the results of Chapter II to spectral measures. The spectral theorem for normal operators associates with each normal operator a spectral measure E . Each vector x of the Hilbert space H , together with this spectral measure gives us a measure in the usual sense, $\mu_x(M) = (E(M)x, x)$. We show that A

has property (P) if and only if there is an $x \in H$ such that μ_x has property (P). This is the same as a result due to Wermer [14, p.274]. A generalization of a result due to Wermer is obtained. Wermer has shown that if $\sigma(A)$ contains a disc D and if m is a measure which assigns to each set S the Lebesgue measure of the intersection of S and D , and if m is absolutely continuous with respect to the spectral measure of A , then property (P) fails for A [14, p.274]. We have shown that his result is independent of Lebesgue measure. If $\sigma(A)$ contains a disc, then property (P) fails for A .

It is shown that if A does not have property (P), then A is the minimal normal extension of a subnormal operator.

The result attributed above to Bram is reproved and generalized to include operators on a Banach space. We show that if $\sigma(A)$ has holes which are sufficiently well surrounded by spectrum, then associated with each hole there is an invariant non-reducing subspace K of A such that the spectrum of $A|_K$ contains the hole. If D_1 and D_2 are any two holes then the subspaces K_1 and K_2 associated with these holes can be chosen so that $K_1 \perp K_2$.

It is well known that $(R_\lambda x, y)$ represents a holomorphic function on any component of the complement of

the spectrum of A . Paragraph 5 of Chapter III deals with functions representable by $(R_\lambda x, y)$.

The last paragraph of Chapter III is an application of the results of Chapter II to a particular operator on l_1 .

In Chapter IV a different characterization of property (P) for an operator is given.

The author wishes to express his appreciation to Professor Arlen Brown for his guidance and criticism during the preparation of this thesis.

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1. Definitions. The definitions of terms which are used in the work that follows, but which are left undefined, can be found in Halmos[4] or Stone[12]. Three theorems are stated without proof. The proofs can be found in Halmos[4].

In the work that follows, we will use measure to mean a positive measure. Complex measure will be used to designate a complex valued, countably additive set function such that the real and imaginary parts of the function are signed measures in the usual sense.

We state without proof the following

Theorem. If μ is a signed measure on the measurable space (X, S) , then there exist two disjoint sets A and B whose union is X such that for any set $E \in S$, $\mu(A \cap E)$ is positive and $\mu(B \cap E)$ is negative.

The decomposition of X described in this theorem is a Hahn decomposition of X . Although this decomposition is not unique, it can be shown that if A_1, B_1 and A_2, B_2 are two such decompositions then for every measurable set E ,

$$\mu(E \cap (A_2 - A_1)) = \mu(E \cap (A_1 - A_2)) = 0,$$

and

$$\mu(E \cap (B_2 - B_1)) = \mu(E \cap (B_1 - B_2)) = 0.$$

Thus, without any ambiguity, we can define two set functions μ^+ and μ^- by the equations

$$\mu^+(E) = \mu(E \cap A) \text{ and } \mu^-(E) = -\mu(E \cap B).$$

μ^+ and μ^- will be called the upper variation and lower variation respectively for the measure μ . The set function defined by $|\mu|(E) = \mu^+(E) + \mu^-(E)$ is the total variation of μ .

Jordan Decomposition Theorem. The upper, lower, and total variations of a signed measure μ are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$ for every measurable set E . If μ is totally finite then so also are μ^+ and μ^- .

If f is integrable with respect to μ^+ and μ^- , then we say that f is integrable with respect to μ and write

$$\int_E f d\mu = \int_E f d\mu^+ - \int_E f d\mu^-.$$

If ν is a complex measure with $\nu = \nu_1 + i\nu_2$, and if f is integrable with respect to ν_1 and ν_2 , then we say that f is integrable with respect to ν and write

$$\int_E f d\nu = \int_E f d\nu_1 + i \int_E f d\nu_2.$$

Theorem (Radon-Nikodym). If $(X, \mathcal{S}, \mu)$ is a totally finite measure space and if a totally finite signed measure ν is absolutely continuous with respect to μ , then

there exists a μ -summable function f on X such that

$$\nu(E) = \int_E f d\mu$$

for every measurable set E . The function is unique in the sense that if also $\nu(E) = \int_E g d\mu$, $E \in S$, then $f = g$ a.e. $[\mu]$.

This theorem is usually stated in a more general form; see Halmos [4].

The space X under consideration in what follows will be the complex plane. Let C be the class of all compact subsets of X . Let S be the σ -ring generated by C . The sets of S are the Borel sets of X . Henceforth, it will be understood that the σ -ring on which the measure under consideration is defined is the σ -ring generated by C . Throughout the work that follows, we will consider only measures μ such that $\mu(E) < \infty$ for $E \in C$. We will call these Borel measures.

A measure μ is said to be inner regular if

$$\mu(E) = \sup_{C \subset E} \mu(C)$$

where the C 's are compact. A measure is said to be outer regular if

$$\mu(E) = \inf_{U \supset E} \mu(U)$$

where the U 's are open. If μ is both inner and outer

regular, then it is regular. Every totally finite Borel measure is regular [4, p. 220, 229].

Given a measure μ on X , the support of μ , written $\Sigma(\mu)$, is the complement of the union of all open subsets U of X such that $\mu(U) = 0$. If μ is a signed measure, then $\Sigma(\mu) = \Sigma(\mu^+) \cup \Sigma(\mu^-)$. A measure is said to be compact if its support is compact. It should be noted that a compact measure is totally finite.

An atom is a set $E \in \mathcal{S}$ such that $\mu(E) \neq 0$, but if $F \subset E$, then either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. For the measures under consideration, an atom is a point P such that $\{P\}$ has non-zero measure. A measure is non-atomic if it has no atoms. A measure is purely atomic if every measurable set with non-zero measure is the union of atoms.

Hilbert space will be used to mean a complex Hilbert space, i.e., we use the complexes as the field of scalars. Also, subspace will be used to mean closed subspace.

1.2. The continuous functions in $L_p(\mu)$. As usual, $L_p(\mu)$ is the space of functions such that

$$\int |f|^p d\mu = \int_{\Sigma(\mu)} |f|^p d\mu < \infty.$$

(Throughout the work that follows, the absence of a limit on an integral indicates that the integral is being taken over the entire complex plane.) Defining $\|f\|_p =$

$(\int |f|^p d\mu)^{1/p}$, $L_p(\mu)$ is a Banach space. If we define $(f, g) = \int f \bar{g} d\mu$, then $L_2(\mu)$ is a Hilbert space.

Theorem 1.2.1. If μ is a measure then $L_p(\mu)$ is spanned by the set of functions which are continuous on $\Sigma(\mu)$.

Proof. It suffices to show that the set of functions continuous on $\Sigma(\mu)$ is dense in $L_p(\mu)$.

We show first that the set of characteristic functions of measurable sets (Borel sets) spans $L_p(\mu)$. To prove this it is sufficient to show that the simple functions defined on measurable sets are dense in $L_p(\mu)$.

Let f be a function belonging to $L_p(\mu)$. Given $\epsilon > 0$, we will show that there is a simple function f_0 such that $\|f - f_0\|_p < \epsilon$. For any positive integer n , define f_n by

$$f_n = \begin{cases} f & \text{if } |f| < n, \\ -n & \text{if } f \leq -n, \\ n & \text{if } f \geq n. \end{cases}$$

Since the $|f_n|$ are bounded by $|f|$, the Lebesgue Dominated Convergence Theorem tells us that $\lim_{n \rightarrow \infty} f_n = f$. Choose n_0 such that $\|f - f_{n_0}\|_p < \epsilon/2$. Consider the ladder

$$-n_0 = l_0 < l_1 < l_2 < \dots < l_k = n_0$$

partitioning the range of f_{n_0} , with $l_1 - l_{1-1} = 2n_0/k$.

Since f is measurable, f_{n_0} is measurable. Consequently, the set $E_1 = \{z \mid l_1 \leq f_{n_0}(z) \leq l_{1+1}\}$ is measurable. Letting χ_1 be the characteristic function of E_1 and taking k sufficiently large

$$\|f_{n_0} - \sum_{i=0}^k l_i \chi_i\|_p < \epsilon/2.$$

Thus we have

$$\|f - \sum_{i=0}^k l_i \chi_i\|_p < \epsilon.$$

Since f was an arbitrary function of $L_p(\mu)$, it follows that the simple functions are dense in $L_p(\mu)$.

The theorem will be proved if we can show that the characteristic function of a measurable set is the limit in the mean of order p of a sequence of continuous functions. Let E be a measurable set. Then there is an open set U and a closed set F such that $U \supset E \supset F$, and $\mu(U) - \mu(F) < \epsilon$. Let G be the complement of U . According to Urysohn's Lemma, there is a continuous function f such that f maps X onto the unit interval $[0,1]$ with $f = 1$ on F and $f = 0$ on G . Now $|f(z) - \chi_E(z)| \leq 1$.

Hence

$$\int |f - \chi_E|^p d\mu < \epsilon.$$

Theorem 1.2.2. If μ is a signed measure such that $\int f d\mu = 0$ for all functions f which are continuous on

$\Sigma(\mu)$, then $\mu = 0$.

Proof. Observe that

$$\left| \int_E (f - g) d\mu \right| \leq \int_E |f - g| d|\mu|.$$

This theorem then follows from the application of the last part of the proof of the preceding theorem to μ .

CHAPTER II

PROPERTY (P) AND PROPERTY (R)

2.1. Definition of property (P). Let μ be a compact measure. Then μ has property (P) if and only if for every complex measure ν with $\nu \ll \mu$, we have $\int z^n d\nu = 0$ for $n \geq 0$ implies that $\nu = 0$. Let $\mathcal{P}(z)$ be the set of polynomials in the complex variable z .

Theorem 2.1.1. If μ has property (P) then $\mathcal{P}(z)$ spans $L_2(\mu)$.

Proof. Suppose $\mathcal{P}(z)$ does not span $L_2(\mu)$. Then there is a function $g(z) \in L_2(\mu)$ such that $g(z) \perp \mathcal{P}(z)$ and $g(z)$ is not the zero vector in $L_2(\mu)$. Thus $\int p \bar{g} d\mu = 0$ for all $p(z) \in \mathcal{P}(z)$. However, from Theorem 1.2.1, it follows that there is a continuous function f such that $\int f \bar{g} d\mu \neq 0$. Let $\nu(E) = \int_E \bar{g} d\mu$. Then $\int z^n d\nu = 0$ for $n \geq 0$, but $\nu \neq 0$, and $\nu \ll \mu$. Hence property (P) fails for μ .

It is not known whether the converse of this theorem is true or false. Notice, however, that if property (P) fails for μ , then there is a signed measure ν with $\nu \ll \mu$ and $\int z^n d\nu = 0$, $n \geq 0$, but $\nu \neq 0$. From the Radon-Nikodym Theorem, this tells us that there is a function g which is μ -summable such that $\nu(E) = \int_E g d\mu$ for any $E \in \mathcal{S}$. If $g \in L_2(\mu)$, then according to Theorem 1.2.2, there is a

continuous function f such that $\int f g d\mu \neq 0$. But since $\int p g d\mu = 0$ for all $p \in \mathcal{P}(z)$, it follows that f is not in the weak closure of $\mathcal{P}(z)$. Hence f is not in the strong closure of $\mathcal{P}(z)$, and $\mathcal{P}(z)$ does not span $L_2(\mu)$.

Another view point of property (P) is afforded by the following remarks. Suppose $\nu \ll \mu$. Then by the Radon-Nikodym Theorem there is a function g such that $\nu(E) = \int_E g d\mu$, where g is summable μ . If μ has property (P), and $\int z^n d\nu = 0$, $n \geq 0$, then $g = 0$ a.e. $[\mu]$. But $\int z^n d\nu = 0$ is the same as $\int p d\nu = \int p g d\mu$ for all $p \in \mathcal{P}(z)$. Now $\mathcal{P}(z) \subset L_\infty(\mu)$ and $L_1^*(\mu) = L_\infty(\mu)$. From what we have said, it follows that μ has property (P) if and only if $\mathcal{P}(z)$ is weakly dense in $L_\infty(\mu)$.

2.2. Property (P) and $\Sigma(\mu)$. We give two examples to point out that property (P) is connected with the support of the particular measure under consideration.

Suppose $\Sigma(\mu)$ is a compact set without interior which does not separate the plane. According to a known theorem due to Lavrentieff [9], any function continuous on $\Sigma(\mu)$ can be approximated uniformly on $\Sigma(\mu)$ by polynomials in z . Let $\nu \ll \mu$. Then, according to the Radon-Nikodym Theorem, there is a μ -summable function g such that $\nu(E) = \int_E g d\mu$ for all $E \in \mathcal{S}$. Suppose $\int z^n d\nu = 0$, $n \geq 0$. Then if $p \in \mathcal{P}(z)$, $\int p d\nu = 0$. Let f be continuous on $\Sigma(\mu)$.

By Lavrentieff's theorem, there is a sequence of polynomials p_n such that if $\epsilon > 0$, arbitrary,

$$|p_n(z) - f(z)| < \epsilon/M$$

for $n > n_0$ and all $z \in \Sigma(\mu)$, where $M = \int |g| d\mu$. Thus

$$\begin{aligned} \left| \int f d\nu \right| &\leq \left| \int (f(z) - p_n(z))g(z) d\mu \right| + \left| \int p_n(z)g(z) d\mu \right| \\ &\leq \int |f(z) - p_n(z)| |g(z)| d\mu < \epsilon \end{aligned}$$

for $n > n_0$. Therefore,

$$\int f d\nu = 0 \text{ for all continuous } f.$$

By Theorem 1.2.2, $\nu = 0$. Hence μ has property (P).

