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INTEGRAL EQUATIONS AND THE COOLING PROBLEM FOR SEVERAL MEDIA

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Math. Dept. 5-26-31

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INTEGRAL EQUATIONS AND THE COOLING PROBLEM FOR SEVERAL MEDIA

By W.M. RUST, JR.

PART I. SOLUTION OF A GENERALISED ABEL INTEGRAL EQUATION.

In a simple problem concerning the cooling of castings, the determination of the temperature may be reduced to the solution of a certain type of generalised Abel integral equation. The purpose of this first part is to discuss a method of solving this equation.

The equation can be put in the form

$$(1.1) \quad \int_0^t \left\{ \frac{1 + K(t, t')}{(t - t')^{1/2}} \right\} u(t') dt' = f(t)$$

with $K(t, t')$ and $f(t)$ known functions.

In the actual problem, the function $K(t, t')$ vanishes to a high order for $t = t'$. For the method used here it is sufficient that $K(t, t')$ satisfy the conditions;

(A) $\frac{K(t, t')}{(t - t')}$ is bounded and is absolutely continuous in t , uniformly for all $t' \leq t$, and is summable in each of the variables. This implies the vanishing to at least the first order of $K(t, t')$ for $t = t'$.

(B) $\frac{\partial K(t, t')}{\partial t'}$ exists, is bounded and is absolutely continuous in t , uniformly for all $t' \leq t$, and is summable in each of the variables.[#]

[#] The form (1.1) was used because this is the form needed in the sequel, however no essential change is made if we write the equation in the form

$$\int_0^t \frac{G(t, t')}{(t - t')^\lambda} u(t') dt' = f(t) \quad 0 < \lambda < 1.$$

Where $G(t, t')$ satisfies the following conditions;

(1) $G(t, t')$ is bounded and is absolutely continuous in t , uniformly for all $t' \leq t$, and is summable in each of the variables.

(Footnote continued at bottom of next page)

The function $f(t)$ is assumed bounded and absolutely continuous, it is not assumed that $\lim_{t \rightarrow 0} f(t) = 0$, as is usually a condition for solution.

We shall assume that the equation (1.1) has a solution which is summable Lebesgue and solve for $\int_0^t u(t') dt'$, rather than for $u(t)$ itself. The method of solution is to multiply through by $(t''-t)^{-\frac{1}{2}}$ and integrate from $t =$ to $t = t''$, as in the usual treatment. If (1.1) is satisfied by $u(t)$, this gives

$$(1.2) \quad \int_0^{t''} \frac{dt''}{(t''-t)^{\frac{1}{2}}} \int_0^t \left\{ \frac{1+K(t,t')}{(t-t')^{\frac{1}{2}}} \right\} u(t') dt' \equiv \int_0^{t''} \frac{f(t) dt}{(t''-t)^{\frac{1}{2}}}$$

In the left hand member we change the order of integration, this is possible since $K(t,t')$ is bounded. The result is

$$\int_0^{t''} \frac{dt''}{(t''-t)^{\frac{1}{2}}} \int_0^t \left\{ \frac{1+K(t,t')}{(t-t')^{\frac{1}{2}}} \right\} u(t') dt' \equiv \pi \int_0^{t''} u(t') dt' + \int_0^{t''} u(t') dt' \int_t^{t''} \frac{K(t,t') dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}}$$

At this point we vary from the usual treatment by integrating the last term by parts. This gives

$$\left[\left\{ \int_0^{t'} u(t) dt \right\} \left\{ \int_t^{t'} \frac{K(t,t') dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right\} \right]_{t'=0}^{t'=t''} - \int_0^{t''} \left\{ \int_0^{t'} u(t) dt \right\} \frac{\partial}{\partial t'} \left(\int_t^{t''} \frac{K(t,t') dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right) dt'$$

Footnote continued from preceding page.

(2) For t' near enough to t , we have

$$G(t,t') = g(t) + h(t)(t-t') + \eta(t-t')$$

where $|g(t)| \geq m > 0$, $\eta(t-t')$ vanishes to higher than the first order for $t = t'$, and $g(t)$, $h(t)$ and $\eta(t-t')$ are bounded and absolutely continuous.

(3) $\frac{\partial G}{\partial t'}$ exists, is bounded and is absolutely continuous in t , uniformly for all $t' \leq t$, and is summable in each of the variables.

λ is a constant between zero and one.

The term outside the sign of integration vanishes since the factor $\int_0^{t'} u(t) dt$ vanishes for $t' = 0$ and the factor $\int_0^{t''} \frac{K(t, t') dt}{(t''-t)^{1/2}(t-t')^{1/2}}$ vanishes for $t' = t''$. This follows from condition (A), for if $K(t, t')$ vanishes to the order α for $t = t'$, this integral is less than

$$\int_0^{t''} \frac{M dt}{(t''-t)^{1/2}(t-t')^{1/2-\alpha}} \equiv M(t''-t')^\alpha B(\frac{1}{2}, \frac{1}{2} + \alpha)$$

and vanishes to the order α for $t' = t''$.

Formal differentiation gives

$$(1.3) \quad \frac{\partial}{\partial t'} \left(\int_0^{t''} \frac{K(t, t') dt}{(t''-t)^{1/2}(t-t')^{1/2}} \right) \equiv \left[\frac{K(t, t')}{(t''-t)^{1/2}(t-t')^{1/2}} \right]_{t=t'} + \int_0^{t''} \left\{ \frac{\frac{\partial}{\partial t'} K(t, t')}{(t''-t)^{1/2}(t-t')^{1/2}} + \frac{K(t, t')}{2(t''-t)^{1/2}(t-t')^{3/2}} \right\} dt$$

The first term is zero by condition (A). The second term is bounded by condition (B). The third term is bounded, for by condition (A) the function $K(t, t')$ vanishes to at least the first order for $t = t'$ and so the integral is less than

$$M \int_0^{t''} \frac{dt}{(t''-t)^{1/2}(t-t')^{3/2-\alpha}}$$

with α at least as great as one.

This establishes the fact that the derivative (1.3) is bounded and justifies the integration by parts.

We thus have as a necessary condition on summable solutions of (1.1),

$$(1.4) \quad \pi \int_0^{t''} u(t') dt' \equiv \int_0^{t''} \frac{f(t') dt'}{(t''-t')^{1/2}} + \int_0^{t''} \left\{ \int_0^{t'} u(t) dt \right\} \frac{\partial}{\partial t'} \left(\int_0^{t''} \frac{K(t, t') dt}{(t''-t)^{1/2}(t-t')^{1/2}} \right) dt'$$

We wish to show that the equation

$$(1.5) \quad \pi V(t'') = \int_0^{t''} \frac{f(t') dt'}{(t''-t')^{1/2}} + \int_0^{t''} V(t') \frac{\partial}{\partial t'} \left(\int_0^{t''} \frac{K(t, t') dt}{(t''-t)^{1/2}(t-t')^{1/2}} \right) dt'$$

has a unique absolutely continuous solution, whose derivative satisfies the equation (1.1) nearly everywhere.

Since equation (1.5) is a Volterra integral equation of the second sort with a bounded kernel, (1.3), it may be solved by a process of successive approximations, the process necessarily converging and the solution is summable and unique.[#]

Since equation (1.4) is a necessary condition ~~on~~ summable solutions of (1.1), the uniqueness of the solution of (1.5) shows that the solutions of (1.1) can, at most, differ on a point set of zero measure.

To show that the solution of (1.5) is absolutely continuous, we need the following lemma.

LEMMA I : The function $F(t) = \int_0^t f(t, t')g(t')dt'$ is an absolutely continuous function of t if $f(t, t')$ is absolutely continuous in t, uniformly for all t' and if $g(t')$ is a summable function.

Proof. By the λ -variation of the function $f(t, t')$ we shall mean the function $T_f(\lambda)$ defined as the upper limit, for all sets of non-overlapping intervals (a_i, b_i) such that $\sum |a_i - b_i| \leq \lambda$ and for all t' , of the numbers

$$\sum |f(a_i, t') - f(b_i, t')|$$

To say that $f(t, t')$ is absolutely continuous in t , uniformly for all t' , means that the λ -variation approaches zero with λ .

The λ -variation of $F(t)$, defined in an analogous way, is less than or equal to

$$T_f(\lambda) \int_0^1 |g(t')| dt'$$

[#] Volterra, Lecons sur les Equations Integrales, page 40 ff.

This approaches zero with λ , so that $F(t)$ is absolutely continuous. This establishes the lemma.

By means of the transformation $t' = t'y$, the first term on the right hand side of equation (1.5) becomes

$$(t)^{1/2} \int_0^1 \frac{f(t'y) dy}{(1-y)^{1/2}}$$

which satisfies the conditions of Lemma I and so is absolutely continuous.

