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**Presented to the Faculty of
The Rice Institute in May 1939
in partial fulfillment of
their requirements for the
degree of
Doctor of Philosophy,**

**Describing the graduate
research done by**

FRED TERRY ROGERS, JR.

**while in residence at The
Rice Institute, Houston,
Texas, during
Sept. 1935 - May 1939.**

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FOREWORD

By way of orientation, it should be pointed out about the several papers in this volume that:

Number One is a reprint of that portion of my Master of Arts thesis which was published. Numbers Seven and Seven A constitute the chief work done since Number One was finished.

Of the remaining papers, many appeared as incidentals. Thus, the material in Number Three was prepared for some other elementary computations which required it. Numbers Four, Six, Eight, and Nine were incidental to Numbers Seven. Number Five was prepared in connection with, or as an aftermath of, Number One.

The sequence composed of Numbers Two, Ten, Eleven, and Twelve forms to some extent a separate, planned program. As such, these papers should be considered together.

F. T. R., Jr.

May 1939

1952-1953

THE UNITED STATES OF AMERICA

DEPARTMENT OF AGRICULTURE

The Precise Measurement of Three Radium B Beta-Particle Energies

F. T. ROGERS, JR., *The Rice Institute*

(Received July 1, 1936)

The $H\rho$ of the three most intense Ra B β -particle lines have been measured by a precise magnetic spectrograph method and found to be 1406.0, 1671.1, and 1931.5 gauss cm. An explicit analysis has been made of the Cotton balance equations for H in this experiment and on the basis of this a complete analysis was made of the maximum relative error to be expected in the accepted values of $H\rho$. It is shown that the maximum relative error to be expected in the least $H\rho$ is 1 part in 3000 and that within this possible error the greatest $H\rho$ agrees with the value obtained

by Scott. It is suggested that the accuracy of 1 part in 10,000 hoped for by Scott is rather too optimistic. The energies of these three β -particles are 1.512, 2.045, and 2.612 in units of 10^6 electron volts. It is shown that the least of these cannot be expected to be any more accurate than 1 part in 1000. The conclusion is that except for systematic errors which might have been present and undetected throughout this experiment and that of Scott, these experiments have established the values of these $H\rho$'s to within 1 part in 3000.

INTRODUCTION

IN 1924 Ellis and Skinner¹ measured $H\rho$ (H is the magnetic field necessary to produce a radius ρ of curvature in the path of a β -particle) for many of the β -particle energies of Ra B; for the three most intense lines of Ra B they got, correctly to within what they thought was 1 part in 500,

$$H\rho = 1410, 1677, 1938 \text{ gauss cm.}$$

In 1934 Ellis² repeated this work with more precision and got

$$H\rho = 1400.4, 1665.9, 1925.5 \text{ gauss cm}$$

(in which the last figures are not entirely significant) for these same lines. In 1934 also, Scott³ performed a precise experiment on the most intense of these lines using the Cotton balance method of measuring the magnetic field rather than an inductive method; he got for this most intense line

$$H\rho = 1931.8 \text{ gauss cm}$$

and hoped it to be correct to 1 part in 10,000.

The present experiment was undertaken to check Scott's work and method and to extend the measurements to other lines in the Ra B β -particle spectrum. The values of

$$H\rho = 1406.0, 1671.1, 1931.5 \text{ gauss cm}$$

were obtained, the last of which is in good agreement with Scott's value for the most intense line. The $H\rho$'s for the other two lines as found by this

experiment yield slightly different relative values of $H\rho$ from those previously obtained.

A more thorough investigation of the theory of the Cotton balance method of measuring H was carried out for this experiment. On the basis of it an analysis was made of the errors which might be present in the finally accepted values of $H\rho$. The conclusion was reached that in this experiment an accuracy of 1 part in 3000 is attainable if the value of H used in the finally accepted $H\rho$'s is determined from the mean of at least forty separate, equally weighted determinations of the mass used with the Cotton balance and if the values of ρ used finally are the means of seven separate, equally weighted determinations.

The good agreement (within 1 part in 6400 in $H\rho$) of the present experiment with that of Scott would seem to indicate that this type of measurement has at last established the absolute values of the $H\rho$'s for the three most intense lines in the Ra B β -particle spectrum to within 1 part in 3000 except for possible systematic errors which may have been present and undetected in both experiments.

The energies corresponding to the three $H\rho$'s we have determined are found to be

$$V = 1.512 \times 10^6, 2.045 \times 10^6, 2.612 \times 10^6 \text{ electron volts}$$

and can be considered as accurate to not better than 1 part in 1000.

APPARATUS

The apparatus used in this experiment was, except for a few small changes, that used by

¹ Ellis and Skinner, Proc. Roy. Soc. A105, 165 (1924).

² Ellis, Proc. Roy. Soc. A143, 352 (1934).

³ Scott, Phys. Rev. 46, 633 (1934).

Scott in his work and described by him in his paper.³ It consists of a large permanent magnet with cobalt steel poles and with a 1.52 cm air gap faced on each side by soft iron disks 15 cm in diameter and 7.5 cm thick; the faces are quite nearly parallel and vertical. The flux density in the air gap can easily be adjusted to the desired value to within 1.5 gauss by means of a controlled current in two solenoids around the pole pieces. An investigation of the direction of the field in the gap showed that it deviates from the normal to the pole faces by less than $0^{\circ} 05'$. Computation by the series for $\cos 0^{\circ} 05'$ shows that the horizontal component of H differs from H by less than 1 part in 500,000. Thus the force which the Cotton balance measures, i.e., the horizontal component of H , is sensibly H .

The field in the air gap could be explored with search coils and galvanometers calibrated as fluxmeters. H as a function of the distance from the center of the field was determined with a coil and galvanometer designed to give an accuracy of one percent. Small variations of H near the center of the field were obtained with a 1000 turn coil of diameter 0.9 cm and an 18 ohm, high sensitivity galvanometer, the combination having a sensitivity of 0.0241 gauss per cm.

Supported above the air gap was a new Becker precision balance of which one pan had been removed. In place of the pan was hung a Pyrex glass plate about 30 cm long and 3 cm wide with its longer edges optically worked to be parallel to within 1/500 mm for 15 cm from one end. This plate hung from the lower ends of two parallel aluminum strips which hung in turn from a support-assembly attached to a knife edge of the balance. The plate-assembly was free to swing in two dimensions and was protected from external air currents by a surrounding brass case. Two straight silver strips about 1/10 mm thick were cemented with very dilute white shellac around the lower end and the two long edges of the plate; they were joined by a flat fold at the middle of the lower edge and formed the conductor of current in the magnetic field. This lower edge was placed along the x axis of Fig. 1 while measurements of H were in progress; the center line of the plate was along the y axis.

Measurements of plate and strips could be made with a Brown and Sharpe gauge which was

FIG. 1. PP' are photographic plates; O is the source of β -particles.

graduated in 10^{-4} 's of an inch and which could be reliably estimated to a 10^{-5} of an inch, with a Gaertner comparator and invar steel standard scale 10 cm long, and with a new Brown and Sharpe 31 mm standard disk. The invar scale was stated by the Bureau of Standards to be accurate to 1 part in 10^6 . All measurements of the plate, strips, and photographic plates are expressed in terms of this standard scale when used as absolute lengths.

When the current in the strips was reversed, the change in force on them was measured by balancing it with a mass m on the balance. m was nearly 400 mg and was obtained in the main by combining the 200, 100 and 100 mg weights of a new and previously unused set of platinum, assayers' weights by Becker; each of these weights is quoted by the manufacturer as being accurate to within 0.005 mg. To get m really accurately, small balance deflections (occurring when the current in the strips was reversed) could be measured by the motion of a spot of light reflected from a small mirror on the balance beam to a scale about two meters from the balance. The location of the spot of light could be read to within 1/5 mm with a magnifying glass. When fully loaded in the experiment, the balance had a sensitivity of 0.409 mg/cm, which was entirely sufficient to get m accurately.

Current was supplied to the strips around the plate by two 160 ampere-hour lead storage batteries connected in parallel. The current was sent through two standard resistors of 4.0001 international ohms at 25°C and having negligible temperature corrections near 25°C. These resistors were connected in parallel by heavy copper bars and the current through them adjusted until a galvanometer in series with a standard cell (the two being connected across the resistors) showed no deflection. A measurement of H using one resistor balanced by two standard cells yielded the same value of H as this arrangement, so the resistance between the resistors and the copper bars was entirely negligible. The resistors were calibrated to 0.005 percent by the Bureau of Standards in February, 1934, and, except for Scott's work then, have not been used until the present experiment. The standard cell along with two others was calibrated to 0.01 percent by the Bureau of Standards at the same time and was found to have an e.m.f. of 1.01884 international volts at 25°C; at the time of this experiment these cells had not changed their relative e.m.f.'s by more than $\frac{1}{2} \times 10^{-5}$ volts, so it was assumed that their absolute values had not altered appreciably. The galvanometer used to indicate the existence of the correct current was of sensitivity 1.84×10^{-5} amp. cm, readable to 1.5 mm easily. The current I could be adjusted by means of a variable resistance network to within $\frac{1}{2} \times \text{mm}$ scale deflection, or to within 10^{-6} amp.

The Ra B β -particles (from the active deposit from the radon of 23 mg of radium) were emitted from the surface of a platinum wire placed at 0 (Fig. 1); they traveled in opposite directions as allowed by slits in the camera and fell on photographic plates PP' . The distances between the outer edges of the three sets of clearly defined "images" on the plates could be measured with the Gaertner comparator and standard scale. The camera and its enclosing box are of brass tested previously for magnetic effects and found to have none; Scott fully describes them in his paper.³ The vacuum in the box was got by a Cenco Hyvac pump running continuously during the three and one-half hour exposures.

All observations and measurements were made in two rooms which were maintained at temperatures between 23° and 25°C.

THEORY

Preliminary measurements have shown that when the glass plate is in position for field measurements, the center of the silver strip around the glass plate lies, in the large, along a curve C of the form (exaggerated in the diagram) of $abcdefgh$ in Fig. 1; this curve is sufficiently nearly symmetric to be treated for our purposes as symmetric. In Fig. 1 the magnetic field H is perpendicular to the plane of the paper and directed into the paper; x and y axes perpendicular to H are chosen with their origin in line with the centers of the pole faces and half way between them, the x axis horizontally; ds with x component dx is an element of length along C . The width l of the current loop is measured at some convenient location (as bg) along the glass plate.

To get the field H_0 at the origin by the Cotton balance method, note that the force acting on the current I (e.m.u.) in the strip is, besides gravity, $I \int_c H dx$; when I is reversed, the change in force on the strip is measured by the weight of the mass m which will just counterbalance it; then

$$mg = 2I \int_c H dx.$$

Since H and C are necessarily continuous, we can write

$$\int_c = \int_a^c + \int_c^d + \int_d^e + \int_e^f + \int_f^h;$$

and since subsequent experiment shows that

$$\int_a^c, \int_c^d, \int_d^e, \int_e^f < 10^{-3} \int_f^h,$$

approximate evaluation of these integrals is satisfactory. Let

$$\begin{aligned} H &= H_0 + h(x) \text{ along } de, \\ H &= H_0 F(y) \text{ along } ac \text{ and } fh, \\ H &= H_0 \text{ along } cd \text{ and } ef; \end{aligned}$$

then

$$\begin{aligned} mg = 2I \bigg\{ & H_0 \int_a^c F(y) dx + H_0 \int_c^d dx \\ & + \int_d^e [H_0 + h(x)] dx + H_0 \int_e^f dx + H_0 \int_f^h F(y) dx \bigg\}. \end{aligned}$$

Also let C be represented from a to c by $x=f(y)$, and let

$$\int_c^d dx = \frac{1}{2}\Delta l;$$

then in view of the symmetry of C with respect to the y axis,

$$mg = 2I \left[2H_0 \int_a^c F(y)f'(y)dy + H_0\Delta l + H_0\Delta l + \int_{-l/2}^{l/2} h(x)dx \right].$$

By solving for the coefficient of l ,

$$H_0 = \frac{1}{l} \left[\frac{mg}{2I} - 2H_0 \int_a^c F(y)f'(y)dy - H_0\Delta l - \int_{-l/2}^{l/2} h(x)dx \right];$$

here m , g , l and I are measurable with high accuracy; the last three terms in the right member

are of the nature of corrections to the first and (as previously stated) are fairly small, so in them only the approximate value of H_0 need be used.

To this Cotton balance equation for H_0 must be added for this experiment the Hartree correction for non-uniformity of H along the β -particle paths. In Fig. 1 two semi-circular paths for the same β -particle energy are Okj and Omn ; measuring θ along them as shown in the figure, let

$$H = H_0 + h_1(\theta) \text{ along } Okj,$$

$$H = H_0 + h_2(\theta) \text{ along } Omn.$$

Then according to Hartree,⁴ the first order correction to be applied to H_0 to give the constant value of H which, if it were acting all along the β -particle path Okj , would produce the observed deviation Oj is

$$\frac{1}{2} \int_0^\pi h_1(\theta) \sin \theta d\theta.$$

Applying to the above equation for H_0 the mean of the two Hartree corrections for the two paths of the same energy of β -particle, we get

$$H_0 = \frac{1}{l} \left[\frac{mg}{2I} - 2H_0 \int_a^c F(y)f'(y)dy - H_0\Delta l - \int_{-l/2}^{l/2} h(x)dx \right] + \frac{1}{4} \left(\int_0^\pi h_1(\theta) \sin \theta d\theta + \int_0^\pi h_2(\theta) \sin \theta d\theta \right), \quad (1)$$

which is the expression adopted for H_0 to be used in $H\rho$.

To get ρ , the distance d between the outer, clearly defined edges of corresponding images on the photographic plates is measured; from it is subtracted the diameter D of the platinum wire supporting the source at O . Then

$$\rho = (d - D)/4. \quad (2)$$

$H\rho$ for a β -particle energy is the product of the appropriate values of expressions (1) and (2).

The relative error to be expected in a single determination of H_0 follows from two principles concerning errors:⁵ the relative error in a product of quantities having known relative errors is the sum of the relative errors of the multiplicands, and the absolute error of a sum of quantities having known absolute errors is the

sum of the absolute errors of the summands. Let ϵ_i , $i=1$ through 10, be the absolute errors which might possibly be present in single determinations of d , D , l , m , g , I , $H_0 \int Ff'dy$, $H_0\Delta l$, $\int h(x)dx$, and $\int h(\theta) \sin \theta d\theta$ respectively; these must be estimated from the precisions of the observations on the several quantities entering into expressions (1) and (2). Then since (by the definition of relative error) there corresponds to the relative error in a quantity an absolute error whose magnitude is equal to the product of the magnitude of the quantity into the relative error,

$$(mg/2I)(\epsilon_4/m + \epsilon_5/g + \epsilon_6/I) + 2\epsilon_7 + \epsilon_8 + \epsilon_9$$

is the absolute error to be expected in the bracketed part of expression (1); to it corresponds quite nearly the relative error

$$\epsilon_4/m + \epsilon_5/g + \epsilon_6/I + (2\epsilon_7 + \epsilon_8 + \epsilon_9)/(mg/2I).$$

Hence the absolute error in the first term of the

⁴ Hartree, Proc. Camb. Phil. Soc. 21, 746 (1923).

⁵ Gibbs, *Adjustment of Errors in Practical Science* (Oxford Press, 1929), pp. 76-78.

right member of (1) is approximately

$$H_0 \left(\frac{\epsilon_3}{l} + \frac{\epsilon_4}{m} + \frac{\epsilon_5}{g} + \frac{\epsilon_6}{I} + \frac{2\epsilon_7 + \epsilon_8 + \epsilon_9}{mg \cdot 2I} \right);$$

added to the absolute error

$$\frac{1}{2}(\frac{1}{2}\epsilon_{10} + \frac{1}{2}\epsilon_{10}) = \epsilon_{10} \cdot 2,$$

in the second term and divided by H_0 , we have as the relative error to be expected in a single determination of H_0 :

$$e_1 = \frac{\epsilon_3}{l} + \frac{\epsilon_4}{m} + \frac{\epsilon_5}{g} + \frac{\epsilon_6}{I} + \frac{2\epsilon_7 + \epsilon_8 + \epsilon_9}{mg \cdot 2I} + \frac{\epsilon_{10}}{2H_0}. \quad (3)$$

The relative error to be expected in a single determination of ρ is obviously

$$e_2 = (\epsilon_1 + \epsilon_2) \cdot (d - D). \quad (4)$$

If ϵ_i , $i=1$ through 10, are estimated conservatively as the maximum absolute errors to be expected in their parent magnitudes, the corresponding maximum relative error to be expected in an accepted $H\rho$ is

$$e_3 = e_1 + e_2. \quad (5)$$

To get the energy E of a β -particle of known $H\rho$, combine with the relativistic expression

$$E = m_0 c^2 [1 - v^2/c^2]^{-1/2}$$

for E the relativistic force equation

$$H e v = m v^2 \cdot \rho = m_0 v^2 \cdot \rho (1 - v^2/c^2)^{-1/2}$$

so as to eliminate v . Here v is the velocity of the β -particles, m_0 their rest mass, e their charge, and c the velocity of light. The resulting expression

$$E = m_0 c^2 [(A+1)^{1/2} - 1], \quad A = (H\rho e/m_0)^2 (1/c)^2,$$

is the one commonly used. If e/m_0 is in e.m.u. and if e is in e.s.u., then $m = (m_0/e)(e/c)$; also if E is expressed (as V) in electron volts, its value in ergs must be multiplied by $10^{-8} e/e$; thus if $H\rho$ is in gauss cm, the energy in electron volts is

$$V = (m_0/e)c^2 [(A+1)^{1/2} - 1] \times 10^{-8}. \quad (6)$$

This can be put more conveniently for computation if we define ϕ by

$$\phi = \tan^{-1} A^{1/2} = \tan^{-1} (H\rho e/m_0)(1/c),$$

then

$$V = (m_0/e)c^2 (\sec \phi - 1) \times 10^{-8}. \quad (7)$$

If ϵ_{11} and ϵ_{12} are the absolute errors to be expected in e/m_0 and c respectively, the relative error to be expected in $(A+1)$ (see Eq. (6)) is

$$\frac{A}{A+1} \left(2e_3 + \frac{2\epsilon_{11}}{e/m_0} + \frac{2\epsilon_{12}}{c} \right).$$

Since the extraction of the square root of a quantity with a known relative error yields roots with relative errors only half that of their square,⁶ the absolute error in $(A+1)^{1/2}$ is

$$\frac{A}{(A+1)^{1/2}} \left(e_3 + \frac{\epsilon_{11}}{e/m_0} + \frac{\epsilon_{12}}{c} \right),$$

which is also to be expected in $[(A+1)^{1/2} - 1]$. Hence by addition of the relative errors in m_0/e , c^2 , and $[(A+1)^{1/2} - 1]$, the relative error to be expected in V is

$$e_4 = \frac{\epsilon_{11}}{e/m_0} + \frac{2\epsilon_{12}}{c} + \frac{A}{(A+1) - (A+1)^{1/2}} \left(e_3 + \frac{\epsilon_{11}}{e/m_0} + \frac{\epsilon_{12}}{c} \right). \quad (8)$$

The values of ϵ_{11} and ϵ_{12} are preferably the maximum errors judged to be possible in the accepted values of their parent magnitudes; if they are not, but are the probable errors quoted with the accepted values, e_4 is of the nature of an hybrid which is only a lower bound of the maximum relative error to be expected in V .

It should be remembered that expressions (3), (4), (5) and (8) do not take into account any systematic errors which might have been present and undetected in the determinations of the quantities appearing in expressions (1), (2) and (7).

MEASUREMENTS

The experiment consisted in measuring the quantities appearing in expressions (1) and (2) so as to determine, respectively, H and the ρ 's

⁶ For if a is a quantity with a small known absolute error Δa , then

$$(a + \Delta a)^{1/2} = a^{1/2} + \Delta a/2a^{1/2} + \dots \leq a^{1/2} + \Delta a/2a^{1/2}.$$

Thus the relative error in $a^{1/2}$ is

$$(1/a^{1/2}) \times (\Delta a/2a^{1/2}) = 1/2(\Delta a/a),$$

or is one-half the relative error in a .

for the three lines. H_0 was determined from forty determinations of m and each ρ was determined seven times in order to reduce the maximum possible relative error to be expected in $H\rho$ as far as is reasonably possible. The accepted $H\rho$ for an energy is the product of the H_0 value (from the forty determinations of m) as corrected for the proper path by the Hartree correction, into the appropriate radius of curvature of the path. The correction quantities were in general determined first. The error quantities ϵ_i were estimated as each quantity was measured. All measurements were repeated at least once, usually more than once. Data are usually recorded with more figures than are strictly significant, in order to prevent the entrance of errors in computation.

The functions $h(x)$ and $h(\theta)$ were determined from changes in H revealed by the 1000 turn coil and the sensitive galvanometer. Table I is the data obtained for $\int h(x)dx$ at $H_0=1250$ gauss. Integration was effected numerically by assuming that $h(x)$ is linear between adjacent experimentally determined points; the result is that

$$\int_{-1.5}^{1.5} h(x)dx = -0.319 \text{ gauss cm.}$$

It is estimated that $\epsilon_9=0.01$. Table II is the data used in calculating $\int h(\theta) \sin \theta d\theta$ for $H_0=1250$ gauss and $\rho=1.54$ cm; $h(\theta)$ are given in gauss. Integration is exactly as before and the results for this radius and for the two other radii used are tabulated in Table III; the Hartree correction as tabulated is the mean of the two corrections for the two paths from 0. It is estimated from experience with these that $\epsilon_{10}=0.01$.

The value of l was measured at the location bg (Fig. 1), which is between $y=8.0$ and $y=9.0$. It was measured first with the comparator and standard scale and found to be 3.07918 cm. After the experiment was terminated, l was measured again, but with the 31 mm disk and the Brown and Sharpe gauge. The value of 3.07891 cm was obtained in this way. The finally accepted value of l is the mean of these two: $l=3.07905$ cm. These values of l are got by subtracting from the overall width of the glass plate and silver strips at bg , half the thicknesses

TABLE I. Data for $\int h(x)dx$.

x (cm)	$h(x)$ (gauss)
-1.5	-0.018
-1.0	-0.106
-0.5	-0.076
0.0	-0.000
0.5	-0.113
1.0	-0.187
1.5	-0.263

TABLE II. Data for $\int h(\theta) \sin \theta d\theta$, $\rho=1.54$ cm.

θ	$h_1(\theta)$	$h_2(\theta)$
$\rho=1.54$ cm		
0°	0.000	0.000
30°	-0.027	-0.263
60°	-0.115	-0.566
90°	-0.110	-0.667
120°	-0.234	-0.501
150°	-0.130	-0.314
180°	-0.189	-0.150

TABLE III. Hartree corrections.

ρ (cm)	$\int h_1 \sin \theta d\theta$ (gauss)	$\int h_2 \sin \theta d\theta$ (gauss)	Correction (gauss)
1.12	-0.098	-0.352	-0.225
1.34	-0.128	-0.418	-0.273
1.54	-0.122	-0.471	-0.296

of the silver strips at that location. To accompany l we assign $\epsilon_3=0.00015$.

To evaluate $f'dy$ and Δl , the total width of the glass plate and of the silver strips in place was measured at many locations along the plate with the 10^{-4} inch gauge; as above, from these widths were subtracted half the thicknesses at the same locations of the silver strips. In Table IV are representative data from the large amount actually obtained by these measurements. As before it is assumed that $f(y)$ is linear between points determined experimentally. Since l was chosen as the width of the current loop between $y=8.0$ and $y=9.0$, Δl is obtained by subtracting the width of the current loop at $y=8.5$ from the width at $y=0.0$; from Table IV we have $\Delta l=3.08251-3.07905=0.00346$ cm.

The function $F(y)$ was obtained with the less sensitive fluxmeter arrangement; its values are given in Table V. For purposes of integration, $F(y)$ was also assumed to be linear between observed points. It is found that $2\int Ff'dy=$

TABLE IV. Width of current loop.

y (cm)	Width of strips and plate (cm)	1/2 sum of strip thicknesses (cm)	2f(y) = width of current loop (cm)
0.0	3.09195	0.00944	3.08251
0.5	3.08914	0.00945	3.07969
1.0	3.08901	0.00945	3.07956
2.0	3.08853	0.00947	3.07906
2.3	3.08835	0.00948	3.07887
2.4	3.08876	0.00949	3.07927
2.8	3.08905	0.00950	3.07955
2.9	3.08862	0.00951	3.07911
3.0	3.08849	0.00952	3.07897
4.3	3.08966	0.00952	3.08014
4.5	3.08953	0.00953	3.08000
5.5	3.08878	0.00952	3.07926
6.0	3.08877	0.00951	3.07926
6.7	3.08903	0.00949	3.07954
7.0	3.08911	0.00950	3.07961
8.5	3.08853	0.00948	3.07905
9.5	3.08837	0.00946	3.07891
11.5	3.08861	0.00950	3.07911
12.0	3.08873	0.00949	3.07924
12.4	3.08854	0.00948	3.07906
12.6	3.08876	0.00946	3.07932
13.2	3.08905	0.00944	3.07963
13.5	3.08861	0.00944	3.07917
15.0	3.08861	0.00945	3.07916
17.0	3.08869	0.00945	3.07924
17.5	3.08903	0.00943	3.07960
18.0	3.08854	0.00943	3.07911
20.0	3.08847	0.00942	3.07905
25.0	3.08872	0.00944	3.07928

-0.00354 using the entire set of data of which Table IV is a representative portion; the Table IV data give a slightly lower value than this.