For our second example, we merely quote a result due to Wermer [14, p.275]. Let d be a circular disk contained in the support of μ and let m be the measure which assigns to each set S the Lebesgue measure of the intersection of S and d . If $m \ll \mu$, then property (P) fails for μ .

These two examples indicate that the topological structure of the support of a measure μ plays an important role in the determination of whether or not μ has property (P). An easy counter example shows that property (P) is not completely determined by the support of the measure. Suppose μ is purely atomic with its atoms dense on the unit circle. Wermer has shown that μ has property (P) [14, p.272]. On the other hand, if ν is lin-

ear Lebesgue measure on the unit circle. Wermer has shown that property (P) fails for ν [14,p.276]. Thus, we have two measures with the same support. One of these measures has property (P), and the other one does not have property (P). We will proceed to further investigate the connection between the topology of the support of a measure and property (P) for the measure.

A point p is said to be accessible from a domain R provided for $x \in R$ there exists a simple arc xp lying in $R \cup p$. If there is no arc xp lying in $R \cup p$ for any $x \in R$, then p is said to be inaccessible from R .

If R is a point set in the complex plane, the principal component of R is the component containing the point at infinity.

Wermer comments that the unicity problem for Borel series is the same as the problem of the determination of whether or not a purely atomic measure has property (P) [14,p.272]. This comment led us to the review of some of the results of Denjoy in connection with Borel series [2]. We have the following

Theorem 2.2.1. Let μ be a compact purely atomic measure. Then a necessary condition that property (P) fail for μ is that there exist a set $E \subset \Sigma(\mu)$ such that

if R is the principal component of the complement of $\bar{E}$, then every point of E is inaccessible from R .

Proof. Suppose μ does not have property (P).

According to the definition, there is a totally finite measure $\nu \ll \mu$ such that $\int z^n d\nu = 0$, $n \geq 0$, but $\nu \neq 0$.

Since μ is atomic, ν must be atomic also. Let $\{\lambda_i\}_{i=1}^{\infty}$ be $\Sigma(\nu)$, and let $\omega_i = \nu(\{\lambda_i\})$. Since ν is totally finite $\sum_{i=1}^{\infty} |\omega_i| < \infty$. Now

$$(2.2.1) \quad \sum_{i=1}^{\infty} \lambda_i^n \omega_i = \int z^n d\nu = 0 \text{ for } n \geq 0.$$

Denjoy has shown [2] that if a Borel series

$$(2.2.2) \quad \sum_{n=1}^{\infty} \frac{A_n}{z - \alpha_n}$$

represents a given function $f(z)$ for z in a domain D with D in the complement of $\bar{E}$, where $E = \{\alpha_i\}_{i=1}^{\infty}$, and if any α_i , $i = 1, 2, \dots$, is accessible from D , then α_i is a singularity of $f(z)$. Equation (2.2.1) gives us information about a series of the form (2.2.2) for certain values of z . Let $A_i = \omega_i$ and $\alpha_i = \lambda_i$. Since $\Sigma(\nu)$ is compact, we know that $\{|\lambda_i|\}_{i=1}^{\infty}$ is bounded, say $|\lambda_i| \leq M$ for $i = 1, 2, \dots$. Then for $|z| > M$, we can expand each term of the Borel series as a geometric series,

$$\frac{\omega_1}{z - \lambda_1} = \frac{1}{z} \left(\frac{\omega_1}{1 - \frac{\lambda_1}{z}} \right) = \frac{\omega_1}{z} \sum_{n=0}^{\infty} \left(\frac{\lambda_1}{z} \right)^n.$$

An application of the Weierstrass Double Series Theorem then gives

$$(2.2.3) \quad \sum_{i=1}^{\infty} \frac{\omega_i}{z - \lambda_i} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \sum_{i=1}^{\infty} \lambda_i^n \omega_i.$$

Applying (2.2.1), all of the coefficients of powers of z on the right are zero. Hence for z in the principal component of the complement of $\bar{E}$, $E = \{\lambda_i\}_{i=1}^{\infty}$, the series (2.2.3) represents the constant function zero. Since zero has no singularities, each point of E must be inaccessible from R .

The theorem that we have just proved generalizes two theorems due to Wermer [14, p.272].

If Γ is a closed curve which separates the plane, we will call the principal component of the complement of Γ the exterior or outside of Γ . If R is the exterior of Γ then the complement of $R \cup \Gamma$ will be called the interior or inside of Γ .

2.3. A sufficient condition that property (P) fail for μ . In this section, Runge's method of approximation will be employed to prove that a certain condition on $\Sigma(\mu)$ is sufficient to ensure that property (P) fails for μ .

Lemma 2.3.1. Let Γ_1 , Γ_2 , and C be non-intersecting

simple closed rectifiable Jordan curves such that C is in the interior of Γ_2 , and Γ_1 is in the interior of C . Let $f(z)$ be a function which vanishes at infinity and is holomorphic for z outside or on Γ_1 . Suppose μ is a compact measure such that $C \subset \Sigma(\mu)$. Then given $\epsilon > 0$, arbitrary, there exists a complex measure ν with $\nu \ll \mu$ such that for z on or exterior to Γ_2 ,

$$\left| f(z) - \int \frac{1}{t-z} d\nu(t) \right| < \epsilon.$$

Proof. Since $f(z)$ is holomorphic for z outside Γ_1 , and vanishes at infinity, according to the Cauchy integral formula, for z exterior to C

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt,$$

where the integral is taken around C in a positive direction with respect to the exterior of C . Let L be the length of C . Let $M = \max. |f(z)|$ for z on or exterior to C . For t on C and z on or exterior to Γ_2 , $\frac{f(t)}{t-z}$ is a uniformly continuous function of t independent of z , i.e., given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(t)}{t-z} - \frac{f(\hat{t})}{\hat{t}-z} \right| < \frac{\epsilon}{2L}$$

for $|t - \hat{t}| < \delta$ and for all z on or exterior to Γ_2 . Since C is a rectifiable Jordan curve, there is a partition of C such that the norm of the partition is less than δ .

Let $c_1, c_2, \dots, c_n$ be the partition points on C , where the subscripts of the c_1 's increase as we proceed around C in a positive direction with respect to the exterior of C . Let

$$a_1 = \frac{-f(c_1)(c_{1+1} - c_1)}{2\pi i},$$

$i = 1, 2, \dots, n$, where we take $c_{n+1} = c_1$. Note that, from the definition of the a_1 , $\sum_{i=1}^n |a_i| < ML$. Define

$$g_1(z) = f(z) - \sum_{j=1}^n \frac{a_j}{z - c_j}.$$

Then $g_1(z)$ is holomorphic on or exterior to Γ_2 (in fact $g_1(z)$ is holomorphic exterior to C). Furthermore, for z on or exterior to Γ_2 ,

$$\begin{aligned} |g_1(z)| &= \left| \frac{1}{2\pi i} \int_C \frac{f(t) dt}{t - z} - \sum_{j=1}^n \frac{a_j}{z - c_j} \right| \\ &= \left| \frac{1}{2\pi i} \sum_{j=1}^n \int_{c_j}^{c_{j+1}} \left(\frac{f(t)}{t - z} - \frac{f(c_j)}{c_j - z} \right) dt \right| \\ &\leq \frac{1}{2\pi} \sum_{j=1}^n \int_{c_j}^{c_{j+1}} \left| \frac{f(t)}{t - z} - \frac{f(c_j)}{c_j - z} \right| |dt| \\ &< \frac{\epsilon}{4\pi L} \sum_{j=1}^n \int_{c_j}^{c_{j+1}} |dt| \\ &\leq \frac{\epsilon}{4\pi} < \epsilon/2. \end{aligned}$$

About each c_j we can construct a closed disc of radius δ_j such that for $|t - c_j| \leq \delta_j$, we have

$$\left| \frac{1}{c_j - z} - \frac{1}{t - z} \right| < \frac{\epsilon}{2ML}$$

for any z on or exterior to Γ_2 . Call this disc F_j . We assume that the δ_j are chosen so that $F_1 \cap F_j = \emptyset$ for $1 \neq j$. Define ν on F_j by

$$\nu(E) = \int_E \frac{a_j}{\mu(F_j)} d\mu$$

for $E \subset F_j$. This defines ν for any subset of the set $F = \bigcup_{j=1}^n F_j$. Define $\nu(E) = 0$ if $E \cap F = \emptyset$. Then $\nu \ll \mu$. Consider

$$f_1(z) = f(z) - \int \frac{1}{t - z} d\nu(t).$$

For z on or exterior to Γ_2 ,

$$|f_1(z)| \leq \left| f(z) - \sum_{j=1}^n \frac{a_j}{z - c_j} \right| + \left| \sum_{j=1}^n \frac{a_j}{z - c_j} - \int \frac{1}{t - z} d\nu(t) \right|.$$

The first term on the right has already been shown to be less than $\epsilon/2$. Consider

$$\begin{aligned} \left| \sum_{j=1}^n \frac{a_j}{z - c_j} - \int \frac{1}{t - z} d\nu(t) \right| &= \left| \sum_{j=1}^n \left(\frac{a_j}{z - c_j} + \int_{F_j} \frac{1}{z - t} d\nu(t) \right) \right| \\ &\leq \sum_{j=1}^n \int_{F_j} \left| \frac{1}{z - c_j} - \frac{1}{z - t} \right| \left| \frac{a_j}{\mu(F_j)} \right| d\mu(t) \\ &< \frac{\epsilon}{2ML} \sum_{j=1}^n \int_{F_j} \frac{|a_j|}{\mu(F_j)} d\mu \\ &\leq \epsilon/2. \end{aligned}$$

Therefore, $|f_1(z)| < \epsilon$ for z on or exterior to Γ_2 .

According to the definition of ν , it follows that X is measurable and $|\nu(X)| \leq \sum_{j=1}^n |a_j| < ML$. Thus ν is totally finite. This completes the proof of the lemma.

Lemma 2.3.2. The function $f_1(z)$ defined in Lemma 2.3.1 is holomorphic for z on or exterior to Γ_2 , and $f_1(z)$ vanishes at infinity.

Proof. From the definition of $f_1(z)$, it suffices to prove that $F(z) = \int \frac{1}{t-z} d\nu(t)$ is holomorphic on and exterior to Γ_2 and vanishes at infinity.

Since $\frac{1}{t-z}$ tends to zero as z tends to infinity for any $t \in \Sigma(\nu)$, it follows that the function $F(z)$ vanishes at infinity.

We show that $F(z)$ is holomorphic for z in the complement of the closed set F . It suffices to show that $F(z)$ is differentiable at every point z in the complement of F . Now for any z_1 in the complement of F , there is a neighborhood N of z_1 such that for $z \in N$ and $t \in \Sigma(\nu)$ $\frac{1}{t-z}$ is bounded, say $|\frac{1}{t-z}| < K$. Hence $|F(z)| < KLM$ for $z \in N$. Consider

$$\begin{aligned} \lim_{z \rightarrow z_1} \frac{F(z_1) - F(z)}{z_1 - z} &= \lim_{z \rightarrow z_1} \frac{1}{z_1 - z} \int \left(\frac{1}{t - z_1} - \frac{1}{t - z} \right) d\nu(t) \\ &= \lim_{z \rightarrow z_1} \int \frac{1}{(t - z_1)(t - z)} d\nu(t). \end{aligned}$$

This integral is uniformly bounded for $z \in N$. Hence by the Lebesgue Dominated Convergence Theorem,

$$\lim_{z_1 \rightarrow z} \frac{F(z_1) - F(z)}{z_1 - z} = \int \frac{1}{(t - z)^2} d\nu(t).$$

Hence $F(z)$ is differentiable at every point in the complement of F .

Given a set of simple closed Jordan curves $\{C_n\}_{n=1}^{\infty}$, we say that the sequence $\{C_n\}_{n=1}^{\infty}$ increases monotonically if each C_n is interior to C_{n+1} . If we replace the word "interior" by "exterior", then we say that the sequence decreases monotonically.

Theorem 2.3.1. Let μ be a compact measure. Suppose $\Sigma(\mu)$ includes a monotonically increasing sequence of simple closed rectifiable Jordan curves $\{C_n\}_{n=0}^{\infty}$. Then property (P) fails for μ .

Proof. Let $f(z)$ be a function which is holomorphic for z on or exterior to C_0 and such that $f(z)$ vanishes at infinity. Since μ is compact, $\Sigma(\mu)$ is bounded, i.e., there is a positive number r such that for $z \in \Sigma(\mu)$, $|z| \leq r$. It is well known, by using the Runge method of approximation, that $f(z)$ can be represented for $|z| > r$ by a Borel series [13, p. 20]. Using essentially the same method of approximation, we will show that there is a

complex measure ν and a set S , with $\nu \ll \mu$, such that

$$\int_S \frac{1}{t-z} d\nu(t) = f(z) \text{ for } |z| > r.$$

Let L_n be the length of C_n . Choose a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n > 0$ for $n = 1, 2, \dots$, and

$$\sum_{n=1}^{\infty} \epsilon_n L_n < \infty.$$

We proceed by applying Lemma 2.3.1 to each of the curves with an odd subscript. Let C be C_1 , $\Gamma_1 = C_0$, $\Gamma_2 = C_2$, $\epsilon = \epsilon_3$. Then by Lemma 2.3.1, we have a complex measure $\nu_1 \ll \mu$ and

$$\left| f(z) - \int \frac{1}{t-z} d\nu_1(t) \right| < \epsilon_3$$

for z on or exterior to Γ_2 . To indicate the curve of the sequence to which they belong, for convenience in our future calculations, we add a superscript to the coefficients in the rational approximation and to the symbol for the closed sets. Thus $\sum_{j=1}^{n_1} |a_j^1| < M L_1$, and $F^1 = \bigcup_{j=1}^{n_1} F_j^1$.