By means of the transformation $t = t' + y(t'' - t')$, the expression (1.3) becomes

$$\int_0^1 \left\{ \frac{K_2(t' + y[t'' - t'], t')}{(1-y)^{1/2} y^{1/2}} + \frac{K(t' + y[t'' - t'], t')}{2(1-y)^{1/2} y^{1/2} (t'' - t')} \right\} dy$$

where $K_2(t, t') = \frac{\partial K(t, t')}{\partial t'}$. By the condition (3) and the Lemma I, the first term is absolutely continuous in t'' . By condition (A) $\frac{K}{t-t'} = \frac{K}{y(t''-t')}$ is absolutely continuous in t'' , and is bounded so that by the Lemma I the second term is absolutely continuous in t'' . In each case the absolute continuity is uniform in t' . Thus the expression (1.3) is absolutely continuous in t'' , uniformly in t' .

We need now the further lemma.

Lemma II . The function $G(t'') = \int_0^{t''} g(t') k(t'', t') dt'$ is an absolutely continuous

The condition is a sufficient one, regardless of the manner in which $f(t, t')$ involves t , but in general it is not a necessary condition. If $f(t, t')$ is a function of t alone, the condition is obviously necessary, but if $f(t, t')$ is a function of the product tt' , the condition may be modified to admit a set of points of discontinuity of the first sort, if this set is of zero measure.

function of t'' , if $k(t'', t')$ is absolutely continuous in t'' uniformly for all t' and is bounded, and if $g(t')$ is summable.

Proof. The λ -variation, $T_\lambda(\lambda)$, of $g(t'')$ is the upper limit for all sets of nonoverlapping intervals (a_i, b_i) such that $\sum |a_i - b_i| \leq \lambda$ of the numbers

$$\sum \left| \int_{a_i}^{b_i} g(t') k(b_i, t') dt' - \int_{a_i}^{b_i} g(t') k(a_i, t') dt' \right| \\ \equiv \sum \left| \int_{a_i}^{b_i} g(t') k(b_i, t') dt' + \int_{a_i}^{b_i} g(t') \{k(b_i, t') - k(a_i, t')\} dt' \right|$$

Thus

$$T_\lambda(\lambda) \leq \overline{\text{Limit}} M \sum \int_{a_i}^{b_i} |g(t')| dt' + T_k(\lambda) \sum \int_{a_i}^{b_i} |g(t')| dt'$$

where M is the maximum absolute value of $k(t, t')$ and T'' is the greatest of the numbers a_i and b_i . Since $g(t')$ is summable, each of these terms approaches zero with λ . This establishes the lemma.

The second term in the right hand member of (1.5) satisfies the conditions of Lemma II and so is absolutely continuous. Since each of the terms in the right hand member of (1.5) is absolutely continuous, $V(t'')$ is absolutely continuous, as was to be shown.

Since $V(t)$ is absolutely continuous, it has a derivative $V'(t)$ nearly everywhere and $\int_0^t V'(t') dt' = V(t)$. We shall now show that $u(t) = V'(t)$ satisfies equation (1.1) nearly everywhere. To do this we write

$$(1.6) \quad D(t) \equiv \int_0^t \left\{ \frac{1 + K(t, t')}{(t - t')^{1/2}} \right\} V'(t') dt' - f(t)$$

Multiplying through by $(t'' - t)^{-1/2}$, integrating from $t = 0$ to $t = t''$, changing the order of integration and integrating by parts, as above to obtain equation (1.4), gives

$$\begin{aligned}
 (1.7) \quad \int_0^{t''} \frac{D(t) dt}{(t''-t)^{1/2}} &\equiv \pi \int_0^{t''} V'(t) dt - \int_0^{t''} \frac{f(t) dt}{(t''-t)^{1/2}} \\
 &- \int_0^{t''} \left\{ \int_0^t V'(t) dt \right\} \frac{\partial}{\partial t'} \left(\int_t^{t''} \frac{K(t, t') dt'}{(t''-t)^{1/2} (t-t')^{1/2}} \right) dt' \equiv \pi V(t'') \\
 &- \int_0^{t''} \frac{f(t) dt}{(t''-t)^{1/2}} - \int_0^{t''} V(t') \frac{\partial}{\partial t'} \left(\int_t^{t''} \frac{K(t, t') dt'}{(t''-t)^{1/2} (t-t')^{1/2}} \right)
 \end{aligned}$$

But since $V(t)$ satisfies equation (1.5), the right hand side of (1.7) is zero. Hence, as Tonelli shows[#], $D(t)$ is zero nearly everywhere, that is the derivative of the solution of equation (1.5) satisfies (1.1) nearly everywhere.

We have the following theorem.

THEOREM. Under the conditions specified with respect to $K(t, t')$ and $f(t)$ in (A) and (3), the equation (1.1) has one and only one summable solution $u(t)$, and this solution is given as the derivative of the solution of equation (1.5). Two summable functions differing only on a point set of zero measure are, of course, to be considered as equivalent.

PART II. A PROBLEM IN THE FLOW OF HEAT.

As an example of a physical problem that may be solved by such an equation, we consider the following problem in heat conduction, one which has both theoretical and intrinsic interest.

A quantity of one material is heated and placed between two masses

L. Tonelli, "Su un problema di Abel", Mathematische Annalen, Vol. 99 (1928) page 187.

of a different material, as a casting in its mold. The problem is : given the initial temperatures of the three regions and the temperatures at the two outer boundaries, find the temperature at any point at any time.

For convenience we choose very simple conditions. We take the three regions to be bounded by parallel infinite planes and take the thicknesses of the two outer regions to be the same. We take the initial temperatures in the outer regions to be constant and equal to each other. We take the initial temperature on the inner region to be constant, not necessarily the same as in the outer regions. We take the temperatures on the outer bounding surfaces to be equal, and at any instant constant over the entire bounding plane. With these conditions the problem is symmetric about the central plane and at any instant the temperature is constant over any plane parallel to the central plane, thus only one spatial coordinate, the distance from the central plane, is involved. Some generalisations of this problem are considered in Part V.

The conductivity of the material in the inner regions is K_1 and the conductivity of the material in the outer regions is K_2 . The quantities a^2 and b^2 are positive constants, equal to the ratios of the conductivity to the product of the specific heat by the density, for the inner and outer regions respectively. Also x is the distance from the central plane and t is the time after the initial time.

We take the bounding planes between the inner and outer regions to be at $x = m$ and $x = -m$ and the outer boundaries to be at $x = l$ and $x = -l$.

At the interior points $u_1(x,t)$, the temperature in the inner regions, and $u_2(x,t)$, the temperature in the outer regions, satisfy the partial differential equations

$$(2.1) \begin{cases} \frac{\partial^2 u_1(x,t)}{\partial x^2} - a^2 \frac{\partial u_1(x,t)}{\partial t} = 0, & -m < x < m \\ \frac{\partial^2 u_2(x,t)}{\partial x^2} - b^2 \frac{\partial u_2(x,t)}{\partial t} = 0, & -l < x < -m \text{ or } m < x < l \end{cases}$$

respectively.

If the temperature in the inner region is initially u_1 , a constant, and in the outer regions u_2 , a constant, we have

$$(2.2) \begin{cases} \lim_{t \rightarrow 0^+} u_1(x,t) = u_1, & \text{for } -m < x < m \\ \lim_{t \rightarrow 0^+} u_2(x,t) = u_2, & \text{for } -l < x < -m \text{ or } m < x < l. \end{cases}$$

At the outer boundaries the temperature is taken to be a known, bounded continuous function of the time, say $f(t)$, with a bounded, continuous derivative. We have

$$(2.3) \lim_{t \rightarrow 0^+} u_2(x,t) = f(t), \text{ for } t > 0.$$

At the separating boundaries we have two conditions, first, the temperature is continuous, in x , across the boundaries, that is,

$$(2.4) \lim_{x \rightarrow m-0} u_1(x,t) = \lim_{x \rightarrow m+0} u_2(x,t), \text{ for } t > 0,$$

and, second, the partial derivatives with respect to x satisfy the equation

$$(2.5) \lim_{x \rightarrow m-0} K_1 \frac{\partial u_1(x,t)}{\partial x} = \lim_{x \rightarrow m+0} K_2 \frac{\partial u_2(x,t)}{\partial x}, \text{ for } t > 0.$$

Similar conditions are imposed at $x = -m$ and at $x = -l$, but due to

For the derivation of the equations (2.1) see any standard work on the conduction of heat, for example, Carslaw, Conduction of Heat, Chapter One, or Riemann-Weber, Differentialgleichungen der Physik, (1912) Vol.II, page 82.