The actual values of the corrections $H_0\Delta$ and $2H_0\int Ff'dy$ are obtained directly as

$$H_0\Delta = 1250 \times 0.00346 = 4.33 \text{ gauss cm,}$$

$$2H_0\int Ff'dy = 1250 \times (-0.00354) = -4.43 \text{ gauss cm.}$$

It is estimated that $\epsilon_7 = \epsilon_8 = 0.05$.

Since the resistors used in getting I are calibrated to only 0.005 percent and since the standard cells, to only 0.01 percent, the maximum error to be expected in I is at least 0.015 percent. The error inherent in adjusting for I with the resistance network is at most 10^{-7} e.m.u. By Ohm's law I is 0.509403 international ampere; the conversion⁷ to e.m.u. gives $I = 0.0509433$ e.m.u. Thus $\epsilon_8 = 0.0000076 + 0.0000025 = 0.00001$; the second term is added because of the uncertainty in the factor of conversion from international to electromagnetic units.

⁷ The Smithsonian Physical Tables, eighth edition (1933), p. 81, gives one international ampere equal to (0.999995 ± 0.000005) absolute ampere.

Because three weights are required to produce the 400 mg of m , ϵ_4 must contain 0.015 mg. Because of errors possible in locating the ends of the balance deflections when I is reversed, which come to $2 \times 0.02 \times 0.41 = 0.0164$ mg, ϵ_4 is larger than 0.015. Since m was measured nine times in each of the forty determinations of it, we take⁸ $\epsilon_4 = (0.015 + 0.0164/9)^{1/2} = 0.0032$ mg.

The value of g in the locality of the experiment is 979.28 cm sec.^2 and $\epsilon_5 = 0.005 \text{ cm sec.}^2$.

By observing the balance deflections when I was reversed and when the 400 mg were placed on the balance, the correction to be applied to the 400 mg to give m accurately could be found. These deflections were observed nine times for each determination of m . The forty mean observations of the balance deflection are recorded with their mean in Table VI. The evaluation of H_0 by taking for m the mean of a large number

TABLE V. Function $F(y)$.

y (cm)	F(y)
0.0	1.00
5.6	1.00
7.5	0.91
8.5	0.50
9.5	0.26
10.5	0.17
11.5	0.13
12.5	0.11
15.0	0.07
17.5	0.04
22.5	0.02

TABLE VI. Observations of balance deflections for m .

TRIAL	DEFL. (cm)	TRIAL	DEFL. (cm)	TRIAL	DEFL. (cm)	TRIAL	DEFL. (cm)
1	1.53	11	1.52	21	1.52	31	1.52
2	1.52	12	1.54	22	1.50	32	1.52
3	1.52	13	1.53	23	1.52	33	1.52
4	1.53	14	1.54	24	1.51	34	1.52
5	1.52	15	1.54	25	1.51	35	1.51
6	1.53	16	1.50	26	1.51	36	1.51
7	1.51	17	1.51	27	1.52	37	1.54
8	1.52	18	1.51	28	1.53	38	1.51
9	1.53	19	1.52	29	1.52	39	1.52
10	1.54	20	1.52	30	1.52	40	1.50

Mean = 1.520

⁸ Analogously to the customary procedure used in the operation of classical error computations, we assume that the relative error of the mean of n determinations of a quantity varies inversely as $n^{1/2}$. See, e.g., Bartlett, *The Method of Least Squares*, third edition (1915), p. 44, Eq. (46).

of observations which extended over several days is valid because Scott³ showed in his experiment that over a period of weeks the value of H_0 remained sensibly constant. The value of m to

be used is thus $0.4000000 + 1.52 \times 0.000409 = 0.400622$ gm. By use of Eq. (1) and the previously evaluated correction quantities it is seen that, except for the Hartree corrections,

$$H_0 = \frac{1}{3.07905} \left(\frac{0.400622 \times 979.28}{2 \times 0.0509433} + 4.43 - 4.33 + 0.32 \right) = 1250.70 \text{ gauss.}$$

Values of d were read from the photographic plates with the Gaertner comparator corrected by the standard scale; they could be repeated to within 0.0001 cm usually. Since each was done seven separate times, $\epsilon_1 = 0.0001$, $\epsilon_2 = 0.000038$. The value of D was found with a direct-reading 10⁻⁴-inch Brown and Sharpe micrometer caliper and is 0.02405 cm to within $\epsilon_2 = 0.0001$ cm. In Table VII are presented the seven values of each ρ got by Eq. (2) and their three means. Since H_0 is constant, these should be constant within the proper groups in the table.

Knowledge of the H_0 from Table VI, of the mean ρ 's of Table VII, of the Hartree corrections of Table III, of the various correction terms, and of the estimates of the possible errors is sufficient to provide the desired values of $H\rho$ and of the possible errors present.

RESULTS

Tables VIII, IX, X, and XI contain the results of the experiment. The accepted $H\rho$'s of Table

TABLE VII.

Trial	ρ (cm)	Mean ρ (cm)
1	1.12442	1.12440
2	1.12446	
3	1.12442	
4	1.12440	
5	1.12442	
6	1.12440	
7	1.12426	
1	1.33642	1.33644
2	1.33642	
3	1.33650	
4	1.33645	
5	1.33657	
6	1.33635	
7	1.33642	
1	1.54471	1.54472
2	1.54476	
3	1.54468	
4	1.54462	
5	1.54478	
6	1.54470	
7	1.54471	

VIII are from the value of H_0 from Table VI and corrected by Table III and from the ρ 's of Table VII. In view of the good agreement (1 part in 6400) of the present value of $H\rho$ for the most intense β -particles and that of Scott, it seems safe to consider the $H\rho$'s listed in Table VIII as sufficiently adequately determined in absolute value (within the possible limits of relative error given below) to serve as rather reliable standards of β -particle $H\rho$'s.

In Table IX are the possible relative errors necessary to the computations based on Eqs. (3), (4) and (5) for the least $H\rho$; they are collected from the section on Measurements. Thus the values of $H\rho$ in Table VIII cannot be considered as any more accurate than 1 part in 3000, even though each measurement used in getting $H\rho$ is more accurate than 1 part in 3000. Evidently some 60 percent of the uncertainty is accounted for by the uncertainty in the calibrations of the standard cells and standard resistors and in the uncertainty in the conversion factor for international and electromagnetic units of current. Within this limit of 1 part in 3000 the present experiment yields the same value of $H\rho$ for the most intense Ra B β -particles as Scott's experiment, though it is seen that Scott's hopes for an accuracy of 1 part in 10,000 are rather too optimistic.

The energies corresponding to the $H\rho$'s we have established are obtained from Eq. (7) using⁹

TABLE VIII. $H\rho$'s of three intense Ra B β -ray lines.

Line	$H\rho$ (gauss cm)
Least intense	1406.0
Of medium intensity	1671.1
Most intense	1931.5

⁹ The value of e/m_0 is that given by Birge, Phys. Rev. 49, 204 (1936). c is taken from *The Smithsonian Physical Tables*, eighth edition (1933), p. 74.

TABLE IX. Maximum relative errors to be expected in least $H\rho$.

Error	Magnitude	Magnitude (parts per 10,000)
$\epsilon_1 l$	0.00015 3.079	0.49
$\epsilon_4 m$	0.00002 .4007	0.08
$\epsilon_3 g$	0.005 979.3	0.05
$\epsilon_6 I$	.00001 .05094	1.97
$2I(2\epsilon_7 + \epsilon_8 + \epsilon_9) mg$	0.16 3850	0.41
$\epsilon_{10} 2H_0$	0.01 2500	0.04
$\epsilon_1 (d-D)$	.000038 4.496	0.08
$\epsilon_2 (d-D)$	.0001 4.496	0.22
Sum = ϵ_4		3.34

$$e/m_0 = 1.7562 \times 10^7 \text{ e.m.u. g.}$$

$$c = 2.99796 \times 10^{10} \text{ cm sec.}$$

with $\epsilon_{11} = 0.00027 \times 10^7$ and $\epsilon_{12} = 0.00004 \times 10^{10}$. They are in Table X.

In Table XI are the numbers necessary for the calculation of the lower bound e_4 of the relative error to be expected in the least V according to Eq. (8). They follow from the preceding two paragraphs. Eq. (8) becomes by substitution of these numbers

$$e_4 = \frac{1.54}{10,000} + \frac{0.24}{10,000} + \frac{0.678}{0.383} \left(\frac{3.34}{10,000} + \frac{1.54}{10,000} + \frac{0.24}{10,000} \right) = \frac{1.0}{1000}$$

Thus the energies V cannot be considered as more accurate than 1 part in 1000. In fact, since ϵ_{11} and ϵ_{12} are only classical probable errors, e_4 may be even larger than 1 part in 1000, the increase depending only upon how much larger than the probable errors the corresponding absolute errors possible in e/m_0 and c are.

TABLE X. Energies of three intense Ra B β -rays.

$H\rho$ (gauss cm)	V (electron volts)
1406.0	1.513×10^6
1671.1	2.047×10^6
1931.5	2.610×10^6

TABLE XI. Quantities for calculation of e_4 for least V .

Quantity	Magnitude	Magnitude (parts per 10,000)
$\epsilon_{11} (e/m_0)$	0.00027 1.7562	1.54
$\epsilon_{12} c$	0.00004 2.998	0.12
e_4	3.34 10,000	3.34
A	$\left(\frac{1.406 \times 1.756}{2.998} \right)^2 = 0.679$	

It is also noteworthy that even though $H\rho$ is accurate to within 1 part in 3000, the major part of the 1 part in 1000 accuracy allowed for V is accounted by possible errors in $H\rho$.

Finally, it should be noted that the improvements of this experiment over that of Scott lie in the facts that: a new balance of higher sensitivity, a new set of weights for m , straight silver strips, the standard scale, and a new gauge graduated in 10^{-4} 's of an inch were available; twice as large a current, determined by a far simpler potentiometer, was used in the Cotton balance; it includes measurements for the three most intense Ra B lines rather than for only one; and it provides a conservative, explicit estimate of the precision of which it is capable, 1 part in 3000 in $H\rho$, 1 in 1000 in V .

The writer also wishes in conclusion to record his appreciation both for the suggesting of this experiment and for the valuable aid given him during the experiment, by Dr. H. A. Wilson.

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On the Theory of the Electrostatic Beta-Particle Energy Spectrograph

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The necessary equations for the use of the electrostatic velocity analyzer as an energy spectrograph for β -particles are derived. A table is presented in which the conversion factor for converting particle-counts as a function of the electric deflecting potential to particle-counts as a function of energy is shown for electron energies of less than 10^6 electron volts. At low energies the conversion factor does not cause the excessive distortion of the observed curve that the conversion factor for the magnetic spectrograph causes in the same low energy region. It is thus concluded that the electrostatic analyzer is suitable for the investigation of the lower reaches of the continuous β -particle spectrum.

INTRODUCTION

IN view of the marked success of the new mass spectrograph of Bainbridge and Jordan,¹ which utilizes an electrostatic velocity analyzer after Hughes and Rojansky,² it is of interest to investigate the electrostatic velocity analyzer as an apparatus for the study of continuous β -particle spectra. In their paper² Hughes and Rojansky have shown that a radial, inverse first-power electrostatic field will partially focus a slightly divergent beam of electrons of the same energy, provided the electrons diverge from a source in the field and provided they revolve through an angle $\Phi = \pi/2^{\frac{1}{2}} = 127^{\circ} 17'$ in this field. In Fig. 1 let AB and CD be the two electrodes (cylindrical about O and shown in cross section) which form the electric field. Let the potentials be $\psi = \psi_1 > 0$ on AB and $\psi = \psi_2 < 0$ on CD ; in general let the potential at the radial distance ρ from O be ψ . Let the outer radius of AB and the inner radius of CD be r_1 and r_2 , respectively. Let S_1 and S_2 be the source of β -particles and the detector slit, respectively; they are situated at $\rho = r_0 = \frac{1}{2}(r_1 + r_2)$ and are $127^{\circ} 17'$ apart angularly. Finally let

FIG. 1. Schematic cross section of apparatus.

¹ Bainbridge and Jordan, Phys. Rev. 50, 282 (1936).

² Hughes and Rojansky, Phys. Rev. 34, 284 (1929).

r = distance from the origin O to the intersection of an electron path (deflection, $127^{\circ} 17'$) with the focal plane at S_2 ;

v = velocity of an electron which intersects the focal plane at radial distance r ;

v_0 = velocity of an electron which enters S_1 with $r = r_0$;

P = potential difference (e.s.u.) of plates AB and CD ;

E = energy (ergs) of an electron with velocity v ; corresponding to E , let V be the same energy expressed in electron volts;

e/m = ratio of charge (e.s.u.) to rest mass (g) of an electron;

c = velocity of light.

Hughes and Rojansky have shown² that for any given P , there exists a $r = r_0$ such that all electrons which have their $v = v_0$ and which are projected from S_1 perpendicularly to OY travel along the path $\rho = r_0$. They have also shown that the focusing produced by the radial, inverse first-power electrostatic field at $127^{\circ} 17'$ is very nearly as good as that produced by a magnetic field at the familiar 180° . Considering for the moment only those β -particles which leave S_1 perpendicularly to OY , Hughes' and Rojansky's Eq. (7)² gives

$$r_0 - r = 2r_0(v_0 - v)[1 - 2(v_0 - v)/v_0]/v_0 \quad (1)$$

for r in the neighborhood of r_0 ; this relates r and v at constant P . Hence as a function of r in this neighborhood and at constant P ,

$$v = \frac{1}{2}v_0\{3 + [1 - 4(r_0 - r)/r_0]^{\frac{1}{2}}\}; \quad (2)$$

the radical is taken with a positive sign in order that $r = r_0$ when $v = v_0$.

ENERGY AS A FUNCTION OF P AND r

The relativistic centrifugal force equation for the circular path of radius r_0 is

$$mv_0^2 [r_0(1-v_0^2/c^2)^{1/2} + (-Ae/r_0)] = 0,$$

where A is the electric field (F) constant:
 $F = A/\rho$.³ Direct solution for v_0 gives

$$v_0 = [(D^2 + 4c^2D)^{1/2} - D]^{1/2} \quad (3)$$

where $D = A^2(e/m)^2/c^2 > 0$;

here the radicals are taken positive in order to make v_0 real and positive. Evidently Eq. (3) is a relation between v (as v_0) and P (through A) for particles traversing the path $\rho = r_0$. Eq. (3) substituted into Eq. (2) yields

$$v = \frac{1}{4}[(D^2 + 4c^2D)^{1/2} - D]^{1/2} \times \{3 + [1 - 4(r_0 - r)/r_0]^{1/2}\}^{1/2} \quad (4)$$

which is the value of v possessed by particles found at r in the neighborhood of r_0 and under the effect of the known potential difference P . Here we neglect the small radial changes in v caused by the action of F when the particles are not on the path $\rho = r_0$; this is in accord with the procedure of Hughes and Rojansky.

The kinetic energy of a particle at velocity v is

$$E = mc^2[(1 - v^2/c^2)^{-1/2} - 1].$$

Using Eq. (4) we have

$$\begin{aligned} 1 - v^2/c^2 &= 1 - [(D^2 + 4c^2D)^{1/2} - D] \\ &\quad \times \{3 + [1 - 4(r_0 - r)/r_0]^{1/2}\}^2 / 32c^2 \\ &= 1 - x/32c^2 \end{aligned}$$

(which serves to define x), whence the energy in electron volts⁴ is

$$V = 10^{-8}(m/e)c^2[(1 - x/32c^2)^{-1/2} - 1]. \quad (5)$$

This is still valid only for r in the neighborhood of r_0 .

In view of the focusing effect at angle Φ , Eq. (5) holds for all electrons arriving at r in the neighborhood of r_0 regardless of the directions of their initial velocities at S_1 .

CONVERSION FACTORS

Now in practice we can count the number N_1 of β -particles passing through S_2 in a convenient

³ To evaluate A , note that F requires the potential function $\psi = -A \ln \rho + B$, where B is a constant. Thus from the equations

$$\psi_1 = -A \ln r_1 + B, \quad \psi_2 = -A \ln r_2 + B,$$

we have that $P = \psi_1 - \psi_2 = A \ln (r_2/r_1)$. Thus $A = P/\ln (r_2/r_1)$.

⁴ It is well known that $V = 10^{-8}(c/e)E$.

interval of time; let N_1 be determined as a function of P experimentally. Suppose S extends a distance $\frac{1}{2}\Delta r$ on each side of $r = r_0$, where Δr is small. To convert N_1 to the number N_2 of particles counted in the chosen time interval and with V in the appropriate ΔV , we set

$$N_1 \Delta r = N_2 \frac{\partial V}{\partial x} \frac{\partial x}{\partial r}$$

and obtain

$$\begin{aligned} N_1 &= N_2 \frac{10^{-8}(m/e)c}{64(1-x/32c^2)^{1/2}} \\ &\quad \times \frac{4[(D^2 + 4c^2D)^{1/2} - 1]^{1/2} \{3 + [1 - 4(r_0 - r)/r_0]^{1/2}\}^{1/2}}{r_0[1 - 4(r_0 - r)/r_0]^{1/2}} \end{aligned}$$

from Eq. (5) and from the definition of x . Since we take all observations at $r = r_0$, however, this reduces immediately to

$$N_2 = \frac{4 \times 10^8 (e/m) r_0 \{1 - [(D^2 + 4c^2D)^{1/2} - D] 2c^2\}^{1/2}}{c[(D^2 + 4c^2D)^{1/2} - D]} N_1$$

if we put $r = r_0$. Furthermore, if we define θ by

$$4c^2/D = \tan^2 \theta, \quad (6)$$

we have $(D^2 + 4c^2D)^{1/2} - D = D(\sec \theta - 1)$ and, finally,

$$N_2 = \frac{4 \times 10^8 (e/m) r_0 [1 - D(\sec \theta - 1) 2c^2]^{1/2}}{cD(\sec \theta - 1)} N_1. \quad (7)$$

In case the relativity correction for m as a function of v is deemed unnecessary, we take $E = \frac{1}{2}mv^2$ and find directly, as a counterpart to Eq. (5),

$$V = 10^{-8}(m/e)cx/64.$$

Hence in this case

$$N_2 = [4 \times 10^8 (e/m) r_0 / cD(\sec \theta - 1)] N_1. \quad (8)$$

TABLE I. A typical arrangement.

P (e.s.u.)	$300P$ (volts)	$D \times 10^{-19}$	$D(\sec \theta - 1)$ $\times 10^{-19}$	V (ev)	N_2/N_1 $\times 10^4$
175	52,500	11.57	5.40	99,400	0.538
150	45,000	8.53	4.76	84,900	0.656
125	37,500	5.90	4.06	69,800	0.829
100	30,000	3.77	3.33	54,900	1.09
75	22,500	2.13	2.57	41,000	1.53
50	15,000	0.945	1.755	26,000	2.41
25	7,500	0.236	0.900	13,000	5.09

Eqs. (7) and (8) are the desired conversion equations for N_1 to N_2 .

Note in passing that Eq. (5) when $r=r_0$ and in the light of Eq. (6) is

$$V = 10^{-8}(m_0/e)c^2 \times \{ [1 - D(\sec \theta - 1) / 2c^2]^{1/2} - 1 \}, \quad (9)$$

a result to be used along with Eq. (7) in getting N_2 as a function of V .

In Table I are given several values of the conversion factor N_2/N_1 computed for an hypothetical but typical set of apparatus and for several deflecting potentials. Here, for the sake of simplified calculations, we have taken $r_0 = 7$ cm, $e/m = 5.3 \times 10^{17}$ e.s.u. per g, $c = 3 \times 10^{10}$ cm sec., and $r_2/r_1 = 8/6 = 1.333$; we use $\ln 1.333 = 0.2874$. V is from Eq. (9) and N_2/N_1 is from Eq. (7). The values of N_2/N_1 are correct to not more than three significant figures and the last three values of V are correct to only two significant figures.

DISCUSSION

It is noteworthy in Table I that the conversion factor N_2/N_1 increases by only a factor of about five, from 10^5 ev to $\frac{1}{4} \times 10^6$ ev. Thus a good set of data giving N_1 as a function of P is not excessively distorted in its conversion to N_2 as a

function of V . It is well known that at energies as low as this the conversion factor of the conventional magnetic spectrograph becomes very large and so renders even good data practically noninformative. It would thus seem that the electrostatic spectrograph, being more suitable for energies less than 10^6 ev, might yield important information about the shapes of continuous β -particle spectra curves.³

It should, however, be pointed out that the use of the electrostatic spectrograph at low energies is complicated by two factors: the uncertain β -particle reflection coefficient of the source support, and the possible δ -ray emission caused by any α -particles that may have been present in the source.³ Complications caused by these factors may be greatly minimized (if not eliminated) by appropriate, reasonable extrapolations, to the end that really worthwhile information might be obtained with the spectrograph.

In conclusion, the writer is indebted to Dr. H. A. Wilson for suggesting certain references which have been useful in connection with this work.

³ For other statements on this matter, see Rutherford, Chadwick, and Ellis, *Radiations from Radioactive Substances* (Cambridge University Press, 1930), pp. 407-408.

On the Ionization of Gases by High Energy Beta-Particles

It has been customary to express the experimentally determined relation between the number (I) of ions of a gas (at STP) formed by a β -particle in a centimeter length of its path, by an equation of the form

$$I = B'\beta^{-C'}, \quad (1)$$

where β is the ratio of the velocity of the β -particle to the velocity of light, and where B' and C' are suitable constants for a given gas; B' and C' , of course, change from gas to gas. A more precise representation of the experimental results is an equation of the form

$$I = A + B\beta^{-C}, \quad (2)$$

where A is another constant (which also changes from gas to gas). For almost all calculations based on these experi-

mental results, equations of the form of Eq. (2) are fully as easy to use as equations of the form of Eq. (1). In Table I are given the values of A , B and C as determined by the method of least squares,¹ for the four gases for which I is well known as a function of β ; in the last column are the ranges of β for which each trio of constants is known to be valid (i.e., the ranges of β for which I has been determined experimentally).

The data used in getting A , B and C are those of Williams and Terroux² on H_2 and O_2 and of Skramstad and Loughridge³ on N_2 and Ne . In each of the four cases at hand, the sum of the squares of the residuals got by using Eq. (2) is (on the average) only 0.4 of the sum of the squares of the residuals got by using Eq. (1).

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February 23, 1937

TABLE I. Values of the constants of Eq. (2) for various gases.

GAS	A	B	C	β RANGE OF VALIDITY
H_2	4.14	1.46	2.87	0.454-0.950
N_2	6.72	12.20	1.64	0.874-0.980
O_2	14.08	8.65	1.95	0.538-0.972
Ne	4.95	7.70	2.01	0.836-0.980

¹ See, e.g., the Sokolnikoffs, *Higher Mathematics for Physicists and Engineers* (McGraw-Hill, 1934), pp. 407-409.

² Williams and Terroux, *Proc. Roy. Soc. A* **126**, 289 (1930).

³ Skramstad and Loughridge, *Phys. Rev.* **50**, 677 (1936).

Note on the Absolute Determination of Magnetic Field Strength

For determinations of the energies of charged particles by means of a focusing magnetic spectrograph it is becoming increasingly desirable to know not the average value of the strength of the magnetic field of the spectrograph over a sizable area, but the strength of the field at the source of the particles. This is especially true for absolute determinations of energies with high precision involving precise absolute determinations of the strength of the given magnetic field, because the necessary corrections for the small nonuniformities which may occur in even the most nearly uniform magnetic fields found in practice require a knowledge of the field strength at the source of particles.¹ Since these corrections also require a knowledge (obtained by small exploring coils, e.g.) of the variations of the magnetic field from perfect uniformity in the region traversed by the particles, it is but a little more trouble to determine absolutely the field strength at the source of the particles, utilizing a knowledge of these variations from uniformity. The following derivations are for the inductometric, "flip-coil" method of measuring magnetic field strengths.²

Let the given magnetic field (everywhere parallel to some fixed direction) be H , and let it be required to determine the value H_0 of H at some point O in H . If a flat circular coil having N turns of wire, having an average effective area A , and having its center at O is rotated 180° about an axis in its plane and through O , and if the breaking of a current I produces through a mutual inductance M an equal effect, then the average value of H over the area of radius $(A/\pi)^{1/2}$ about O is

$$\bar{H} = MI/2IN. \quad (1)$$

However, let H' defined by $H = H_0 + H'$ be determined by moving in suitable jumps a coil of a much smaller area a than A but having a much greater number n of turns of wire than N ; it would be convenient to construct a set of coordinate x and y axes perpendicularly to H and with origin at O and then to determine H' as a function of the two variables x and y . Then we can use instead of Eq. (1),

$$H_0 = (MI - 2N \int \int_A H' ds)/2IN, \quad (2)$$

where ds is an element of area in the xy plane. Here M , I , A and N are quantities which can be measured with high precision; the term in $(\int \int_A H' ds)/A$ is just a small correction term which of course grows smaller as H becomes more nearly uniform over A . Eq. (2) is the chief equation to be used in getting H_0 instead of $\bar{H}$ experimentally. Note in passing that

$$\bar{H} - H_0 = (\int \int_A H' ds)/A. \quad (3)$$

It will of course be of interest to know the conditions under which Eq. (2) should be used instead of the simpler Eq. (1). We see directly by Eq. (3) that the relative error introduced into H_0 by taking $H_0 = \bar{H}$ is

$$E_1 = (\int \int_A H' ds)/H_0 A, \quad (4)$$

which would exist in addition to the effects of the experimental errors in M , I , A and N . Also, a glance at Eq. (2) will show that the relative error introduced into H_0 by the

use of Eq. (2) is equal to the product of $1/H_0$ into the absolute error in $(\int \int_A H' ds)/A$, which would also exist in addition to the effects of the experimental errors in $MI/2IN$; letting e_1 and e_2 be the relative errors possibly present in $(\int \int_A H' ds)$ and in A respectively, this additional relative error caused by Eq. (2) is thus

$$E_2 = (e_1 + e_2)(\int \int_A H' ds)/H_0 A,$$

whence by Eq. (4)

$$E_2/E_1 = e_1 + e_2. \quad (5)$$

Now the computation of the volume-like $\int \int_A H' ds$ may be considered to involve the product of the average value H_m' of H' into A ; so if e' is the possible relative error present in H' , we have that

$$e_1 = e' + e_2. \quad (6)$$

But e' will be composed of first, the experimental uncertainties in the determinations of H' , and second, the effects of taking as the value of H at a point the average value of H over the area a placed with its center at that point. Thus if ϵ is the absolute, purely experimental error considered possible in H' , we see by Eq. (3) that

$$e' \leq [\epsilon + 2(\int \int_a h' ds)/h_m' a] H_m' \quad (7)$$

the factor of two being used to account for the fact that the experimental determination of an H' involves the difference of exploring coils effects for two points in H ; here h' is the difference between the actual value of H and the value of H at the center of the small coil, and " Av " refers to the average value for many locations well scattered over A . Defining h_m' by

$$(\int \int_a h' ds)/h_m' a = h_m' a \quad (8)$$

and substituting Eqs. (6), (7) and (8) into Eq. (5), we get

$$E_2/E_1 = 2e_2 + (\epsilon + 2h_m') H_m'.$$

Thus, especially if

$$2e_2 + (\epsilon + 2h_m') H_m' < 1, \quad (9)$$

it would be an advantage to use Eq. (2) instead of Eq. (1). A little consideration will show that in many cases it would be easy to satisfy Eq. (9).