Consider the function

$$f_1(z) = f(z) - \int \frac{1}{t-z} d\nu_1(t).$$

According to Lemma 2.3.2, $f_1(z)$ is holomorphic for z on or exterior to C_2 . Letting $C = C_3$, $\Gamma_1 = C_2$, $\Gamma_2 = C_4$, $\epsilon = \epsilon_5$, we again apply Lemma 2.3.1. We obtain a function

$$f_2(z) = f_1(z) - \int \frac{1}{t-z} d\nu_2(t)$$

where ν_2 is a complex measure with $\nu_2 \ll \mu$ and for z on or exterior to C_4 $|f_2(z)| < \epsilon_5$. From Lemma 2.3.2, $f_2(z)$ is holomorphic outside and on C_4 and vanishes at infinity. Notice that $\sum_{j=1}^{n_3} |a_j^3| < \epsilon_3 L_3$.

Now $\Sigma(\nu_1)$ is in the annular region bounded by C_0 and C_2 while $\Sigma(\nu_2)$ is in the annular region bounded by C_2 and C_4 . Furthermore, from the definition of $\Sigma(\nu_1)$, it follows that $\Sigma(\nu_1) \cap C_2 = \emptyset$. Hence $\Sigma(\nu_1) \cap \Sigma(\nu_2) = \emptyset$. Thus, without ambiguity, we define a measure ν such that $\nu(E) = \nu_1(E)$ for $E \subset \Sigma(\nu_1)$ and $\nu(E) = \nu_2(E)$ for $E \subset \Sigma(\nu_2)$. Then we have

$$f_2(z) = f(z) - \int_{F^1 \cup F^3} \frac{1}{t-z} d\nu(t).$$

Proceeding by induction, suppose $f_{m-1}(z)$ has been defined and ν has been defined on $\bigcup_{j=1}^{m-1} F^{2j-1}$. We apply Lemma 2.3.1 to the function $f_{m-1}(z)$ with $C = C_{2m-1}$, $\Gamma_1 = C_{2m-2}$, $\Gamma_2 = C_{2m}$, and $\epsilon = \epsilon_{2m+1}$. We obtain a function,

$$f_m(z) = f_{m-1}(z) - \int \frac{1}{t-z} d\nu_m(t)$$

with $|f_m(z)| < \epsilon_{2m+1}$ for z on or exterior to C_{2m} ,

$\sum_{j=1}^{n_{2m-1}} |a_j^{2m-1}| < \epsilon_{2m-1} L_{2m-1}$, and $\Sigma(\nu_m) \subset F^{2m-1}$. According

to Lemma 2.3.2, $f_m(z)$ is holomorphic for z on or exte-

rior to C_{2m} and vanishes at infinity. Defining $\nu(E) = \nu_m(E)$ for $E \subset \sum(\nu_m)$, we have

$$f_m(z) = f(z) - \int_K \frac{1}{t-z} d\nu(t),$$

where $K = \bigcup_{j=1}^m F^{2j-1}$. Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, it follows that $\lim_{n \rightarrow \infty} |f_n(z)|^{j=1} = 0$. Hence

$$f(z) = \int_S \frac{1}{t-z} d\nu(t)$$

for $|z| > r$, where $S = \bigcup_{j=1}^{\infty} F^{2j-1}$.

We now repeat the approximation process. This time we use the curves with even subscripts and approximate $-f(z)$.

The function $-f(z)$ is holomorphic for z on or exterior to C_1 and $-f(z)$ vanishes at infinity. We apply Lemma 2.3.1 and Lemma 2.3.2. Let $C = C_2$, $\Gamma_1 = C_1$, $\Gamma_2 = C_3$, $\epsilon = \epsilon_4$. Then there is a function

$$g_1(z) = -f(z) - \int \frac{1}{t-z} d\lambda_1(t)$$

with $g_1(z)$ holomorphic for z outside or on C_3 , $g_1(z)$ vanishes at infinity, and $|g_1(z)| < \epsilon_4$ for z outside or on C_3 . Also $\sum_{j=1}^{n_2} |a_j^2| < M L_2$. We choose the F_j^2 's so that $F^2 \cap S = \emptyset$. Then we define ν on F^2 by $\nu(E) = \lambda_1(E)$ for $E \subset F^2$.

After applying the preceding lemmas m times, we have

$$g_m(z) = -f(z) - \int_M \frac{1}{t-z} d\nu(t),$$

where $M = \bigcup_{j=1}^m F^{2j}$ and $M \cap S = \emptyset$. For z on or outside C_{2m+1} ,

$g_m(z)$ is holomorphic and $|g_m(z)| < \epsilon_{2m+2}$. Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, it follows that $\lim_{n \rightarrow \infty} |g_n(z)| = 0$ for $|z| > r$. Thus

$$-f(z) = \int_{\hat{S}} \frac{1}{t-z} d\nu(t),$$

where $\hat{S} = \bigcup_{j=1}^{\infty} F^{2j}$. Define $\nu(E) = 0$ if $E \cap (\hat{S} \cup S) = \emptyset$. Then we have

$$(2.3.2) \quad \int \frac{1}{t-z} d\nu(t) = \int_S \frac{1}{t-z} d\nu(t) + \int_{\hat{S}} \frac{1}{t-z} d\nu(t) \\ = f(z) - f(z) = 0,$$

for $|z| > r$. By definition ν is compact, and if $t \in \Sigma(\nu)$ then $|t| \leq r$. Then for $|z| > r$ and $t \in \Sigma(\nu)$,

$$(2.3.3) \quad \frac{1}{t-z} = - \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}.$$

Furthermore,

$$\left| \int t^n d\nu \right| \leq r^n |\nu|(\hat{S} \cup S).$$

Hence for $|z| > r$, we have

$$\sum_{n=0}^{\infty} \left| \int \frac{t^n}{z^{n+1}} d\nu(t) \right| < \infty.$$

Thus integration of the series (2.3.3) term by term is justified, giving

$$(2.3.4) \quad \int \frac{1}{t-z} d\nu(t) = - \sum_{n=0}^{\infty} \int \frac{t^n}{z^{n+1}} d\nu(t)$$

for $t \in \Sigma(\nu)$ and $|z| > r$. By equation (2.3.2), the left

side of equation (2.3.4) is identically zero for $|z| > r$. But the right side of (2.3.4) cannot be identically zero for $|z| > r$ unless the coefficient of each power of z is zero. Consequently, we have

$$(2.3.5) \quad \int t^n d\mu = 0 \text{ for } n \geq 0.$$

From the definition of ν , it follows that $\nu \ll \mu$ and $\nu \neq 0$. Also, $|\nu(X)| \leq \sum_{j=1}^{\infty} \sum_{j=1}^{n_1} |a_j^1|$. It has been pointed out above that $\sum_{j=1}^{n_1} |a_j^1| \leq ML_1$, $\sum_{j=1}^{n_2} |a_j^2| \leq ML_2$, and $\sum_{j=1}^{n_1} |a_j^1| < \epsilon_1 L_1$ for $1 \neq 1, 2$. Hence from the choice of the ϵ_n ,

$$|\nu(X)| \leq ML_1 + ML_2 + \sum_{n=3}^{\infty} \epsilon_n L_n < \infty.$$

Therefore, ν is totally finite. This proves the theorem.

Corollary 2.3.11. If μ is compact and $\sum^0(\mu) \neq 0$, then property (P) fails for μ .

Proof. Clear.

Corollary 2.3.12. Let $f(z)$ be holomorphic for $|z| > r_0$, where r_0 is such that the domain $|z| > r_0$ contains all but a finite number of the curves described in the hypothesis of the theorem. Suppose that $f(z)$ vanishes at infinity. For $|z|$ sufficiently large, $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$. Then there is a measure ν such that $\nu \ll \mu$ and

$$\int z^n d\nu = a_n.$$

Proof. From The proof of the theorem, we can write

$$f(z) = \int_S \frac{1}{t - z} d\nu(t),$$

for $|z| > r$. Define $\nu(E) = 0$ if $E \cap S = \emptyset$. Then ν is defined for all Borel sets. Thus, as in the last part of the proof of the theorem,

$$f(z) = - \sum_{n=0}^{\infty} \int \frac{t^n}{z^{n+1}} d\nu(t)$$

for $|z| > r$. The corollary then follows from the uniqueness of the series representation of a function.

2.4. Property (R). We will say that a compact measure μ has property (R) if given any complex measure ν with $\nu \ll \mu$ then $\int z^n d\nu = 0$ for $n \geq 0$ implies

$$\int (z - \lambda)^n d\nu(z) = 0$$

for $n < 0$ and any λ in the complement of $\Sigma(\mu)$. Throughout this section, it will be assumed that all measures under consideration are compact.

Theorem 2.4.1. A necessary condition that property (R) fail for a measure μ is that $\Sigma(\mu)$ separate the plane.

Proof. Suppose $\Sigma(\mu)$ does not separate the plane. Then every function holomorphic on $\Sigma(\mu)$ can be approximated uniformly on $\Sigma(\mu)$ by polynomials in z [13, p.47]. Now if λ belongs to the complement of $\Sigma(\mu)$, then $(z - \lambda)^n$ for $n < 0$ is holomorphic on $\Sigma(\mu)$. Hence, $(z - \lambda)^n$ can

be approximated uniformly by polynomials on $\Sigma(\mu)$. Thus $\int z^n d\mu = 0$, $n \geq 0$, implies $\int (z - \lambda)^n d\mu = 0$ for $n < 0$. Therefore, μ has property (R).

Observe that if μ has property (P), then μ has property (R). The preceding theorem shows that the converse is not true. For, suppose $\Sigma^0(\mu) \neq \emptyset$ but $\Sigma(\mu)$ does not separate the plane. Then according to Theorem 2.3.1, μ does not have property (P). However, according to the preceding theorem μ has property (R).

Theorem 2.4.2. If μ has property (R) then the closure of $\mathcal{P}(z)$ in $L_2(\mu)$ includes all functions holomorphic on $\Sigma(\mu)$.

Proof. We will assume that the closure of $\mathcal{P}(z)$ in $L_2(\mu)$ does not include all functions holomorphic on $\Sigma(\mu)$ and show that this implies that property (R) fails for μ .

Since the closure of $\mathcal{P}(z)$ in $L_2(\mu)$ does not include all functions holomorphic on $\Sigma(\mu)$, there is a function $g(z) \perp \overline{\mathcal{P}(z)}$ such that $g(z)$ is holomorphic on $\Sigma(\mu)$ and $g(z)$ is not the zero element of $L_2(\mu)$. Now by Runge's Theorem [1, p. 176] $g(z)$ can be approximated uniformly on $\Sigma(\mu)$ by a rational function whose poles lie in the complement of $\Sigma(\mu)$. From what we have said $\int z^n \overline{g(z)} d\mu = 0$, $n \geq 0$. Runge's Theorem then tells us

that for some λ in the complement of $\Sigma(\mu)$,

$$\int (z - \lambda)^n \overline{g(z)} d\mu \neq 0$$

for some $n < 0$. Let $\nu(E) = \int_E \overline{g(z)} d\mu$. Then $\nu \ll \mu$. Thus μ does not have property (R).

It is not known if the converse of this theorem is true or false.

Theorem 2.4.3. Let $\Sigma(\mu)$ contain a set of simple closed rectifiable Jordan curves $\{C_n\}_{n=-\infty}^{\infty}$ such that
 1) the sequence $\{C_n\}_{n=0}^{\infty}$ increases monotonically, 11) the sequence $\{C_n\}_{n=0}^{-\infty}$ decreases monotonically, 111) there is a disc D which is contained in the interior of each curve of the sequence. Then property (R) fails for μ .

Proof. For convenience we assume that $z = 0$ belongs to D . It then suffices to show that there is a complex measure ν such that $\nu \ll \mu$ and $\int z^n d\nu = 0$ for $n \geq 0$, but $\int z^n d\nu \neq 0$ for some $n < 0$.

The proof depends on an approximation process differing very little from the one employed to prove Theorem 2.3.1. Instead of repeating the whole process, we will only indicate the manner in which the present procedure differs from the one used in the proof of Theorem 2.3.1.

Let $f(z)$ be a function which is holomorphic for z outside C_0 and such that $f(z)$ vanishes at infinity. Let

$g(z)$ be holomorphic for z inside C_0 . Let $M_1 = \max. |f(z)|$ for z on or outside C_1 . Let $M_2 = \max. |g(z)|$ for z on or interior to C_{-1} . Choose a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n > 0$ for all n and $\sum_{n=1}^{\infty} \epsilon_n (L_n + L_{-n}) < \infty$, where L_n is the length of C_n , $n = \pm 1, \pm 2, \dots$. Let C_1 and C_{1+1} be any two successive curves of the given doubly infinite sequence. We construct Γ_1 such that Γ_1 is a simple closed Jordan curve and Γ_1 is in the interior of C_{1+1} , but Γ_1 is in the exterior of C_1 . The construction is illustrated in the sketch below.