For the derivation of equation (2.5) see, for example, Riemann-Weber, loc. cit., Vol.II, page 85.

the symmetry of the problem, these are equivalent to those given by equations (2.3) to (2.5).

We now establish the following uniqueness theorem.

UNIQUENESS THEOREM A. There can not be more than one solution of the problem as given by equations (2.1), subject to the conditions (2.2) to (2.5), which is bounded everywhere - (including $t = 0$) - and is continuous for $t > 0$, except at the boundaries, and has first derivatives with respect to each of the variables, which are continuous for $t > 0$, except across the boundaries- the derivative with respect to x being bounded everywhere for $t > 0$.

Suppose there were two such solutions. Call the difference between these solutions $V_1(x, t)$ and $V_2(x, t)$ in the inner and outer regions respectively. $V_1(x, t)$ and $V_2(x, t)$ satisfy the conditions of the problem and take on the boundary and initial values zero.

Consider the integrals

$$(a) \quad J_\epsilon(t) = \frac{K_2 b^2}{2} \int_{-l+\epsilon}^{-m-\epsilon} (V_2(x, t))^2 dx \\ + \frac{K_1 a^2}{2} \int_{m+\epsilon}^{m-\epsilon} (V_1(x, t))^2 dx + \frac{K_2 b^2}{2} \int_{m+\epsilon}^{l-\epsilon} (V_2(x, t))^2 dx$$

For $t > 0$, $V_1(x, t)$ and $\frac{\partial V_1}{\partial t}$ are continuous and bounded in the region of integration, so that we may differentiate under the integral sign. We have

$$\frac{dJ_\epsilon(t)}{dt} = K_2 b^2 \int_{-l+\epsilon}^{-m-\epsilon} V_2(x, t) \frac{\partial V_2}{\partial t}(x, t) dx \\ + K_1 a^2 \int_{m+\epsilon}^{m-\epsilon} V_1(x, t) \frac{\partial V_1}{\partial t}(x, t) dx + K_2 b^2 \int_{m+\epsilon}^{l-\epsilon} V_2(x, t) \frac{\partial V_2}{\partial t}(x, t) dx$$

and

$$(b) \quad \int_{t_1}^{t_2} \frac{dJ_\epsilon}{dt} dt = J_\epsilon(t_2) - J_\epsilon(t_1), \quad t_1, t_2 > 0.$$

Applying (2.1), we have

$$\frac{dJ_\epsilon(t)}{dt} = K_2 \int_{-l+\epsilon}^{-m-\epsilon} V_2(x, t) \frac{\partial^2 V_2}{\partial x^2}(x, t) dx \\ + K_1 \int_{m+\epsilon}^{m-\epsilon} V_1(x, t) \frac{\partial^2 V_1}{\partial x^2}(x, t) dx + K_2 \int_{m+\epsilon}^{l-\epsilon} V_2(x, t) \frac{\partial^2 V_2}{\partial x^2}(x, t) dx$$

In the region of integration each $\frac{\partial^2 V_i}{\partial x^2}$ is continuous and bounded, so that we may integrate by parts.

$$\begin{aligned}
 (c) \quad \frac{dJ_\epsilon(t)}{dt} &= K_2 V_2(x, t) \frac{\partial V_2}{\partial x}(x, t) \Big]_{x=-l+\epsilon}^{x=-m-\epsilon} \\
 &+ K_1 V_1(x, t) \frac{\partial V_1}{\partial x}(x, t) \Big]_{x=-m+\epsilon}^{x=m-\epsilon} + K_2 V_2(x, t) \frac{\partial V_2}{\partial x}(x, t) \Big]_{x=m+\epsilon}^{x=l-\epsilon} \\
 &- K_2 \int_{-l+\epsilon}^{-m-\epsilon} \left(\frac{\partial V_2}{\partial x}(x, t) \right)^2 dx - K_1 \int_{-m+\epsilon}^{m-\epsilon} \left(\frac{\partial V_1}{\partial x}(x, t) \right)^2 dx \\
 &- K_2 \int_{m+\epsilon}^{l-\epsilon} \left(\frac{\partial V_2}{\partial x}(x, t) \right)^2 dx, \quad t, t_1 > 0.
 \end{aligned}$$

We have then from (b) and (c),

$$\begin{aligned}
 (d) \quad \lim_{\epsilon \rightarrow 0} [J_\epsilon(t_2) - J_\epsilon(t_1)] &= \lim_{\epsilon \rightarrow 0} \int_{t_1}^{t_2} \frac{dJ_\epsilon}{dt} dt \\
 &= \lim_{\epsilon \rightarrow 0} \int_{t_1}^{t_2} \left\{ K_2 V_2(x, t) \frac{\partial V_2}{\partial x}(x, t) \Big]_{x=-l+\epsilon}^{x=-m-\epsilon} + K_1 V_1(x, t) \frac{\partial V_1}{\partial x}(x, t) \Big]_{x=-m+\epsilon}^{x=m-\epsilon} \right. \\
 &+ K_2 V_2(x, t) \frac{\partial V_2}{\partial x}(x, t) \Big]_{x=m+\epsilon}^{x=l-\epsilon} - K_2 \int_{-l+\epsilon}^{-m-\epsilon} \left(\frac{\partial V_2}{\partial x}(x, t) \right)^2 dx \\
 &- K_1 \int_{-m+\epsilon}^{m-\epsilon} \left(\frac{\partial V_1}{\partial x}(x, t) \right)^2 dx - K_2 \int_{m+\epsilon}^{l-\epsilon} \left(\frac{\partial V_2}{\partial x}(x, t) \right)^2 dx \Big\} dt, \quad t, t_1 > 0.
 \end{aligned}$$

Since each $V_i(x, t)$ is bounded

$$\begin{aligned}
 (e) \quad \lim_{\epsilon \rightarrow 0} J_\epsilon(t) &= J(t) = \frac{K_2 b^2}{2} \int_{-l}^{-m} (V_2(x, t))^2 dx \\
 &+ \frac{K_1 a^2}{2} \int_{-m}^m (V_1(x, t))^2 dx + \frac{K_2 b^2}{2} \int_m^l (V_2(x, t))^2 dx \quad t, t_1 > 0
 \end{aligned}$$

The lefthand member of (d) is thus

$$J(t_2) - J(t_1)$$

In the right hand member of (d), we may pass to the limit under the sign of integration in the first three terms, since $V_i(x, t)$ and $\frac{\partial V_i}{\partial x}(x, t)$ are bounded.¹¹

¹¹ See Lebesgue, Lecons sur l'integration, page 125 (1928).

That is

$$(f) \quad \lim_{x \rightarrow x_0} \int_{t_1}^{t_2} K_1 V_1(x,t) \frac{\partial V_1}{\partial x}(x,t) dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0} \{ K_1 V_1(x,t) \frac{\partial V_1}{\partial x}(x,t) \} dt$$

x_0 being the x coordinate of any boundary. By the boundary conditions the limit of the first three terms is zero.

Since K_1 and K_2 are positive constants, the remaining three terms are negative, or zero, if $t_2 > t_1$, so that in the limit the right hand member of (d) is negative or zero. We have then

$$(g) \quad J(t_2) - J(t_1) \leq 0 \quad \text{if } t_2 > t_1 > 0.$$

Since $V_1(x,t)$ is bounded and $\lim_{t \rightarrow 0} V_1(x,t) = 0$, we have

$$\lim_{t \rightarrow 0} J(t) = 0$$

Applying this to (g) we have

$$(h) \quad J(t) \leq 0 \quad \text{if } t > 0.$$

But in (e) the terms are all positive, since K_1 and K_2 are positive, so that

$$(j) \quad J(t) \geq 0 \quad \text{if } t > 0.$$

The two equations (h) and (j) can only be satisfied simultaneously if

$$J(t) = 0.$$

But from (e) we see that this can only be the case if $V_1(x,t)$ is zero nearly everywhere and since $V_1(x,t)$ is continuous except at the boundaries for $t > 0$, it must be identically zero for $t > 0$ and the two solutions are identical for $t > 0$.

This establishes the theorem.

For infinite outer regions, we may replace (2.3) by the condition

$$\lim_{x \rightarrow \pm\infty} \frac{\partial u}{\partial x}(x,t) = 0, \quad t > 0.$$

We now establish a uniqueness theorem with less restrictive conditions.