In conclusion, two facts should be noted. First, Eq. (7) is not entirely rigorous, for its second term is only an approximation, so Eq. (9) is not entirely rigorous; to be more nearly consistent until Eq. (7) is improved, we should consider Eq. (9) as requiring only that the left member be of the order of magnitude of unity. And second, although Eq. (9) requires h_m' and H_m' , it is not necessary to obtain very many values of H' in order to obtain reliable estimates of them; a few values for points well scattered over A should suffice.

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Houston, Texas,
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¹ E.g., for the method of Hartree, Proc. Camb. Phil. Soc. 21, 746 (1923).

² An analogous method has already been presented for the Cotton balance; see Phys. Rev. 50, 515 (1936).

A Method for Magnetic Spectrograph CalculationsF. T. ROGERS, JR.
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A method is presented for calculating the kinetic energies of charged particles from magnetic spectrograph data. The method is of some generality in that it is valid for a magnetic field which may depart appreciably from perfect uniformity, provided only that the field focus the particles properly. It is suggested that, although the method is quite lengthy, it might be of use for calculations based on measurements of the highest precision and for calculations of ordinary precision based on data taken with a magnetic field which is not at all uniform.

INTRODUCTION

IT is well known that in the 180° focusing magnetic spectrograph, used for determining experimentally the kinetic energies of charged particles, it is desired to have as nearly uniform a steady magnetic field as is reasonably attainable, for then these energies may be calculated approximately by means of the simple theory for the motion of charged particles in an uniform magnetic field. Since it has not been possible to obtain a suitable perfectly uniform magnetic field, however, the simple theory has in no actual experiment been strictly valid, though in most cases it has been sufficiently nearly valid to cause no errors greater than the inevitable experimental errors. But in precision spectrograph work, in which the variations in strength of the magnetic field over the region traversed by the charged particles are often not negligibly small, the simple theory is in general not sufficiently nearly valid. It was to care for the small effects of the variations in field strength that Hartree¹ developed a first-order correction which, calculated from the observed variations and applied to the magnetic field strength at the source of the particles, in effect renders the simple theory much more nearly valid for nearly uniform magnetic fields. Now, in this paper, we present a more general method for calculating the kinetic energies of charged particles from magnetic spectrograph data, a method valid for magnetic fields which depart quite appreciably from perfect uniformity.

In all the following considerations of the motion of charged particles in a magnetic field

we shall neglect all induced effects which might be caused by the motion of the particles in a nonuniform magnetic field; this is consistent with past and present practice.² We shall, also, carry through the following derivations only for a point source of particles, for the line source used in all actual experiments (the line being parallel to the direction of the magnetic field of the spectrograph) may be considered as just a set of many point sources.

PRELIMINARY DERIVATIONS

Given a steady magnetic field H which is everywhere parallel to some fixed direction but which may vary continuously in strength from place to place, consider a particle having charge e and mass m moving with a velocity v in H and perpendicularly to H . Choose x and y axes in H and directed perpendicularly to H , taking their origin at the source of the particle. Then the equations of motion of the particle are

$$\begin{aligned} m_0 c \ddot{x} - e(c^2 - v^2)^{1/2} H \dot{y} &= 0, \\ m_0 c \ddot{y} + e(c^2 - v^2)^{1/2} H \dot{x} &= 0, \end{aligned}$$

where $m_0 = m(1 - v^2/c^2)^{1/2}$ is the rest mass of the particle; thus

$$\ddot{x} - A(c^2 - v^2)^{1/2} H \dot{y} = 0, \quad \ddot{y} + A(c^2 - v^2)^{1/2} H \dot{x} = 0 \quad (1)$$

where

$$A = e/m_0 c. \quad (2)$$

Of course

$$H = H(x, y); \quad (3)$$

and for further use, $H(x, y)$ for the given field must be known explicitly, preferably as a power series in x and y .

¹ Hartree, Proc. Camb. Phil. Soc. 21, 746 (1923).

² See, e.g., Hartree, reference 1.

To solve Eqs. (1) in the general case in which $H \neq$ constant, we assume an ordinary power series solution and then by substitution evaluate the coefficients in the series. Assume as a possible solution

$$x = \sum_0^{\infty} a_i t^i, \quad y = \sum_0^{\infty} b_i t^i,$$

which of course contains four arbitrary constants. Supposing that the particle starts from the origin at $t=0$, whence $a_0 = b_0 = 0$, the assumed solution becomes

$$x = \sum_1^{\infty} a_i t^i, \quad y = \sum_1^{\infty} b_i t^i, \quad (4)$$

which has two arbitrary constants, a_1 and b_1 , which are respectively the x and y components of the initial velocity of the particle. If we substitute these series for x and y in Eq. (3), we get in general an infinite power series in t ; this we denote by

$$H = \sum_0^{\infty} c_k t^k, \quad (5)$$

where c_k is the coefficient of t^k as obtained by this substitution: for example, if

$$H = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$

then we have directly that

$$\begin{aligned} c_0 &= a_{00}, \\ c_1 &= a_{10}a_1 + a_{01}b_1, \\ c_2 &= a_{10}a_2 + a_{01}b_2 + a_{20}a_1^2 + a_{11}a_1b_1 + a_{02}b_1^2, \\ c_k &= a_{10}a_k + a_{01}b_k \\ &\quad + \sum_{u=1}^{k-1} (a_{20}a_u a_{k-u} + a_{11}a_u b_{k-u} + a_{02}b_u b_{k-u}) \\ &\quad + \sum_{s=1}^{k-2} [(a_{30}a_s + a_{21}b_s) (\sum_{w=1}^{k-s-1} a_w a_{k-s-w}) \\ &\quad + (a_{12}a_s + a_{03}b_s) (\sum_{w=1}^{k-s-1} b_w b_{k-s-w})] \quad \text{for } k \geq 3. \end{aligned}$$

Also, since the force on the particle due to the magnetic field is perpendicular to the direction of v , v is constant in magnitude and equal in magnitude to v at $t=0$, i.e., to $(a_1^2 + b_1^2)^{1/2}$; so

we have¹

$$(c^2 - v^2)^{1/2} = [c^2 - (a_1^2 + b_1^2)]^{1/2} \equiv B/A, \quad (6)$$

which serves to define B . Thus the substitution of Eqs. (4), (5) and (6) in Eqs. (1) gives

$$\left. \begin{aligned} \sum_2^{\infty} i(i-1)a_i t^{i-2} - B(\sum_1^{\infty} j b_j t^{j-1})(\sum_0^{\infty} c_k t^k) &= 0 \\ \sum_2^{\infty} j(j-1)b_j t^{j-2} + B(\sum_1^{\infty} i a_i t^{i-1})(\sum_0^{\infty} c_k t^k) &= 0 \end{aligned} \right\} \quad (7)$$

Since it is true that

$$(\sum_1^{\infty} p \alpha_p t^{p-1})(\sum_0^{\infty} \beta_q t^q) = \sum_0^{\infty} \gamma_r t^r,$$

where

$$\gamma_r = \sum_{s=1}^{r-1} s \alpha_s \beta_{r+1-s},$$

we can easily equate to zero the coefficients of the powers of t in Eqs. (7). For the r th power of t we have

$$(r+2)(r+1)a_{r+2} - B \sum_1^{r+1} j b_j c_{r+1-j} = 0,$$

$$(r+2)(r+1)b_{r+2} + B \sum_1^{r+1} i a_i c_{r+1-i} = 0;$$

replacing $(r+2)$ by n , these equations give the coefficients of t^n which make Eqs. (4) be a solution of Eqs. (1). The desired solution of Eqs. (1) thus has

$$\left. \begin{aligned} a_n &= [B/n(n-1)] \sum_1^{n-1} j b_j c_{n-1-j} \\ b_n &= [B/n(1-n)] \sum_1^{n-1} i a_i c_{n-1-i} \end{aligned} \right\} n \geq 2, \quad (8)$$

by which the coefficients of t^n are expressible directly in terms of a_1 and of b_1 . For use in computations, these coefficients a_n and b_n are to be expressed in terms of a_1 and of v , b_1 having been eliminated by the relation

$$b_1 = (v^2 - a_1^2)^{1/2}. \quad (9)$$

In the particular case in which $H \equiv$ constant, say

¹ If we use $v^2 = (dx/dt)^2 + (dy/dt)^2$ and substitute for x and y by Eqs. (4), we get for $(c^2 - v^2)^{1/2}$ an infinite series of the form $B/A + \sum d_l t^l$, $l=1$ to ∞ . If this series is then used in the subsequent analysis, we eventually find that d_l vanishes for all $l \geq 1$, leaving just Eq. (6).

h , the Eqs. (8), substituted in Eqs. (4), give

$$\begin{aligned}x &= (a_1/Bh) \sin Bht + (b_1/Bh)(1 - \cos Bht), \\y &= (b_1/Bh) \sin Bht - (a_1/Bh)(1 - \cos Bht),\end{aligned}$$

and these combine to give

$$(x - b_1/Bh)^2 + (y + a_1/Bh)^2 = (a_1^2 + b_1^2) B^2 h^2,$$

the familiar circular path. By Eq. (9) and since from Eq. (6)

$$B = (e/m_0)(1 - v^2/c^2)^{1/2},$$

the circular path becomes

$$\begin{aligned}\left[x - \frac{m_0}{eh} \left(\frac{v^2 - a_1^2}{1 - v^2/c^2} \right)^{1/2} \right]^2 + \left[y + \frac{a_1 m_0 eh}{(1 - v^2/c^2)^{1/2}} \right]^2 \\= \frac{v^2 (m_0 eh)^2}{1 - v^2/c^2} \quad (10)\end{aligned}$$

in terms of a_1 and of v as parameters.

Finally we consider the partial derivative at constant v of the series

$$x = \sum_1^{\infty} a_n t^n,$$

a_n being expressed in terms of a_1 and v ; it will be another power series in t , which we denote by

$$\partial x / \partial a_1 = \sum_1^{\infty} e_t t^t, \quad e_t = \partial a_t / \partial a_1. \quad (11)$$

Each e_t will of course contain the parameters a_1 and v and be directly calculable once a_t is determined.

ON THE x AXIS FOCUSING PROPERTIES OF $H(x, y)$

Since most spectrographic work is done with the 180° focusing magnetic spectrograph, we shall from now on confine our attention only to considerations for a spectrograph of the same general structure as that of the 180° spectrograph. Such a spectrograph will, however, be further restricted to having the source (at 0) of charged particles in the plane of the detecting apparatus (e.g., photographic plates), the customary construction for precision work. Take the x axis to be in the plane of the detecting apparatus. Of $H(x, y)$ we require as yet only that it in some manner and to some useful degree

focus the given particles on the x axis. Let there be many like particles of velocity v emerging from 0 perpendicularly to H but with their initial velocities otherwise directed at random and consider the first intersections of their trajectories with the x axis. Let the intersection corresponding to a_1 be at a distance d from 0. Then the behavior of d as a_1 varies gives information about the focusing of the trajectories on the x axis.

The x axis focusing produced by the magnetic field $H \equiv h$ can be seen from Eq. (10) to be partially characterized by

$$\left. \begin{aligned}\partial d / \partial a_1 &= 0 \\ \partial^2 d / \partial a_1^2 &< 0\end{aligned} \right\} \text{ at } d = D, \quad (12)$$

where D is a maximum value of d , which occurs when $a_1 = 0$. Such focusing is useful in precision spectrograph work because the maximum focusing effect occurs at $d = D$, thus rendering D accurately measurable. In case $H \neq \text{constant}$, x axis focusing is of course still possible for many forms of $H(x, y)$ and for many values of v and e/m_0 , though it may not always be characterized by Eqs. (12). For precision work, however, in which D for the given particles must be determined with high accuracy, it will be of much convenience if $H(x, y)$ is such that the focusing of the given particles does satisfy Eqs. (12). We will thus require that $H(x, y)$ be such that the first of Eqs. (12) is satisfied for the given particles.

In any actual case the rigorous procedure of course includes an investigation of Eqs. (12) for the given $H(x, y)$ and the given particles. One possible numerical method for this investigation, based on Eqs. (4) and (8), is as follows: knowing⁴ an approximate value of v , first put $y = 0$ in the second of Eqs. (4) (as determined by Eqs. (8)) and for several values of a_1 , calculate the corresponding values of t which satisfy the resulting equation; then substitute each a_1 and its corresponding value of t in the first of Eqs. (4) (as determined by Eqs. (8)) and compute the resulting values of $x = d$; and, finally, plot the values of d against those of a_1 . From the graph so formed it will be possible to see by inspection

⁴ This approximate value of v may be got in any manner whatever.

how nearly Eqs. (12) are satisfied. If it is not certain whether the approximate value is fairly nearly equal to the true value of v in any case, it might be advisable for that case to construct two such graphs, one for a value of v known to be slightly larger than the true value, and one for a value known to be slightly smaller than the true value.

THE DETERMINATION OF v

Given a group of like particles emerging from a source at 0 in an appropriate steady magnetic field with a certain but as yet unknown velocity v , we focus them along the x axis and measure D accurately. Having determined $II(x,y)$ (preferably as a power series in x and y) by suitable measurements, having been satisfied that the first of Eqs. (12) is valid for the given particles, and knowing e/m_0 for the particles, we are able to determine v , for the three equations

$$\sum_1^{\infty} a_n t^n = D, \quad \sum_1^{\infty} b_n t^n = 0, \quad \sum_1^{\infty} e_n t^n = 0, \quad (13)$$

are satisfied simultaneously by the proper values of a_1 , t and v , for the point $(0, D)$; in these equations a_n , b_n and e_n are to be expressed in terms of the parameters a_1 and v , b_1 having been eliminated by Eq. (9). The first of these equations is the first of Eqs. (4) as determined by Eqs. (8), expressing the observation of focusing at $x=D$; the second equation is the second of Eqs. (4) as determined by Eqs. (8), expressing the fact that the focusing occurs and is observed on the x axis. The third equation is Eq. (11) expressing the fact that the focusing of particles on the x axis satisfies the first of Eqs. (12). These three equations, if solved, yield the desired value of the quantity v .

The formation of Eqs. (13) in any case may be conveniently done in four steps, as follows:

- (1) substitute Eqs. (4) in the function $II(x,y)$ and obtain as many terms of Eq. (5) as may be necessary;³
- (2) determine as many terms of Eqs. (4) by the Eqs. (8) as may be necessary, in terms of a_1 and b_1 ;

(3) eliminate b_1 from these terms of Eqs. (4) by means of Eq. (9);

(4) by partial differentiation of the a_n terms with respect to a_1 , determine as many terms of Eq. (11) as may be necessary.

These four steps are sufficient to establish Eqs. (13) to as many terms as may be considered necessary for any actual experiment. Obviously the less precise an experiment is, the fewer terms there will be required in Eqs. (13).

The solution of Eqs. (13), after they have been formed, may be accomplished by any convenient method. Such method will in general be a numerical one of successive approximations. While v is the only quantity desired from Eqs. (13), a_1 and t must also be found in order to effect the complete simultaneous solution of Eqs. (13). The value of v as found thus is the desired velocity of the given particles.

Having got v , the corresponding kinetic energy V of the particles can be got by

$$V = 10^{-8} (m_0/e) c^2 [(1 - v^2/c^2)^{-1/2} - 1] \text{ ev}, \quad (14)$$

in which e/m_0 is in e.m.u./g. This is the standard relativistic expression for the kinetic energy (in units of electron volts) of mass m moving with velocity v .

COMMENTS

We have seen how a magnetic spectrograph can be made to give the kinetic energies of charged particles regardless of a lack of constancy of the steady magnetic field of the spectrograph, provided only that the field focus the particles on the x axis according to the first of Eqs. (12). This means that it is not necessary to have as nearly uniform a magnetic field as is possible in a spectrograph in order to use the spectrograph for energy determinations. Also, since we have made no approximations in the foregoing theory (except that mentioned in the introduction), a value of v got by the method we have outlined is completely (except for possible effects of induction) "corrected" for variations in the value of II along the particles' paths in the magnetic field.

As apparent compensations for these advantages offered by the above method of calculating v , we note the necessity of a knowledge of $II(x,y)$, of certain knowledge about Eqs. (12),

³ "Necessary" here refers to the necessity involved in the sufficiently complete convergence of the series in Eqs. (13).

and of a much greater length of computation than the simple, uniform-field theory requires. Because of these disadvantages, the above method is to be recommended for use not in work of ordinary precision with a nearly uniform magnetic field,⁶ but in the two following important classes of work in which the Hartree

correction might not be sufficient:

- (1) work based on measurements of the highest precision, in which a nearly uniform magnetic field will in general be used to simplify the determination of $H(x,y)$;
- (2) work of ordinary precision and in which a magnetic field departing appreciably from uniformity is used.

⁶ The Hartree method should be sufficient for such work.

The Energy-Range Relations for Deuterons, Protons and Alpha-Particles

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The complete deuteron, proton and α -particle energy-range relations as given by Livingston and Bethe and as subsequently revised by Bethe to include the latest data, are presented as a set of eleven equations. Given the range of a known particle, the corresponding kinetic energy of the particle can be computed easily and quickly from these equations.

IN 1937 Livingston and Bethe¹ published a very extensive set of kinetic energy vs. range relations for deuterons, protons and α -particles. These relations were in the form of graphs, in which the particle energies were plotted as functions of particle ranges. Later in 1937 these graphical energy-range relations were revised² to include the recent data of Parkinson, Herb, Bellamy and Hudson³ on proton ranges and of

Blewett and Blewett² on α -particle ranges. Now, to provide these latest revised energy-range relations in a complete and compact form suitable for the easy and rapid computation of energies, we present the following.

Let r be the mean range in cm of a particle in air at a temperature of 15°C and at a pressure of 760 mm Hg; let E be the corresponding energy of the particle expressed in millions of electron volts. Let the subscripts D , P , and α refer the symbols to which they are affixed to deuterons, protons and α -particles respectively. Then, given a value of r_D , r_P , or r_α (by observa-

¹ Livingston and Bethe, Rev. Mod. Phys. **9**, 266-269 (1937).

² See Bethe, Phys. Rev. **53**, 313 (1938).

³ Parkinson, Herb, Bellamy and Hudson, Phys. Rev. **52**, 75 (1937).

TABLE I. *Precisions of Eqs. (1)–(11). Average deviations.*

Eq.	1	2	3	4	5	6	7	8	9	10	11
A	0.004	0.002	0.004	0.003	0.002	0.001	0.004	0.005	0.002	0.003	0.004

tion or otherwise), the corresponding value of V_D , V_P , or V_α can be obtained from the following equations:⁴

for deuterons having $r_D \leq 12$,

$$\log_{10} V_D = -0.0835 + (0.595 + 0.45e^{-2r} + 0.0025 \sin^2(\pi r/9) \cos(\pi r/6)) \log_{10} r; \quad (1)$$

for protons having $r_P \leq 6$,

$$\log_{10} V_P = -0.3845 + (0.595 + 0.45e^{-4r} + 0.0025 \sin^2(2\pi r/9) \cos(\pi r/3)) \log_{10} 2r; \quad (2)$$

for protons having $6 \leq r_P \leq 30$,

$$V_P = 1.819 + 0.16758(r-6) - 0.003(r-6)^2 + 0.00004(r-6)^3; \quad (3)$$

for protons having $30 \leq r_P \leq 90$,

$$V_P = 4.661 + 0.08715(r-30) - 0.000454(r-30)^2 + 0.00000217(r-30)^3; \quad (4)$$

for protons having $90 \leq r_P \leq 140$,

$$V_P = 8.714 + 0.054964(r-90) - 0.00014(r-90)^2 + 0.00000047(r-90)^3; \quad (5)$$

for protons having $140 \leq r_P \leq 236$,

$$V_P = 11.174 + 0.044137(r-140) - 0.0000429(r-140)^2 - 0.000000108(r-140)^3; \quad (6)$$

for α -particles having $r_\alpha \leq 0.9$,

$$V_\alpha = 0.9851r + 3.4958r^2 - 2.4681r^3 + 0.03r \sin(2\pi r/0.9); \quad (7)$$

for α -particles having $0.9 \leq r_\alpha \leq 2.4$,

$$V_\alpha = 1.919 + 1.3167(r-0.9) + 0.076 \sin(2\pi(r-0.9)/3); \quad (8)$$

TABLE II. *Estimated accuracies of Eqs. (1)–(11).*

PARTICLES	RANGE (cm)	POSSIBLE ERROR IN V	EQUATIONS INVOLVED
Deuterons	0 to 0.6	10%	1
Deuterons	0.6 to 12	0.06×10^6 ev	1
Protons	0 to 0.3	10%	2
Protons	0.3 to 236	0.03×10^6 ev	2–6
α -particles	0 to 0.35	10%	7
α -particles	0.35 to 1.05	0.05×10^6 ev	7–8
α -particles	1.05 to 3.5	0.03×10^6 ev	8–9
α -particles	3.5 to 10.5	0.01×10^6 ev	9–10
α -particles	10.5 to 21	0.03×10^6 ev	10–11

⁴In these equations the subscripts of the r 's are, for simplicity, omitted.

for α -particles having $2.4 \leq r_\alpha \leq 7.2$,

$$V_\alpha = 3.894 + 1.0906(r-2.4) - 0.07025(r-2.4)^2 + 0.00357(r-2.4)^3; \quad (9)$$

for α -particles having $7.2 \leq r_\alpha \leq 12$,

$$V_\alpha = 7.897 + 0.67375(r-7.2) - 0.0108(r-7.2)^2 - 0.00082(r-7.2)^3; \quad (10)$$

for α -particles having $12 \leq r_\alpha \leq 20.8$,

$$V_\alpha = 10.799 + 0.50865(r-12) - 0.000786(r-12)^2 - 0.000549(r-12)^3. \quad (11)$$

Each of Eqs. (1)–(6) and (9)–(11) is associated with only one of the graphs of Livingston and Bethe. Eqs. (7) and (8) are together associated with only one of the graphs.

The majority of the Eqs. (1)–(11) were obtained by fitting to the revised energy-range graphs, functions of suitable forms by a method of zero sums;⁵ Eqs. (1) and (2) were obtained by a "trial and error" method, while Eq. (8) was obtained simply by inspection. As measures of the precisions with which Eqs. (1)–(11) represent these revised energy-range graphs of Livingston and Bethe, Table I shows for each equation the average deviation A (in millions of electron volts) of the computed energy-range graph from the corresponding revised energy-range graph. Obviously the representations afforded by Eqs. (1)–(11) are quite good.

As estimates of the accuracies with which Eqs. (1)–(11) give the appropriate energies, we may legitimately adopt the estimates by Livingston and Bethe¹ of the accuracies of their graphical energy-range relations. Accordingly in Table II we have tabulated as functions of ranges, the possible errors which (from Livingston's and Bethe's estimates) might be present in the energies computed from Eqs. (1)–(11). Table II should always be taken into consideration when Eqs. (1)–(11) are used, because it acts as a set of necessary restrictions which must be applied to computations made by Eqs. (1)–(11).

Finally, it is noteworthy that Eqs. (1)–(11) (of course restricted by Table II) are well suited for the simple and rapid numerical computation of V 's when the corresponding r 's are given, especially if tables of x^2 , x^3 , e^{-x} , $\log_{10} x$, and $\sin x$ are available.

⁵See, e.g., Campbell, Phil. Mag. 39, 177 (1920).

AN INDEPENDENT DETERMINATION OF THE BINDING
ENERGY OF THE DEUTERON

by

Fred Terry Rogers, Jr.

A Major Thesis Presented to the Faculty
of The Rice Institute
in Partial Fulfillment
of Their Requirements for the Degree of
Doctor of Philosophy

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(ABSTRACT)

This paper is an account of an experiment designed to determine the binding energy of the deuteron by a method which is relatively insensitive to uncertainties in the energy-range relation for protons of low energy. The protons, produced by the disintegration of deuterium by Th C'' γ -radiation, were observed in a low-pressure cloud chamber in a strong magnetic field. The curvatures of the tracks allowed the calculation of the corresponding kinetic energies. The final value of the binding energy as got by this experiment is

$$\overline{V}_B = (2.17 \pm 0.05) \times 10^6 \text{ electron-volts,}$$

which is in excellent agreement with the values previously got by Bethe (from the data of Chadwick, Feather and Bretscher) and by J. R. Richardson and Emo.