Let K be the point set consisting of Γ_1 , Γ_{-2} , the interior of Γ_{-2} , and the exterior of Γ_1 . For $t \in C_1$ and $z \in K$, $\frac{f(t)}{t-z}$ is uniformly continuous as a function of t . Thus, there is a $\delta_1 > 0$ such that

$$\left| \frac{f(t)}{t-z} - \frac{f(\hat{t})}{\hat{t}-z} \right| < \frac{\epsilon_2}{4L_1},$$

for $t, \hat{t} \in C_1$, $|t - \hat{t}| < \delta_1$, and any z in K . Similarly, for $t \in C_{-1}$ and $z \in K$, $\frac{g(t)}{t-z}$ is uniformly continuous as a function of t . Hence, there is a $\delta_{-1} > 0$ such that

$$\left| \frac{g(t)}{t-z} - \frac{g(\hat{t})}{\hat{t}-z} \right| < \frac{\epsilon_2}{4L_{-1}},$$

for $t, \hat{t} \in C_{-1}$, $|t - \hat{t}| < \delta_{-1}$, and any $z \in K$. Partition C_1 and C_{-1} so that the norm of the partition of C_1 is less than δ_1 , and the norm of the partition of C_{-1} is less than δ_{-1} . Order the partition points on C_1 proceeding in a positive direction with respect to the exterior of C_1 . Order the partition points of C_{-1} proceeding in a positive direction with respect to the interior of C_{-1} . Define a_j^1 and a_j^{-1} in analogy with the definition of the coefficients in the proof of Lemma 2.3.1. The function $g(z)$ is used in defining a_j^{-1} and $f(z)$ is used to define the a_j^1 . Consider

$$h_1(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{t-z} dt = \sum_{j=1}^{n_1} \frac{a_j^1}{z - c_j^1},$$

and

$$h_{-1}(z) = \frac{1}{2\pi i} \int_{C_{-1}} \frac{g(t)}{t-z} dt = \sum_{j=1}^{n_{-1}} \frac{a_j^{-1}}{z - c_j^{-1}},$$

where the integral is taken around C_1 in a positive direction with respect to the exterior of C_1 and the integral is taken around C_{-1} in a positive direction with respect to the interior of C_{-1} . We select the sets F_j^1 and F_j^{-1} similar to the way they were chosen in the proof of Lemma 2.3.1. F_j^1 is a closed disc of radius r_j^1 and center c_j^1 , where r_j^1 is chosen so that for $t \in F_j^1$ and

any z in the set K ,

$$\left| \frac{1}{c_j^1 - z} - \frac{1}{t - z} \right| < \frac{\epsilon_2}{4M_1L_1}.$$

For $t \in F_j^{-1}$ and $z \in K$,

$$\left| \frac{1}{c_j^{-1} - z} - \frac{1}{t - z} \right| < \frac{\epsilon_2}{4M_2L_{-1}}.$$

It is also assumed that $F_j^1 \cap F_i^1 = \emptyset$ for $i \neq j$, and that each F_j^1 is contained in the annular region bounded by Γ_1 and Γ_2 . Similarly, $F_j^{-1} \cap F_i^{-1} = \emptyset$ for $i \neq j$, and F_j^{-1} is contained in the annular region bounded by Γ_{-2} and Γ_{-1} . Define ν on $F^1 = \bigcup_{j=1}^{n_1} F_j^1$ by

$$\nu(E) = \int_E \frac{-a_j^1}{\mu(F_j^1)} d\mu$$

for $E \subset F_j^1$. Define ν on $F^{-1} = \bigcup_{j=1}^{n_1} F_j^{-1}$ by

$$(E) = \int_E \frac{-a_j^{-1}}{\mu(F_j^{-1})} d\mu$$

for $E \subset F_j^{-1}$. Let

$$k_1(z) = \sum_{j=1}^{n_1} \frac{a_j^1}{z - c_j^1} - \int_{F^1} \frac{1}{t - z} d\nu(t),$$

and

$$k_{-1}(z) = \sum_{j=1}^{n_{-1}} \frac{a_j^{-1}}{z - c_j^{-1}} - \int_{F^{-1}} \frac{1}{t - z} d\nu(t).$$

In the same manner as in the proof of Lemma 2.3.1, for z in K ,

$$|h_1(z)| < \epsilon_2/4, \quad |h_{-1}(z)| < \epsilon_2/4, \quad |k_1(z)| < \epsilon_2/4, \quad |k_{-1}(z)| < \epsilon_2/4.$$

Let

$$\varphi_1(z) = \frac{1}{2\pi i} \left\{ \int_{C_{-1}} \frac{g(t)}{t-z} dt + \int_{C_1} \frac{f(t)}{t-z} dt \right\} - \int_{F^1 \cup F^{-1}} \frac{1}{t-z} d\nu(t).$$

Then, applying the obvious generalization of Lemma 2.3.2, $\varphi_1(z)$ is holomorphic for $z \in K$, vanishes at infinity and $|\varphi_1(z)| < \epsilon_2$ for $z \in K$.

Proceeding by the method used in the proof of Theorem 2.3.1, but using the curves in succession, and with the changes indicated above, we have at the n -th stage,

$$\varphi_n(z) = \frac{1}{2\pi i} \int_{C_{-1}} \frac{g(t)}{t-z} dt + \frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{t-z} dt - \int_{T_n} \frac{1}{t-z} d\nu(t),$$

where $T_n = \bigcup_{j=1}^n (F^j \cup F^{-j})$. $\varphi_n(z)$ is holomorphic and $|\varphi_n(z)| < \epsilon_n$ for z exterior to Γ_n or z interior to Γ_{-n-1} . Also $\varphi_n(z)$ vanishes at infinity. Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, it follows that

$$\frac{1}{2\pi i} \int_{C_{-1}} \frac{g(t)}{t-z} dt + \frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{t-z} dt = \int_T \frac{1}{t-z} d\nu(t),$$

where $T = \bigcup_{j=1}^{\infty} (F^j \cup F^{-j})$ and z is in D or z is in the exterior of all the curves of the sequence. (The curves are bounded, because $\Sigma(\mu)$ is compact and contains them.)

But $\frac{1}{2\pi i} \int_{C_{-1}} \frac{g(t)}{t-z} dt = g(z)$ for z interior to C_{-1} and the integral is zero for z exterior to C_{-1} . Also,

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{t-z} dt = f(z)$$

for z exterior to C_1 , and this integral is zero for z interior to C_1 . Hence, $\int_T \frac{1}{t-z} d\nu(t) = g(z)$ for z in D , and $\int_T \frac{1}{t-z} d\nu(t) = f(z)$ for z in the exterior of all of the curves. To complete the definition of the measure ν , we define $\nu(E) = 0$ if $E \cap T = \emptyset$.

For future reference, we have purposefully used an arbitrary holomorphic function $f(z)$ in the above approximation. Let $f(z) = 0$. Then for z outside all the curves, say for $|z| > r$,

$$(2.4.1) \quad \int \frac{1}{t-z} d\nu(t) = 0.$$

In obtaining equation (2.3.5), we showed that (2.4.1) implies $\int t^n d\nu(t) = 0$ for $n \geq 0$. Assuming that $g(z) \neq 0$, it follows that $\int \frac{1}{t-z} d\nu(t) \neq 0$ for $z \in D$. Using the same method that was used to obtain equation (2.3.4), it follows that for some $n < 0$, $\int t^n d\nu(t) \neq 0$. From the definition of ν , we have $|\nu(X)| \leq \sum_{i=-\infty}^{\infty} \sum_{j=1}^{n_i} |a_j^i|$. But

$$\sum_{i=-\infty}^{\infty} \sum_{j=1}^{n_i} |a_j^i| < M_1 L_1 + M_2 L_{-1} + \sum_{n=1}^{\infty} \epsilon_n (L_n + L_{-n}) < \infty.$$

Hence ν is totally finite and $\nu \ll \mu$. This proves the theorem.

APPENDIX TO CHAPTER II

The results of Theorem 2.3.1 and Theorem 2.4.3 may be generalized by using the following theorem which was proved by G. R. MacLane.

Theorem. Consider two sets of concentric circles $\{C_n\}_{n=1}^{\infty}$ and $\{\Gamma_n\}_{n=1}^{\infty}$. Let r_1 be the radius of C_1 and let ρ_1 be the radius of Γ_1 , $i = 1, 2, \dots$. Let $z = 0$ be the common center of both sets of circles. Suppose $r_1 < \rho_1 < r_2 < \rho_2 < \dots < r_n < \rho_n < \dots < 1$, with

$$(1) \quad \begin{aligned} \rho_n - r_n &= o((r_n - \rho_{n-1})^3) \\ &= o((r_{n+1} - \rho_n)^3) \text{ as } n \text{ tends to infinity.} \end{aligned}$$

Let E be a point set with angular density in $r_n \leq |z| \leq \rho_n$ defined as follows:

(2) one point of E in every angular opening $A(\rho_n - r_n)$, where A is an arbitrary constant.

Then if $f(z)$ is holomorphic for $|z| \geq r_1$ and $f(z)$ vanishes at infinity, $f(z)$ can be represented for $|z| > 1$ by a Borel series $\sum_{n=1}^{\infty} \frac{\omega_n}{z - \lambda_n}$, where $\{\lambda_n\}_{n=1}^{\infty} \subset E$ and $\sum_{n=1}^{\infty} |\omega_n| < \infty$.

Proof. Let $f(z)$ be holomorphic for $|z| \geq r_1$ and vanish at infinity. Then

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)}{t-z} dt, \quad |z| > r \geq r_1,$$

where the integral is taken around $|t| = r$ in a positive

direction with respect to the exterior of $|t| = r$. Let $t = re^{i\theta}$. Then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta} - z} d\theta.$$

Thus

$$\int_{r_1}^{r_2} f(z) dr = \frac{1}{2\pi} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta} - z} dr d\theta.$$

Hence

$$(3) \quad f(z) = \frac{1}{2\pi(r_2 - r_1)} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{re^{i\theta} - z} dr d\theta.$$

Let $\sigma_n = \frac{1}{2}(\rho_{n-1} + r_n)$. We will define a sequence of functions $\{f_n(z)\}$ such that $f_n(z)$ is holomorphic for $|z| \geq \sigma_n$, $f_n(z)$ vanishes at infinity and $|f_n(z)| \geq M_n$ for $|z| \geq \sigma_n$. Using the integral (3), we approximate $f(z)$ on the annulus bounded by r_1 and ρ_1 . Divide the circle $|z| = \rho_1$ into sectors of angular opening $2\pi/\mu_1$ such that the intersection of each sector with the annulus contains at least one point of the point set E . We order the sets in which the sectors intersect the annulus, call them $F_1^1, F_2^1, \dots, F_{\mu_1}^1$ proceeding around the annulus in a positive direction with respect to the exterior of Γ_1 . Let $c_k^1 \in F_k^1 \cap E$. Define

$$A_k^1 = - \frac{f(c_k^1)}{2\pi(\rho_1 - r_1)} \iint_{F_k^1} re^{i\theta} dr d\theta.$$

Then $\sum_{k=1}^{\mu_1} A_k^1 = \frac{M(\rho_1^2 - r_1^2)\pi}{2\pi(\rho_1 - r_1)} = \frac{M(\rho_1 + r_1)}{2} < M$, where $M = \max. |f(z)|$ for $|z| \geq r_1$. Define

$$f_2(z) = f(z) - \sum_{k=1}^{\mu_1} \frac{A_k^1}{z - c_k^1}.$$

Then $f_2(z)$ is holomorphic for $|z| \geq \sigma_2$ and $f_2(z)$ vanishes at infinity. Let M_2 be such that $|f_2(z)| \leq M_2$ for $|z| \geq \sigma_2$.

Proceeding by induction, suppose $f_n(z)$ has been defined so that $f_n(z)$ is holomorphic for $|z| \geq \sigma_n$, $f_n(z)$ vanishes at infinity and $|f_n(z)| \leq M_n$ for $|z| \geq \sigma_n$.