UNIQUENESS THEOREM B. There can not be more than one solution of the problem satisfying the conditions of Theorem A, except that the derivative with respect to x is not assumed bounded, but summable in t and satisfying the condition

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} \left| \frac{\partial u_i}{\partial x}(x, t) \right| dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} \left| \frac{\partial u_i}{\partial x}(x, t) \right| dt, \quad t_1, t_2 > 0,$$

x_0 being the x coordinate of any boundary.

The proof of this theorem is the same as that of Theorem A except in the proof of the equation (f)

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} K_i V_i(x, t) \frac{\partial V_i}{\partial x}(x, t) dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} \{ K_i V_i(x, t) \frac{\partial V_i}{\partial x}(x, t) \} dt$$

In Theorem A this was a consequence of the boundedness of $V_i(x, t)$ and $\frac{\partial V_i}{\partial x}(x, t)$. Having removed the latter restriction, we need a different proof.

Each $V_i(x, t)$ is bounded, say less in absolute value than M_i , hence

$$\left| K_i V_i(x, t) \frac{\partial V_i}{\partial x}(x, t) \right| \leq K_i M_i \left| \frac{\partial V_i}{\partial x}(x, t) \right|$$

and by the condition of our theorem

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} \left| \frac{\partial V_i}{\partial x}(x, t) \right| dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} \left| \frac{\partial V_i}{\partial x}(x, t) \right| dt$$

and so ^{##}

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} K_i V_i(x, t) \frac{\partial V_i}{\partial x}(x, t) dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} K_i V_i(x, t) \frac{\partial V_i}{\partial x}(x, t) dt$$

^{##} That is to say, the absolute continuity of $\int_0^t \left| \frac{\partial u_i}{\partial x}(x, t) \right| dt$ - and hence of $\int_0^t \frac{\partial u_i}{\partial x}(x, t) dt$ - is uniform in x . From this it follows that

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} \frac{\partial u_i}{\partial x}(x, t) dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} \frac{\partial u_i}{\partial x}(x, t) dt$$

^{###} See Hobson, "Theory of Functions of a Real Variable," (1926),

This establishes the equation (f) and the remainder of the proof is unaltered.

We will now show that a solution of the problem satisfying the conditions of the Uniqueness Theorem B, is given by

$$(2.6) \quad u_1(x, t) = u_1 + \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} + \frac{1}{(t-t')^{1/2}} e^{\frac{-a^2(x-m)^2}{4(t-t')}} \right\} \psi_1(t') dt'$$

for $-m < x < m$, $t > 0$

and

$$(2.7) \quad u_2(x, t) = u_2 + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-b^2(x-m)^2}{4(t-t')}} \psi_2(t') dt' + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-b^2(x-l)^2}{4(t-t')}} \psi_3(t') dt'$$

for $m < x < l$ and $u_1(-x, t) = u_2(x, t)$. $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are summable functions satisfying the following integral equations, nearly everywhere.

$$(2.8) \quad \int_0^t \frac{1}{(t-t')^{1/2}} \psi_3(t') dt' = f_2(t)$$

$$- \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\rho^2}{t-t'}} \psi_2(t') dt'$$

$$(2.9) \quad \psi_2(t) = -c \psi_1(t) + \frac{c\alpha}{\sqrt{\pi}} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \psi_1(t') dt' + \frac{\beta}{\sqrt{\pi}} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\rho^2}{t-t'}} \psi_3(t') dt'$$

$$(2.10) \quad \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} + \frac{1}{(t-t')^{1/2}} \right\} \psi_1(t') dt' = f_1 + \int_0^t \frac{1}{(t-t')^{1/2}} \psi_2(t') dt' + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\rho^2}{t-t'}} \psi_3(t') dt'$$

The integrals are to be Lebesgue integrals. The following abbreviations are used,

$$(i) \quad f_2(t) = f(t) - u_2$$

$$(ii) \quad \alpha = a m$$

$$(iii) \quad \beta = \frac{b(l-m)}{2}$$

$$(iv) \quad c = \frac{K_1 a}{K_2 b} > 0$$

$$(v) \quad f_1 = u_2 - u_1$$

By actual differentiation we see that

$$\frac{1}{(t-t')^{1/2}} e^{\frac{-a^2(x-x')^2}{4(t-t')}}$$

when considered as a function of x and t is a solution of the first equation (2.1) for any $x' \neq x$ and for any $t' < t$. If we replace the a^2 in the exponent by b^2 , we have a solution of the second equation (2.1).[†]

[†] For a full discussion of the properties of these and similar functions, see E.E. Levi, "Sull' equazione del calore," Annali di Matematica, S.III, Vol. XIV, (1908), page 187.

Since the equations (2.1) are linear, any linear combination of solutions, such as $u_1(x,t)$ and $u_0(x,t)$ as given by equations (2.6) and (2.7), is also a solution. The integrals converge, except at the boundaries, for any summable functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, since the expression

$$\frac{1}{(t-t')^{1/2}} e^{-\frac{a^2(x-m)^2}{4(t-t')}}$$

is bounded if $x \neq m$ and similarly for the other terms.

Since in each of the integrals the first factor is bounded, except at the boundaries, each integrand is summable and the limit of each integral as t approaches zero is zero, except at the boundaries. The initial conditions (2.2) are thus satisfied.

We now show that any set of summable functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ satisfying the equations (2.8) to (2.10) must be of the form

$$\frac{A}{t^{1/2}} + \phi(t)$$

where $\phi(t)$ is bounded and continuous.

Equation (2.8) can be written

$$A: \int_0^t \frac{1}{(t-t')^{1/2}} \psi_3(t') dt' = g(t)$$

$$\text{where } g(t) = f_2(t) - \int_0^t \frac{1}{(t-t')^{1/2}} e^{-\frac{a^2}{t-t'}} \psi_2(t') dt'$$

$$\text{Its derivative is } g'(t) = f_2'(t) - \int_0^t \frac{\partial}{\partial t} \left\{ \frac{1}{(t-t')^{1/2}} e^{-\frac{a^2}{t-t'}} \right\} \psi_2(t') dt'$$

Since $f(t)$ is bounded and continuous with a bounded and continuous derivative and $\psi_2(t)$ is summable, $g(t)$ and $g'(t)$ are also bounded and continuous.

Equation (A) is an Abel's equation of the usual type and has the solution[#]

$$\psi_3(t) = \frac{g(0)}{\pi \sqrt{t}} + \frac{1}{\pi} \int_0^t \frac{g'(t') dt'}{(t-t')^{3/2}}$$

[#] See Volterra, Leçons sur les Équations Intégrales, page 37.

* The terms in the derivative, with respect to t , introduced by the change in the upper limit are zero.

Since $g'(t)$ is bounded and continuous, $\int_0^t \frac{g'(t')}{(t-t')^{1/2}} dt'$ is also bounded and continuous and $\psi_3(t)$ has the form given.

To show that $\psi_1(t)$ and $\psi_2(t)$ have this form, we eliminate $\psi_2(t)$ from the equations (2.9) and (2.10). Thus from (2.9) we have

$$(B) \quad \psi_2(t) = -c \psi_1(t) + h(t)$$

where
$$h(t) = \frac{c\beta}{\pi} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \psi_1(t') dt' + \frac{\beta}{\pi} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_3(t') dt'$$

So that

$$\int_0^t \frac{\psi_2(t') dt'}{(t-t')^{1/2}} = -c \int_0^t \frac{\psi_1(t') dt'}{(t-t')^{1/2}} + \int_0^t \frac{h(t') dt'}{(t-t')^{1/2}}$$

Changing the order of integration in the last term gives

$$\int_0^t \frac{h(t') dt'}{(t-t')^{1/2}} = c \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \psi_1(t') dt' + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_3(t') dt'$$

If we substitute in (2.10) we have

$$(C) \quad (1+c) \int_0^t \frac{\psi_1(t') dt'}{(t-t')^{1/2}} = t_1 + h_1(t)$$

where
$$h_1(t) = \int_0^t \frac{h(t') dt'}{(t-t')^{1/2}} + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_3(t') dt' - \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \psi_1(t') dt' = (c-1) \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \psi_1(t') dt' + 2 \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_3(t') dt' \quad , (h_1(0) = 0),$$

and is bounded and continuous and has a bounded and continuous derivative.

Equation (C) thus has the solution

$$\psi_1(t) = \frac{f_1}{(1+c)\pi\sqrt{t}} + \frac{1}{(1+c)\pi} \int_0^t \frac{h_1(t') dt'}{(t-t')^{1/2}}$$

and since $h_1'(t)$ is bounded and continuous $\int_0^t \frac{h_1'(t') dt'}{(t-t')^{1/2}}$ is bounded and continuous and $\psi_1(t)$ has the required form.

It follows from (B) that $\psi_2(t)$ also has this form.