I. INTRODUCTION

1. Previous work on the binding energy of the deuteron. In 1935 Chadwick and Goldhaber¹ determined the binding energy V_B of the deuteron from the disintegration of deuterium by the Th C'' γ -radiation of high energy. The particles taking part in this disintegration react according to the equation

Chadwick and Goldhaber detected the product protons with a proportional counter and linear amplifier and estimated the kinetic energies of these protons by the amounts of ionization they produced in the counter. In this way they got

$$V_B = (2.14 \pm 0.16) \times 10^6 \text{ electron volts.}$$

Subsequently Chadwick, Feather and Bretscher² performed with a cloud chamber a more precise experiment designed to measure the ranges of the protons which are liberated when H^2 disintegrates. After converting the observed proton ranges to the corresponding proton kinetic

energies by means of the best range-energy data available at that time, they got

$$V_B = (2.25 \pm 0.05) \times 10^6 \text{ electron volts.}$$

Their data, recalculated by Bethe³ in the light of more recent data on the range-energy relation for protons, yields, however,

$$V_B = (2.17 \pm 0.04) \times 10^6 \text{ electron volts.}$$

The discrepancy between these last two values of V_B is caused by previous uncertainties in the range-energy relation for protons of low energy.

At about the same time, J. R. Richardson and Emo⁴ obtained

$$V_B = (2.18 \pm 0.07) \times 10^6 \text{ electron volts}$$

from the disintegration of H^2 by the high energy γ -radiation from radioactive Na^{24} . This value of V_B also involves the use of the energy-range relation for protons, for Richardson and Emo determined the range of the product protons with a cloud chamber and then got the kinetic energies of the protons by the energy-range relation. Richardson and Emo used almost the same energy-range relations as that which Bethe³ used.

2. The present experiment. Since these previous values of V_p depend upon the somewhat uncertain energy-range relation for protons of low energy (except for the first value quoted above, which is not of high precision), it was thought desirable to determine V_p by a method that yields results which are relatively insensitive to this dependence. An attempt to do this with an ordinary magnetic spectrograph using photographic recording (1936-1937) was not successful because the available Th C" γ -ray source was not sufficiently strong. It now appears that this magnetic spectrographic method might be entirely feasible if a proportional counter and linear amplifier were used to detect the protons.

It was decided, however, for the present experiment to use a deuterium-filled cloud chamber in which the total pressure was 8 cm. Hg and to place this cloud chamber in a strong magnetic field. Then the observable curvatures of the proton tracks formed in the chamber should enable the kinetic energies of the protons to be determined. Such an apparatus seemed to present

several advantages in addition to the relative freedom it should provide from the energy-range relation, among them being:

- (1) a cloud chamber mixture of deuterium and heavy water, with a total pressure of about 8 cm. Hg at room temperature, has a fairly low stopping power (approximately .17 that of air) for low-energy protons; thus the protons which we should observe would be able to make moderately long tracks;
- (2) the low average atomic number (approximately 1.1) of a cloud chamber mixture of deuterium and heavy water vapor (at a total pressure of 8 cm. Hg at room temperature) should allow the γ -source to be placed very near to the cloud chamber without producing an excessive fog on the chamber; this is so because the intensity of the background fog produced in a cloud chamber by heavy γ -radiation increases quite rapidly with the average number of electrons per atom of the gas, i. e., with the average atomic number of the gas.

In the following sections of this paper we present the details of such an experiment. We give first the necessary theoretical considerations, and then the appropriate experimental details. In the latter portion of this paper we give the measurements which were made and the calculations which have been carried out for the determination of V_B by this method.

II. THEORETICAL CONSIDERATIONS FOR BINDING ENERGY

Since a proton having a kinetic energy of 2×10^5 e. v. (which is of the same order of magnitude as the kinetic energy of the protons with which we deal in the present experiment) has a velocity v' of only 6×10^8 cm./sec., the quantity $(1 - v'^2/c^2)$ (in which c is the velocity of light) differs from unity by only 4 parts in 10^4 . The quantity $(1 - v'^2/c^2)^{1/2}$, which is characteristic of any relativistic mechanical consideration of this proton, thus differs from unity by only 2 parts in 10^4 . In view of the fact that the present experiment cannot even remotely approach a precision of 2 parts in 10^4 , we need not consider a relativistic collision theory for the photoelectric disintegration of the deuteron.

3. Definition of some symbols. Let

E_γ = energy in ergs of the Th C" gamma ray
which disintegrates a deuteron;

E_B = binding energy of the deuteron in ergs;

E_1 = kinetic energy in ergs of the proton
liberated by the disintegration of the
deuteron;

E_2 = kinetic energy of the accompanying neu-
tron;

m_1, m_2 = masses in grams of the product proton and
neutron respectively;

$$E = m_2 - m_1;$$

$$p = (E/c)(2m_1 E_1)^{1/2};$$

v_1, v_2 = velocities in cm./sec. of the product
proton and neutron, respectively, immedi-
ately after the disintegration;

θ, ϕ = angles made by the paths of the product
proton and neutron, respectively, with
the direction of the incident gamma ray
(see Fig. 1);

e = protonic charge expressed in em. u.;

e_s = protonic charge expressed in es. u., $= ec$;

F = the Faraday, in units of em. u. of charge
per gram-equivalent.

Corresponding to any E in ergs, let V be the same
energy expressed in electron volts:

$$E = 10^8 e_s V / c = 10^6 \text{ eV.} \quad (1)$$

Fig. 1. Essential features of the collision-disintegration process.

Corresponding to E and V , let f be the same energy expressed in atomic-weight units:

$$f = 10^8 FV/c^2 = EF/ec^2. \quad (2)$$

4. Derivations. Then, assuming the deuteron to have been sensibly at rest before the disintegrating collision, the following equations express the conservation of energy and of two perpendicular components of momentum by the disintegration process (see Fig. 1):

$$\left. \begin{aligned} E_Y &= E_1 + \frac{1}{2}m_2 v_2^2 + E_B, \\ E_Y/c &= m_1 v_1 \cos \theta + m_2 v_2 \cos \phi, \\ 0 &= m_1 v_1 \sin \theta - m_2 v_2 \sin \phi. \end{aligned} \right\} \quad (3)$$

Here we consider m_1 , m_2 and E_Y (E_Y through the experimentally known V_Y) as known constants, θ and E_1 as observable quantities, and v_2 and ϕ as essentially unobservable quantities. Eliminating v_2 from the first pair of Eqs. (3) by means of the last of Eqs. (3), we have

$$\left. \begin{aligned} E_Y &= E_1 \left[1 + (m_1/m_2)(\sin \theta / \sin \phi)^2 \right] + E_B, \\ E_Y/c &= m_1 v_1 (\cos \theta + \sin \theta \cot \phi). \end{aligned} \right\} \quad (3')$$

Then, eliminating ϕ from the former of these Eqs. (3') by means of the latter and substituting $(2m_1 E_1)^{1/2}$ for $m_1 v_1$, we have

$$E_V = E_1 \left\{ 1 + (m_1/m_2) \left[1 - 2(E_V/c)(2m_1 E_1)^{-1/2} \cos \theta + (E_V/c)^2 / 2m_1 E_1 \right] \right\} + E_B \quad (3'')$$

Finally, substituting Eq. (1) into Eq. (3''), we get the desired expression for V_B :

$$V_B = V_V - \left[1 + (1-\beta/m_1)(1-\beta \cos \theta + \beta^2) \right] V_1 \quad (4)$$

In atomic weight units, by Eq. (2) this binding energy is simply

$$f_B = 10^{-6} F V_B / c^2. \quad (5)$$

A mean value $\overline{V_B}$ of n independent determinations of V_B can thus be computed from Eq. (4) after several (n) pairs of values of V_1 and θ have been found by suitable observations.

III. THEORETICAL CONSIDERATIONS FOR V_1 AND θ

In the present experiment, several pairs of values of V_1 and θ were got from stereoscopic photographs of the several proton tracks observed in our low pressure cloud chamber, the cloud chamber having been located in a nearly uniform magnetic field of known strength. The magnetic field was directed perpendicularly to the "top" of the cloud chamber, as indicated in Fig. 1. We shall let H , directed parallel to OZ (in Fig. 1), represent the magnetic field in the cloud chamber. OZ is the direction of the z -axis of a set of mutually perpendicular x -, y -, and z -axes with origin at O in the centre of the cloud chamber; the x - and y -axes are perpendicular to the magnetic field. We seek now a formula, or set of formulae, by which V_1 for a proton can be got from the observed track of the proton in our cloud chamber.

5. Motion of a proton in an uniform mag-

Fig. 2. Coordinate axes in the cloud chamber. O is at the center of the chamber.

magnetic field and in a perfect vacuum.

First, then, consider the motion of a proton in an uniform magnetic field H' and in a perfect vacuum. The equations of motion (referred to coordinate axes such as those just established in the cloud chamber) of this proton are

$$\left. \begin{aligned} m(d^2x/dt^2) &= eH'(dy/dt), \\ m(d^2y/dt^2) &= -eH'(dx/dt), \\ m(d^2z/dt^2) &= 0. \end{aligned} \right\} \quad (6)$$

To solve these equations, try as a solution the set of functions

$$\left. \begin{aligned} x &= A + \rho' \sin(Bt + C), \\ y &= D + \rho' \cos(Bt + C), \\ z &= Et + F, \end{aligned} \right\}$$

in which A , B , C , D , E , F , and ρ' are constants. Substitution of this set of functions back into Eqs. (6) shows that it forms a satisfactory solution of Eq. (6), involving the six arbitrary constants A , C , D , E , F , and ρ' , provided B is taken thus:

$$B = eH'/m.$$

The solution of Eqs. (6) is thus

$$\left. \begin{aligned} x &= A + \rho' \sin(eH't/m + C), \\ y &= D + \rho' \cos(eH't/m + C), \\ z &= Et + F, \end{aligned} \right\} \quad (7)$$

which is the equation of a right circular-cylindrical helix; the axis of this helix is the straight line

$$x = A, \quad y = D;$$

the radius of the right circular-cylinder is just ρ' ; and the pitch of this helix in the z -direction is $\hbar m/eH'$. If observed along OZ , the path of the proton would thus appear to be a circle of radius ρ' and in a plane perpendicular to $\vec{H}$ or parallel to the xy -plane. The path would, of course, be characterized by the following Eq. (8), which is derived thus: since by Eqs. (7),

$$\begin{aligned} dx/dt &= \rho' (eH'/m) \cos(eH't/m + C), \\ dy/dt &= -\rho' (eH'/m) \sin(eH't/m + C), \end{aligned}$$

we have

$$m \left[\left(dx/dt \right)^2 + \left(dy/dt \right)^2 \right] = \tau \left[\left(\rho' e H' / m \right)^2 \right] = \rho' e H';$$

letting

$$v_n'^2 = \left(dx/dt \right)^2 + \left(dy/dt \right)^2, \quad (8)$$

it follows that

$$m v_n' = \rho' e H'. \quad (9)$$

An observer of such a freely moving proton could thus determine its kinetic energy. For by Eq. (9)

$$m v_n'^2 = (1/2m) (\rho' e H');$$

therefore defining ψ by the equation

$$v_n' / v' = \cos \psi, \quad (10)$$

where v' is the velocity $\left[\left(dx/dt \right)^2 + \left(dy/dt \right)^2 + \left(dz/dt \right)^2 \right]^{1/2}$ of the proton, we have

$$E \equiv \frac{1}{2} m v_n'^2 = (1/2m) (\rho' e H' \sec \psi)^2 \quad (10')$$

in ergs, or in electron-volts by Eq. (1),

$$V' = 5 \times 10^{-9} (e/m) (H' \rho' \sec \psi)^2. \quad (11)$$

If ρ' , the radius of the cylinder of the helical path, were determined and if ψ , which is a partial measure of the pitch of the helix, were determined, the observer could by Eq. (11) compute the kinetic energy V of the proton, provided of course e/m and H were known.

3. Motion of a proton in the present experiment. Now in the present experiment, the protons did not move in a perfect vacuum. Instead, they moved through the gases in the cloud chamber and in moving lost energy by ionization of these gases (and by excitation of them). Accordingly the proton paths observed in the cloud chamber were (except for singular points produced by collisions of the protons with the nuclei of gas atoms) segments of smooth spirals which, though far from being helical, were suggestive of the helical path we have just considered for a proton moving in a vacuum. Eq. (11) is therefore not applicable in the large to the proton paths actually observed in the cloud chamber, but it is ap-

applicable to these paths in the small; that is, at each point on a proton path the magnetic field has a certain value H corresponding to V ; the projection of the path on a plane perpendicular to H has a definite radius of curvature ρ corresponding to ρ' ; and the kinetic energy has a definite value V corresponding to V' ; the quantity v_n/v (corresponding in every way with the v'_n/v' for the free proton) remains constant (barring collisions of the proton with gas nuclei) over the path, for v and v_n both decrease at the same rate. Therefore, Eq. (11) can be applied in the small at any point on an observed proton path, thus:

$$V = 5 \times 10^{-6} (e/m_p) (H \rho \sec \psi)^2. \quad (12)$$

7. An expression for V_1 . To render Eq. (12) suitable for our purposes, multiply through it by $1/V$, integrate from $1/V = \infty$ to $1/V = V_1$, and divide by V_1 ; thus

$$\begin{aligned} (1/V_1) \int_0^{\psi} V d_1 V &= 5 \times 10^{-6} (e/m_p) \sec^2 \psi (1/V_1) \int_0^{\psi} (H \rho)^2 d_1 V \\ &= (1/V_1) (V_1^2/2) = \frac{1}{2} V_1. \end{aligned}$$

See also p. 21, para 2.

Letting

$$(1/Y_1) \int_0^{V_1} (H\rho)^2 dV = \overline{(H\rho)^2}, \quad (13)$$

we get the following expression for the initial kinetic energy of a proton in terms of the properties of its path in the cloud chamber:

$$V_1 = 10^{-3} (e/n_1) \sec^2 \psi \overline{(H\rho)^2}. \quad (14)$$

This expression involves only the directly observable ψ and the quantity $\overline{(H\rho)^2}$ which can be got from the shape of the path and from a knowledge of the variation of H with the coordinates x, y, z . In the next section we shall derive a more useful form for $\overline{(H\rho)^2}$.

2. Determination of θ . - for a proton path is a directly observable entity.

IV. CONCERNING THE ENERGY-RANGE RELATION AND ITS APPLICATION TO THE DETERMINATION OF V_1 .

1. A possible simple procedure for V_1 . It might be thought, at first glance, that Eq. (12) would be sufficient for the determination of V_1 and that any further manipulation of it, such as to produce Eq. (14), is superfluous. Thus, we might denote the value of V at the beginning of a proton track by V_I , the value of ρ at the beginning of the track by ρ_I , and then simply write down Eq. (12) for the beginning of the track:

$$V_I = 5 \times 10^{-8} (e/m_e)(N_I \rho_I \sec \psi)^{1/2}. \quad (15)$$

And, in theory, this equation (15) might conceivably be adequate, provided ρ_I could be accurately determined from the stereoscopic photographs of the tracks.

2. Impracticability of this possible procedure. One factor, however, stands out to prevent the use of Eq. (15) for the calculation of V_1 in the present experiment, viz: it

is extremely difficult, and in fact often impossible, to determine ρ_I with any useful degree of precision, for several reasons. First, the strongest magnetic field strength which was available was not great enough to produce more than just a small amount of curvature in the proton paths in the cloud chamber. This difficulty would not have been insuperable, had it been possible to reduce the pressure inside the cloud chamber to a very low value which would have allowed the protons in the chamber to make very long tracks. Such extremely long tracks were, of course, out of the question because:

- (1) the magnetic field available was not very nearly uniform over a region more than 11 cm. in diameter;
- (2) the very low gas pressure needed for long tracks in the cloud chamber would not have yielded many deuteron disintegrations, and as it was, disintegrations were very scarce;
- (3) proton tracks at very low pressures may, because of diffusion, become very broad and

indistinct rapidly.

In the second place, the proton tracks as photographed were often quite broad, which means that the honest measurement of them for ρ_I would be at times an extremely uncertain process. This observed broadness of tracks was caused by:

- (1) the diffusion of water droplets forming the protons' tracks;
- (2) the fact that some tracks may have been formed in a short interval of time just immediately preceding the expansion of the cloud chamber;
- (3) the camera which was used to photograph the proton tracks had lenses with a rather small depth of focus, probably due to the fact that the lenses had the quite low numerical aperture of 1.9.

While the broadness of some of the proton tracks was thus not an insuperable barrier to the satisfactory measurement of many tracks for ρ_I , it would nevertheless have constituted

an objectionable source of uncertainty.

Third, even if it had been possible to measure what would have appeared to be ρ_I for a proton track, one could have no assurance that the proton had not suffered small-angle scatterings in the short segment of its track which was used to determine ρ_I . This phenomenon of the very frequent small-angle scattering of low-energy protons is well known⁶ and probably needs no further amplification here. The small-angle scattering which occurred in the proton tracks we have photographed produced only a few singularities which could be recognized as such, probably produced very many which were too small to be visible. Also, the inescapable process of diffusion of the proton tracks probably helped to obscure the very small scatterings.

11. Outline and preliminary details of a satisfactory method. It is thus evident that in the present experiment ρ_I could not have been reliably determined and that the simple

Eq. (15) could not have been safely used. The next possible mode of determining V_1 which presented itself is that embodied in Eq. (14) and its subsequent modifications. This method, by virtue of the integration with respect to V on $[r, V_1]$ as performed in getting Eq. (14), obviously yields V_1 in terms of the properties of the entire proton path rather than of only the beginning of the path. As will be seen later, it thus allows the determination of the curvature factor with good precision (because the determination is based on the entire track) and it allows the effects of nuclear scattering of the protons to "average itself out" over the entire lengths of several tracks. To develop further this method of getting V_1 , we proceed to study the quantity $(E\rho)^2$ from the point of view of the energy-range relation for low-energy protons. In Section V we shall see that the value of V_1 thus got, using the energy-range relation as we shall use it, will not be very sensitive to uncertainties in the energy-range relation.

Suppose we have a proton in the cloud chamber moving along its path. Since we wish to investigate the quantity $(E\rho_x)^{-1}$, which contains ρ_x , consider the projection of this proton track on a plane which is parallel to the xy -plane and which contains the end-point P_2 of the path. In this plane establish mutually perpendicular ξ - and η - axes with origin at P_2 and with the ξ -axis tangent to the ^{projection of the} proton path at P_2 ; see Fig. 3. Let P_1 be the $\xi\eta$ -projection of the beginning point P_1' of the path, and let P be the $\xi\eta$ -projection of any point P' on the path. To be associated with the projection point P are the proton's kinetic energy $\frac{1}{2}mv^2$ at P' , the radius of curvature of the projected path at P , and the magnetic field strength H at P' . Let the range of the proton corresponding to the point P' be r' , the total length of the path be L' ; then let

$$r = r' \cos \psi \quad (16)$$

be the projected (on the $\xi\eta$ -plane) range corresponding to r' ; also let

Fig. 3. α is measured from the P_2 direction as shown. $J_1 P_2 J_2$ is the direction perpendicular to the f_n -plane.

$$L = L' \cos \psi. \quad (16)$$

Since the proton loses energy as it traverses the path P_2P_1 , the value of ρ decreases as P moves from P_2 to P_1 , a fact which we may indicate by the functional relation

$$\rho^2 = g(r) \quad (17)$$

between ρ and r .⁷ This Eq. (17) is in reality the intrinsic equation of the curve P_2PP_1 . To get the equation of the curve in terms of the rectangular coordinates ξ and η , note that

$$\left. \begin{aligned} d\xi &= dr \cos \alpha, \\ d\eta &= dr \sin \alpha, \\ d\alpha &= dr/\rho \end{aligned} \right\}$$

where α is an angle (in the $\xi\eta$ -plane) defined adequately by Fig. 5. Since

$$\alpha = \int_0^r dr/\rho = \int_0^r du/\sqrt{g(u)}, \quad (18')$$

the equation of the curve P_2PP_1 in terms of the coordinates ξ and η and of the parameter r is

$$\left. \begin{aligned} \xi &= \int_0^r \cos\left(\int_0^v du/\sqrt{g(u)}\right) dv, \\ \eta &= \int_0^r \sin\left(\int_0^v du/\sqrt{g(u)}\right) dv. \end{aligned} \right\} \quad (22)$$

Here u and v are simply auxiliary variables which have no further purpose than to aid in the unambiguous representation of ξ and η as functions defined by definite integrals. If ρ approaches zero as r approaches zero, then the definite integral in the right member of E_1 . (19') is improper and must be treated as such.

Let the chord E_2E_1 , from E_2 to E_1 , be of length d ; see Fig. 4. The perpendicular to this chord erected at the mid-point M of the chord will intersect the ^{projected} path E_2EP_1 at some point N ; let $MN=h$. Then consider the circle (part of which is shown by the dashed line of Fig. 4) through the points E_2 , N , and E_1 ; let the radius of this circle be R . Since

$$\begin{aligned} R^2 &= (R-h)^2 + d^2 \\ &= R^2 - 2Rh + h^2 + d^2, \end{aligned}$$

we have on solving for R ,

$$R = (h^2 + d^2)/2h \quad (21)$$

Fig. 4. MN is perpendicular to P_2P_1 .
This figure lies in the β_1 -plane.

Also let $H = H_1$ when the projection of P' coincides with H . We now seek a relation between first, the quantity $(H\rho)^2$, and second the value of $\overline{(H\rho)^2}$ which would be expected for a given energy-range relation such as Eq. (18).

12. The case of $H \equiv \text{constant}$ and $\rho^2 \propto r^1$.
We shall consider first the relatively simple case for which $H \equiv \text{constant}$ and $\rho^2 \propto r^1$. Since $\rho^2 \propto r \sec \psi$, this case is characterized by

$$\rho^2 = C_1 r \sec \psi, \quad (21)$$

where C_1 is a suitable constant. We have by Eq. (13) that

$$\begin{aligned} \overline{(H\rho)^2} &= (1/V_1) \int_0^{V_1} (H\rho)^2 dV \\ &= (H^2/V_1) \int_0^{V_1} \rho^2 dV \quad \text{since } H \equiv \text{const.}, \\ &= (H^2/V_1) \int_0^{V_1} \rho^2 (10^{-8} (e/m_1) H^2 \sec^2 \psi) \rho d\rho, \end{aligned}$$

by Eq. (12),

$$\begin{aligned} &= \frac{H^2 \times 10^{-8} (e/m_1) H^2 \sec^2 \psi}{5 \times 10^{-7} (e/m_1) H^2 \rho_I^2 \sec^2 \psi} \int_0^{\rho_I} \rho^3 d\rho \quad \text{by Eq. (15),} \\ &= (CH^2/\rho_I^2) (\rho_I^4/4) \quad \text{on integration,} \end{aligned}$$

whence

$$(\overline{E\rho})^2 = \frac{1}{2} E^2 \rho_I^2. \quad (22)$$

Also, for this special case

$$\begin{aligned} \alpha &= \int_0^r du / (C_1 u \sec \psi) \quad \text{by Eq. (19')} \\ &= (C_1 \sec \psi)^{-1} \int_0^r u^{-1} du = 2(C_1 \sec \psi)^{-1} (u)^{\frac{1}{2}}_0^r \end{aligned}$$

whence

$$\alpha = 2r^{1/2} (C_1 \sec \psi)^{-1}; \quad (23)$$

so the equation of the η -projection of the path is, by Eqs. (21),

$$\left. \begin{aligned} \xi &= \int_0^r \cos \left[2v^{1/2} / (C_1 \sec \psi) \right] dv, \\ \eta &= \int_0^r \sin \left[2v^{1/2} / (C_1 \sec \psi) \right] dv. \end{aligned} \right\}$$

Since $dv = 2v^{1/2} d(v^{1/2})$, these are just

$$\left. \begin{aligned} \xi &= 2 \int_0^r v^{1/2} \cos \left[2v^{1/2} / (C_1 \sec \psi) \right] d(v^{1/2}), \\ \eta &= 2 \int_0^r v^{1/2} \sin \left[2v^{1/2} / (C_1 \sec \psi) \right] d(v^{1/2}). \end{aligned} \right\}$$

Integrating these, using the limits indicated, and then replacing r by α according to Eq. (23), we get simply

$$\left. \begin{aligned} \xi &= C_1 \sec \psi (\cos \alpha + \alpha \sin \alpha - 1), \\ \eta &= C_1 \sec \psi (\sin \alpha - \alpha \cos \alpha), \end{aligned} \right\} \quad (24)$$

which is the equation of an evolute of a circle of radius $C_1 \sec \psi$. In Table I is a set of values of the quantities $2\xi \cos \psi / C_1$ and $2\eta \cos \psi / C_1$ for the evolute of Eqs. (24) for several selected values of $\alpha \leq 1$. The graph shown in Fig. 5 is of these values $2\eta \cos \psi / C_1$ plotted as functions of $2\xi \cos \psi / C_1$.