Define

$$A_k^n = - \frac{f_n(c_k^n)}{2\pi(\rho_n - r_n)} \iint_{F_k^n} re^{i\theta} dr d\theta.$$

Then

$$\sum_{k=1}^{\mu_n} |A_k^n| = \frac{M_n(\rho_n^2 - r_n^2)\pi}{2\pi(\rho_n - r_n)} \leq M_n.$$

For $|z| \geq \sigma_{n+1}$

$$|f_{n+1}(z)| = \frac{1}{2\pi(\rho_n - r_n)} \left| \sum_{k=1}^{\mu_n} \iint_{F_k^n} \left\{ \frac{f_n(re^{i\theta})}{re^{i\theta} - z} - \frac{f_n(r_k e^{i\theta_k})}{r_k e^{i\theta_k} - z} \right\} re^{i\theta} dr d\theta \right|$$

Now

$$\left| \frac{d}{dz} \frac{f_n(\xi)}{\xi - z} \right| \leq \left| \frac{f_n'(\xi)}{\xi - z} \right| + \left| \frac{f_n(\xi)}{(\xi - z)^2} \right|.$$

Using the integral form for the derivative,

$$f_n'(z) = \frac{1}{2\pi i} \int_{|t|=\sigma_n} \frac{f(t) dt}{(t - z)^2} \quad \text{for } |z| > \sigma_n.$$

Since $\rho_n \leq |\xi| \leq r_n$, we have

$$|f'_n(\xi)| \leq \frac{M_n \sigma_n}{(|\xi| - \sigma_n)^2} < \frac{4M_n}{(r_n - \rho_{n-1})^2}.$$

Therefore,

$$\begin{aligned} \left| \frac{d}{dz} \frac{f_n(z)}{z - z} \right| &\leq \frac{4M_n}{(r_n - \rho_{n-1})^2 (\sigma_{n+1} - \rho_n)} + \frac{M_n}{(|z| - \rho_n)^2} \\ &\leq \frac{8M_n}{(r_n - \rho_{n-1})^2 (r_{n+1} - \rho_n)} + \frac{4M_n}{(r_{n+1} - \rho_n)^2} \\ &= \tau_n. \end{aligned}$$

Now for $t \in F_K^n$ and $|z| > \sigma_n$

$$\left| \frac{f_n(t)}{t - z} - \frac{f_n(c_K^n)}{c_K^n - z} \right| \leq \tau_n (\rho_n - r_n + \frac{2\pi}{\mu_n}).$$

Hence

$$\begin{aligned} M_{n+1} &\leq \frac{1}{2\pi(\rho_n - r_n)} \pi(\rho_n^2 - r_n^2) \tau_n (\rho_n - r_n + \frac{2\pi}{\mu_n}) \\ &\leq \frac{\rho_n + r_n}{2} \tau_n (\rho_n - r_n + \frac{2\pi}{\mu_n}) \\ &< \tau_n (\rho_n - r_n + \frac{2\pi}{\mu_n}). \end{aligned}$$

Substituting for τ_n and making use of hypothesis (2),

$$\frac{M_{n+1}}{M_n} < \left\{ \frac{8}{(r_n - \rho_{n-1})^2 (r_{n+1} - \rho_n)} + \frac{4}{(r_{n+1} - \rho_n)^2} \right\} (1+A) (\rho_n - r_n).$$

From hypothesis (1),

$$\lim_{n \rightarrow \infty} \frac{M_{n+1}}{M_n} = 0.$$

Hence M_n tends to zero as n tends to infinity. Consequently,

$$f(z) = \sum_{n=1}^{\infty} \sum_{k=1}^{\mu_n} \frac{A_k^n}{z - c_k^n}.$$

Also by the ratio test for infinite series and from what we have said above,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\mu_n} |A_k^n| \leq \sum_{n=1}^{\infty} M_n < \infty.$$

This proves the theorem.

Using the method of this theorem instead of Runge's method as the basis of our approximation, we can obtain results concerning property (P). Suppose the set $E \subset \Sigma(\mu)$, where μ is a compact measure. Then we may take discs G_k^n contained in the sets F_k^n with $c_k^n \in G_k^n \cap F_k^n$ and such that for $t \in G_k^n \cap F_k^n$ and $|z| > \sigma_n$

$$\left| \frac{1}{t - c_k^n} - \frac{1}{t - z} \right| < \frac{M_n}{2^n M_{n-1}}.$$

Defining ν in analogy with its definition in Theorem 2.3.1, we can show, by the same technique used there, that

$$\int \frac{1}{t - z} d\nu(t) = 0$$

for $|z| > 1$. Then $\int t^n d\nu(t) = 0$, $n \geq 0$, but $\nu \neq 0$. We can conclude then that μ does not have property (P) if $\Sigma(\)$ contains a set E satisfying the hypothesis of the theorem. This is a generalization of the previous result

due to the fact that it does not require that the points of E lie on any prescribed set of curves. This is of particular interest if μ is purely atomic.

Generalizing the above to include doubly infinite sequences of curves, we can obtain a similar extension of Theorem 2.4.3 concerning property (R).

CHAPTER III

INVARIANT NON-REDUCING SUBSPACES OF A NORMAL OPERATOR

3.1. Definitions. If T is a linear transformation whose range and domain are in the same Hilbert space H , then T is an operator on H . We will use operator to mean bounded operator. If T sets up a one-to-one correspondence between its range and its domain, then we define T^{-1} , the inverse of T , to be the transformation whose domain is the range of T and whose range is the domain of T and such that for x in the domain of T , $x = T^{-1}Tx$. If T is an operator on H , we define the adjoint of T , T^* , to be the operator such that $(Tx, y) = (x, T^*y)$ for all $x, y \in H$. This uniquely defines T^* and $\|T^*\| = \|T\|$ [3, p.39]. An operator is normal if $TT^* = T^*T$. Hereafter we shall write A for a normal operator. If $T^* = T^{-1}$, i.e., $T^*T = TT^* = I$, then T is unitary; we use U to denote a unitary operator. If $T^* = T$, then T is Hermitian. A projection P on a subspace $K \subset H$ is the operator such that if $z = x + y$, $x \in K$, $y \in K^\perp$, then $Pz = x$.

If T is an operator, consider the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity transformation. The values of λ for which T_λ has a bounded inverse R_λ with domain H is the resolvent set, $\rho(T)$, of T . The operator R_λ for all $\lambda \in \rho(T)$ is called

the resolvent of T . The complement of $\rho(T)$ in the complex plane is the spectrum of T , written $\sigma(T)$.

Suppose K is a subspace of H . K is said to be invariant with respect to an operator T provided $Tx \in K$ for all $x \in K$. If K and $K^\perp$ are both invariant then K reduces T . If K is invariant with respect to T , then a necessary and sufficient condition that K reduce T is that K be invariant with respect to T^* [3,p.40]. We shall say that a normal operator A has property (P) if every invariant subspace of A also reduces A . The connection between property (P) for an operator and the property (P) for a measure, which we explored in the last chapter, will be discussed in the next paragraph.

3.2. Conditions on $\sigma(A)$ which insure the failure of property (P) for A . In this paragraph we shall apply the results of Chapter II to obtain information about the failure of property (P) for A .

According to the spectral theorem for normal operators [3,p.71], corresponding to A there is a spectral measure E defined for every Borel set M and such that $E(M)$ is a projection on H . For a complete list of the properties, see Halmos [3,p.58]. If $x \in H$, then the set function μ_x defined by $\mu_x(M) = (E(M)x, x)$ is a compact measure. If $x, y \in H$, then the set function defined by

$\mu_{xy}(M) = (E(M)x, y)$ for all Borel sets M is a compact complex measure. The connection between A and E is given by

$$(Ax, y) = \int \lambda d(E_\lambda x, y),$$

for all $x, y \in H$. The operator $E(M)$ commutes with A . We also have

$$(A^*x, y) = \int \bar{\lambda} d(E_\lambda x, y),$$

and

$$(f(A)x, y) = \int f(\lambda) d(E_\lambda x, y),$$

for all f continuous on $\sigma(A)$.

Lemma 3.2.1. For a given $x \in H$, let K be the subspace of H spanned by the set of all vectors of the form $E(M)x$ for all Borel sets M . If μ_x does not have property (P), then there are $x', x'' \in K$ such that $\int \lambda^n d(E_\lambda x', x'') = 0$ for $n \geq 0$, but the complex measure $\mu_{x', x''}$ is not identically zero.

Proof. If μ_x does not have property (P), then there is a measure ν such that $\int \lambda^n d\nu = 0$ for $n \geq 0$ but $\nu \neq 0$. In general, ν will be a complex measure. The proof of the lemma is completed by generalization of a known result; see Halmos [3, p.104]. Since $\nu \ll \mu_x$, by the Radon-Nikodym Theorem, there is a function $g(\lambda)$ which is summable μ_x , and such that

$$\nu(E) = \int_E g d\mu_x.$$

for all Borel sets E . Let $f(\lambda)$ be one branch of $(g(\lambda))^{\frac{1}{2}}$. Then f and $\bar{f}$ belong to $L_2(\mu_x)$. It is known that there is an isomorphism U between K and $L_2(\mu_x)$ [3, p.95]. Let $x' = Uf$ and $x'' = U\bar{f}$. Then

$$\nu(E) = \int_E d(E_\lambda x', x'').$$

This proves the lemma.

Theorem 3.2.1. Let A be a normal operator on the Hilbert space H . Then a necessary and sufficient condition that property (P) fail for A is that there exist an $x \in H$ such that property (P) fails for μ_x .

Proof. Suppose property (P) fails for A . Then there is a subspace $K \subset H$ such that for all $y \in K$, $Ay \in K$ but for some $x \in K$, $A^*x \notin K$. Thus there is a $z \in K^\perp$, such that $(A^n x, z) = 0$ for all $n \geq 0$, but $(A^*x, z) \neq 0$. Hence

$$\int \lambda^n d(E_\lambda x, z) = 0, \quad n \geq 0,$$

but

$$\int \bar{\lambda} d(E_\lambda x, z) \neq 0.$$

Define ν by $\nu(M) = (E(M)x, z)$. Then $\nu \ll \mu_x$. By definition, μ_x does not have property (P).

Suppose there is an $x \in H$ such that μ_x does not have property (P). Applying Lemma 3.2.1, there are $x', x'' \in H$ such that $\int \lambda^n d(E_\lambda x', x'') = 0$ for $n \geq 0$, but $(E(M)x', x'') \neq 0$

for all Borel sets M . Observing that

$$(A^n x', x'') = \int \lambda^n d(E_\lambda x', x''),$$

we see that x'' is orthogonal to the smallest invariant subspace containing x' . The set of all vectors of the form $E(M)x'$ belong to the smallest reducing subspace containing x' . Since $(E(M)x', x'') \neq 0$, x'' is not orthogonal to the smallest subspace of H which reduces A and contains x' . Hence, the smallest invariant subspace containing x' is not reducing. Therefore, property (P) fails for A , and the theorem is proved.

Theorem 3.2.2. Suppose the eigenvectors of A span H . Then a necessary condition that A not have property (P) is that there exist a set of eigenvalues $E = \{\lambda_n\}_{n=1}^\infty$ such that every point of E is inaccessible from the principal component of the complement of $\bar{E}$.

Proof. Suppose property (P) fails for A . Then according to Theorem 3.2.1, there is an $x \in H$ such that μ_x does not have property (P). Now μ_x is a compact purely atomic measure. This theorem then follows from Theorem 2.2.1.

The preceding theorem generalizes two results of Wermer [14, p.272]. The results to which we refer are:

- 1) If the eigenvectors of A span H and the eigenvalues

lie on a Jordan curve, then A has property (P). 2) If the eigenvectors of A span H and there is not an infinite sequence of eigenvalues with strictly increasing modulus, then A has property (P).

The spectrum of an operator A on H is said to be simple if it has uniform multiplicity one. If H is separable, it is well known that there exists a vector $x \in H$ such that $E(M)x$, where M ranges over all Borel sets, span H [12, p.275]. If the spectrum of A on H is not simple, then there is a subspace of H which reduces A and such that the spectrum of A restricted to this subspace is simple. On the other hand, if H is not separable and the spectrum of A on H is simple, then there can be no x such that the $E(M)x$, where M ranges over the Borel sets, span H . For if such a vector existed, then H would be isomorphic to $L_2(\mu_x)$ [3, p.95]. Since $L_2(\mu_x)$ is separable, this contradicts the non-separability of H . However, in the presence of non-separability, we have the following result.

Lemma 3.2.2. Any Hilbert space H has a subspace K which is separable, which reduces A , and such that $\sigma(A|K) = \sigma(A)$.

Proof. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a denumerable dense subset

of the complex plane, say the points with rational real and imaginary parts. Let λ_k be a particular one of the sequence. With λ_k as center, construct a sequence of discs, say $\{S_j\}_{j=1}^{\infty}$, so that if r_j is the radius of S_j , $\lim_{j \rightarrow \infty} r_j = 0$. For each disc choose a vector x_{kj} belonging to the range of the projection $E(S_j)$. If $S_j \cap \sigma(A) = \emptyset$, then $x_{kj} = 0$; otherwise, $x_{kj} \neq 0$. In this way we obtain an infinite sequence of vectors $\{x_{kj}\}_{j=1}^{\infty}$ associated with λ_k . Repeating this process for each λ_k , $k = 1, 2, \dots$, we obtain a doubly indexed sequence of vectors $\{x_{kj}\}_{k,j=1}^{\infty}$. The cycle generated by each x_{kj} , i.e., the subspace spanned by the $E(M)x_{kj}$, for all Borel sets M , is a separable subspace of H . Hence, the denumerable union of all such cycles is separable. Let K be the subspace spanned by these cycles. Then K is separable. Since each of the cycles is reducing for A , it follows from the linearity and continuity of A that K reduces A .

We show that $\sigma(A) = \sigma(A|K)$. It suffices to show that $\sigma(A) \subset \sigma(A|K)$. Suppose $\lambda \in \sigma(A)$. Then any neighborhood $N(\lambda)$ contains one of the λ_k 's, say λ_k , such that some $S_1(\lambda_k)$ has the properties: 1) $S_1(\lambda_k) \subset N(\lambda)$, 11) $S_1(\lambda_k) \cap \sigma(A) \neq \emptyset$. By definition, there is an $x_{k1} \neq 0$ such that $E(S_1)x_{k1} = x_{k1}$. But this means that $S_1(\lambda_k)$ contains a point of the spectrum of $\sigma(A|K)$. If this

point is λ then λ belongs to $\sigma(A|K)$; if this point is not λ then this shows that λ is a limit point of $\sigma(A|K)$. Since $\sigma(A|K)$ is closed, $\lambda \in \sigma(A|K)$ in either case. Hence $\sigma(A) \subset \sigma(A|K)$.

Theorem 3.2.3. If $\sigma(A)$ contains a sequence of simple rectifiable closed Jordan curves $\{C_1\}_{1=0}^{\infty}$ such that the sequence increases monotonically, then property (P) fails for A.