From (2.6) and (2.7) we see that

$$|u_1(x,t)| \leq |u_1| + 2 \int_0^t \frac{1}{(t-t')^{1/2}} |\psi_1(t')| dt' \leq M_1$$

$$|u_2(x,t)| \leq |u_2| + \int_0^t \frac{1}{(t-t')^{1/2}} \{ |\psi_2(t')| + |\psi_3(t')| \} dt' \leq M_2$$

so that $u_1(x,t)$ and $u_2(x,t)$ are bounded.

Since $\left| \frac{1}{(t-t')^{1/2}} e^{\frac{-b^2(x-l)^2}{4(t-t')}} \psi_1(t') \right| \leq \frac{|\psi_1(t')|}{(t-t')^{1/2}}$, which is summable, we have

$$\lim_{x \rightarrow l-0} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-b^2(x-l)^2}{4(t-t')}} \psi_1(t') dt' = \int_0^t \frac{\psi_1(t') dt'}{(t-t')^{1/2}}$$

so that

$$(2.11) \quad \lim_{x \rightarrow l-0} u_2(x,t) = u_2 + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_2(t') dt' + \int_0^t \frac{\psi_2(t') dt'}{(t-t')^{1/2}} = f(t), \quad t > 0,$$

in virtue of equation (2.6) and the boundary condition (2.3) is satisfied.

Likewise we can show that

$$\lim_{x \rightarrow m+0} u_2(x,t) = u_2 + \int_0^t \frac{\psi_2(t') dt'}{(t-t')^{1/2}} + \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_1(t') dt'$$

and

$$\lim_{x \rightarrow m-0} u_1(x,t) = u_1 + \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} + \frac{1}{(t-t')^{1/2}} \right\} \psi_1(t') dt'$$

Recalling (v), we see that in virtue of equation (2.10)

$$\lim_{x \rightarrow m-0} u_1(x,t) = \lim_{x \rightarrow m+0} u_2(x,t), \quad t > 0$$

and the boundary condition (2.4) is satisfied.

By direct differentiation we have

$$(2.12) \quad \frac{\partial u_1(x,t)}{\partial x} = -\frac{a^2}{2} \int_0^t \left\{ \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} + \frac{x-m}{(t-t')^{3/2}} e^{\frac{-a^2(x-m)^2}{4(t-t')}} \right\} \psi_1(t') dt'$$

and

$$(2.13) \quad \frac{\partial u_2(x,t)}{\partial x} = -\frac{b^2}{2} \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{\frac{-b^2(x-m)^2}{4(t-t')}} \psi_2(t') dt' - \frac{b^2}{2} \int_0^t \frac{x-l}{(t-t')^{3/2}} e^{\frac{-b^2(x-l)^2}{4(t-t')}} \psi_1(t') dt'$$

The integrals involved all converge for $x \neq m$ or 1.

To evaluate the $\lim_{x \rightarrow m+0} \frac{\partial u_1(x,t)}{\partial x}$, we need to know the value of

$$\lim_{x \rightarrow m+0} \frac{b}{2\sqrt{\pi}} \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{-\frac{b^2(x-m)^2}{4(t-t')}} \phi(t') dt', \text{ for } \phi(t) \text{ summable.}$$

Hobson² considers the general question of such convergence. Restating his results in terms applicable here, we have the theorem;

Let $F(t', t, x)$ be defined for $t', t > 0$ and $m < x < 1$. Let μ denote a positive number and let $F(t', t, x)$ satisfy the following conditions:

(1) For each pair of values t and x , and for all values of $t' > 0$, such that $|t-t'| > \mu$, the function $F(t', t, x)$ is equivalent to a function that does not exceed in absolute value a positive number $K\mu$, independent of the values of t and x .

(2) If α and β are two numbers such that $0 < \alpha \leq \beta \leq t$, $\int_{\alpha}^{\beta} F(t, t', x) dt'$ exists as a Lebesgue integral, for all values of x ($m < x < 1$) and for all those values of t such that $t-\mu > \alpha, \beta$; and as x approaches $m+0$, the integral converges to zero, uniformly for all such values of t .

Let $F(t', t, x)$ be a function of $t-t'$, say $F(t-t', x)$ and satisfy the conditions:

$$(a) \lim_{x \rightarrow m+0} \int_0^{\mu} F(z, x) dz = 1$$

$$(b) \int_0^{\mu} |F(z, x)| dz < A$$

where A is independent of μ and x , for all sufficiently small values of $\mu > 0$.

(I) $\lim_{x \rightarrow m+0} F(t-t', x) = 0$, when $t' \neq t$.

(II) $zF(z, x)$ has a total variation in the interval $(0, \mu)$ less than a fixed number independent of x .

Hobson, loc.cit. Vol. II, pages 443, 447 and 452.

Then

$$\lim_{x \rightarrow m+0} \int_0^t \phi(t') F(t-t', x) dt' = \phi(t)$$

whenever

$$\int_0^{t'} \{ \phi(t+t') - \phi(t) \} dt'$$

has a differential coefficient, with respect to t' , equal to zero for $t' = 0$. This is the case for almost all values of t , since $\phi(t)$ is summable.[#]

We must show that

$$F(t-t', x) = \frac{b}{2\sqrt{\pi}} \frac{x-m}{(t-t')^{3/2}} e^{-\frac{b^2(x-m)^2}{4(t-t')}}$$

satisfies the conditions (1), (2), (a), (b), (I), (II).

(1) If $|t-t'| \geq \mu$

$$|F(t-t', x)| \leq \frac{b}{2\sqrt{\pi}} \frac{(x-m)}{\mu^{3/2}} \leq \frac{b}{2\sqrt{\pi}} \frac{(l-m)}{\mu^{3/2}}, \quad m < x < l$$

(2) Put $y = \frac{b(x-m)}{2(t-t')^{1/2}}$

$$\begin{aligned} & \int_a^\beta F(t, t', x) dt' \\ &= \int_{\frac{b(x-m)}{2(t-d)^{1/2}}}^{\frac{b(x-m)}{2(t-\beta)^{1/2}}} \frac{2}{\sqrt{\pi}} e^{-y^2} dy \leq \frac{2}{\sqrt{\pi}} \int_0^{\frac{b(x-m)}{2\sqrt{\mu}}} e^{-y^2} dy \end{aligned}$$

and converges to zero as x approaches $m+0$, uniformly for all $t > \mu+d, \mu+\beta$.

(a) By the substitution $y = \frac{b(x-m)}{2\sqrt{z}}$

$$\text{we have } \int_0^\mu F(z, x) dz = \frac{2}{\sqrt{\pi}} \int_{\frac{b(x-m)}{2\sqrt{\mu}}}^0 e^{-y^2} dy$$

Hobson, loc.cit. Vol. I, paragraph 432.

and for any $\mu > 0$

$$\lim_{x \rightarrow \infty} \int_0^\mu F(z, x) dz = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y^2} dy = 1$$

$$(b) \int_0^\mu |F(z, x)| dz = \frac{2}{\sqrt{\pi}} \int_{\frac{b(x-\mu)}{2\sqrt{\mu}}}^\infty e^{-y^2} dy \leq \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y^2} dy = 1$$

independent of x and μ , for $\mu > 0$.

(I) Limit $F(t-t', x) = 0$ for $t \neq t'$.

(II)

$$z F(z, x) = \frac{b}{2\sqrt{\pi}} \frac{x-\mu}{z^{1/2}} e^{-\frac{b^2(x-\mu)^2}{4z}}$$

$$\begin{aligned} \frac{\partial}{\partial z} \{ z F(z, x) \} &= \frac{b}{2\sqrt{\pi}} \frac{(x-\mu)}{z^{3/2}} \left\{ \frac{-1}{2z} + \frac{b^2(x-\mu)^2}{4z^2} \right\} e^{-\frac{b^2(x-\mu)^2}{4z}} \\ &= \frac{b(x-\mu)}{2\sqrt{\pi} z^{3/2}} \left\{ b^2(x-\mu)^2 - 2z \right\} e^{-\frac{b^2(x-\mu)^2}{4z}} \end{aligned}$$

$$\left| \frac{\partial}{\partial z} \{ z F(z, x) \} \right| = \frac{\partial}{\partial z} \{ z F(z, x) \} \quad \text{if } z \leq \frac{b^2(x-\mu)^2}{2}$$

$$\left| \frac{\partial}{\partial z} \{ z F(z, x) \} \right| = -\frac{\partial}{\partial z} \{ z F(z, x) \} \quad \text{if } z \geq \frac{b^2(x-\mu)^2}{2}$$

The total variation of $z F(z, x)$ is

$$\begin{aligned} \int_0^z \left| \frac{\partial}{\partial z'} \{ z' F(z', x) \} \right| dz' &= \int_0^z \frac{\partial}{\partial z'} \{ z' F(z', x) \} dz' \\ &= - \int_{\frac{b^2(x-\mu)^2}{2}}^z \frac{\partial}{\partial z'} \{ z' F(z', x) \} dz' = \frac{2b}{2\sqrt{\pi}} \frac{(x-\mu)\sqrt{2}}{b(x-\mu)} e^{-1/2} \\ &= \frac{b}{2\sqrt{\pi}} \frac{(x-\mu)}{z^{1/2}} e^{-\frac{b^2(x-\mu)^2}{4z}} \leq \frac{3\sqrt{2}}{2\sqrt{\pi}} e^{-1/2}, \quad \text{if } z \leq \frac{b^2(x-\mu)^2}{2} \end{aligned}$$

independent of x . This is on the assumption that $z \geq \frac{b^2(x-\mu)^2}{2}$, but since

the total variation is a monotone, nondecreasing function, if z is

less than this the total variation will be less than the value above.