To effect the comparison of $\overline{(H\rho)^2}$ and $(H_0 R)^2$ for this case, we now proceed to determine the ratio $\overline{(H\rho)^2} / (H_0 R)^2$ for each of several typical proton paths or evolute curves defined thus: one such curve is the segment of the curve (Fig. 5) of Eqs. (24) from P_2 out to the point $P = P_0$ at angle $\alpha = \alpha_0$. We saw in Eq. (22) that for the case of $H = \text{constant}$ and $\rho^2 \propto r^4$, ρ_1^2 is a partial measure of $\overline{(H\rho)^2}$; thus by Eq. (21') for the curve from P_2 to the point P_0 ,

$$\rho_0^2 = C_1 L_0 \sec \psi,$$

TABLE I. ξ and η from Eqs. (24)

α	$2\xi \cos \psi / C_1$	$2\eta \cos \psi / C_1$
0.000	0.0000	0.00000
0.050	0.0013	0.00004
0.100	0.0050	0.00024
0.125	0.0079	0.00064
0.150	0.0112	0.00112
0.175	0.0152	0.00173
0.200	0.0198	0.00250
0.225	0.0250	0.00376
0.250	0.0308	0.00517
0.275	0.0371	0.00687
0.300	0.0440	0.00892
0.325	0.0514	0.01131
0.350	0.0594	0.01411
0.375	0.0679	0.01727
0.400	0.0768	0.02100
0.450	0.0958	0.03000
0.500	0.1174	0.04070
0.550	0.1400	0.05380
0.600	0.1641	0.06950
0.650	0.1895	0.08790
0.700	0.2150	0.10870
0.750	0.2430	0.13270
0.875	0.3130	0.20700
1.000	0.3810	0.30100

Fig. 5. The heavy curve is for the path of a proton moving in a uniform magnetic field where the range is proportional to the kinetic energy of the proton.

where ρ_0 is the value of ρ for the point P_0 and where R_0 is the value of r for P_0 . But by Eq. (25) for R_0 ,

$$\alpha_0 = 2R_0(C_1 \sec \psi)^{-1}; \quad (25')$$

$$\text{therefore} \quad \frac{1}{2}\rho_0^2 = \frac{C_1^2 \sec^2 \psi \alpha_0^2}{2} \quad (25'')$$

Also, since R^2 is a similar partial measure of $(H_N R)^2$, we also need to know the value R_0 of R for the curve $P_2 P P_0$: i.e., we construct the chord $P_2 M_0 P_0$ (the length of which we shall call $2d_0$) from P_2 to P_0 , and erect a perpendicular (the length of which we shall call h_0) to this chord at the mid-point M_0 of the chord and ending at a point N_0 on the evolute curve from P_2 to P_0 ; then the radius of the circle through P_2 , N_0 , and P_0 is R_0 , where

$$R_0^2 = (h_0^2 + d_0^2)^2 / 4h_0^2, \quad (26)$$

as in Eq. (21). Therefore,

$$\frac{1}{2}\rho_0^2 / R_0^2 = \frac{1}{2}C_1^2 h_0^2 \alpha_0^2 \sec^2 \psi / (h_0^2 + d_0^2)^2. \quad (27)$$

This quantity $\frac{1}{2}\rho_0^2 / R_0^2$ is equal to the quantity $\overline{(H\rho)^2} / (H_N R)^2$ for the proton track beginning at P_0

and α_0 and ending at P_0 , for the special case which we are considering.

The further evaluation of Eq. (27) obviously requires knowledge of d_0 and of h_0 , preferably in terms of α_0 . This we can get by Eqs. (24), for

$$\begin{aligned} d_0^2 &= [(\cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1)^2 + (\sin \alpha_0 - \alpha_0 \cos \alpha_0)^2] \frac{C^2 \sec^2 \psi}{4} \\ &= (\alpha_0^2 - 2 \cos \alpha_0 - 2 \alpha_0 \sin \alpha_0) \frac{C^2 \sec^2 \psi}{4}; \end{aligned}$$

thus

$$d_0^2 = \frac{1}{4} [\alpha_0^2 - 2(\cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1)] \frac{C^2 \sec^2 \psi}{4} \quad (28)$$

To get h_0 , let α' be the angle α corresponding to the point P_0 ; then from the definition of h_0 ,

$$\begin{aligned} h_0^2 &= \left\{ [(\cos \alpha' + \alpha' \sin \alpha' - 1) - (\cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1)]^2 \right. \\ &\quad \left. + [\sin \alpha' - \alpha' \cos \alpha' - (\sin \alpha_0 + \alpha_0 \cos \alpha_0)]^2 \right\} \frac{C^2 \sec^2 \psi}{4}. \quad (29) \end{aligned}$$

Finally, for α' we use the excellent approximation

$$\alpha' = \alpha_0 / \sqrt{2}, \quad \alpha_0 \leq 1; \quad (30)$$

this approximation is valid to the precision of 1 part in 1000 and is verified in Mathema-

tical Appendix I. Therefore, Eqs. (27), (28), (29), and (30) taken together give an expression (which, because of its great length, we shall not write out) for ρ_0/P_0^2 involving neither C_1 nor ψ but only α_0 . Several representative values² of ρ_0/P_0^2 have been computed and are given in Table II as functions of α_0 ; these were computed directly by means of Eqs. (27)-(30). The unexpected near-constancy at about 1.23 of the ratio ρ_0/P_0^2 is quite noteworthy and extremely fortunate.

Thus for proton tracks of length L' formed where $H \approx \text{constant}$ and $\rho^2 \propto r'$, we see that

$$(\overline{H\rho})^2 = (1.23 \pm 0.05)(H_T R)^2, \quad (31)$$

provided L' is of such a length that (see Eq. (25'))

$$0.2 \leq 2L'^{1/2}(C_1 \sec \psi)^{-1/2} \leq 1 \quad (31')$$

Eqs. (31) and (31') are obviously got from an examination of Table II. If the protons we used in the present experiment moved in

TABLE II. $\frac{1}{2}\rho_0^2/R_0^2$ vs. α_0

α_0	$\frac{1}{2}\rho_0^2/R_0^2$
0.150	1.29
0.200	1.26
0.300	1.22
0.450	1.21
0.600	1.21
0.875	1.23
0.750	1.22
1.000	1.24

an uniform magnetic field and if their ranges were proportional to their respective kinetic energies, it would be sufficient to substitute Eqs. (51) and (51') into Eq. (14) to get an entirely satisfactory and practical expression for V_1 .

13. The more general case for $H \equiv \text{const.}$
 Actually, the range of a proton may not be proportional to the kinetic energy of the proton, though it may be nearly so over small intervals of kinetic energy. Therefore, the results of the previous Section 12, as embodied in Table II and in Eqs. (51) and (51') could not be used rigorously for any experiment without knowledge that Eqs. (51') were true for the conditions under which the experiment was performed.

In case Eq. (21') happens not to be valid for an experiment, the ratio $\overline{(H\rho)^2}/(H_H R)^2$ for the known Eq. (17), (i.e., for that particular form of Eq. (17) which instead of Eq. (21') would be known to hold) can in

practice best be determined for $H \equiv \text{constant}$ numerically in several steps, thus:

- (1) Knowing $f(r)$ or $g(r)$ substitute in Eqs. (10) and perform the first integration.
- (2) Evaluate the integrals (Eqs. (10)) ~~for~~ numerically for several values of r .
- (3) Plot accurately the graph of η vs. r as got in Step (2).
- (4) Construct, on this graph, several chords for several values of r_0 (as $P_0M_0P_0$ in Fig. 12) and their mid-point perpendiculars (as M_0N_0 in Fig. 12) to the η curve.
- (5) By suitable measurements, compute R_0 for each chord $P_0M_0P_0$ by Eq. (11).
- (6) From Eq. (12), calculate $\overline{P_0^2}$ for each chord $P_0M_0P_0$.
- (7) Form the ratio $\overline{P_0^2}/R_0^2$ for each chord $P_0M_0P_0$. This gives the desired ratio $(\overline{H\rho})^2/(H_{\text{TM}}R)^2$ as a function of r_0 .

This procedure may seem to be quite laborious and inaccurate, but it will often be the only practical one available. The

difficulties involved in the mathematically rigorous determination of the ratio $(\overline{E\rho})^2/(\overline{E} \overline{\rho})^2$ for any given Eq. (10') are chiefly those involved in the integration of Eq. (10') to get a formal expression for the η vs. ϕ path.

In Section V we shall make use of this set of seven steps.

14. A better expression for V_1 . As was pointed out at the end of Section 12, we cannot rigorously use Eqs. (51) and (51') in Eq. (14) for the present experiment unless we can show that Eq. (51') holds. Actually, there are two lines of reasoning which indicate that it is valid in the present experiment to use these Eqs. (51) and (51') with Eq. (14) to get an expression for V_1 .

In the first place, the energy-range relation for protons of low energy is, within five or ten percent (of energy), that given in Table III and plotted in Fig. 6. These energy-range data are those of Living-

TABLE III. Energy-range relation for low-energy protons

Range of proton in cm. in dry air at 15° C. and 760 mm. of Hg	Kinetic energy of the proton in m. e. v.
0.0	0.000
0.2	0.202
0.4	0.355
0.6	0.463
0.8	0.550
1.0	0.628

Fig. 6. Energy-range relation for protons of low energy.

ston and Bethe⁹ as revised by them¹⁰ to include the recent experimental data of Parkinson, Herb, Bellamy, and Hudson.¹¹ It is easy to see by reference to the graph in Fig. 6 that the strictly linear energy-range curve (the dashed curve which we have interposed is entirely consistent with the true energy-range relation.

In the second place, consider the loss of energy by a proton of low energy moving in the gas in the cloud chamber. Suppose that in the increment of distance ds this proton with kinetic energy E and range r loses an increment of kinetic energy de . It is reasonable to suppose that very nearly,

$$de/ds = \tau,$$

where τ is a suitably chosen constant. Thus

$$\int_E^0 de = \tau \int_r^0 ds,$$

or $E = \tau r.$

But since $E \propto \rho^2,$

this means that $\rho^2 \propto r.$

We are thus led to the conclusion that within the limits of experimental uncertainties, the range of a low energy (i.e., below 150,000 e.v.) proton either is proportional to the kinetic energy of the proton or is so very nearly proportional to it that we shall not be in great error in taking Eq. (31') to be true and correct.

In allowing for the fact that $H \neq \text{const.}$ (over the path of a proton in our cloud chamber) when we seek to apply Eqs. (31) and (31'), we note that over the length of any proton's track which we have available for study, the value of H does not vary more than one percent. This fact comes from measurements made on the homogeneity of the magnetic field which we used in our cloud chamber. We shall therefore not make an error of more than 1.5 percent due to H if we use Eqs. (31) and (31').

$$\left. \begin{aligned} \overline{(H\rho)^2} &= (1.23 \pm 0.05)(H_{\text{H}}R)^2, \\ 0.2 \leq 2H'/(C_1 \sec \psi)^{-1} &\leq 1.0 \end{aligned} \right\} \quad (36)$$

for our case in which $H \neq \text{constant}$.

Therefore, combining Eqs. (36) with Eq. (14), we get the expression

$$\left. \begin{aligned} V_1 &= 1.23 \times 10^{-5} (e/n_1) \sec^2 \psi (E_1 R)^2, \\ 0.1 &\leq 2\pi^{1/2} (C_1 \sec \psi)^{-1/2} \leq 1, \end{aligned} \right\} \quad (37)$$

which we set to be an improvement over Eq. (15).

We shall show in Section 7 that whether Eq.

(37) is true or not, Eqs. (31) and (34) and

(37) cannot be far wrong, and we will if

necessary modify them still further there.

In Section VI we shall make allowance for the

variation of e/n_1 at low energies, and in

Section VII we shall develop expressions

for the accuracy which can be expected of

calculations made by Eq. (37) or by its

successor in Section VI

V. THE ACCURACY OF EQUATION (37)

To investigate the accuracy of Eq. (37) we must study essentially the accuracy of the equation

$$\overline{(H\rho)^2} = 1.23(E_H R)^2 \quad (38)$$

for a proton track — i.e., we must find out how near this ~~Eq.~~ Eq. (38) is valid for all likely laws giving ρ^2 as a function of r' . Since we saw in §13 that a general study of the ratio $\overline{(H\rho)^2}/(E_H R)^2$ would be exceedingly complicated and difficult, if at all possible, we consider only two typical cases.

15. $\rho^2 \propto r'$. We saw in §12 that if the kinetic energy of a proton is proportional to the range of the proton, Eqs. (27) and (30) gave

$$\overline{(H\rho)^2} = 1.23(E_H R)^2 \quad (38')$$

independently of C_1 provided

$$0.2 \leq 2E_H^{1/2}(C_1 \sec \psi)^{-1/2} \leq 1. \quad (38'')$$

It is difficult to over-emphasize the importance of this result, i.e., that Eq. (38')

is independent of C_1 (except, of course, for the bounding conditions on C_1 expressed in Eq. (36)). This result in itself gives to Eq. (36) a large measure of independence of the energy-range relation followed by the protons we used in the present experiment.

For example, suppose that we were to calculate the energy of our protons simply from the ranges of the protons. Assuming, as we could, that within the limits of experimental error

$$V_1 \propto \rho_1^2 = C_1 E^2,$$

we would determine V_1 from the known E^2 , C_1 , and other proportionality constants. Suppose, however, that C_1 were in error by a certain percent¹²; then the V_1 thus calculated would be in error by a similar amount. But on the other hand, if V_1 were got by Eq. (37), then it would be the correct value regardless of what C_1 really might be, provided, of course, that Eq. (36) were satisfied.¹³

16. The case for $\rho^2 \propto r/(a_0/r^5 + a_1 + a_2 r)^2$.

To find out how nearly valid Eq. (36) is when ρ^2 is not proportional to r' but does not differ excessively from the function $C_1 r'$ which we have previously considered (Ch. 12 and 14), we shall take the special case of the function

$$r(r) = \frac{C_1 r}{(a_0/r + a_1 + a_2 r)^2}. \quad (39)$$

We shall for convenience use $\Psi = 0$ so that $r = r'$. Here a_0 , a_1 , and a_2 are suitable constants so chosen that this function does not differ too much from $C_1 r$.

To determine $\overline{(H\rho)^2}/(H_T R)^2$ for this case, we follow the seven steps of procedure given in Ch. 13. We have as the path of a proton obeying Eq. (39)

$$\left. \begin{aligned} \xi &= \int_0^r \cos \left[\int_0^v \frac{1}{K_1} \left(\frac{a_0}{u^{3/2}} + \frac{a_1}{u^{1/2}} + a_2 u^{1/2} \right) du \right] dv \\ \eta &= \int_0^r \sin \left[\int_0^v \frac{1}{K_1} \left(\frac{a_0}{u^{3/2}} + \frac{a_1}{u^{1/2}} + a_2 u^{1/2} \right) du \right] dv \end{aligned} \right\}$$

by Eq. (39). Since

$$\int_0^v \frac{1}{K_1} \left(\frac{a_0}{u^{3/2}} + \frac{a_1}{u^{1/2}} + a_2 u^{1/2} \right) du = \lim_{s \rightarrow 0} \int_s^v \frac{1}{K_1} \left(\frac{a_0}{u^{3/2}} + \frac{a_1}{u^{1/2}} + a_2 u^{1/2} \right) du$$

$$\begin{aligned}
&= \lim_{s \rightarrow 0} \frac{1}{\kappa_1} \left(10 a_0 u^{0.1} + 2 a_1 u^{1/2} + \frac{2}{3} a_2 u^{3/2} \right) \Big|_s^v \\
&= \frac{1}{\kappa_1} \left(10 a_0 v^{0.1} + 2 a_1 v^{1/2} + \frac{2}{3} a_2 v^{3/2} \right) \\
&\quad - \lim_{s \rightarrow 0} \left(10 a_0 s^{0.1} + 2 a_1 s^{1/2} + \frac{2}{3} a_2 s^{3/2} \right) \frac{1}{\kappa_1} \\
&= \frac{1}{\kappa_1} \left(10 a_0 v^{0.1} + 2 a_1 v^{1/2} + \frac{2}{3} a_2 v^{3/2} \right),
\end{aligned}$$

we have

$$\left. \begin{aligned}
\xi &= \int_0^r \cos \frac{1}{\kappa_1} \left(10 a_0 v^{0.1} + 2 a_1 v^{1/2} + \frac{2}{3} a_2 v^{3/2} \right) dv, \\
\eta &= \int_0^r \sin \frac{1}{\kappa_1} \left(10 a_0 v^{0.1} + 2 a_1 v^{1/2} + \frac{2}{3} a_2 v^{3/2} \right) dv,
\end{aligned} \right\} \quad (40)$$

which completes essentially the first step of the procedure in §3.

To finish the remaining six steps in this procedure, we of course need to know the values of the constants a_0 , a_1 , a_2 , and C_1 . In §17 we supply a set of constants and complete the remaining six steps.

17. The case for $C_1 = 120$, $a_0 = 0.236$, $a_1 = 0.717$, $a_2 = 0.0368$. We shall now work out the ratio $(H\rho)^2 / (H_{\infty}R)^2$ for ρ^2 given as a

function of r by the solid curve of Fig. 7;
 $C_1 r$ is indicated on Fig. 7 by the dashed curve.
 This particular $C_1 r$ is the same as the $C_1 r$
 that appears in Fig. 6 except that it is adapted
 to a gas having a stopping power 0.875 that
 of air; such a gas is very nearly that which
 was used in the present experiment. Using H
 $= 3520$ gauss', a value quite near to the one
 which we used in the present experiment, we
 get¹⁵

$$C_1 = 126. \quad (41)$$

The shape of the solid curve in Fig. 7 for ρ^2
vs. r is similar to that suggested for protons
 of very low energy by Livingston and Bethe⁹ on
 their energy-range graph -- it is for this
 reason that we are considering the function in
 Eq. (39) for ρ^2 : if ρ^2 departs from the law of
 proportionality to r^1 , it is likely to obey a
 law similar to that in Eq. (39). For this case

$$a_0 = 0.236, \quad a_1 = 0.717, \quad a_2 = 0.068. \quad (42)$$

Substituting these values for C_1 , a_0 , a_1

Fig. 7. Energy-range relation for protons of low energy as investigated in #17. Here $C_1 = 120$, and stopping power of gas is 0.375 that of air.

and a_2 into Eqs. (40), we get as the path of a proton obeying Eqs. (39), (41), and (42),

$$\left. \begin{aligned} \xi &= \int_0^r \cos(0.215 v^{0.1} + 0.131 v^{\frac{1}{2}} + 0.00224 v^{\frac{1}{2}}) dv, \\ \eta &= \int_0^r \sin(0.215 v^{0.1} + 0.131 v^{\frac{1}{2}} + 0.00224 v^{\frac{1}{2}}) dv. \end{aligned} \right\} \quad (42')$$

In Table III' are given several values of the integrands of these two expressions for ξ and η ; in them we let

$$\left. \begin{aligned} \Delta(v) &= \cos(0.215 v^{0.1} + 0.131 v^{\frac{1}{2}} + 0.00224 v^{\frac{1}{2}}), \\ \delta(v) &= \sin(0.215 v^{0.1} + 0.131 v^{\frac{1}{2}} + 0.00224 v^{\frac{1}{2}}). \end{aligned} \right\} \quad (43)$$

To effect the second step of the procedure of ¶13, we plot in Fig. 9 the functions $\Delta(v)$ and $\delta(v)$ as functions of v , and then graphically obtain the definite integrals (Eqs. (42')) of them from $v=0$ to $v=r$ for several values of r . In this manner we get the values of ξ and η as functions of r as tabulated in Table IV. Fig. 9 is a graph (Step (3) of ¶13) of η vs. ξ from Table IV.

The remaining four steps of ¶13 are now to be considered and carried out. On Fig. 9

TABLE III'. $\Delta(v)$ and $\delta(v)$

v	$\Delta(v)$	$\delta(v)$
0.0	1.000	0.000
0.1	-----	0.211
0.2	-----	0.239
0.3	-----	0.265
0.4	-----	0.276
0.5	0.957	0.290
1.0	0.940	0.341
1.5	0.925	0.379
2.0	0.910	0.410
2.5	0.899	0.437
3.0	0.888	0.461
3.5	0.876	0.483
4.0	0.864	0.503

42°

$\Delta(v)$

0.5

0

1

2

3

4

FIG. 8.

$\delta(v)$

0.75

0

1

2

3

4

TABLE IV. ξ and η as functions of r , as
obtained from Fig. 8

ξ	η
0.00	0.000
0.25	0.052
0.49	0.120
0.72	0.196
0.96	0.279
1.20	0.369
1.43	0.461
1.66	0.557
1.89	0.656
2.34	0.868
2.78	1.088
3.68	1.578

Fig. 2. Here $d_0 = 1.49$ and $h_0 = 0.094$.

the chord (length being $2h_0'$ for $I_0 = 3$) and the half-point perpendicular (of length h_0') are constructed as an example: we find that for this example $I_0 = 1.48$ cm. and $h_0 = 1.224$ cm., whence by Eq. (38),

$$I_0 = 11.6\%.$$

By Eqs. (38), (41), (42), however,

$$\rho^2 = 100r / (1.224/r^2 + 0.612 + 1.55r)^2; \quad (43')$$

for $I_0 = 3$ this gives (see Eq. (23))

$$\overline{\rho^2} = \rho_0^2 = (100) / 1.16 = 86.$$

Therefore for protons with an initial range of 3 cm.,

$$(\overline{\rho^2}) / (E_{HD})^2 = 100 / 1.16 = 86$$

if ρ^2 varies with r by Eq. (43'). In Table V

are several values of this ratio ρ_0^2 / I_0^3 found in this same way for several values of I_0 . From Table V we see that for protons with initial energies E in the range from 0.175×10^8 e. v. to 0.275×10^8 e. v.,

$$\begin{aligned} (\overline{\rho^2}) &= (1.37 \pm 0.05)(E_{HD})^2, \\ 2.5 &< I_0 \leq 4 \end{aligned} \quad (44)$$

TABLE V. $\frac{1}{2} \rho_0^2 / R_0^2$ vs. L_0 for $\omega 17$

L_0	$\frac{1}{2} \rho_0^2 / R_0^2$
1.50	1.40
1.75	1.38
2.00	1.36
2.50	1.33
3.00	1.29
4.00	1.22

for the case at hand. This is to be compared with the result expressed in Eqs. (38') and (38'') for the case in which

$$\rho^2 \propto r'.$$

Obviously Eq. (37) is quite a good expression even if $\rho^2 \neq r'$.

13. A better expression for V_1 . If we now make use of the fact that, by Eqs. (38) and (44),

$$\overline{(H\rho)^2} = v(H_N R)^2,$$

where

$$1.20 \leq v \leq 1.33,$$

for protons such as ours, we can take

$$\overline{(H\rho)^2} = (1.26 \pm 0.06)(H_N R)^2 \quad (44')$$

for protons such as ours. If we use this in Eq. (14) instead of Eq. (36), we get

$$\left. \begin{aligned} V_1 &= (1.26 \pm 0.06) 10^{-8} (e/m_1)(H_N R)^2 \sec^2 \psi, \\ 0.175 \times 10^6 &< V_1 \leq 0.275 \times 10^6 \end{aligned} \right\} (45)$$

instead of Eq. (37). In the next section, Section VI, we shall modify this equation to allow for the variation of e/m_1 with V_1 .

III. THE EFFECT OF THE VARIATION OF e/m_1

All our previous considerations, now, have been based on the assumption that $e/m_1 = \text{const.}$ It is known, however, that the charge of a proton travelling through a gas may be neutralized for short intervals of time if the kinetic energy of the proton is small. As Bethe⁶ points out, only if a proton has a kinetic energy of 0.1×10^6 e. v. or more, can one be sure that its charge is not ever zero. Since the protons with which we deal in the present experiment have a kinetic energy less than 0.1×10^6 e. v. over almost half of their paths, we must make an allowance for their reduced e/m_1 at low energies if we intend to utilize their entire tracks for calculations. We may of course allow for the variations of protonic charge by allowing for the corresponding variations in e/m_1 .

Caring for variations in e/m_1 obviously involves either the alteration or the further establishment of Eq. (23) as well as of Eq. (44'). For

simplicity we shall investigate only the case for which $H \equiv \text{constant}$.

19. Validity of Eq. (22) for variable e/m_1 .

As in deriving Eq. (14), we start from the beginning with

$$E' = (1/2m_1)(H'e\rho' \sec \psi)^2. \quad (46)$$

If we put this into electron-volts by Eq. (1), we have for this case

$${}_1V = 5 \times 10^{-9} (c/m_1 e_s)(H'e\rho \sec \psi)^2, \quad (47)$$

where e_s is the electronic (or protonic) charge in electrostatic units at energy ${}_1V$. Taking $H \equiv \text{const.}$, we then have

$$\begin{aligned} (1/V_1) \int_0^V {}_1V d({}_1V) &= \frac{1}{2} V_1 \\ &= \frac{5 \times 10^{-9} c(H \sec \psi)^2}{m_1 e_s V_1} \int_0^V (e\rho)^2 d({}_1V). \end{aligned} \quad (48)$$

Letting

$$\pi^2 = (1/V_1) \int_0^V (e\rho)^2 d({}_1V), \quad (49)$$

we obviously have

$$V_1 = 10^{-8} (H\pi \sec \psi)^2 c/m_1 e_s. \quad (50)$$

Now from Eq. (47),

$$(e\rho)^2 = \frac{1}{2} V m_1 e_s / 5 \times 10^{-9} c (H \sec \psi)^2; \quad (51)$$

therefore

$$\begin{aligned} \pi^2 &= (1/V_1) \int_0^{V_1} V d_1 V m_1 e_s / 5 \times 10^{-9} c (H \sec \psi)^2 \\ &= \frac{1}{2} m_1 e_s V_1^2 / V_1 c \times 5 \times 10^{-9} (H \sec \psi)^2 \\ &= V_1 m_1 e_s / 10^{-8} c (H \sec \psi)^2. \end{aligned}$$

Denoting ρ at the beginning of a proton's path by ρ_I and e/m_1 at the beginning of the path by $(e/m_1)_I$ (this value of e/m_1 is of course the maximum possible value of e/m_1 , the value got by the usual experiments for specific charge), we have

$$V_1 = 5 \times 10^{-9} (e/m_1)_I \rho_I^2 (H \sec \psi)^2 \quad (52)$$

by Eq. (47); thus substituting this into the last expression for π^2 , we get

$$\pi^2 = \frac{1}{2} (e/m_1)_I \rho_I^2. \quad (53)$$

This result is the analog for the present case of Eq. (22).