Proof. Suppose H is separable. Then there is a subspace of H, say K, such that K reduces A and the spectrum of $B = A|K$ is simple. Also, $\sigma(B) = \sigma(A)$. Further, K has the property that it is spanned by the vectors $E(M)x$, where M ranges over the Borel sets, for some x in H. (The existence of K and x for a self-adjoint operator is proved by Stone [12, p.250]. His proof is valid for normal operators.) A simple application of a known result [3, p.64], shows that $\sigma(B) = \Sigma(\mu_x)$. It then follows from Theorem 2.3.1 that μ_x does not have property (P). The desired result then follows from Theorem 3.2.1.

If H is not separable, then an application of Lemma 3.2.2 gives us a separable subspace of H, K_1 . Replacing H by K_1 in the above proof gives the desired result.

Corollary 3.2.31. If $\overset{\circ}{\sigma}(A) \neq \emptyset$, then property (P)

fails for A.

Proof. If $\sigma^0(A) \neq 0$, then $\sigma(A)$ certainly contains curves such as those described in the hypothesis of the theorem.

3.3. Subnormal operators. Let A be a normal operator on the Hilbert space H. Let K be a subspace of H which is invariant with respect to A. Let B be the restriction of A to K. Then B is said to be subnormal. The only case of a subnormal operator which interests us is when K is invariant but not reducing, i.e., B is not a normal operator. If A does not have property (P), then it is the extension of such a subnormal operator. Suppose B is a subnormal operator on the Hilbert space K. Then A on H is the minimal normal extension of B if there is no proper subspace M of H with $K \subset M \subset H$ such that M reduces A.

If A does not have property (P) then A on H is the minimal normal extension of a subnormal operator B. For if A does not have property (P) then there is a subspace $K_2 \subset H$ such that K_2 is invariant under A but K_2 does not reduce A. Let K_1 be the smallest subspace of H which reduces A and contains K_2 . Then we can write $H = K_1 \oplus K_1^\perp$. Let $\hat{H} = K_2 \oplus K_1^\perp$. Then $\hat{H}$ is invariant under A but does

reduce A . Let $B = A|_{\hat{H}}$. B is subnormal. Furthermore, A on H is the minimal normal extension of B on $\hat{H}$.

If $\sigma(A)$ contains a sequence of simple closed rectifiable Jordan curves $\{C_1\}_{1=0}^{\infty}$ such that the sequence increases monotonically, then from Theorem 3.2.3 we know that A does not have property (P). In this case the following discussion gives a construction of B in terms of A and a particular invariant non-reducing subspace K_2 of A . Since A does not have property (P), by Theorem 3.2.1, there is a vector $x \in H$ such that μ_x does not have property (P). This means that there is a complex measure $\nu \ll \mu_x$ such that $\int \lambda^n d\nu = 0$, $n \geq 0$, but $\nu \neq 0$. The complex measure ν is constructed from μ_x in Theorem 2.3.1. We shall show how the smallest reducing subspace K_1 containing K_2 is related to the construction of ν from μ_x .

Suppose H is separable. We can write $H = H_1 \oplus H_2$, where H_1 reduces A and $A|_{H_1}$ has simple spectrum with $\sigma(A|_{H_1}) = \sigma(A)$. H_1 is separable with simple spectrum, hence, as above, there is a vector $x \in H_1$ such that the vectors $E(M)x$, where M ranges over the Borel sets, span H_1 . Then the measure μ_x satisfies the hypothesis of Theorem 2.3.1. Hence, there is a complex measure ν constructed from μ_x by the method used to prove that theorem, such that $\nu \ll \mu_x$, $\int \lambda^n d\nu = 0$ for $n \geq 0$, but $\nu \neq 0$. Let

F be the point set associated with the curves $\{C_i\}_{i=0}^{\infty}$ used to define ν as in the proof of Theorem 2.3.1. Now since $\nu \ll \mu_x$, the Radon derivative $\frac{d\nu}{d\mu_x}$ exists a.e. $[\mu_x]$.

Let G be the subset of F for which $\frac{d\nu}{d\mu_x} \neq 0$. Let K_1 be the spectral subspace associated with G , i.e., the range of $E(G)$.

According to Lemma 3.2.1, there are vectors y, z such that $\nu = \mu_{yz}$. Let K_2 be the subspace spanned by $\{A^n y\}_{n=0}^{\infty}$. Since $\int \lambda^n d(E_{\lambda} y, z) = 0$, $n \geq 0$, it follows that $z \perp K_2$. However, as we have pointed out before, the fact that $\mu_{yz} \neq 0$ implies that z is not orthogonal to the smallest reducing subspace containing y . Hence K_2 is an invariant non-reducing subspace of H_1 . Let $M_1 = \sigma(A) - G$. Then the spectral subspace of $A|_{H_1}$ associated with M_1 is $K_1^{\perp}$. Since K_2 is invariant non-reducing, it follows that

$$\hat{H} = K_2 \oplus K_1^{\perp} \oplus H_2$$

is invariant non-reducing for A . Hence $A|_{\hat{H}}$ is subnormal. We wish to show that $A|_H$ is the minimal normal extension of $A|_{\hat{H}}$. It suffices to show that $A|_{K_1}$ is the minimal normal extension of $A|_{K_2}$.

It is well known that if a subspace K reduces an operator A , then the projection on K must commute with A .

But the commutative ring generated by $A|H_1$ is maximal abelian. This means that the projections which commute with $A|H_1$ are spectral projections. Therefore, the desired subspace is spectral. Now the spectrum of $A|K_1$ contains $G \cap \sigma(A)$, which is the spectral set used to define ν . Hence K_1 is a reducing subspace containing K_2 . On the other hand, if $M \subset G \cap \sigma(A)$, $E(M)y \neq 0$. For if $E(M)y = 0$, then $(E(M)y, z) = \nu(M) = 0$ which is impossible because of the definition of G . Thus if $K \supset K_2$ and K reduces A , then $\sigma(A|K) \supset G$. Hence K_1 is the smallest reducing subspace of H which contains K_2 . Therefore, $A|K_1$ is the minimal normal extension of $A|K_2$.

Suppose H is not separable. Applying Lemma 3.2.2, there is a separable subspace K of H and K reduces A . Also $\sigma(A|K) = \sigma(A)$. We write $H = K \oplus K^\perp$. Replace H in the above construction by K . All of the construction goes through as before except $K^\perp$ must be added as a direct summand to determine $\hat{H} = \hat{K}$.

Scholium. If A does not have property (P) then there is an $x \in H$ such that μ_x does not have property (P). We can choose x so that $\Sigma(\mu_x) = \sigma(A)$. From the definition of property (P), there is a complex measure ν associated with μ_x . If $\sigma(A)$ contains a monotonically increasing sequence of simple closed rectifiable

curves $\{C_1\}_{i=0}^{\infty}$, then a construction is given by the proof of Theorem 2.3.1 for ν in terms of μ_x . Let H_1 be the subspace of H spanned by the $E(M)x$'s, where M ranges over all Borel sets. We can identify an invariant non-reducing subspace K_2 of A such that the smallest reducing subspace K_1 containing K_2 is the spectral subspace of $A|_{H_1}$ determined by the support of ν .

3.4. Holes in the spectrum. We will refer to all components of $\rho(A)$ except the principal component as holes in the spectrum of A . It is our purpose here to discover conditions under which there is an invariant subspace K such that the spectrum of $A|_K$ includes $\sigma(A)$ and, also, at least one hole in $\sigma(A)$. Now if K is such a subspace then K is also non-reducing. This follows from the fact that if K is reducing for A then $\sigma(A|_K) \subset \sigma(A)$ [6, p.45]. Hence if such a subspace exists, then property (P) fails for A .

We will show first that if D is a hole in the spectrum, the problem of finding an invariant subspace K such that $\sigma(A|_K)$ includes D is the same as the problem of finding an invariant subspace K such that K is not invariant with respect to R_{λ_0} for some fixed $\lambda_0 \in D$. The proof of the equivalence of these two problems depends in no way on the fact that the operator is normal.

In fact, the proof does not depend on any of the properties of a Hilbert space which distinguish it from any other Banach space. In other words, our proof will show that the two problems are equivalent for an arbitrary bounded operator on a Banach space. For a normal operator on a Hilbert space, this result has been proved by Bram [1, p. 82].

If λ_0 is a point such that the range of $A - \lambda_0 I$ is not dense, then λ_0 belongs to the residual spectrum of A , written $R\sigma(A)$.

Lemma 3.4.1. If A^{-1} is defined for all $x \in H$ and if $K \neq 0$ is an invariant subspace with respect to A , then $(A|K)^{-1}$ is defined for all $x \in K$ if and only if K is invariant with respect to A^{-1} .

Proof. Suppose A^{-1} is defined for all $x \in H$ and K is invariant with respect to A . If also K is invariant with respect to A^{-1} , then $A|K$ and $A^{-1}|K$ are clearly inverses of each other. In any case A^{-1} is defined on $A(K)$ and must carry it onto K . Hence, if $A|K$ has an inverse, then $A(K) = K$. Thus $(A|K)^{-1} = A^{-1}|K$ and K is invariant with respect to A^{-1} .

Lemma 3.4.2. Suppose K is an invariant subspace of H with respect to A . If $\lambda \in \rho(A)$ and $\lambda \in \sigma(A|K)$, then

$\lambda \in R\sigma(A|K)$.

Proof. Since $\lambda \in \rho(A)$, R_λ is defined for all $x \in H$. By the previous lemma, $(A|K - \lambda I)^{-1}$ is defined for all x in K if and only if K is invariant with respect to R_λ . Now if K is invariant with respect to $(A - \lambda I)^{-1} = R_\lambda$, then $(A|K - \lambda I)^{-1}x = R_\lambda x$ for all $x \in K$. Since $\lambda \in \rho(A)$, this would imply $\lambda \in \rho(A|K)$. Hence, K is not invariant with respect to R_λ if $\lambda \in \sigma(A|K)$. Therefore, there is an $x \in K$ such that $R_\lambda x$ does not belong to K . Consequently, $(A|K - \lambda I)^{-1}x$ is undefined, i.e., $\lambda \in R\sigma(A|K)$.

Theorem 3.4.1. Suppose D is a hole in $\sigma(A)$ and K is an invariant subspace of A . A necessary and sufficient condition that $\sigma(A|K) \supset D$ is that there exist a $\lambda_0 \in D$ such that K is not invariant with respect to R_{λ_0} .

Proof. Let K be an invariant subspace of A . Suppose $\lambda \in \sigma(A|K)$ with $\lambda \in D$. Since $\lambda \in \rho(A)$, by the preceding lemma, $\lambda \in R\sigma(A|K)$. As was pointed out in the proof of Lemma 3.4.2, this implies that K is not invariant with respect to R_λ .

On the other hand, suppose K is not invariant with respect to R_{λ_0} for some $\lambda_0 \in D$. Then there is an $x \in K$ and a $y \in K^\perp$ such that $(R_{\lambda_0}x, y) \neq 0$. Now $(R_\lambda x, y) = g(\lambda)$ is holomorphic on D . On D , $g(\lambda) \neq 0$, therefore, it is

zero at most at a denumerable nowhere dense subset of D . But for any λ_1 such that $g(\lambda_1) \neq 0$ with $\lambda_1 \in D$, K is not invariant with respect to R_{λ_1} . This means that, in particular, $R_{\lambda_1}x \notin K$, i.e., $(A|_K - \lambda_1 I)^{-1}x$ is not defined. Hence $\lambda_1 \in R\sigma(A|_K)$. Therefore, since $g(\lambda) \neq 0$ except at a denumerable nowhere dense subset of D , every $\lambda \in D$ except a denumerable nowhere dense set belongs to $R\sigma(A|_K)$. Since $\sigma(A|_K)$ is closed, $D \subset \sigma(A|_K)$.

Corollary 3.4.11. Let A be an operator on the Hilbert space H . If K is invariant under A and λ belongs to the principal component of $\rho(A)$, then λ cannot belong to $\sigma(A|_K)$.

Proof. The proof of the theorem shows that if $\lambda \in \sigma(A|_K)$ then every point of the principal component of $\rho(A)$ belongs to $\sigma(A|_K)$. But the fact that A is bounded on H implies that $A|_K$ is bounded. Then, according to a well known fact, $\sigma(A|_K)$ is a bounded point set. Hence $\sigma(A|_K)$ cannot contain the point at infinity. Thus the assumption that $\lambda \in \sigma(A|_K)$ leads to a contradiction.

As we have pointed out before, the preceding theorem and corollary are valid for any bounded operator on a Banach space. Bram proved the result for a normal operator on a Hilbert space [1, p.82].

Theorem 3.4.2. Let A be a normal operator on the Hilbert space H . If $\sigma(A)$ has a hole D and K is an invariant subspace of H such that $\sigma(A|_K)$ includes D , then there is an $x \in H$ such that μ_x does not have property (R).

Proof. Suppose the hypothesis of the theorem is satisfied. Then there is a $\lambda_0 \in D$, and $x \in K$, $y \in K^\perp$ such that $((A - \lambda_0 I)^{-1}x, y) \neq 0$. Then

$$(A^n x, y) = \int \lambda^n d(E_\lambda x, y) = 0, \quad n \geq 0.$$

But

$$((A - \lambda_0 I)^{-1}x, y) = \int (\lambda - \lambda_0)^{-1} d(E_\lambda x, y) \neq 0.$$

Let $\nu = \mu_{xy}$. Then $\nu \ll \mu_x$. From the definition of property (R), it follows that μ_x does not have property (R).

It is not known if the converse of this theorem is true or false.