The conditions of the theorem are thus satisfied and so

$$\lim_{x \rightarrow \infty} \frac{b}{2\sqrt{\pi}} \int_0^t \frac{x-\mu}{(t-z')^{1/2}} e^{-\frac{b^2(x-\mu)^2}{4(t-z')}} \phi(t') dt' = \phi(t)$$

almost everywhere.

Similarly we can show

$$\lim_{x=m-0} \left(\frac{-a}{2\sqrt{\pi}} \right) \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{\frac{-a^2(x-m)^2}{4(t-t')}} \phi(t') dt' = \phi(t)$$

nearly everywhere, if $\phi(t)$ is suitable.

Applying these results to equation (2.10) we have

$$\lim_{x=m-0} \frac{\partial u_1(x,t)}{\partial x} = a\sqrt{\pi} \psi_1(t) - \int_0^t \frac{a^2 m}{(t-t')^{3/2}} e^{\frac{-x^2}{t-t'}} \psi_1(t') dt'$$

and similarly from equation (2.10) we have

$$\lim_{x=m+0} \frac{\partial u_2(x,t)}{\partial x} = -b\sqrt{\pi} \psi_2(t) - \int_0^t \frac{b^2(m-l)}{2(t-t')^{3/2}} e^{\frac{-x^2}{t-t'}} \psi_2(t') dt'$$

In virtue of equation (2.9) we have then

$$\lim_{x=m-0} K_1 \frac{\partial u_1(x,t)}{\partial x} = \lim_{x=m+0} K_2 \frac{\partial u_2(x,t)}{\partial x} \quad \text{for } t > 0,$$

and the boundary condition (2.4) is satisfied.

The functions $u_1(x,t)$ and $u_2(x,t)$ defined by equations (2.6) and (2.7) thus satisfy the differential equations (2.1), the initial condition (2.2) and the boundary conditions (2.3) to (2.5). We have already shown that they are bounded everywhere and must now show that the other conditions of the Uniqueness Theorem B are satisfied.

From the equations (2.6) and (2.7) we see that $u_1(x,t)$ and $u_2(x,t)$

are continuous for $t > 0$, except at the boundaries, since the integral terms are the integrals of the product of a bounded, continuous function by a summable function.

The first derivatives with respect to x , given by equations (2.12) and (2.13) are continuous except at the boundaries by the same reasoning that applying to $u_1(x, t)$ and $u_0(x, t)$.

For the first derivatives with respect to t , we have

$$\frac{\partial u_1(x, t)}{\partial t} = \frac{1}{a^2} \frac{\partial^2 u_1(x, t)}{\partial x^2} = -\frac{1}{2} \int_0^t \left\{ \left[\frac{1}{(t-t')^{3/2}} - \frac{a^2(x+m)}{(t-t')^{5/2}} \right] e^{-\frac{a^2(x+m)^2}{4(t-t')}} + \left[\frac{1}{(t-t')^{3/2}} - \frac{a^2(x-m)^2}{(t-t')^{5/2}} \right] e^{-\frac{a^2(x-m)^2}{4(t-t')}} \right\} \psi_1(t') dt'$$

and an analogous expression for $\frac{\partial u_2(x, t)}{\partial t}$. By the same reasoning as above this is continuous for $t > 0$, except at the boundaries.

Finally we must show that

$$\lim_{x \rightarrow x_0 \pm 0} \int_{t_1}^{t_2} \left| \frac{\partial u_i}{\partial x}(x, t) \right| dt = \int_{t_1}^{t_2} \lim_{x \rightarrow x_0 \pm 0} \left| \frac{\partial u_i}{\partial x}(x, t) \right| dt,$$

x_0 being the x coordinate of any boundary.

As a typical case we consider

$$\begin{aligned} & \lim_{x \rightarrow -m+0} \int_{t_1}^{t_2} \left| \frac{\partial u_1}{\partial x}(x, t) \right| dt \\ &= \lim_{x \rightarrow -m+0} \int_{t_1}^{t_2} \left| -\frac{a^2}{2} \int_0^t \frac{x+m}{(t-t')^{3/2}} e^{-\frac{a^2(x+m)^2}{4(t-t')}} \psi_1(t') dt' - \frac{a^2}{2} \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{-\frac{a^2(x-m)^2}{4(t-t')}} \psi_1(t') dt' \right| dt \end{aligned}$$

The term

$$G(x, t) = -\frac{a^2}{2} \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{-\frac{a^2(x-m)^2}{4(t-t')}} \psi_1(t') dt'$$

is bounded and continuous in x , as x approaches $-m$, so that

$$\lim_{x \rightarrow -m+0} \int_{t_1}^{t_2} |G(x, t)| dt = \int_{t_1}^{t_2} \lim_{x \rightarrow -m+0} |G(x, t)| dt$$

Consider the term

$$H(x, t) = -\frac{a^2}{2} \int_0^t \frac{x+m}{(t-t')^{3/2}} e^{-\frac{a^2(x+m)^2}{4(t-t')}} \psi_1(t') dt'$$

We define

$$\bar{H}(x, t) = \frac{a^2}{2} \int_0^t \frac{x+m}{(t-t')^{3/2}} e^{-\frac{a^2(x+m)^2}{4(t-t')}} |\psi_1(t')| dt'$$

and have

$$\lim_{x \rightarrow -\infty} \bar{H}(x, t) = \frac{\sqrt{\pi}}{a} |\psi_1(t)|$$

almost everywhere, by the considerations above, since $|\psi_1(t)|$ is summable.

We have $\bar{H}(x, t) \geq |H(x, t)|$ for all x and t .

Consider now

$$\lim_{x \rightarrow -\infty} \int_0^T \bar{H}(x, t) dt = \lim_{x \rightarrow -\infty} \int_0^T dt \frac{a^2}{2} \int_0^t \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} |\psi_1(t')| dt'.$$

By a change of order of integration we have

$$\begin{aligned} \int_0^T dt \frac{a^2}{2} \int_0^t \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} |\psi_1(t')| dt' \\ = \int_0^T dt' |\psi_1(t')| \frac{a^2}{2} \int_{t'}^T \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} dt = \int_0^T g(t', T, x) |\psi_1(t')| dt' \end{aligned}$$

where

$$\begin{aligned} g(t', T, x) &= \frac{a^2}{2} \int_{t'}^T \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} dt, \quad x > m \\ &\leq \frac{a^2}{2} \int_{t'}^0 \frac{x+m}{(t-t')^{3/2}} e^{\frac{-a^2(x+m)^2}{4(t-t')}} dt \end{aligned}$$

Putting $\xi = \frac{a(x+m)}{2(t-t')^{1/2}}$ we have

$$g(t', T, x) \leq 2a \int_0^{\infty} e^{-\xi^2} d\xi = a\sqrt{\pi}, \text{ independent of } t', T, x.$$

We have then

$$\begin{aligned} \lim_{x \rightarrow -\infty} \int_0^T g(t', T, x) |\psi_1(t')| dt' &= \int_0^T \lim_{x \rightarrow -\infty} g(t', T, x) |\psi_1(t')| dt' \\ &= a\sqrt{\pi} \int_0^T |\psi_1(t')| dt' = \int_0^T \lim_{x \rightarrow -\infty} \bar{H}(x, t') dt' \end{aligned}$$

that is

$$\lim_{x \rightarrow -\infty} \int_0^T \bar{H}(x, t) dt = \int_0^T \lim_{x \rightarrow -\infty} \bar{H}(x, t') dt'$$

and since $\bar{H}(x, t) \geq |H(x, t)|$ and $\lim_{x \rightarrow -\infty} H(x, t)$ exists nearly everywhere

$$\lim_{x \rightarrow -\infty} \int_0^T |H(x, t)| dt = \int_0^T \lim_{x \rightarrow -\infty} |H(x, t)| dt.$$

Hence
$$\lim_{x \rightarrow -\infty} \int_0^T \left| \frac{\partial u_1}{\partial x}(x, t) \right| dt = \int_0^T \lim_{x \rightarrow -\infty} \left| \frac{\partial u_1}{\partial x}(x, t) \right| dt$$

as was to be shown.