20. V_1 in terms of R^2 for an observed

variation of e/m_1 with ${}_1V$. Just as in Section IV, we now seek to replace ρ_I^2 and $(e/m_1)_I$ in the preceding equations by the more easily measured R^2 and K . To do this, we must determine the ratio W^2/KR^2 as suggested by the procedure in §13.

We need to know, then, how e/m_1 varies with ${}_1V$. This variation, unfortunately, has not been studied directly; but a very nearly related phenomenon, the variation of charge of hydrogen positive-rays passing through hydrogen, has been investigated for proton energies as great as 0.05×10^6 e. v.¹⁶ From these investigations we can gain some idea of the variation of e/m_1 with ${}_1V$ when ${}_1V \leq 0.1 \times 10^6$ e. v. In Fig. 10 is plotted the behaviour of e/m_1 obtained by calculation of the positive-ray data of Bartels¹⁶; see also Appendix IV. The dashed portion of the curve in Fig. 10 is the extrapolation to energies greater than 0.05×10^6 e. v. The data used to prepare this curve are for a pressure of 1 mm. Hg. Now we let

FIG. 10. $(e/m_e)/K$ vs. V got from the computation (Appendix IV) of Bartels' ¹⁶ data on canal rays in H_2 at a pressure of 1 mm. Hg.

$$e/m_1 = K \delta({}_1V) \quad (54)$$

define $\delta({}_1V)$; $\delta({}_1V)$ is the average fraction of K which a proton possesses when its kinetic energy is in the neighborhood of ${}_1V$. The values of $\delta({}_1V)$ plotted in Fig. 10 are given in Table VI. Strictly speaking, this quantity $\delta({}_1V)$ is the quantity $n_+/(n_0 + n_+)$ defined in Appendix IV and computed as indicated there from the data of Bartels; note that Table VI is much more nearly complete than is Table B.

We now proceed with the computation of π^2/KR^2 for the elimination of ρ_I in the equations of 19. For simplicity we shall also restrict our considerations to the cases for which

$$\psi = 0; \quad (55)$$

this we do after Eq. (56). Since by Eqs. (47) and (54)

$${}_1V = 5 \cdot 10^{-9} K (H \delta \rho \sec \psi)^2, \quad (56)$$

we can write

$$\rho^2 = {}_1VC_2/\delta^2, \quad (57)$$

TABLE VI. $\delta(\text{,}V)$ vs. $\text{,}V$ for canal-rays in H_2 at a pressure of 1 mm. Hg. The values for $\text{,}V < .06 \times 10^6$ e. v. are computed from Bartels' ¹⁶ data; the remaining values are extrapolations.

$10^{-6} \text{,}V$	$\delta(\text{,}V)$
0.0	0.0
0.0178	0.20
0.0357	0.39
0.0535	0.56
0.0713	0.69 ₅
0.0891	0.82
0.1248	0.95
0.1783	0.99
0.2317	1.00
> 0.2400	1.00

where C_2 is a suitable constant. For the case in which $H \approx 3520$ gauss (which we have used earlier in Section V)^{16'} $C_2 = 0.001687$. Furthermore, assuming for further simplicity that

$$V = C_3 r \quad (58)$$

(where C_3 is another constant), Eq. (57) becomes

$$\rho^2 = C_2 C_3 r / s^2. \quad (59)$$

We shall allow for departures from Eq. (58) later. For the example used in Section V, we take C_3 from the dashed curve in Fig. 7; we get^{16'} $C_3 = 7.13 \times 10^4$. Thus for this example we take as $s^{-\frac{1}{2}}$ (cf. Eq. (17))

$$1/\rho = s / 10.97 r^{\frac{1}{2}}. \quad (60)$$

Substituting this into Eq. (19'), we have for our example

$$\alpha = (1/10.97) \int_0^r (s/u^{\frac{1}{2}}) du; \quad (61)$$

the performance of this integration for the given by Table VI must be (for convenience) done numerically. A graph which will be of aid

in the effecting of such integration is given in Fig. 11.

From the values of α got by Eq. (61) we can get the equations of the path of a proton moving under the conditions of this example, by Eqs. 20. Here again the integrations must be effected numerically. Fig. 12 contains the graphs of the sine and cosine functions of the α got by Eq. (61). The results of these integrations are given in Table VII.

To get the ratio π^2/KR^2 for this example, we must first get R_0 , as was done in Section V. E. g., for $L_0 = 3$, we have from Table VII that $d_0 = 1.505$ and $h_0 = 0.0714$; hence by Eq. (26)

$$R_0 = 15.9.$$

The remaining values of R_0 can be expressed by the equations

$$\left. \begin{array}{l} R_0 \cong 16.5 \pm 0.5, \\ 2.5 \leq L_0 \leq 4.0. \end{array} \right\} \quad (62)$$

In view of the fact that the observed values of R_0 in the present experiment were in the

$\cos \alpha$

32"

0.5

0

1

2

3

4

r

FIG 12

$\sin \alpha$

0.2

0

1

2

3

4

r

TABLE VII. f and η for $b=20$

r	f	η
0.0	0.0	0.0
0.25	0.25	0.0005
0.50	0.50	0.0032
0.75	0.75	0.0092
1.00	1.00	0.0190
1.25	1.25	0.0328
1.50	1.50	0.0508
1.75	1.74	0.0728
2.00	1.99	0.0987
2.25	2.24	0.1287
2.50	2.48	0.1624
2.75	2.73	0.200
3.00	2.97	0.241
3.25	3.22	0.285
3.50	3.47	0.332
3.75	3.71	0.381
4.00	3.96	0.433

neighborhood of only 12 or 13 cm., it is thus useless to proceed further with this example.

21. V in terms of R^2 for a more nearly correct variation of e/m_1 with ${}_1V$. In view of the fact that the results given by Eq. (62) do not agree with the experimentally observed values, we must examine the discrepancy and see what is to be made of it. In the first place, as we have indicated in §15, the results got from the foregoing type of calculations may be expected to be practically independent of C_3 . It may be, therefore, that the $\delta({}_1V)$ assumed in Fig. 10 is not correct. In the second place, the experiments were performed at a pressure of about 8 cm. Hg in the presence of moderately intense γ -radiation. Hence we cannot expect that Fig. 10 will be the correct behaviour of $\delta({}_1V)$ for the present experiment.

In Fig. 13 we have plotted a function $\delta({}_1V)$ of ${}_1V$ which departs appreciably from that in Fig. 10, but not unreasonably so. The departure was chosen so that the calculated values

FIG. 13. $\delta(V)$ vs. V as it probably is,
as given by Eq. (63).

to
of R_0 come about the same as the observed values.
This function $g({}_1V)$ can be represented by the
equation

$$g({}_1V) = 1 - e^{-{}_1Va}, \quad (63)$$

in which a is a constant taken to be 40×10^{-6} .

Substituting this into Eq. (59), we have

$$\rho^2 = {}_1VC_2 / (1 - e^{-{}_1Va})^2; \quad (64)$$

working again with the simple case for which

$${}_1V = C_3 r, \quad (65)$$

this last gives

$$1/\rho = (1/C_2 C_3)^{1/2} (r^{-1/2} - r^{-1/2} e^{-b_1 r}), \quad (66)$$

where

$$b_1 = a C_3. \quad (67)$$

Thus by Eq. (19')

$$\alpha = \int_0^r (1/C_2 C_3)^{1/2} (u^{-1/2} - u^{-1/2} e^{-b_1 u}),$$

or simply

$$\alpha = (1/C_2 C_3)^{1/2} \left(\int_0^r u^{-1/2} du - \int_0^r u^{-1/2} e^{-b_1 u} du \right). \quad (68)$$

$$\begin{aligned}
\text{Now } \int_0^r u^{-\frac{1}{2}} du &= \lim_{s \rightarrow 0} \int_s^r u^{-\frac{1}{2}} du \\
&= \lim_{s \rightarrow 0} (2r^{\frac{1}{2}} - 2s^{\frac{1}{2}}) \\
&= 2r^{\frac{1}{2}}. \tag{69}
\end{aligned}$$

$$\begin{aligned}
\text{Also } \lim_{a \rightarrow 0} (e^{-b_1 u} / u^{\frac{1}{2}}) &= \lim_{a \rightarrow 0} \frac{d}{du} (e^{-b_1 u}) / \frac{d}{du} (u^{\frac{1}{2}}) \\
&= \lim_{a \rightarrow 0} (-b_1 e^{-b_1 u}) / (\frac{1}{2} u^{-\frac{1}{2}}) \\
&= \lim_{a \rightarrow 0} (-2b_1 u^{\frac{1}{2}} e^{-b_1 u}) \\
&= 0
\end{aligned}$$

by L'Hospital's rule; therefore

$$\int_0^r u^{-\frac{1}{2}} e^{-b_1 u} du = 2 \int_0^{\sqrt{r}} e^{-b_1 w^2} dw.$$

Putting this into the form of the well-known probability integral, we have for it

$$(2/b_1^{\frac{1}{2}}) \int_0^{\sqrt{b_1 r}} e^{-q^2} dq,$$

or, further,

$$(2/b_1)^{\frac{1}{2}} (2/\sqrt{\pi}) \int_0^{\sqrt{b_1 r}} e^{-q^2} dq;$$

then, denoting the probability integral by

$I(\sqrt{b_1 r})$, this last reduces to just

$$\int_0^r u^{-\frac{1}{2}} e^{-b_1 u} du = (\pi/b_1)^{\frac{1}{2}} I(\sqrt{b_1 r}). \quad (70)$$

Referring back to α in Eq. (68), we may now substitute into it the Eqs. (69) and (70) to get

$$\alpha = (1/C_2 C_3)^{\frac{1}{2}} [2r^{\frac{1}{2}} - (\pi/b_1)^{\frac{1}{2}} I(\sqrt{b_1 r})]. \quad (71)$$

For the example in which $H \approx 3520$ gauss' and the stopping power of the gas through which the protons move is 0.075 that of air, several values of $\cos \alpha$ and $\sin \alpha$ are tabulated in Table VIII for some typical values of r . This example, the one which obtains in #17 and in #20, has very nearly the same conditions as those which actually held in the present experiment.

C_3 is taken from the dashed curve in Fig. 9. Using¹⁶ $C_2 = 0.001687$ and $C_3 = 7.13 \times 10^4$ with the additional Eq. (72):

$$b_1 = 2.85, \quad (72)$$

Eq. (71) reduces to

$$\alpha = 0.183 [r^{\frac{1}{2}} - 0.525 I(\sqrt{b_1 r})]. \quad (73)$$

TABLE VIII. $\cos \alpha$ and $\sin \alpha$ vs. r ,
from Eq. (73)

r	$\cos \alpha$	$\sin \alpha$
0.0	1.000	0.000
0.1	1.000	0.005
0.2	1.000	0.013
0.3	1.000	0.023
0.4	1.000	0.032
0.5	0.999	0.042
1.0	0.996	0.080
1.5	0.992	0.127
2.0	0.987	0.161
2.5	0.982	0.191
3.0	0.976	0.219
3.5	0.970	0.244
4.0	0.964	0.267

The data in Table VIII were computed from Eq. (53). In Fig. 14 are graphs of the $\cos \alpha$ and $\sin \alpha$ tabulated in Table VIII.

Integrating $\cos \alpha$ and $\sin \alpha$ numerically from $v=0$ to $v=r$ as required by Eqs. (20), we get the ξ - and η -coordinates of a proton path formed in accord with the conditions laid down in Eqs. (63), (65), and (32). These ξ - and η -coordinates are tabulated in Table IX as functions of r . The path itself is obtained by plotting η as a function of ξ ; this is done in Fig. 15.

To get the ratio π^2/KR^2 , we proceed as in the following example for $L_0=3$. Since $d_0=1.49$ and $h_0=0.0056$, we have by Eq. (26)

$$R_0 = 12.86.$$

Now, Eqs. (64), (63), (56) give

$$\rho_0^2 = 120r_0/(1-e^{-2/35r_0})^2; \quad (74)$$

for $L_0 = r_0 = 3$ this yields

$$\frac{1}{2}\rho_0^2 = 180.$$

Since $e/m_1 \approx K$ for $r \geq 1.4$ cm., we thus have for

FIG. 14

Table IX. β and η for 21

r	η	β
0.00	0.0000	0.00
1.00	0.0416	1.00
1.25	0.0656	1.25
1.50	0.0950	1.50
1.75	0.1288	1.74
2.00	0.1670	1.99
2.50	0.255	2.48
3.00	0.360	2.97
3.50	0.473	3.46
4.00	0.600	3.95

Fig. 15. Proton path, η vs. ϕ for 421.

$$L_0 = 3,$$

$$\overline{v}^2/KR^2 = \frac{1}{2}(e/m_1)\rho_0^2/KR^2 = 180K/168K = 1.07.$$

In Table X are several values of $\overline{v}^2/KR^2$ for several other typical values of L_0 . From Table X we can get a good expression for $\overline{v}^2/KR^2$, hence for $\overline{v}^2$ in terms of K and R^2 , which can be used in Eqs. (52) and (53) for V_1 , provided of course that e/m_1 varies according to Eq. (63).

22. Final expression for V_1 . In view of the fact that the value of R_0 as got in 421 just before Eq. (74) is about 13 and of the fact that the observed values of R were of the order of 12 or 13 cm. in this experiment, we may conclude that the variation of e/m_1 given by Eq. (63) is essentially correct. It may in a few regions differ from the true variation of e/m_1 by a small amount, but it seems safe to conclude that these differences must not be great. In any event, the true variation of e/m_1 must be such that it predicts values of R in the neighborhood of 12 or 13, a thing which Eq. (63)

TABLE X. π^2/KR^2 for 421

L_0	π^2/KR^2
2.00	0.94
2.50	1.02
3.00	1.07
4.00	1.20

does quite well.

Incidentally, it is not entirely surprising that the variation of ϵm_1 hypothesized in Fig. 10 should fail to hold for the present experiment. In the first place, the gases traversed by the protons in the present experiment were under moderately intense γ -ray action caused by the Th C'' γ -ray source. In the second place, the gases here were at about 80 times the pressure for which the data in Fig. 10 are actually valid. It might be argued that the data in Fig. 10 ought to hold for the conditions of present experiment, since it would not be easy to quote reasons why they should not; but on the other hand there is available no reliable evidence that Fig. 10 does hold under the conditions of the present experiment.

Referring to Table X, we thus see that for protons such as those we shall use, i. e., such as those treated in 16, 20 and 21, moving in a magnetic field of 3520 gauss', moving in

a gas with a stopping power 0.075 that of air, and losing energy according to the equation ${}_1V = C_3r$, we may take

$$\left. \begin{aligned} \pi^2 &= (1.11 \pm 0.09)KR^2, \\ 2.5 \leq L \leq 4.0. \end{aligned} \right\} \quad (75)$$

This is to be compared with Eqs. (31) and (31').

To allow for possible deviations from the law ${}_1V = C_3r$, we notice that the coefficient 1.23 in Eq. (31) became 1.26 (an increase of about three percent) if the energy-range relation were modified to the one shown as the solid curve in Fig. 7; see Eq. (44'). Applying such a "correction" to the coefficient in Eq. (75), we get as the final equation for π^2 ,

$$\left. \begin{aligned} \pi^2 &= (1.15 \pm 0.09)KR^2, \\ 2.5 \leq L \leq 4.0. \end{aligned} \right\} \quad (76)$$

For the present experiment in which $H \neq \text{const.}$, we may replace H in Eq. (50) et seq. by H_N and not cause any error greater than 0.5 per cent (see 4.14), an amount which is well

over-shadowed by the uncertainty 0.09 allowed for the coefficient in Eq. (76). Also, since for these derivations which we have carried through the ranges 2.5 and 4.0 correspond to energies of about 0.175×10^6 e. v. and 0.275×10^6 e. v. respectively, we can change from the bounding condition in L in Eqs. (76) to a bounding condition in terms of V_1 . Bearing all this in mind, we substitute Eqs. (76) into Eqs. (52) and (53) and get the expression

$$\left. \begin{aligned} V_1 &= (1.15 \pm 0.09) \times 10^{-8} K(H_N R \sec \psi)^2, \\ 0.175 \times 10^6 &\leq V_1 \leq 0.275 \times 10^6, \end{aligned} \right\} \quad (77)$$

which we shall use in computations based on the data which were taken in the present experiment.

VII. ERROR THEORY

23. Some constants. Before beginning to investigate the effects which experimental errors in observable entities will have on the value of $\overline{V}_2$ which we shall compute later, we shall establish the values of four necessary constants.

Since¹⁷, 18

$$c = (2.99776 \pm 0.00004) \times 10^{10} \text{ cm./sec.}, \quad (76)$$

$$F = 9648.9 \pm 0.7 \text{ em.u./gm.-equivalent}, \quad (77)$$

c and F can be considered for the present experiment to be exact quantities. The quantity

$$K = \lim_{v \rightarrow \infty} e/m_1 \quad (\text{see (51)})$$

is given in terms of the electronic specific charge e/m by

$$K = (m/m_1)(e/m);$$

Since¹⁹

$$e/m = (1.7584 \pm 0.0003) \times 10^7 \text{ em. u./gm.},$$

and since²⁰

$$m/m_1 = 1/(1 - 30.2 \pm 1.6),$$

It is simply

$$K = 9577 \pm 5 \text{ cm. u./gm.}; \quad (80)$$

we may thus consider K also as a sensibly exact constant for the present experiment.

The quantity k is known with sufficient accuracy already. It is¹⁷

$$k = (1.1 \pm 0.1) \times 10^{-3} \text{ at. wt. units.} \quad (81)$$

34. Error in V_1 . Referring to Eq. (77), which is

$$\left. \begin{aligned} V_1 &= (1.15 \pm 0.09) \times 10^{-3} K (H_{\perp} R \sec \psi)^2, \\ 0.175 \times 10^{-6} &\leq V_1 \leq 0.275 \times 10^{-6}, \end{aligned} \right\} \quad (82)$$

let ϵ_i , $i=1$ through 3, be the possible absolute errors which might occur in the determinations of $H_{\perp}$, R and ψ respectively. Then the absolute error which might be present in V_1 is, by a total differential of Eq. (82), ΔV_1 where

$$|\Delta V_1| = |\partial R_1 / \partial \sigma| \Delta \sigma + |\partial R_1 / \partial H_1| \epsilon_1 + |\partial R_1 / \partial R| \epsilon_2 + |\partial R_1 / \partial \psi| \epsilon_3;$$

here σ is the constant 1.15 and R_1 is the right member of Eq. (22). Substituting the values of these partial derivatives, we get on dividing through by V_1 ,

$$|\Delta V_1 / V_1| = \Delta \sigma / \sigma + \epsilon_1 / H_1 + \epsilon_2 / R + \epsilon_3 \tan \psi; \quad (23)$$

the values of these partial derivatives are worked out in the Mathematical Appendix III.

Thus the absolute error ϵ_4 which might be present in a determination of V_1 is

$$\epsilon_4 = V_1 \left(\frac{\Delta \sigma}{\sigma} + \frac{\epsilon_1}{H_1} + \frac{\epsilon_2}{R} + \epsilon_3 \tan \psi \right). \quad (24)$$

25. Error in V_2 . Let ϵ_i , $i=5$ through 7, be the absolute errors which might be present in V_T , $(1-k/m_2)$, and θ respectively. The corresponding absolute uncertainty which they may produce in V_2 is, by a total differential of Eq. (4) (which is reproduced here as Eq. (25))

$$V_2 = V_T - [1 + (1-k/m_2)(1-2p \cos \theta + p^2)] V_1, \quad (25)$$

simplify $|\Delta V_B|$ where

$$|\Delta V_B| = |\partial R_2 / \partial V_Y| \epsilon_5 + |\partial R_2 / \partial (1 - \beta / m_2)| \epsilon_6 \\ + |\partial R_2 / \partial \theta| \epsilon_7 + |\partial R_2 / \partial V_1| \epsilon_4.$$

Here R_2 is the right member of Eq. (65). These partial derivatives are also worked out in the Mathematical Appendix III. Substituting them and dividing by V_B , we get

$$|\Delta V_B / V_B| = \epsilon_5 / V_B + V_1 \epsilon_6 (1 - \beta \cos \theta + \beta^2) / V_B \\ + 2 \beta V_1 \epsilon_7 \sin \theta / V_B \\ + [1 + (1 - \beta / m_2)(1 - \beta \cos \theta + \beta^2)] \epsilon_4 / V_B. \quad (66)$$

Simplifying this, we get

$$|\Delta V_B / V_B| = \epsilon_5 / V_B + (V_Y - V_B - V_1) \epsilon_6 / V_B (1 - \beta / m_2) \\ + 2 \beta V_1 \epsilon_7 \sin \theta / V_B + \epsilon_4 (V_Y - V_B) / V_1 V_B.$$

Thus

$$|\Delta V_B| = \epsilon_5 + \frac{V_Y - V_B - V_1}{1 - \beta / m_2} \epsilon_6 + 2 \beta V_1 \epsilon_7 \sin \theta + \frac{V_Y - V_B}{V_1} \epsilon_4. \quad (67)$$

Now since $V_Y \cong 2.6 \times 10^6$ e. v., $V_B \cong 2.2 \times 10^6$ e. v. and $V_1 \cong 0.2 \times 10^6$ e.v., we have, in view of Eq. (61),

$$\frac{V_1 - V_B - V_1}{1 - k/m_2} = \frac{0.2 \times 10^6}{1 - k/m_2} \approx 0.2 \times 10^6, \quad (88)$$

for $m_2 \approx 1$ atomic weight unit. Also

$$p = (E_1/c)(2m_1 E_1)^{-1/2} \approx 0.15;$$

$$\text{so} \quad 2pV_1 \approx 5 \times 10^4. \quad (89)$$

$$\text{And, finally,} \quad (V_1 - V_B)/V_1 \approx 2. \quad (90)$$

Substituting Eqs. (88)-(90) into Eq. (87), we find

$$|\Delta V_B| = \epsilon_5 + 0.2 \times 10^6 \epsilon_6 + 5 \times 10^4 \epsilon_7 \sin \theta + 2 \epsilon_4. \quad (91)$$

But from Eq. (81), $\epsilon_6 = 0.0001$ atomic weight units. And from Eq. (2), 0.2×10^6 e. v. $\approx 0.5 \times 10^{-3}$ atomic weight unit. Therefore the term in Eq. (91) in ϵ_6 is essentially negligible. We thus have left as the absolute error ϵ_8 which might be present in V_B ,

$$\epsilon_8 = \epsilon_5 + 5 \times 10^4 \epsilon_7 + 2 \epsilon_4. \quad (92)$$

26. The error in $\overline{V_B}$. We actually seek in the present experiment not V_B but the mean value $\overline{V_B}$ of n determinations of V_B of equal

weight. Let ϵ_5 be the absolute error to be expected in $\overline{V_B}$. By combining Eqs. (92) and (94), we have

$$\epsilon_1 = \epsilon_5 + 5 \times 10^{-4} \epsilon_7 + 27_1 \Delta r / r + 47_1 (\epsilon_1 / \overline{V_B} + \epsilon_2 / 2 + \epsilon_5 \tan \psi). \quad (95)$$

The first and third terms in the right member of Eq. (95) represent what might be called intrinsic errors in $\overline{V_B}$ or $\overline{V_B}$; i. e., no amount of measuring or of averaging the results can in any way reduce them. They are, from the strictly experimental point of view, completely unalterable, although of course a separate experiment to that effect might reduce ϵ_5 .

The remaining terms of the right member of Eq. (95), however, are of a different kind. If there is a certain probability that a positive error of this type exists for one of the n determinations (of $\overline{V_B}$), then there is an equal chance that one of the remaining $(n-1)$ determinations will possess a negative

error of the same magnitude. Thus in the mean value $\overline{V}_B$ of the n determinations of V_B , the errors corresponding to the second and fourth terms of Eq. (93) will tend toward good deal smaller values than the values of these second and fourth terms themselves. We may, in fact, thus apply to them the classical theory of errors in the mean. According to this theory, the second and fourth terms in Eq. (93) are reducible by a factor of $n^{1/2}$ in the error expression for $\overline{V}_B$. We therefore take

$$\epsilon_9 = \epsilon_5 + 2V_1 \Delta\sigma/\sigma + \frac{4V_1(\epsilon_1/H_N + \epsilon_2/R + \epsilon_3 \tan \bar{\psi}) + 5 \times 10^4 \epsilon_7}{n^{1/2}}. \quad (94)$$

In this expression $\bar{\psi}$ will be taken to be the mean value of the n values of ψ observed in the taking of the necessary data.

VIII. APPARATUS

Having derived the three essential equations, Eqs. (4), (77) and (94), for the present experiment, we now leave theoretical discussions and turn to the more practical experimental details.

27. The cloud chamber. As has been intimated, we used a cloud chamber to observe the protons in the present experiment. This cloud chamber^{was} released automatically and^{was} reset in the contracted position manually after each expansion. The walls of the chamber were of glass to admit light for photographing the ion tracks which appeared inside the chamber.

To begin with, the cloud chamber which we used was an hybrid type designed as a compromise between the two preferable types; see Fig. 16. The first preferable type is the one which is most often used at present. It contains as the movable bottom of the chamber a rubber diaphragm which (while fixed around the edges) is free from all mechanical restraint except (see Fig. 16b) from that of

(a)

(b)

(c)

Fig. 16.

the difference of gas pressure in the chambers B and C. To "set" such a chamber, the pressure in B is increased until the volume in C is the desired amount and is held there. To expand such a chamber, the pressure in B is very quickly reduced (through exit-tube A) to a value very much less than that in C by quickly connecting B to still another chamber D which is partly evacuated. This type of cloud chamber is very convenient to operate if sufficiently large differences of pressure can be established in the systems B-C and B-D; for example if the pressure in C can be about atmospheric, these pressure differences can be readily obtained with only a little apparatus. Unfortunately, however, at pressures of 20 cm. Hg or less in C, these pressure differences cannot be made large enough to operate a rubber diaphragm (of moderate weight) sufficiently quickly, i. e., sufficiently quickly to produce sharp ion tracks.