Theorem 3.4.3. Suppose $\sigma(A)$ contains a doubly infinite sequence of simple closed, rectifiable Jordan curves $\{C_1\}_{1=-\infty}^{\infty}$ such that the sequence $\{C_1\}_{1=0}^{\infty}$ increases monotonically, the sequence $\{C_1\}_{1=-\infty}^0$ decreases monotonically, and the intersection of the interiors of the $\{C_1\}_{1=0}^{\infty}$ has a non-void interior D . Then there is an invariant non-reducing subspace K of H such that $\sigma(A|_K)$ includes D .

Proof. We again consider the case H separable first. In this case, as we have observed before, there

is a subspace $K \subset H$ such that K reduces A , the spectrum of $A|_K$ is simple, and $\sigma(A|_K) = \sigma(A)$. Let x be a vector such that the set of vectors $E(M)x$, where M ranges over the Borel sets, span K . Then according to Theorem 2.4.3 μ_x does not have property (R). In particular, from the proof of Theorem 2.4.3, there exists a compact complex measure ν with $\nu \ll \mu_x$ such that $\int \lambda^n d\nu = 0$, $n \geq 0$, but for some $\lambda_k \in D$ and some $n_0 > 0$, $\int (\lambda - \lambda_k)^{-n_0} d\nu \neq 0$. If D has more than one component, from the proof of the theorem, it follows that such a λ_k exists for each component. We have seen in the proof of Lemma 3.2.1 that ν can be written $\nu(M) = (E(M)y, z)$ for some $y, z \in K$. Hence z is orthogonal to the smallest invariant subspace K_1 containing y , but z is not orthogonal to the smallest subspace containing y which is invariant under $(A - \lambda_k I)^{-n_0}$. Thus z is not orthogonal to the smallest subspace containing y which is invariant under $(A - \lambda_k I)^{-1}$. Since a λ_k exists for each component of D , from Theorem 3.4.1, it follows that $D \subset \sigma(A|_{K_1})$.

If H is not separable, we apply Lemma 3.2.2 to obtain a subspace H_1 of H such that H_1 reduces A , is separable, and $\sigma(A|_{H_1}) = \sigma(A)$. The above proof is then repeated with H_1 replacing H in the argument.

Corollary 3.4.31. Suppose $\sigma(A)$ contains a doubly infinite sequence of simple closed rectifiable Jordan curves $\{C_1\}_{1=-\infty}^{\infty}$ such that the sequence $\{C_1\}_{1=0}^{\infty}$ increases monotonically, the sequence $\{C_1\}_{1=0}^{-\infty}$ decreases monotonically, and the intersection of the interiors of the $\{C_1\}_{1=0}^{-\infty}$ has a non-void interior D . Then A on H is the minimal normal extension of a subnormal operator B such that $\sigma(B) = \sigma(A) \cup D$.

Proof. Using essentially the same procedure used in paragraph 3.3, we can obtain a subnormal operator B such that A on H is its minimal normal extension. As a matter of fact, the only difference between the proof required here and the procedure used in paragraph 3.3 is that at the appropriate places the results and proof of Theorem 2.4.3 must be used instead of the corresponding results and proof of Theorem 2.3.1. The above theorem, Theorem 3.4.3, shows that B may be chosen so that (B) contains D . An application of a result due to Halmos [6, p.46], then shows that $\sigma(B) = D \cup \sigma(A)$.

Corollary 3.4.32. Suppose $\sigma(A)$ contains a sequence of simple closed rectifiable Jordan curves $\{C_1\}_{1=0}^{\infty}$ which increase monotonically. Let $\{D_j\}_{j=1}^{\infty}$ be the set of all holes in $\sigma(A)$. Suppose $\sigma(A)$ contains a sequence of

curves $\{C_i\}_{i=1}^{\infty}$, which decrease monotonically, associated with each D_j as in the hypothesis of the theorem such that the sequence of curves associated with D_k are all exterior to the sequence of curves associated with D_j for $j \neq k$, and such that each of the decreasing sequences is interior to C_0 . Then 1) associated with each hole D_j there is an invariant non-reducing subspace K_j such that $K_k \perp K_j$ for $j \neq k$ and $\sigma(A|_{K_j}) \supset D_j$, 2) there is a subnormal operator B such that $\sigma(B) = \sigma(A) \cup$ (any finite number of the holes in $\sigma(A)$) and A is the minimal normal extension of B .

Proof. 1) The fact that there is an invariant non-reducing subspace K_j associated with each D_j such that $\sigma(A|_{K_j}) \supset D_j$ is a direct application of the theorem. Changing the procedure of paragraph 3.3 as indicated in the proof of the Corollary 3.4.31, we see that the smallest reducing subspace containing K_j is a spectral subspace, say K_j^i . The spectral sets used to determine K_j^i and K_k^i may be so chosen that $K_j^i \perp K_k^i$. This implies $K_j \perp K_k$.

2) This statement depends on the fact that the approximation process carried out in proving Theorem 2.4.3 can be carried out for any finite number of holes simultaneously. It was carried out for two domains in the proof of Theorem 2.4.3 as a matter of convenience.

Let us recapitulate what this last corollary says. If the holes D are suitably surrounded by spectra, then we can find an invariant non-reducing subspace associated with each hole. If we are judicious in our choice of spectra used in the approximation process, these invariant non-reducing subspaces will be two by two orthogonal. Of course the spectrum of A restricted to one of these subspaces includes the corresponding hole in $\sigma(A)$. The second statement of the corollary asserts that we can build a subnormal operator B such that A is its minimal normal extension and $\sigma(B)$ includes any specified finite number of holes in $\sigma(A)$. In view of Corollary 3.4.31, this merely amounts to saying that the approximation process used in Theorem 2.4.3 can be used to approximate a non-zero function on any finite number of the holes while the zero function is being approximated on the principal component of $\rho(A)$.

In the discussion above, we have found that many different subnormal operators may have the same minimal normal extension. It might be possible that the minimal normal extensions set up an equivalence class decomposition of the subnormal operators. If so, the "natural" equivalence would seem to be unitary equivalence. On the basis of the above results, we can show that the

minimal normal extensions do not set up an equivalence class decomposition of subnormal operators according to unitary equivalence even if the subnormal operators have the same spectrum.

We construct an example of two subnormal operators with the same spectrum, which have the same minimal normal extension, but which are not unitarily equivalent. Let A be a normal operator on the Hilbert space H such that $\sigma(A) \neq \emptyset$. Let C_1 and C_2 be two circles in $\sigma(A)$ such that C_1 is in the exterior of C_2 and the interiors of C_1 and C_2 belong to $\sigma(A)$. Let r_i be the radius of C_i , $i = 1, 2$. Construct circles C'_i concentric with C_i , of radius $r_i/2$. Let G_i be the annulus between C_i and C'_i , and let K_i be the spectral subspace of G_i . Now $\sigma(A|_{K_i})$ has a hole D_i . Applying Corollary 3.4.31, there is a subspace $K'_i \subset K_i$ and a subnormal operator $B'_i = A|_{K'_i}$ such that $\sigma(B'_i) = \sigma(A|_{K_i}) \cup D_i$ and $A|_{K_i}$ is the minimal normal extension of B'_i . By Lemma 3.4.2, D_i is residual spectrum for B'_i . Let $B_i = B'_i \oplus A|_{K_i}$. Then B_i is a subnormal operator on the Hilbert space $K'_i \oplus K_i$ and $\sigma(B_i) = \sigma(A)$. Also $D_i \subset R\sigma(B_i)$. However, the spectral nature of D_2 is the same for B_1 as it is for A . Similarly, D_1 is of the same spectral nature for B_2 as it is for A . Since A can have no residual spectrum, $D_2 \not\subset R\sigma(B_1)$ and $D_1 \not\subset R\sigma(B_2)$. Hence

B_1 and B_2 cannot be unitarily equivalent. However, A on H is the minimal normal extension of both B_1 and B_2 .

The next result properly belongs to Wermer. He did not state the theorem as it is stated here. However, except for a small amount of work which we add, he had all the parts ready for assembly.

Theorem 3.4.4. If $\sigma(A)$ is a simple closed rectifiable curve C , then a necessary and sufficient condition that A not have property (P) is that there exist an invariant subspace $K \subset H$ such that $\sigma(A|K) = C \cup (\text{interior of } C)$.

Proof. We first summarize the results due to Wermer that are applicable here [14, p.276]. He has shown that there is a unitary operator U such that U and A are functions of each other. In particular, $U = f(A)$, where the function $f(\lambda)$ has the properties that it maps the interior of C conformally onto $|f| < 1$, and maps C (interior of C) homeomorphically onto $|f| \leq 1$. It is shown that property (P) fails for A if and only if it fails for U , but property (P) fails for U if and only if there is a subspace K on which U is the one-sided shift operator. Wermer's proof shows that K must also be invariant non-reducing for A .

It is well known that the spectrum of the one-sided shift operator is the unit disc. Hence property (P) fails for U if and only if there is an invariant subspace K for U such that $\sigma(U|_K)$ is the unit disc. It suffices, then, to show that $f \in \sigma(U|_K)$ implies $g(f) \in \sigma(A|_K)$ where $g(f)$ is the inverse function of $f(\lambda)$. The proof of this fact consists of the standard steps used in proving the spectral mapping theorem.

We proceed to show that $g(\sigma(U|_K)) \subset \sigma(A|_K)$. From Corollary 3.4.11, we know that no point exterior to the unit circle belongs to $\sigma(U|_K)$. Now every point on C belongs to the spectrum of $A|_K$. For the spectrum of the minimal normal extension of $A|_K$ must separate the plane, [14, p.275], and must be contained in C [6, p.45]. And the spectrum of $A|_K$ contains the spectrum of the minimal normal extension of $A|_K$ [6, p.46]. Hence it suffices to show that if $f \in \sigma(U|_K)$ and f is interior to the unit circle then $g(f) \in \sigma(A|_K)$. This will be proved by showing that if λ_1 is interior to C and λ_1 does not belong to $\sigma(A|_K)$, then $f(\lambda_1)$ does not belong to $\sigma(U|_K)$.

Let λ_1 be interior to C and suppose λ_1 does not belong to $\sigma(A|_K)$. Then for $x \in K$ and $y \in K^\perp$, $((A - \lambda_1 I)^{-1}x, y) = 0$. Let $f(\lambda_1) = f_1$, or $g(f_1) = \lambda_1$. Define

$$\hat{g}(f) = \frac{g(f) - g(f_1)}{f - f_1},$$

for $\xi \neq \xi_1$, and $\hat{g}(\xi_1) = g'(\xi_1)$. Then $\hat{g}(\xi)$ is holomorphic inside the unit circle and continuous on the unit circle. Thus, $\hat{g}(\xi)$ can be approximated uniformly on the unit circle by polynomials in ξ [13, p.36]. From the definition of A as a function of U ,

$$((U - \xi_1 I)^{-1}x, y) = ((A - \lambda_1 I)^{-1}\hat{g}(U)x, y),$$

for any $x, y \in H$. Let $x \in K$, $y \in K^\perp$. Since $\hat{g}(\xi)$ can be approximated uniformly on the unit circle by polynomials, $\hat{g}(A)x$ belongs to K . Since λ_1 does not belong to the spectrum of $A|_K$,

$$((A - \lambda_1 I)^{-1}\hat{g}(U)x, y) = 0.$$

The elements x, y are arbitrary, thus $((U - \xi_1 I)^{-1}x, y) = 0$ for all $x \in K$, $y \in K^\perp$. Therefore, ξ_1 does not belong to the spectrum of $U|_K$. This proves the theorem.

3.5. Functions representable by $(R_\lambda x, y)$ for x, y in H . The work that we have done gives us some information about what functions are representable by the resolvent $(R_\lambda x, y)$. This paragraph is a resume of these results.

Theorem 3.5.1. If $\sigma(A)$ contains a sequence of simple closed rectifiable Jordan curves $\{C_1\}_{1=0}^\infty$ such that the sequence $\{C_1\}_{1=0}^\infty$ increases monotonically, then every function $f(\lambda)$, which is holomorphic on a region

including the principal component of $\rho(A)$ and the curves C_1 and which vanishes at infinity, is representable as $(R_\lambda x, y)$ for some $x, y \in H$.

Proof. We again make use of the fact that there is an $x \in H$ such that if K is the subspace of H spanned by $E(M)x$, where M ranges over all Borel sets, then $\sigma(A|K) = \sigma(A)$. The result then follows by applying the proof of Theorem 2.3.1.

Theorem 3.5.2. Suppose $\sigma(A)$ contains a doubly infinite sequence of simple closed rectifiable Jordan curves $\{C_1\}_{1=-\infty}^{\infty}$ such that the sequence $\{C_1\}_{1=0}^{\infty}$ increases monotonically, the sequence $\{C_1\}_{1=0}^{-\infty}$ decreases monotonically, and the intersection of the interiors of the $\{C_1\}_{1=0}^{-\infty}$ has a non-void interior D . If $g(\lambda)$ is holomorphic on the interior of C_0 and $f(\lambda)$ is holomorphic on the exterior of C_0 and vanishes at infinity, then there are $x, y \in H$ such that $(R_\lambda x, y)$ represents $f(\lambda)$ on the principal component of $\rho(A)$ and $(R_\lambda x, y)$ represents $g(\lambda)$ on D .