For $T > t'$ $\lim_{x \rightarrow -\infty} g(t', T, x) = a\sqrt{\pi}$

See Hobson, loc.cit. Vol.II Page 299.

We have used the limits 0 and T, by subtraction we can establish the equation for any limits t_1 and t_2 .

Thus the solution given by $u_1(x,t)$ and $u_2(x,t)$ in the inner and outer regions respectively satisfies the conditions of the Uniqueness Theorem B.

It remains to be shown that $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ can be found to satisfy equations (2.8) to (2.10). This will be done in the next two sections.

PART III. CASE OF INFINITE SIDE REGIONS

A case of considerable interest is the one where the outer regions are infinite. In this section we give a method of solution in this case.

Here we have no outer boundary and replace the boundary condition (2.3) by the condition $\lim_{x \rightarrow \pm\infty} \frac{\partial u(x,t)}{\partial x} = 0$. The solution for the outer region differs from that given by equation (2.7) by the absence of the term involving $\psi_3(t)$. The result is that instead of the three integral equations (2.8 to (2.9), we have only two for the two functions $\psi_1(t)$ and $\psi_2(t)$. These are

$$(3.1) \quad \psi_2(t) = -c \psi_1(t) + \frac{c\alpha}{\sqrt{\pi}} \int_0^t \frac{1}{(t-t')^{1/2}} e^{-\frac{x^2}{4(t-t')}} \psi_1(t') dt'$$

§ This problem is treated by A. Sommerfeld, "Zur analytischen Theorie der Wärmeleitung," Mathematische Annalen, 45, (1894), page 270, by

constructing the Green's function by the method of images.

Equation (2.13) becomes $\frac{\partial u_2}{\partial x}(x,t) = -\frac{b^2}{2} \int_0^t \frac{x-m}{(t-t')^{3/2}} e^{-\frac{b^2(x-m)^2}{4(t-t')}} \psi_2(t') dt'$

and Limit $\lim_{x \rightarrow \pm\infty} \frac{\partial u_2}{\partial x}(x,t) = 0$.

and

$$(3.2) \quad \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} + \frac{1}{(t-t')^{1/2}} \right\} \psi_1(t') dt' \\ = f_1 + \int_0^t \frac{1}{(t-t')^{1/2}} \psi_2(t') dt'$$

The equation (3.1) gives a value of $\psi_2(t)$ that may be substituted into (3.2) giving

$$(3.3) \quad \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} + \frac{1}{(t-t')^{1/2}} \right\} \psi_1(t') dt' \\ = f_1 - c \int_0^t \frac{1}{(t-t')^{1/2}} \psi_1(t') dt' \\ + \frac{c}{\sqrt{\pi}} \int_0^t \frac{dt'}{(t-t')^{1/2}} \int_0^{t'} \frac{\alpha}{(t'-t'')^{1/2}} e^{\frac{-\alpha^2}{t'-t''}} \psi_1(t'') dt'' \\ = f_1 - c \int_0^t \frac{1}{(t-t')^{1/2}} \psi_1(t') dt' \\ + \frac{c}{\sqrt{\pi}} \int_0^t \psi_1(t'') dt'' \int_{t''}^t \frac{\alpha}{(t-t')^{1/2} (t'-t'')^{1/2}} e^{\frac{-\alpha^2}{t'-t''}} dt'$$

where we have changed the order of integration in the last term. We may, after carrying out the integration in the last term, rewrite this as

$$(3.4) \quad \int_0^t \left\{ \frac{1 + \frac{1-c}{1+c} e^{\frac{-\alpha^2}{t-t'}}}{(t-t')^{1/2}} \right\} \psi_1(t') dt' = \frac{f_1}{1+c}$$

This is an equation of the sort dealt with in Part One with

$$K(t, t') = \frac{1-c}{1+c} e^{\frac{-\alpha^2}{t-t'}}$$

c is a positive constant. Since $K(t, t')$ vanishes exponentially for $t = t'$ and is bounded elsewhere and is absolutely continuous in t , uniformly for all t' , the conditions of part one are satisfied and we can find a summable solution of (3.4) by the method given in that part. This solution, $\psi_1(t)$, substituted in (3.1) gives a summable function $\psi_2(t)$. These substituted in (2.6) and the modified (2.7) give functions $u_1(x, t)$ and $u_2(x, t)$ that satisfy the conditions of the Uniqueness Theorem B.

PART IV. SOLUTION OF EQUATIONS FOR FINITE SIDE REGIONS

In this section we consider the more general case of finite outer regions. As in part Three, we do not solve for $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ directly, but, assuming the existence of summable solutions, $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, we solve for $\int_0^t \psi_1(t') dt'$, $\int_0^t \psi_2(t') dt'$ and $\int_0^t \psi_3(t') dt'$, and show that the derivatives of the integrals exist and satisfy the equations, nearly everywhere. In order to do this we must modify the equations, (2.8), (2.9) and (2.10) so that they will involve the integrals.

We multiply equation (2.8) by $(t''-t)^{-\frac{1}{2}}$ and integrate from $t = 0$ to $t = t''$, thus

$$(4.1) \quad \int_0^{t''} \frac{dt}{(t''-t)^{1/2}} \int_0^t \frac{1}{(t-t')^{1/2}} \psi_3(t') dt' = \int_0^{t''} \frac{f_3(t) dt}{(t''-t)^{1/2}} - \int_0^{t''} \frac{dt}{(t''-t)^{1/2}} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \psi_2(t') dt'.$$

The first term by a change of order of integration gives

$$\pi \int_0^{t''} \psi_3(t') dt'$$

Integrating the inner integral of the last term by parts gives

$$\left[\left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\beta^2}{t-t'}} \right\} \left\{ \int_0^{t'} \psi_2(t'') dt'' \right\} \right]_{t'=0}^{t'=t} - \int_0^t \left\{ \int_0^{t'} \psi_2(t'') dt'' \right\} \frac{t-t'-2\beta^2}{2(t-t')^{3/2}} e^{\frac{-\beta^2}{t-t'}} dt'$$

The term outside the integral is zero. Putting this in and changing the order of integration gives, in place of (4.1)

$$(4.2) \quad \pi \int_0^{t''} \psi_3(t') dt' = \int_0^{t''} \frac{f_3(t') dt'}{(t''-t')^{1/2}} + \int_0^{t''} \left\{ \int_0^{t'} \psi_2(t'') dt'' \right\} dt' \int_{t'}^{t''} \frac{t-t'-2\beta^2}{2(t-t')^{3/2}} e^{\frac{-\beta^2}{t-t'}} dt \\ = \int_0^{t''} \frac{f_3(t') dt'}{(t''-t')^{1/2}} - \int_0^{t''} \left\{ \int_0^{t'} \psi_2(t'') dt'' \right\} \frac{\beta}{(t''-t')^{1/2}} e^{\frac{-\beta^2}{t''-t'}} dt'$$

We integrate the equation (2.9) from $t = 0$ to $t = t''$ and in the last two terms integrate by parts and then change the order of integration, this gives

$$\begin{aligned}
 (4.3) \quad \int_0^{t''} \psi_2(t') dt' &= -c \int_0^{t''} \psi_1(t') dt' \\
 &- \frac{c\alpha}{\sqrt{\pi}} \int_0^{t''} \left\{ \int_0^{t'} \psi_1(t'') dt'' \right\} dt' \int_{t'}^{t''} \frac{\partial}{\partial t'} \left(\frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} \right) dt \\
 &- \frac{\rho}{\sqrt{\pi}} \int_0^{t''} \left\{ \int_0^{t'} \psi_3(t'') dt'' \right\} dt' \int_{t'}^{t''} \frac{\partial}{\partial t'} \left(\frac{1}{(t-t')^{1/2}} e^{\frac{-\rho^2}{t-t'}} \right) dt \\
 &= -c \int_0^{t''} \psi_1(t') dt' \\
 &+ \frac{c\alpha}{\sqrt{\pi}} \int_0^{t''} \left\{ \int_0^{t'} \psi_1(t'') dt'' \right\} \frac{1}{(t''-t')^{1/2}} e^{\frac{-\alpha^2}{t''-t'}} dt' \\
 &+ \frac{\rho}{\sqrt{\pi}} \int_0^{t''} \left\{ \int_0^{t'} \psi_3(t'') dt'' \right\} \frac{1}{(t''-t')^{1/2}} e^{\frac{-\rho^2}{t''-t'}} dt'
 \end{aligned}$$

In the integration by parts the terms outside the sign of integration vanish as above.