9 Petrova²¹ remarks on this defect at low pressures of the simple rubber - diaphragm

type of cloud chamber, in connection with an experiment of his. To remedy the lack of sharpness found in ion tracks formed in such a low-pressure cloud chamber, he (following Klemperer's earlier work²² with a piston-type cloud chamber) found it necessary to use instead a cloud chamber of the second type; see Fig. 16c. The floor of the chamber in this type is the top of a movable piston. This piston P (see Fig. 16c) is moved by a connecting rod E, which is in turn operated by a suitable spring and lever mechanism. When an expansion is desired, P is simply moved downward very quickly through the necessary distance. Such a cloud chamber can produce extremely sharp and clear ion tracks (in P) at pressures in P of the order of 1 cm. Hg, but it requires a very bulky (and usually heavy) mechanism to actuate the connecting rod E and requires a cylinder of a moderately great length to guide the piston's motion. Whereas the first type of cloud chamber was not suitable to our experiment because sufficient pressure differences

could not be obtained, this second one was not suitable to it because of the large size required for the mechanism and cylinder -- for the cloud chamber to be used in the present experiment has to be fitted between the pole pieces of a medium-sized electromagnet.

Now the usual blend of these types of cloud chambers has been as follows: the use of a "piston" with no depth and which clears the "cylinder" walls by as much as a centimeter, and the connecting of the piston-top to the cylinder-wall with a pliable rubber ring. Cloud chambers of this pattern are used extensively and many of them are well described in modern literature. An essential feature of such chambers is that the "piston" still provides the motive power for the expansions and the rubber ring serves only to isolate effectively the expansion chamber from the surrounding space. The cloud chamber which was used in the present experiment evolved in a manner similar to this one.

The cloud chamber finally adopted for the present experiment is shown in almost complete detail in Fig. 16a. As is evident there, both a complete rubber diaphragm R (as in Fig. 16b) and a rudimentary piston Q (as shown in Fig. 16c) are employed, the piston being behind the rubber diaphragm. This is somewhat suggestive of the usual blend mentioned in the preceding paragraph, but it contains several distinct²³ features, among them being:

- (1) the diaphragm is not attached to the piston in any way;
- (2) the diaphragm is entire, as in Fig. 16b.

The sole purpose of the piston is to push the diaphragm up into the "set" position and to hold it there until an expansion is desired. When an expansion occurs, the piston (which was of a small mass) was snapped downward by the two small brass springs S_1 and S_2 , and the rubber diaphragm was snapped down by the excess of ~~of~~ gas pressure in the expansion chamber over the pressure in the (always) evacuated chamber T. With an arrangement

like this, gum rubber sheet $1/16$ th of an inch can be used successfully and easily with the pressure in the chamber as low (by experiment) as 5 cm. Hg. It is conceivable ~~that~~ an expansion chamber of the first type might be made to operate successfully at these low pressures if a very light rubber diaphragm were used, but in our case such a light diaphragm was ruled out by the fact that the deuterium gas in the chamber would have leaked out through it too rapidly²⁴. The diaphragm was, in use, blackened with India ink.

The dimensions of the cloud chamber are obtainable from Fig. 16a, which was drawn to scale. The inner diameter of the glass cylinder was about 11 cm. When the chamber was expanded, the depth of its sensitive (i. e., to ions) re~~gion~~ion was about an inch ~~in depth~~.

The glass^{top} of the chamber was held in position by a light brass ring which is not shown in Fig. 16a. It was of course held still more firmly in place by the action of the

excess (atmospheric) pressure outside the chamber.

A small brass catch (not shown in the diagram) was arranged to operate against W. This catch held the chamber in the "set" position and, when released, allowed the chamber to expand. An electromagnetic release was arranged to release this catch.

The plate V was used simply to mount the cloud chamber against the coils (K) of the electromagnet which produced the magnetic field H. V was insulated from K by a sheet of fibre as shown in the diagram.

A feature which could not well be shown in Fig. 16a was the mechanism used to re-set the cloud chamber after expansions and to adjust the expansion ratio. X was a brass plate which was held by wedges against the face of the electromagnet's pole piece. On the outer face of X was a mechanical system arranged to operate the block U as part of a second-class lever and to maintain the axis of U always in

the same direction -- Fig. 17 gives a sketch of the essential details. The upper end of U, when in re-setting the chamber or in stopping the expansions, bore against W. When in operation, the chamber cycle was as follows: upon the release of the catch (mentioned above), the piston was snapped back until W hit upon the upper end of U -- U was of course bearing solidly against X; then by means of lever L and the second-class system operating U (see Fig. 17), the chamber was re-set by compressing the piston system up until the catch was again engaged. The expansion ratio of the chamber was adjusted simply by moving the electromagnet's pole piece backward or forward; this could be done easily because the pole piece was movable with a strong screw.

Z (Fig. 16) contains a sliding bearing which served to guide the piston's motion. Σ is an ordinary Sylphon (metallic) bellows which sealed the evacuated chamber T from the atmosphere outside.

The assembly J made it possible to

Fig. 17.

evacuate the expansion chamber, to re-fill it with deuterium and heavy water, and to then seal it off. Its operation is discussed later.

This cloud chamber sometimes contained inside it a small polonium α -particles source on the end of a small piece of copper wire. This source of α -particles was used to check the operation of the chamber.

When preparing the chamber for use, the following steps were carried out:

- (1) the glass top of the chamber was coated on its inner side with a thin coat of gelatin dissolved in H_2O^2 , and the coat was allowed almost to dry;
- (2) the glass top was then put in place, and both the expansion chamber and T were evacuated, the former down to about 2 cm. Hg (the vapor pressure of H_2O^2 at room temperature) -- T was thereafter kept evacuated by a pump running continuously;
- (3) the vacuum line on the J assembly was then turned off and the chamber tested (by a mano-

meter next to J) for leaks -- leaks were stop-
ped by covering them with a "solution" of shel-
lac in ethyl (not methyl) alcohol;

(4) when all leaks had been stopped, nearly
pure H_2^2 from a cylinder (this H_2^2 was supplied
by the Stuart Oxygen Company) was admitted until
the pressure in the cloud chamber was 8 cm.

Hg;

(5) the stopcock in J was then turned to admit
 H_2^2O , under a pressure of about 66 cm. Hg, from
the reservoir K -- the amount of H_2^2O admitted
usually amounted to about one ml.;

(6) the stopcock was closed and Hg was placed
in K;

(7) Hg was allowed, by slightly "cracking" the
stopcock in J, to fill up the capillary tube
extending into the cloud chamber -- this Hg
provided an excellent and effective seal.

This seal was found to be a necessary precau-
tion for preventing turbulence inside the ex-
pansion chamber; a water seal was found to be
entirely ineffective.

After these seven steps, the chamber was

ready for operation.

The chamber was operated on a cycle of from 20 to 30 seconds. After each expansion, light was provided and photographs were made of any ion tracks which happened to occur in the sensitive part of the chamber.

22. The γ -ray source. The γ -radiation used to produce disintegrations was from a contained source of Ra Th . This source was equivalent in γ -ray activity to about 5 mg. of Ra . It was placed in a glass cylinder 6 mm in inside diameter and 1.5 cm (outside) long and was shielded with 4 mm lead. This lead block was placed in front of the chamber in the position shown as the dot-dash object in Fig. 10a.

In spite of the extreme propinquity of the γ -ray source to the sensitive portion of the cloud chamber, no trouble whatever was experienced with any excessive fogging of the chamber by the large amount of γ -rays passing through it. This was due to the low atomic number of the gas in the chamber and to the

fact that the expansion ratio at which the chamber operated was kept just below the value necessary for the appearance of electron tracks.

An electric field, due to 500 volts steady potential difference, was maintained across the chamber except during and just before the expansions.

29. The light source. In Fig. 18 is a schematic diagram of the light source used for the cloud chamber. Two binding posts are shown, one of them being the one shown in Fig. 16a, the other being a post mounted on H and connected electrically to the brass piece U of Fig. 16a. The carbons which are shown in Fig. 18 were kept either in slight contact or only very slightly separated (so as to maintain a very small arc); the current through them was, in between expansions of the cloud chamber, limited by a 10 ohm resistance as shown. In the "set" condition of the cloud chamber, the circuit breaker was closed. The condensing lens

Fig. 18. Light circuit in "set" position.

had a focal length of about 35 cm. and a diameter of about 25 cm; it was so placed that a non-diverging beam of light from the arc could be transmitted into the cloud chamber by way of the Pyrex walls of the chamber (Fig. 16a).

In operation, a releasing of the catch which held the cloud chamber "set" allowed an expansion of the chamber. When W (Fig. 16c) touched U at the end of the expansion, the magnetic switch (Fig. 16) was instantly closed (and the indicator lamp turned on). The closing of the magnetic switch short-circuited the 10 ohm resistance in the direct-current line to the arc and ~~of course~~ allowed a very large current to pass through the arc. This very large current of course produced a blinding flash of light in the arc and also opened the circuit-breaker to stop this large arc. The circuit-breaker, adjusted to open at 100 amperes, opened almost instantly. The effect of all this was to produce a brilliant flash of light lasting for a short fraction of a second immediately after the expansion had

occurred. It is estimated that the flash lasted about $1/5$ th of a second or less. The flash was entirely long enough to allow making good pictures and was not too long.

After the flash of light had occurred, the chamber was re-set, thus releasing the magnetic switch; the circuit-breaker was then closed, and the carbons in the arc were re-adjusted for the next expansion of the chamber.

The arc light was placed some twelve feet from the cloud chamber in order to help in getting a non-diverging beam. No trouble was experienced with any heating effect of the light in the cloud chamber.

It was found that the photographic flame carbon in the arc produced about twice as much photographically useful light as an ordinary projection carbon. Only cored carbons were used.

30. The camera and the electromagnet.
as seen in
Above the cloud chamber (Fig. 16a) was a

flat front-surfaced (chromium over aluminum on glass, the deposition being by evaporation) mirror situated at a 45° angle so as to make the expansion chamber visible (through its glass top) from the side -- i. e., from a direction parallel to the plane of the top of the chamber (see Fig. 16a and Fig. 18). A stereoscopic camera constructed in this laboratory by Dr. E. M. Mott-Smith was trained on the mirror image of the cloud chamber.

This camera had two Leitz lenses having a focal length of 7.5 cm. and a maximum numerical aperture of 1.8. These lenses recorded on 35 mm movie film. In this experiment, Eastman SS Pan film was used, it being about the fastest film available. The camera was cranked by hand after each expansion of the cloud chamber.

The electromagnet which provided the field H had pole pieces 19 cm in diameter. It was of soft cast iron. The current (535 amperes) was passed around the pole pieces in

Fig. 19. Sketch of the lay-out, as seen from above, showing positions of cloud chamber, mirror and camera relative to electromagnet.

the coils K; these coils were of copper tubing through which water was run constantly in order to provide sufficient cooling. The current was kept at about 400 amperes continually and was automatically increased to 535 amperes a few seconds before each expansion of the cloud chamber.

31. The control circuit. To operate all the mechanism associated with the cloud chamber and magnet, a master control circuit was necessary. This circuit, diagrammed in Fig. 20, directed and timed all automatic operations. In every case the manual operations of re-setting and re-adjusting were performed as soon after the expansions as was possible. The diagram is self-explanatory.

These three circuits, Figs. 18, 20 and 21 (Fig. 21, a circuit of the apparatus which provided the electric field in the cloud chamber, is also self-explanatory) provide a fairly complete diagram of the electrical ap-

Fig. 20. Control circuit.

Fig. 21. In this figure,

Tr = Thordarson type T-2645 transformer;

T = RCA-Cunningham type '80 full-wave rectifying vacuum tube;

C = Aerovox 4 mfd. electrolytic condenser.

paratus used in the present experiment. They omit only the electrical connections involved in the electromagnet and its associated motor-generator set.

32. Apparatus for field measurements.

For practical purposes, it was found convenient to represent the magnetic field strength $H(x, y, z)$ at the point (x, y, z) in the cloud chamber (see Fig. 2) by an equation of the form

$$H(x, y, z) = H_0 + h(x, y, z), \quad (95)$$

where H_0 is the value of H at $x=y=z=0$. Then in general,

$$|h(x, y, z)| \ll H_0, \quad (96)$$

and so approximate determinations of h would suffice.

H_0 was measured by the inductive method with allowances²⁵ being made for small variations of H over the area of the "filp" coil used, thus. Let M be the mutual inductance through the primary of which a current I may

be passed and broken (see Fig. 20); let A be the area of the flip coil and let N be the number of turns of wire in this coil. Let ds be an element of the area of A at the point (x, y, z) ; then if the breaking of I produces the same deflection in the galvanometer G as the 180° -rotation of the flip coil, the formula for H_0 is²⁵

$$H_0 = (NI - 2N \iint_A h(x, y, z) ds) / 2AN. \quad (87)$$

In the present experiment a mutual inductance with

$$M = 1.98 \text{ millihenries to within } 0.2\% \quad (88)$$

was used; this value was furnished by the Bureau of Standards. Also,

$$N = 1, \quad (89)$$

and very fine ($\frac{3}{40}$, E. & S. gauge) wire was used to wind the coil. A was the area of an accurately circular coil with a diameter $2a_3$, and

Fig. 22. The battery *B* was a new six-volt lead storage battery. *G₁* was used to judge potentiometer balances.

$$2a_3 = 5.182 \text{ cm.} \quad (100)$$

The current I was measured by a potentiometer method: the potential difference caused by I across a known resistance was measured by a potentiometer and then I was calculated from Ohm's law. The circuit for I is shown in Fig. 22. the resistance R was 4.0001 ohms, as determined by the Bureau of Standards. The potentiometer was a Leeds and Northrup student potentiometer which had not been used much previously. The standard cell was an Eppley cell which had been calibrated by the Bureau of Standards.

$h(x, y, z)$ was calculated from the effect (on a good galvanometer connected to an Ayrton shunt) of the motion in H of a small (outside diameter being 1 cm.) coil having 1000 turns of fine wire. This small coil and its galvanometer were calibrated in a known, small magnetic field; the combination had a sensitivity of 1.0 gauss per cm. scale deflection, which was a sufficient sensitivity.

IX. MEASUREMENTS AND OBSERVATIONS

33. Magnetic field measurements. First of all, $h(x, y, z)$ was determined. It was evident from the beginning that for all our purposes³⁵

$$h \neq \text{fn. } (z)$$

but that

$$h = \text{fn. } (x^2 + y^2),$$

i. e., that h was practically symmetric about the z -axis. The completed observations for h , given in Table XI, showed that h was practically proportional to $(x^2 + y^2)$; when the proportionality constant was determined, we have

$$\left. \begin{aligned} h(x, y, z) &= -(x^2 + y^2) \text{ gauss,} \\ (x^2 + y^2)^{1/2} &\leq 5 \text{ cm.,} \end{aligned} \right\} \quad (101)$$

which was good to within 5%.

It was further found that when the electromagnet current was 535 amperes, the galvano-

TABLE XI. Data for h vs. $(x^2 + y^2)$

$(x^2 + y^2)^{1/2}$ (cm.)	Galv. defl. (cm.)	h (gauss)	$h/(x^2 + y^2)$
0	0.00	0.00	-----
1	-1.02	-1.02	-1.02
2	-3.04	-3.04	-0.96
3	-9.45	-9.45	-1.05
4	-15.9	-15.9	-0.99
5	-24.3	-24.3	-0.95
Mean			-0.99

meter attached to a tiny field-recording flip-coil in H was deflected 14.64 cm. by a 180° flip of the coil. Later, when H_0 was to be determined, the cloud chamber was removed and the electromagnet was again set so that the field recorder produced a 14.64 cm.²⁷ deflection. The flip of the 5.102 cm. coil centered at O then produced a deflection in G (Fig. 22) which was balanced only when the potential difference broken in the M - R circuit (Fig. 22) was 0.635 volts. Thus, since $R = 4.2001$ ohms (see 432), we had for H_0

$$I = 0.01527 \text{ em. u. of current.} \quad (102)$$

34. Observations on the proton tracks.

In all, about 2250 photographs were made in searching for protons in the cloud chamber. Among these were found fifteen proton tracks which were suitable for analysis. In Figs. 23.1 through 23.15 are shown (stereoscopic pairs of) prints of these portions of movie film which photographed the fifteen tracks. Accompanying each photograph is a brief re-

note: these prints do not have as much contrast as the original negatives, hence the faintness of the tracks.

Fig. 23.1. An excellent track.

Fig. 23.2. One of the shorter tracks.

Fig. 23.3. An old, broad track, but an easily and reliably measureable one.

Fig. 23.4. This track, in stereoscopic projection, can be seen not to come from the walls of the cloud chamber. Near its inner end is an electron track which must not be confused with it.

Fig. 23.5. Another fairly old, but reliable, track.

Fig. 23.6. This picture is faint because a smaller amount of light than usual was used to make it.

Fig. 23.7. This picture is faint because a smaller amount of light than usual was used to make it.

Fig. 23.8. An excellent track.

Fig. 23.9. This track originates just beneath the γ -ray source.

Fig. 23.10. One of the best tracks.

Fig. 23.11. This track is faint because it is a very late one.

Fig. 23.12. This track can be seen in stereoscopic projection not to come from the α -particle source or from its immediate vicinity.

Fig. 23.13. A late, but excellent, track.

Fig. 23.14. There is an α -particle track almost parallel to the proton track.

Fig. 23.15. This very faint track begins and ends in the gas in the chamber, as do all the other tracks.

mark of explanation or clarification. The measurement of these tracks was effected as follows:

- (1) the developed film was put back into the camera, the back of the camera was removed, and 60 watt lamp bulbs were placed behind each lens of the camera so as to illuminate the film well;
- (2) the stereoscopic photographs of the tracks were thus projected on a small movable screen which was in the focal region of the two lenses of the camera -- the projections were made through the glass which formed the top of the cloud chamber;
- (3) by properly moving the screen, the two projected images of each track were made to coincide -- this determined the "planes" in which the tracks lay;
- (4) with a Brown and Sharp protractor, ψ and θ for each track were measured and recorded (see Table XII) -- the θ of a track was measured between the line from the center of the γ -ray source to the beginning of the track (the end of a track almost always exhibited tiny deflections

TABLE XII. Data on observed proton tracks

Track	ψ (°)	θ (°)	w (cm)	h (cm)	d (cm)	$(K^2 w^2)^{1/2}$ for N (cm)
1	37	137	1.52	0.086	1.215	3.2
2	30	173	1.25	0.062	1.002	4.1
3	17	63	1.59	0.086	1.520	3.5
4	19	134	1.76	0.125	1.671	3.0
5	35	97	1.61	0.080	1.319	2.8
6	2	47	1.35	0.060	1.350	3.0
7	3	30	1.84	0.097	1.540	3.1
8	42	150	1.33	0.064	0.975	3.2
9	0	76	1.45	0.086	1.450	2.8
10	27	131	1.30	0.062	1.160	3.1
11	12	97	2.00	0.174	1.955	2.3
12	23	96	2.03	0.151	1.869	1.9
13	9	0	1.79	0.110	1.760	3.0
14	7	94	1.20	0.057	1.191	1.6
15	21	93	1.62	0.112	1.512	2.7

which occurred when the proton was nearly stopped) and the tangent to the track at its beginning point;

(5) the value of $(x^2 + y^2)^{1/2}$ for the mid-point N (see Fig. 4) of each proton track was measured and recorded;

(6) the two ends and the mid-point N of each track were carefully recorded on small strips of paper with a very sharp pencil -- this was done several times for each track;

(7) the chord (of length 2ω) P_2P_1' (see Fig. 3) was measured and its half-length recorded (see Table XII);

(8) the mid-chord perpendicular (of length h) from the track to the chord was measured and recorded for each track;

(9) from ω , d was calculated:

$$d = \omega \cos \psi, \quad (103)$$

and recorded.

For all this data, see Table XII.

For the benefit of those who may think

that Step (6) in this track-measuring process is liable to great inaccuracy, it should be stated that on the contrary, experience has proved to the writer that this method affords a very reliable, as well as speedy, method of measuring such curved proton tracks. It is, in fact, believed that the determinations so made of ω are accurate to within 1% and that those so made of h are accurate to within 4%.

In Section X we shall make the necessary calculations to get the $n=15$ values of V_B and their mean value $\overline{V_B}$.

X. CALCULATION OF $\overline{V}_E$

35. Calculation of H_0 . First we shall calculate, from the data quoted in §§ 32 and 33, the value of H_0 which obtained in the present experiment. The remainder of the calculation of $\overline{V}_E$ will follow this. Substituting Eqs. (97), (99), (101), and (102) into Eq. (87), we have

$$H_0 = \frac{0.99 \times 10^6 \times 0.01587 - 2 \iint_h [-(x^2 + y^2)] dx dy}{2\pi (5.182/2)^2}$$

$$= \frac{158,000 - 2 \int_0^{2\pi} \int_0^{250} r^2 r dr d\theta}{13.427 \pi} = \frac{158,700}{42.12}$$

whence

$$H_0 = 3,762 \text{ gauss}. \quad (104)$$

36. Calculation of V_1 . To proceed, we now calculate the values of V_1 for each observed proton by Eq. (77). This obviously requires H_E and R for each track. In Table XIII, in the second column, are the values of R for each track, computed from the h and d values in Table XII by means of Eq. (21). The third

TABLE XIII. Calculation of V_1

Track	R (km.)	sec ψ	R sec ψ (km.)	$h(X, Y, Z)$ for M (gpm-g)	M_H (gpm-g)	$10^{-6} V_1$
1	9.62	1.251	10.83	-10	3752	0.1008
2	9.49	1.155	10.96	-17	3745	0.1855
3	12.09	1.046	12.63	-12	3750	0.2470
4	11.23	1.052	11.83	-9	3753	0.2160
5	9.91	1.221	12.00	-8	3754	0.2253
6	13.22	1.001	13.23	-9	3753	0.2719
7	12.27	1.021	12.38	-10	3752	0.2338
8	7.47	1.346	10.06	-10	3752	0.1566
9	12.27	1.000	12.27	-8	3754	0.2337
10	10.09	1.112	11.12	-10	3752	0.1810
11	11.30	1.022	11.56	-5	3757	0.2077
12	11.5	1.006	12.64	-4	3758	0.2111
13	13.35	1.013	13.50	-9	3753	0.2829
14	12.49	1.008	12.59	-3	3759	0.2461
15	10.27	1.071	11.00	-7	3755	0.2364

column of Table XIII contains the appropriate values of $\sec \psi$, while in the fourth column are the corresponding values of $R \sec \psi$. The fifth column contains the values of $h(x, y, z)$ for the points N for the several tracks; these values were calculated from Eq. (101) and the seventh column of Table XII; in the sixth column are the corresponding values of H_N as got from Eq. (95) and the fifth column (of Table XIII). The seventh column of Table XIII is for the values of V_1 -- these values were got by multiplying H_N^2 by $(R \sec \psi)^2$ (from Table XIII) and this by $K = 9577$ em. u. /gm. (from Eq. (90)) and then, finally, all this by $(1.15 \pm 0.09) \times 10^{-8}$; compare this process with that required by Eq. (77).

37. Calculation of V_B . To compute the 15 values of V_B , we now substitute into Eq. (4). Using the definition in §3 of p , we have

$$p = \frac{10^4}{c} \left(\frac{e}{m_1} \right)^{\frac{1}{2}} \frac{V_r}{(2V_1)^{\frac{1}{2}}} . \quad (105)$$

TABLE XIV. Calculation of $\overline{V_B}$

Track	p	cos θ	$1-2p \cos \theta + p^2$	$-(V_0 - V_1) \times 10^4$	$10^{-4} V_B$	V_i
1	0.142	-0.731	1.238	0.403	2.220	0.05
2	0.141	-0.283	1.307	0.426	2.197	0.03
3	0.122	0.454	0.904	0.470	2.153	0.02
4	0.130	-0.695	1.190	0.478	2.145	0.02
5	0.128	-0.122	1.048	0.457	2.166	0.00
6	0.116	0.682	0.861	0.516	2.117	0.05
7	0.125	0.708	0.817	0.425	2.198	0.03
8	0.153	-0.034	1.309	0.372	2.261	0.09
9	0.125	0.242	0.955	0.456	2.167	0.00
10	0.130	-0.656	1.200	0.422	2.201	0.03
11	0.133	-0.102	1.050	0.406	2.198	0.03
12	0.132	0.070	0.989	0.422	2.201	0.03
13	0.114	0.000	0.785	0.505	2.118	0.05
14	0.122	-0.070	1.032	0.500	2.123	0.05
15	0.124	-0.052	1.028	0.479	2.144	0.03
Mean					2.174	0.03

Thus, using Eq. (80) for e/m_1 and using ²⁸

$$V_Y = 2.625 \times 10^6 \text{ e. v.}, \quad (106)$$

we can get p for each V_1 listed in Table XIII; these values of p are given in the second column of Table XIV. The third column contains values of $\cos \theta$. In the fourth column are the fifteen values of the factor $(1 - 2p \cos \theta + p^2)$. The fifth column contains the quantity $-(V_Y - V_B)$; in computing these numbers we used, of course, Eq. (81). The sixth column gives the corresponding values of V_B .

As is indicated in Table XIV, we find by taking the mean of the fifteen values of V_B ,

$$\overline{V_B} = 2.174 \times 10^6 \text{ e. v.}, \quad (107)$$

with an average deviation of 0.03×10^6 e. v.