Proof. Applying the method of approximation used in the proof of Theorem 2.4.3 to μ_x , where x is the same as in the previous proof. According to Lemma 3.2.1, there are $y, z \in H$ such that

$$(R_\lambda y, z) = \int \frac{1}{t - \lambda} d(E_t y, z) = \begin{cases} f(\lambda) & \text{for } \lambda \text{ in the principal} \\ & \text{component of } \rho(A), \\ g(\lambda) & \text{for } \lambda \text{ in } D. \end{cases}$$

Corollary 3.5.21. Suppose $\{D_i\}_{i=1}^n$ is any finite set of holes in $\sigma(A)$ and suppose $\{C_k^1\}_{k=1}^\infty$, $1 = 1, 2, \dots, n$, are simple closed rectifiable Jordan curves such that D_1 is the interior of the intersection of the interiors of $\{C_k^1\}_{k=1}^\infty$. Suppose the $\{C_k^1\}_{k=1}^\infty$ are all exterior to the $\{C_k^j\}_{k=1}^\infty$ for $1 \neq j$. Let $\sigma(A)$ also contain a sequence of simple closed rectifiable Jordan curves $\{C_k\}_{k=1}^\infty$ which increases monotonically and such that each of the above sequences is interior to C_1 . Then there are vectors $x, y \in H$ such that $(R_\lambda x, y)$ represents a different holomorphic function on each of the components D_i and on the principal component of $\rho(A)$.

Proof. The approximation process used in the proof of Theorem 2.4.3 is applicable to any finite number of suitably surrounded open sets. The use of two sets in the proof of Theorem 2.4.3 was simply an expedient to make the explanation of the construction easier.

Wermer has shown that if K is an invariant subspace of A which is not invariant under A^{-1} , and if there is a vector x such that K is spanned by $\{A^n x\}_{n=1}^\infty$, then K is

homomorphic to a space of functions holomorphic on the component of $\rho(A)$ containing the origin [15, p.862].

3.6. An operator on l_1 . By l_1 space we mean the set of all sequences $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$. This space is a Banach space. The norm for l_1 is defined for any $x = \{a_n\}$ of l_1 by $\|x\|_1 = \sum_{n=1}^{\infty} |a_n|$.

A Vandermonde matrix is a matrix of the form $v = (\lambda_i^{j-1})$, $i, j = 1, 2, \dots, n$. The matrix v can be considered as an operator on Euclidean n -space. By an infinite Vandermonde matrix we mean the infinite matrix defined by $V = (\lambda_i^{j-1})$, $i, j = 1, 2, \dots$. The infinite Vandermonde is an operator on l_1 provided $|\lambda_i| \leq r < 1$. It is well known that the finite Vandermonde determinant is zero if and only if for some $j \neq k$, $\lambda_j = \lambda_k$, i.e., the λ_i are not distinct. Then assuming the λ_i to be distinct, zero is not in the spectrum of v . A different description of some of our earlier results give conditions under which V has zero in its spectrum even though the λ_i are distinct.

Returning to Theorem 2.2.1 and Theorem 2.3.1, we give the results obtained there another interpretation.

Theorem 3.6.1. A necessary condition that V have zero in its point spectrum is that every point of $E = \{\lambda_n\}_{n=1}^{\infty}$ be inaccessible from the principal component of

the complement of $\bar{E}$. A sufficient condition that V have zero in its point spectrum is that there is a set $E = \{\lambda_1\}_{1=1}^{\infty}$ such that $\bar{E}$ contains a set of simple closed rectifiable Jordan curves $\{C_1\}_{1=1}^{\infty}$ which increases monotonically.

Proof of the necessary condition. We say that zero belongs to the point spectrum of V provided there is an $x \neq 0$ such that $Vx = 0$. Suppose $x = \{\omega_1\}$ and E is the set referred to in the proof of Theorem 2.2.1. Then $\sum_{1=1}^{\infty} |\omega_1| < \infty$. Hence $x \in l_1$. Interpreting the result of Theorem 2.2.1, we see that a necessary condition that $Vx = 0$ is that every point of E be inaccessible from the principal component of the complement of $\bar{E}$.

Proof of the sufficient condition. Suppose the measure μ considered in Theorem 2.3.1 is atomic. Then ν is atomic. Let $E = \{\lambda_1\}_{1=1}^{\infty}$ be the atoms of ν . Let $\nu(\{\lambda_1\}) = \omega_1$. Then since ν is totally finite, we know that $\sum_{1=1}^{\infty} |\omega_1| < \infty$. The fact that $\int \lambda^n d\nu = 0, n \geq 0$, implies that

$$(3.6.1) \quad \sum_{1=1}^{\infty} \lambda_1^n \omega_1 = 0, n \geq 0.$$

Let $x = \{\omega_1\}$. Then (3.6.1) is nothing more than the n -th element of Vx . Hence $Vx = 0$.

CHAPTER IV

A CHARACTERIZATION OF PROPERTY (P)

4.1. Property (P) and the direct sum of copies of
H. There are three topologies for the space of operators on a Hilbert space which have received considerable attention. These are: 1) the uniform topology, 2) the strong topology, and 3) the weak topology [10, p.378]. The notion of convergence associated with each of these is as follows: the sequence of operators $\{A_n\}_{n=1}^{\infty}$ is said to converge to the operator A 1) in the uniform topology provided $\lim_{n \rightarrow \infty} \|A_n - A\| = 0$, 2) in the strong topology provided for every $x \in H$, $\lim_{n \rightarrow \infty} \|A_n x - Ax\| = 0$, 3) in the weak topology provided for every $x, y \in H$, $\lim_{n \rightarrow \infty} |(A_n x, y) - (Ax, y)| = 0$. Wermer has shown that a sufficient condition that A have property (P) is the weak convergence of a sequence of polynomials in A to A^* . By a counter example, he has shown that this condition is definitely not necessary. We give here a necessary and sufficient condition that A have property (P) in terms of the more general notion of Moore-Smith convergence. We shall need the following theorem.

Theorem 4.1.1. Let A be a bounded normal operator on a Hilbert space H. Let $K = \sum_{i \in I} \oplus H_i$ where $H = H$

for all $\iota \in I$. Define $\tilde{A} = \sum_{\iota \in I} \oplus A_{\iota}$, where $A_{\iota} = A|_{H_{\iota}}$. If A has property (P) as an operator on H , then $\tilde{A}$ has property (P) as an operator on K .

Proof. According to Theorem 3.2.1 and Lemma 3.2.1, a necessary and sufficient condition that A have property (P) is that $\int \lambda^n d(E_{\lambda}x, y) = 0$ for $n \geq 0$ and for $x, y \in H$ implies that $\int_M d(E_{\lambda}x, y) = 0$ for all Borel sets M . Therefore, it suffices to show that $x, y \in K$ and $\int \lambda^n d(\tilde{E}_{\lambda}x, y) = (A^n x, y) = 0$ for $n \geq 0$ implies $\int d(\tilde{E}_{\lambda}x, y) = 0$ for all M .

Let $x = \sum_{\alpha \in I} \oplus x_{\alpha}$, $y = \sum_{\beta \in I} \oplus y_{\beta} \in K$. Only a denumerable number of the x_{α} or y_{β} may be different from zero. Hence, we can write

$$x = \sum_{i=1}^{\infty} \oplus x_{\alpha_i}, \quad y = \sum_{j=1}^{\infty} \oplus y_{\beta_j}$$

with $x_{\alpha_i} \in H_{\alpha_i}$, $y_{\beta_j} \in H_{\beta_j}$. Now for every Borel set M we have

$$(\tilde{E}(M)x, y) = \sum \delta_{\alpha_i \beta_j} (E(M)x_{\alpha_i}, y_{\beta_j}),$$

where $\delta_{\alpha_i \beta_j}$ is the Kronecker delta. Thus we can write

$$\nu(M) = (E(M)x, y) = \sum_{n=1}^{\infty} (E(M)x_n, y_n),$$

where $x_n, y_n \in H$. From the definition, ν is a totally finite complex measure. Furthermore, ν is absolutely continuous with respect to the measure defined by

$$\mu(M) = \sum_{n=1}^{\infty} (E(M)x_n, x_n),$$

which in turn is totally finite(because $\sum_{n=1}^{\infty} x_n \in K$). It can be shown that there is a sequence of vectors $\{z_n\}_{n=1}^{\infty}$ in H such that $z = \sum_{n=1}^{\infty} z_n \in H$, $(E(M)z_1, z_j) = 0$ for $i \neq j$ and all Borel sets M , and μ is absolutely continuous with respect to the measure μ_z [12,p.249]. But this implies that ν is absolutely continuous with respect to μ_z . Applying Lemma 3.2.1, there are vectors $u, v \in H$ such that $\nu = \mu_{uv}$. Thus if

$$\int \lambda^n d(\tilde{E}_\lambda x, y) = 0 \text{ for } n \geq 0, x, y \in K,$$

we have

$$\int \lambda^n d(E_\lambda u, v) = 0 \text{ for } n \geq 0, \text{ with } u, v \in H.$$

Using the fact that A has property (P), this implies that $\nu = 0$. Hence $\tilde{A}$ has property (P).

4.2. A characterization of property (P). In this paragraph we give a characterization of property (P) in terms of the strong (or weak) operator topology.

Theorem 4.2.1. Let A be a bounded normal operator on the Hilbert space H . A necessary and sufficient condition that A have property (P) is that there exists a directed set of polynomials $\{P_\lambda A\}_{\lambda \in \Lambda}$ converging strongly (or weakly) to the adjoint of A .

Proof. Suppose there exists a directed set of polynomials $\{P_\lambda A\}_{\lambda \in \Lambda}$ converging strongly to A^* . Let K

be an invariant subspace of H with $x \in K$. Then $\{P_\lambda Ax\}_{\lambda \in \Lambda} \subset K$. Since K is closed and $\{P_\lambda Ax\}_{\lambda \in \Lambda}$ converges to A^*x , $A^*x \in K$. We chose x arbitrarily from K , hence K is invariant with respect to A^* . Since K was arbitrary, every invariant subspace of H reduces A , i.e., A has property (P).

Let the directed set $\{P_\lambda A\}_{\lambda \in \Lambda}$ converge weakly to A^* . Choose K and x as above. Let $y \in K^\perp$. Then $(P_\lambda Ax, y) = 0$, for all $\lambda \in \Lambda$. But this implies $(A^*x, y) = 0$. The vector $y \in K^\perp$ was arbitrary; hence, $A^*x \in K$, and, as before, A has property (P).

Now suppose A has property (P). For any $x \in H$, the subspace spanned by $\{A^n x\}_{n=1}^\infty$ is invariant with respect to A . Let K be such a subspace. Since A has property (P), $A^*x \in K$. Thus, there exists a sequence of polynomials $\{Q_n A\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \|Q_n Ax - A^*x\| = 0$.

Let $N(A^*; x_1, x_2, \dots, x_n; \epsilon)$ be a strong neighborhood of A^* . Assuming that A has property (P), we show that there are polynomials in A contained in N . There is no loss in generality in assuming that the set $x_1, x_2, \dots, x_n$ is linearly independent. This will be assumed throughout the proof.

Consider

$$\tilde{K} = H_1 \oplus H_2 \oplus \dots \oplus H_n,$$

where $H_1 = H$, $i = 1, 2, \dots, n$. Let $\tilde{x} \in \tilde{K}$, and write

$$\tilde{x} = \sum_{i=1}^n \oplus y_i.$$

Define $\tilde{A}$ on $\tilde{K}$ by

$$\tilde{A}\tilde{x} = \sum_{i=1}^n \oplus Ay_i.$$

This also defines $\tilde{A}^*$,

$$\tilde{A}^*\tilde{x} = \sum_{i=1}^n \oplus A^*y_i.$$

The normality of A implies that $\tilde{A}$ is normal. According to the preceding theorem, property (P) for A implies property (P) for $\tilde{A}$. The subspace, $\hat{K}$, generated by $\{\tilde{A}^n \tilde{y}\}_{n=1}^{\infty}$, $\tilde{y} \in \tilde{K}$, is invariant with respect to $\tilde{A}$. Since $\tilde{A}$ has property (P), $\tilde{A}^* \tilde{y} \in \hat{K}$. Thus, there exists a sequence of polynomials in $\tilde{A}$, $\{Q_n \tilde{A}\}_{n=1}^{\infty}$, such that $\lim_{n \rightarrow \infty} \|Q_n \tilde{A} \tilde{y} - \tilde{A}^* \tilde{y}\| = 0$. In particular we can let $\tilde{y} = \sum_{i=1}^n \oplus x_i$. Then

$$Q_m \tilde{A} \tilde{y} = \sum_{i=1}^n \oplus Q_m A x_i.$$

And, since $\tilde{K} = \sum_{i=1}^n \oplus H_i$,

$$\|(Q_m \tilde{A} - \tilde{A}^*) \tilde{y}\|^2 = \sum_{i=1}^n \|(Q_m A - A^*) x_i\|^2.$$

Using the fact that $\lim_{n \rightarrow \infty} \|(Q_n \tilde{A} - \tilde{A}^*) \tilde{y}\| = 0$, there is an m_0 such that for $m > m_0$

$$(4.2.1) \quad \|(Q_m A - A^*) y\|^2 = \sum_{i=1}^n \|(Q_m A - A^*) x_i\|^2 < \epsilon^2.$$

Since each of the summands in the above sum is positive,

we have

$$\|(Q_m A - A^*)x_1\| < \epsilon, \quad 1 = 1, 2, \dots, n.$$

Thus for $m > m_0$, $Q_m A \in N(A^*; x_1, x_2, \dots, x_n; \epsilon)$.

Using standard Moore-Smith convergence techniques [8,p.65], the theorem follows from the above discussion.

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