We multiply equation (2.10) by $(t''-t)^{-1/2}$ and integrate from $t = 0$ to $t = t''$

$$\begin{aligned}
 (4.4) \quad \int_0^{t''} \frac{dt}{(t''-t)^{1/2}} \int_0^t \left\{ \frac{1}{(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} + \frac{1}{(t-t')^{1/2}} \right\} \psi_1(t') dt' \\
 = \int_0^{t''} \frac{f \cdot dt}{(t''-t)^{1/2}} + \int_0^{t''} \frac{dt}{(t''-t)^{1/2}} \int_0^t \frac{1}{(t-t')^{1/2}} \psi_2(t') dt' \\
 + \int_0^{t''} \frac{dt}{(t''-t)^{1/2}} \int_0^t \frac{1}{(t-t')^{1/2}} e^{\frac{-\rho^2}{t-t'}} \psi_3(t') dt'
 \end{aligned}$$

The first term by an integration by parts - in which the terms outside the sign of integration vanish - followed by a change of order of integration gives

$$\begin{aligned}
 &- \int_0^{t''} \left\{ \int_0^{t'} \psi_1(t'') dt'' \right\} dt' \int_{t'}^{t''} \frac{t-t'-2\alpha}{2(t''-t)^{1/2}(t-t')^{1/2}} e^{\frac{-\alpha^2}{t-t'}} dt \\
 &= \int_0^{t''} \left\{ \int_0^{t'} \psi_1(t'') dt'' \right\} \frac{\alpha}{(t''-t)^{1/2}} e^{\frac{-\alpha^2}{t''-t}} dt'
 \end{aligned}$$

The second term is $\pi \int_0^{t''} \psi_1(t') dt'$. The third term is $2f, \sqrt{t''}$. The

fourth term is $\pi \int_0^{t''} \psi_2(t') dt'$. The last term by the same treatment as the first gives

$$\int_0^{t''} \left\{ \int_0^{t'} \psi_3(t) dt \right\} \frac{\beta}{(t''-t')^{3/2}} e^{\frac{-\beta^2}{t''-t'}} dt'$$

The equation (4.4) thus becomes

$$(4.5) \quad \int_0^{t''} \left\{ \int_0^{t'} \psi_1(t) dt \right\} \frac{\alpha}{(t''-t')^{3/2}} e^{\frac{-\alpha^2}{t''-t'}} dt' \\ + \pi \int_0^{t''} \psi_1(t') dt' = 2f \sqrt{t''} + \pi \int_0^{t''} \psi_2(t') dt' \\ + \int_0^{t''} \left\{ \int_0^{t'} \psi_3(t) dt \right\} \frac{\beta}{(t''-t')^{3/2}} e^{\frac{-\beta^2}{t''-t'}} dt'$$

By putting

$$F(t'') = \int_0^{t''} \frac{f_2(t') dt'}{(t''-t')^{3/2}}$$

$$V_i(t'') = \int_0^{t''} \psi_i(t') dt' \quad i=1,2,3.$$

$$H_\rho(t'', t') = \frac{\rho}{(t''-t')^{3/2}} e^{\frac{-\rho^2}{t''-t'}} \quad \rho = \alpha, \beta.$$

we can write (4.2), (4.3) and (4.5) as

$$(4.6) \quad \pi V_3(t'') = F(t'') - \int_0^{t''} V_2(t') H_\beta(t'', t') dt'$$

$$(4.7) \quad V_2(t'') = -c V_1(t'') + \frac{c}{\sqrt{\pi}} \int_0^{t''} V_1(t') H_\alpha(t'', t') dt' \\ + \frac{1}{\sqrt{\pi}} \int_0^{t''} V_3(t') H_\beta(t'', t') dt'$$

and

$$(4.8) \quad \int_0^{t''} V_1(t') H_\alpha(t'', t') dt' + \pi V_1(t'') \\ = 2f \sqrt{t''} + \pi V_2(t'') + \int_0^{t''} V_3(t') H_\beta(t'', t') dt'$$

Equation (4.7) multiplied by π and added to the equation (4.8) gives

$$(4.9) \quad V_1(t'') = \frac{1}{\pi(1+c)} \left\{ 2f \sqrt{t''} \right. \\ \left. + (c\sqrt{\pi} - 1) \int_0^{t''} V_1(t') H_\alpha(t'', t') dt' \right. \\ \left. + (1 + \sqrt{\pi}) \int_0^{t''} V_3(t') H_\beta(t'', t') dt' \right\}$$

Equation (4.7) multiplied by π and subtracted from c times equation

(4.8) gives

$$(4.10) \quad V_2(t'') = \frac{1}{\pi(1+c)} \left\{ -2cf \sqrt{t''} \right. \\ \left. + (c + c\sqrt{\pi}) \int_0^{t''} V_1(t') H_\alpha(t'', t') dt' \right. \\ \left. - (c - \sqrt{\pi}) \int_0^{t''} V_3(t') H_\beta(t'', t') dt' \right\}$$

We have in equations (4.6), (4.9) and (4.10) a system of Volterra integral equations of the second kind with bounded kernels. Such a system is solvable by a process of successive approximations that necessarily converges.[#] It may be pointed out that, since the kernels involved vanish exponentially, the process of approximation converges rapidly.

The terms on the right hand sides of equations (4.6), (4.9) and (4.10) are of the types shown to be absolutely continuous in Part One and so the functions $V_1(t)$, $V_2(t)$ and $V_3(t)$ are absolutely continuous. They thus possess derivatives nearly everywhere and are the integrals of these derivatives. We wish to show that the functions $\psi_1(t) = V_1'(t)$, $\psi_2(t) = V_2'(t)$ and $\psi_3(t) = V_3'(t)$ satisfy the equations (2.8), (2.9) and (2.10).

We do this by a device similar to that employed in Part One. That is we substitute $V_2'(t)$ for $\psi_2(t)$ and $V_3'(t)$ for $\psi_3(t)$ in the right and left hand members of equation (2.8) and call the difference between the two members $D_1(t)$. Then multiplying by $(t''-t)^{-\frac{1}{2}}$ and integrating from $t = 0$ to $t = t''$, changing the order of integration and integrating by parts as was done to obtain equations (4.1) and (4.2) we have

$$\int_0^{t''} \frac{D_1(t) dt}{(t''-t)^{1/2}} = \pi \int_0^{t''} V_3'(t) dt - F(t'') \\ + \int_0^{t''} \left\{ \int_0^t V_2'(t) dt \right\} H_p(t'', t') dt'$$

which is zero since $\int_0^t V_2'(t) dt = V_2(t)$ and $\int_0^{t''} V_3'(t) dt = V_3(t'')$ and satisfy the equation (4.6). Hence $D_1(t)$ is zero nearly everywhere and equation

[#] See Volterra, Lecons sur les Equations Integrales, page 71.

(2.8) is satisfied.

Similarly by the methods used to obtain equations (4.3) and (4.5), we can show that equations (2.9) and (2.10) are satisfied nearly everywhere, in virtue of equations (4.9) and (4.10).

PART V. EXTENSION TO GENERAL CASES

The methods of Part Four will apply to any number of regions and to conditions which are not symmetric. The only merit of the symmetric conditions is that they reduce the number of conditions by half.

The initial temperatures were taken to be constant in order to make the form of equations (2.6) and (2.7) as simple as possible. If the initial temperatures are not constant, the constant u_1 in equation (2.6) is replaced by the expression

$$\frac{a}{2\sqrt{\pi}} \int_{-m}^{+m} \frac{1}{t^{1/2}} e^{-\frac{a^2(x-x')^2}{4t}} u_1(x') dx'$$

where $u_1(x)$ is the initial temperature. This expression satisfies equation (2.1) and approaches $u_1(x)$ as t approaches zero, nearly everywhere if $u_1(x)$ is summable.[#] For varying initial temperature in the outer regions the constant u_2 is replaced by an analogous expression. We can show that this solution satisfies the conditions of the Uniqueness Theorem B if $u_1(x)$ and $u_2(x)$ are bounded. The introduction of these expressions complicates the equations, but makes no essential difference in the method.

In closing I should like to acknowledge my indebtedness to Dr.G.C. Evans, both for his proposal of this problem and for his timely criticisms and suggestions.

[#]in Hobson, loc. cit. Vol. II, page 462.