39. Calculation of ϵ_g . In Table XV we have tabulated the estimated values of the quantities which appear in Eq. (94) for ϵ_g . The absolute error ϵ_1 present in the magnetic field measurements is believed to be not more than

TABLE XV. Quantities in Eq. (94)

Quantity	Symbol	Value
Relative error in E_{π}	ϵ_1/E_{π}	0.005
Relative error in R	ϵ_2/R	0.05
Absolute error in Ψ	ϵ_3	0.02
Absolute error in V_{π}	ϵ_5	0.004×10^{-6}
Absolute error in a	ϵ_7	0.02
Average observed proton energy	V_1	0.20×10^{-6}
Coefficient in Eq. (77)	σ	1.15
Absolute uncertainty in σ	$\Delta\sigma$	0.09
Average value of Ψ	$\bar{\Psi}$	19^0
Number of independent determinations of V_1	n	15

% of H_0 . As stated in § 34, the error present in R is believed to be not more than 5% of R . Ψ and θ could be measured to within one degree, so that ϵ_3 and ϵ_7 may be taken to be 0.02 each. For V_Y , we are compelled to allow a possible error of 0.004 million electron-volts. For V_1 we shall use 0.22×10^6 e. v. σ is taken, along with $\Delta\sigma$, from Eq. (87). $\bar{V}$ as got from Table XII is 19° .

Substituting these values into Eq. (94), we get

$$\begin{aligned}\epsilon_9 &= 0.004 \times 10^6 + (0.44 \times 10^6 \times 0.09) / 1.15 \\ &\quad + \frac{0.80 \times 10^6}{(15)^{\frac{1}{2}}} (0.005 + 0.05 + 0.02 \times 1.344) \\ &\quad + 0.02 \times 5 \times 10^4 / 15^{\frac{1}{2}} \\ &= 0.004 \times 10^6 + 0.234 \times 10^6 + 0.014 \times 10^6,\end{aligned}$$

or

$$\epsilon_9 = 0.052 \times 10^6 \text{ e. v.} \quad (108)$$

This ϵ_9 is the maximum absolute error to be expected to be present in V -- it is not the

"probable error", the error quantity usually
 calculated, and ^{it} bears no theoretical relation
 to the "probable error" .

38. Final value for $\overline{V}_2$. Combining the
 results expressed in Eqs. (106) and (107), we
 have for the value of $\overline{V}_2$ as determined by the
 present experiment,

$$\overline{V}_2 = (2.17 \pm 0.05) \times 10^6 \text{ electron-volts. (108)}$$

The uncertainty 0.05 is the maximum uncertainty
 which may be expected in $\overline{V}_2$.

XI. DISCUSSION

In the first place, it is to be noted that the value 2.17×10^6 e. v. (for the binding energy of the deuteron) as got from the present experiment tends definitely to agree with the values got previously (2.17×10^6 e. v. by Bethe¹⁰ from the data of Chadwick, Feather and Bretscher², and 2.18×10^6 by J. R. Richardson and Tho⁴) by using Livingston's and Bethe's revised energy-range relations for low-energy protons, rather than with the value 2.25×10^6 calculated by Chadwick, Feather and Bretscher² from earlier energy-range data. This agreement thus tends to confirm the accuracy of the revised energy-range relation for protons of energies in the neighborhood of 2.2×10^6 e. v.^{29, 30}. This confirmation has yet more weight because of the fact (pointed out in §15) that the results of the present experiment are practically independent of the (nearly linear) energy-range relation for low-energy protons.

In the second place, the agreement of the presently determined value of $\overline{V}_B$ with the 2.17×10^6 e. v. of Chadwick, Feather and Bretschger² and Bethe³ and the 2.16×10^6 of J. R. Richardson and Tho⁴ indicates that the mass of the neutron is probably ~~the~~

$$m_{on} = 1.0089 \text{ at. wt. units}$$

(as calculated by Bethe³) rather than some higher value such as is suggested by Hudspeeth and Penner³¹.

In the third place, it should be noted that the maximum uncertainty 0.65×10^6 e. v. quoted for $\overline{V}_B$ in Eq. (100) may be expected to be two or three times as large as the "probable error" ϵ_{10} calculated by the classical theory of errors. That this disparity actually occurs can be seen for the present case if we calculate ϵ_{10} . It is well known³² that

$$\epsilon_{10} = 0.675 \sqrt{\frac{\sum v_i^2}{n(n-1)}}, \quad (110)$$

where v_i is the residual for the i th determination of $\overline{V}_B$; in the present case

$$\sqrt{n(n-1)} = \sqrt{310} = 14.5$$

while from Table II V

$$\sqrt{\sum_{i=1}^n v_i^2} = \sqrt{0.2297} = 0.545 \times 10^6,$$

whence by Eq. (110)

$$\epsilon_{10} = 0.675 \times 0.545 \times 10^6 / 14.5 = 0.225 \times 10^6.$$

Therefore, using Eq. (108),

$$\epsilon_9 / \epsilon_{10} = 0.052 / 0.225 = 2.0.$$

We thus see that the estimate of 0.25×10^6 e. v. for the uncertainty in $\overline{V}_B$ is, as compared with the conventionally calculated probable error of 0.225×10^6 e. v. in $\overline{V}_B$, a very conservative estimate of the accuracy of our value of $\overline{V}_B$.

Finally, it may be said (in view of the ratio of $\epsilon_9 / \epsilon_{10}$) that the present experiment compares in precision well with the two previous experiments performed for the determination of $\overline{V}_B$.

XIII. ACKNOWLEDGMENTS

The suggestion to the writer that the binding energy of the deuteron be redetermined was made by Dr. H. A. Wilson, who has since provided the writer every desired facility for the prosecution of the necessary experiment. It is indeed a pleasure for the writer to acknowledge these facts.

The writer wishes also to thank Marie M. Rogers for her necessary help in performing the experiment and for her help in making some of the measurements and calculations.

The assistance of Stanley Heaps with the development of the exposed movie film which accumulated during this experiment is also appreciated.

MATHEMATICAL APPENDIX I

It can easily be shown that

$$\alpha' = \alpha_0 / 2^{\frac{1}{2}}, \quad \alpha_0 \leq 1, \quad (A1)$$

to within 1 part in 1,000. By the definition of α' , α' is determined by the relation

$$\frac{\sin \alpha_0 - \alpha_0 \cos \alpha_0}{\cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1} \quad (A2)$$

$$= - \frac{(\cos \alpha' + \alpha' \sin \alpha' - 1) - \frac{1}{2}(\cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1)}{(\sin \alpha' - \alpha' \cos \alpha') - \frac{1}{2}(\sin \alpha_0 - \alpha_0 \cos \alpha_0)}.$$

To begin with, α' can be shown to be equal to within 1 part in 100 to $\alpha_0 / 2^{\frac{1}{2}}$ simply by graphical methods; i. e., given an α_0 , we can draw on a graph like Fig. 5 the appropriate chord and mid-chord perpendicular to N_0 and can then determine α' by interpolation. An example of this process is that for $\alpha_0 = 1$, given in Fig. 5; there we see that $\frac{\alpha_0}{2^{\frac{1}{2}}} = 0.707$ while $\alpha' = 0.71$ for $\alpha_0 = 1$. We may therefore write

$$\alpha' = \alpha_0 / 2^{\frac{1}{2}} + \epsilon, \quad |\epsilon| \ll |\alpha_0 / 2^{\frac{1}{2}}|, \quad \alpha_0 \leq 1 \quad (A3)$$

and seek an approximate expression for $|\epsilon|$ or an

upper bound for it. To this end, let

$$\left. \begin{aligned} \cos \alpha_0 + \alpha_0 \sin \alpha_0 - 1 &\equiv 2f_0, \\ \sin \alpha_0 - \alpha_0 \cos \alpha_0 &\equiv 2\eta_0; \end{aligned} \right\} \quad (\text{A4})$$

then Eq. (A2) for α' becomes, first by Eq. (A4) and then by Eq. (A3),

$$\frac{\eta_0}{f_0} = \frac{-f_0 + [\cos(\frac{\alpha_0}{2} + \epsilon) + (\frac{\alpha_0}{2} + \epsilon) \sin(\frac{\alpha_0}{2} + \epsilon) - 1]}{\eta_0 - [\sin(\frac{\alpha_0}{2} + \epsilon) - (\frac{\alpha_0}{2} + \epsilon) \cos(\frac{\alpha_0}{2} + \epsilon)]}.$$

Expanding the cosine and sine terms in this equation, taking $\cos \epsilon \approx 1$ and $\sin \epsilon \approx \epsilon$, and neglecting terms in ϵ^2 , we get on simplifying

$$|\epsilon| \leq \frac{f_0[f_0 - (\cos \frac{\alpha_0}{2} + \frac{\alpha_0}{2} \sin \frac{\alpha_0}{2} - 1)] + \eta_0[\eta_0 - (\sin \frac{\alpha_0}{2} - \frac{\alpha_0}{2} \cos \frac{\alpha_0}{2})]}{0.707 f_0}$$

or, finally,

$$|\epsilon| \leq \frac{f_0[f_0 - (\cos \frac{\alpha_0}{2} + \frac{\alpha_0}{2} \sin \frac{\alpha_0}{2} - 1)] + \eta_0[\eta_0 - (\sin \frac{\alpha_0}{2} - \frac{\alpha_0}{2} \cos \frac{\alpha_0}{2})]}{0.707 f_0}. \quad (\text{A5})$$

In Table A are given a few values of the right member $|\epsilon|$ of Eq. (A5) for a few representative values of α_0 . The limit of $|\epsilon|$ as $\alpha_0 \rightarrow 0$ can be seen by inspection to be zero because the

TABLE A. R vs. a_0

a_0	R
0.00	0.0000
0.50	-0.0004
1.00	-0.0007

numerator of Eq. (A5) is of a higher degree in α_0 than the denominator. Since $|\epsilon| \leq |\mathcal{R}|$, it is easily seen from Table A that for such values of α_0 we have

$$\alpha' = \alpha_0 / 2^{\frac{1}{2}}, \quad \alpha_0 \leq 1$$

to within 1 part in 1,000.

MATHEMATICAL APPENDIX II

In this section of the Appendix we give the details of the calculation of $\frac{1}{2}p_0^2/R_0^2$ for $\alpha_0 = 0.2$. Since

$$\begin{aligned}\alpha_0 = 0.2 &= 11^\circ 27' 33'', & \frac{\alpha_0}{2^{1/2}} &= 0.1/2^{1/2} = 3^\circ 0' 10'', \\ \sin \alpha_0 &= 0.198674, & \sin \alpha_0/2^{1/2} &= 0.140949, \\ \cos \alpha_0 &= 0.979967, & \cos \alpha_0/2^{1/2} &= 0.99017,\end{aligned}$$

we have

$$\alpha_0^2/2 = 0.02; \quad (A6)$$

and, by Eq. (26),

$$d_0^2 = C_1^2 \sec^2 \nabla = 0.000099/4. \quad (A7)$$

Therefore, using these Eqs. (A6) and (A7) with Eq. (43) (got from Eq. (27))

$$h_0^2 = C_1^2 \sec^2 \nabla = 0.0000001539/4, \quad (A8)$$

we have

$$\frac{1}{2}p_0^2/R_0^2 = 0.02 \times 0.0000001539/(0.000099)^2 = 1.26.$$

This result is that quoted in the second entry in Table II.

MATHEMATICAL APPENDIX III

The partial derivatives used in the error theory in Section VII are tabulated below. From Eq. (82) we have

$$\partial R_1 / \partial R = 10^{-8} K (H_N R \sec \psi)^2,$$

$$\partial R_1 / \partial H_N = 2.30 \times 10^{-8} K H_N (R \sec \psi)^2,$$

$$\partial R_1 / \partial R = 2.30 \times 10^{-8} K R (H_N \sec \psi)^2,$$

$$\partial R_1 / \partial \psi = 2.30 \times 10^{-8} K (H_N R)^2 \sec^2 \psi \tan \psi.$$

From Eq. (85), we have

$$\partial R_2 / \partial V_1 = 1,$$

$$\partial R_2 / \partial (1 - k/m_2) = -V_1 (1 - 2p \cos \theta + p^2),$$

$$\partial R_2 / \partial \theta = -V_1 \times 2p \sin \theta,$$

$$\partial R_2 / \partial V_1 = -[1 + (1 - k/m_2)(1 - 2p \cos \theta + p^2)].$$

APPENDIX IV

Regarding the variation of e/m_1 with $_1V$, there is no direct experimental data available. The experimental data of Bartels¹⁶ on the variation of charge of hydrogen canal rays in hydrogen can be made to give some pertinent information however. If we let n_+/n_0 be the ratio of the number of protons to the number of neutral hydrogen atoms in a beam of hydrogen canal rays, then Bartels' data are summarized in Table B.

TABLE B. n_+/n_0 vs. $10^{-6} _1V$

$_1V$	n_+/n_0	$n_+/(n_0+n_+)$ [#]
0.100	$\gg 1$	1-e
0.050	0.95	.48
0.025	0.35	0.26
0.010	0.15	0.13

[#] $e > 0$ is small.

In the third column of Table B is the ratio (calculated from the second column) $n_+/(n_0+n_+)$, which is proportional to the average value of e/m_1 for the particles in the beam. The num-

bers in the first and third columns of Table B are plotted on the graph in Fig. 10.

REFERENCES

- ¹Chadwick and Goldhaber, Proc. Roy. Soc.,
A151, 409, (1935).
- ²Chadwick, Feather, and Bretscher, ibid,
A163, 566, (1937).
- ³Debye, Phys. Rev., 53, 313, (1943).
- ⁴J.R. Richardson and Tmo, Phys. Rev., 53,
234, (1943).
- ⁵Note that V_1 is the maximum value of V
over a proton track; V takes on this value
at the beginning of the path only.
- ⁶The writer has elsewhere seen proton tracks
(having a range of about 14cm.) the last
centimeter of which had as many as three
visible singularities caused by small-
angle scattering.
- ⁷This functional relation is directly deri-
vable from the energy-range relation for
a low-energy proton, thus: let the energy-
range relation be

$$\rho^2 = f(r'), \quad (18)$$

ρ^2 being directly proportional to the kinetic energy of the proton; by Eq. (16), this is

$$\rho^2 = f(r \sec \psi) \quad (19)$$

which for a given track and hence for a fixed ψ , is a function (which we have called $g(r)$ in Eq. (17)) of r only.

See MATHEMATICAL APPENDIX II.

⁹Livingston and Bethe, Rev. Mod. Phys., 9, 261, (1937).

¹⁰Bethe, Phys. Rev., 53, 313, (1938).

¹¹Parkinson, Herb, Bellamy, and Hudson, Phys. Rev., 52, 75, (1937).

¹²Compare, for example, the energy-range data believed to be nearly correct (solid curve of Fig. 6) with the data previously believed to be valid, as in reference 6 (dotted curve of Fig. 6).

¹³The fact that Eq. (38') holds for any C_1 satisfying Eq. (38'') is of great importance for an additional reason: Eq. (38') is therefore true for a cloud-chamber of

any stopping power; for a change in the stopping power of the gas through which the protons travel simply changes the value of C_1 and leaves $(\overline{E\rho})^2/(E_1-R)^2$ unchanged.

¹⁴Note that $f(r')$ and $g(r)$ are related by Eq. (17), (18), and (19).

¹⁵To get C_1 , we have from Eq. (12) and from the definition of C_1 ,

$$C_1 = \rho'/r = \frac{2 \times 10^8 \pi_1 V}{(e/n_1) H^2} \frac{1}{r} = \frac{2 \times 10^8 \pi_1 \cdot 2.25 \times 10^6}{(3.52 \times 10^3)^2} \frac{1}{4} \\ = 120$$

if we take $H = 3520$ gauss.

¹⁶See, for example, Bartels, Ann. d. Physik, 5-13, 375, (1932).

^{16'}We have for the case at hand

$$C_1 = \frac{1}{5 \times 10^{-8} H^2 \sec^2 \chi \chi} = \frac{1}{5 \times 10^{-8} \times (3.52)^2 \times 1580} \\ = 16.12 \cdot 0.001687.$$

$$\text{and } C_3 = \frac{V}{r} = \frac{2.25 \times 10^6}{4} = 7.13 \times 10^4,$$

$$\text{and } b_1 = 4 \times 10^{-6} \times 7.13 \times 10^4 = 2.85.$$

¹⁷See Birge, Nature, 134, 771, (1934) for c.

¹⁶ See the Smithsonian Physical Tables, Eighth Edition, (1933) for F.

¹⁸ Dunnington, Physical Rev., 52, 475, (1937).

²⁰ Using $(1 + m_e/m) = (e/m)(h/F)$, where h is the atomic weight of hydrogen, with $h = 1.0081$ as obtained in the mass-spectrograph.

²¹ Petrova, Zeits. f. Phys., 55, 521, (1929).

²² Klemperer, Zeits. f. Phys. **45**, 225, (1927).

²³ We have not been able as yet to locate in literature a cloud chamber like the present one.

²⁴ That this is no small matter is well known. If, for example, a metal can were used for the diaphragm, it would leak very rapidly.

²⁵ Rogers, Phys. Rev., 52, 381, (1937).

²⁶ The abbreviation $fn.(q)$ means "a function of q".

²⁷ The current in the electromagnet came to 535 amperes again, showing that nothing had been disturbed the while.

²⁸See, e. g., Bethe, Phys. Rev., 53, 313 (1938).

²⁹An idea of this energy-range relation may be gained from the equations of it given by Rogers and Rogers, Phys. Rev., 53, 713 (1938).

³⁰This energy-range relation at the lower energies was in the main a reinterpretation by Bethe (see reference 10) of the experimental data published by Parkinson, Verb, Bellamy and Hudson, Phys. Rev., 52, 75 (1937).

³¹Hudspeth and Bonner, Phys. Rev., 54, 300 (1938).

³²See, e. g., Merriman, The Method of Least Squares (Wiley and Sons, 1900), p. 89.

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An Independent Determination of the Binding Energy of the Deuteron

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This paper is an account of an experiment designed to determine the binding energy of the deuteron by a method which is relatively insensitive to uncertainties in the energy-range relation for protons of low energy. The protons, produced by the disintegration of Th C'' γ -radiation, were observed in a low pressure cloud chamber in a strong magnetic field. The curvatures of the tracks allowed the calculation of the corresponding kinetic energies. The final value of the binding energy as given by this experiment is

$$V_B = (2.17 \pm 0.05) \times 10^6 \text{ electron volts,}$$

which is in reasonable agreement with previous determinations by others. The uncertainty 0.05 Mev is the *maximum* uncertainty to be expected in this value of the binding energy.

I. INTRODUCTION

IN 1935 Chadwick and Goldhaber¹ determined the binding energy V_B of the deuteron from the disintegration of deuterium by Th C'' γ -radiation of high energy. The particles taking part in this disintegration react according to the equation

Chadwick and Goldhaber detected the product protons with a proportional counter and linear amplifier, and estimated the kinetic energies of these protons by the amounts of ionization they produced in the counter. In this way they got

$$V_B = (2.14 \pm 0.16) \times 10^6 \text{ electron volts.}$$

Subsequently Chadwick, Feather, and Bretscher² performed with a cloud chamber a more precise experiment designed to measure the ranges of the protons which are liberated when H^2 disintegrates. After converting the observed proton ranges to the corresponding proton kinetic energies by means of the best range-energy data available at that time, they got

$$V_B = (2.25 \pm 0.05) \times 10^6 \text{ ev.}$$

Their data, recalculated by Bethe³ in the light of more recent data on the energy-range relation for protons, yield, however,

$$V_B = (2.17 \pm 0.04) \times 10^6 \text{ ev.}$$

The discrepancy between these last two values of V_B is caused by previous uncertainties in the energy-range data for protons of low energy.

At about the same time, J. R. Richardson and L. Emo⁴ obtained

$$V_B = (2.18 \pm 0.07) \times 10^6 \text{ ev}$$

from the disintegration of H^2 by the high energy γ -radiation from radioactive Na^{24} . This value of V_B also involves the use of the energy-range relation for protons, for Richardson and Emo determined the range of the product protons with a cloud chamber and then got the corresponding kinetic energy by the energy-range relation. They used almost the same relation as that which Bethe³ used.

Since these previous values of V_B depend upon the somewhat uncertain energy-range relation for protons of low energy (except for the first value quoted above, which is not of high precision), it was thought desirable to determine V_B by a method which yields results that are relatively insensitive to this dependence. An attempt to do this with an ordinary magnetic spectrograph with photographic recording was not successful, probably because the available Th C'' γ -ray source was not sufficiently strong.

It was decided for the present experiment to use a deuterium-filled cloud chamber in which the total pressure was about 8 cm Hg and to place this cloud chamber in a strong magnetic field. Then the observable curvatures of the pro-

¹J. Chadwick and M. Goldhaber, Proc. Roy. Soc. **A151**, 479 (1935).

²J. Chadwick, N. Feather and E. Bretscher, Proc. Roy. Soc. **A163**, 366 (1937).

³H. A. Bethe, Phys. Rev. **53**, 313 (1938).

⁴J. R. Richardson and L. Emo, Phys. Rev. **53**, 234 (1938).

ton tracks should enable the kinetic energies of the protons to be determined.

II. BINDING ENERGY IN TERMS OF PROTON ENERGY

In the following derivations we shall not consider either the effects of the relativistic variations of mass or the small difference between the mass of the proton and the neutron.

Let V be the energy in Mev of the Th C'' γ -ray which produces the disintegration of a deuteron; V_B the binding energy of the deuteron in Mev; W the kinetic energy in Mev of the proton liberated by the disintegration of the deuteron; p the ratio of the momentum of the incident γ -ray to the momentum of the product proton (obviously $p = 10^4 K^{\frac{1}{2}} V / c(2W)^{\frac{1}{2}}$); $K = e/m$ = specific charge of the proton, in e.m.u. per g; θ = angle between the direction of the incident γ -ray and the direction of emission of the product proton.

Then if the equations for the conservation of energy and for the conservation of two perpendicular components of momentum during the collision-disintegration process are properly solved, we have the simple expression

$$V_B = V - 2W(1 - p \cos \theta + p^2). \quad (1)$$

In this equation we treat V as a known quantity. We determine several pairs of values of θ and W (hence of p also) experimentally by making suitable observations.

III. THEORETICAL CONSIDERATIONS FOR PROTON ENERGY

If a product proton were moving in a vacuum in a uniform magnetic field,⁵ its trajectory would be a right-circular helix, the axis of which would be parallel to the direction of the magnetic field. Actually, in the present experiment, the protons moved in an almost uniform magnetic field (to within 0.7 percent); but, moving in the cloud chamber, they did not traverse a vacuum. The loss of energy by a proton to the chamber gases thus produces a trajectory which is not at all like the right-circular helix mentioned above. Allowance must be made for this.

⁵ See, e.g., J. J. Thomson's *Conduction of Electricity Through Gases* (Cambridge Univ. Press, 1906), p. 106.

Let H be the strength in gauss of the (uniform) magnetic field in which the protons move; π a plane which is perpendicular to the direction of H . Let ψ be the angle between the direction of motion of a proton in H and a π -plane; ψ is of course measured in a plane perpendicular to a π -plane and will be constant (except for the effects of scattering by the chamber gases) over the length of a proton path. Let ρ be the radius of curvature of the π -plane projection of a proton path; ρ will of course decrease as the proton traverses its path. Let ρ' be the value of ρ at the beginning of a proton path; and v the kinetic energy in Mev of a proton when its projected path has a radius of curvature ρ ; $v = W$ when $\rho = \rho'$.

Then if for any point on a proton's path we equate the centrifugal force acting on the proton to the force on it due to its motion in H , and if we solve this equation for v , we get

$$v = 5 \times 10^{-15} K (H \rho \sec \psi)^2. \quad (2)$$

Now if we multiply this by dv and integrate from $v = 0$ to $v = W$, we get

$$W = 10^{-14} K (H \sigma \sec \psi)^2, \quad (3)$$

where

$$\sigma^2 = (1 - W) \int_0^W \rho^2 dv. \quad (4)$$

Eq. (3) gives W in terms of σ , which is a function of the shape of the *entire* proton path, rather than in terms of a radius of curvature (e.g., as ρ') at some one point on the path. The outstanding difficulty in the calculation of W from the "one point" equation,

$$W = 5 \times 10^{-15} K (H \rho' \sec \psi)^2,$$

is the determination of ρ' ; in this experiment the accurate determination of ρ' was not possible because the observed proton tracks were: (1) only slightly curved by the magnetic field; (2) often appreciably broadened by diffusion (characteristic of low pressure cloud chamber containing mostly deuterium gas); (3) occasionally scattered through appreciable angles. Eq. (3) allows these three obstacles to radius of curvature measurements to "average themselves out" over the length of a proton track.

table that the strictly linear energy-range relation assumed in Eq. (9) is entirely consistent, for $v \leq 0.3$, with the presently accepted relation. Therefore, we should not be in great error in taking Eq. (9) to be true. We must admit, however, that ρ^2 may deviate somewhat from the linear relation of Eq. (9).

We can see that even an appreciable deviation from Eq. (9) produces only an insignificant change in the value of σ^2 , R^2 . To see this, we may work out the case in which ($\psi = 0$ for convenience)

$$g(r) = C_1 r (a_0 r^{0.4} + a_1 + a_2 r)^2, \quad (12)$$

in which we shall take

$$\begin{aligned} C_1 &= 120, & a_0 &= 0.236, \\ a_1 &= 0.717, & a_2 &= 0.0368. \end{aligned} \quad (13)$$

This form for $g(r)$ is quite similar to that suggested for protons of very low energy by the energy-range graph of Livingston and Bethe.⁷ Its average deviation from the function $C_1 r$ is, for proton energies up to 0.3 Mev, of the order of 10 percent. The constants in it are chosen to fit the following case: the stopping power of the gas is 0.075 that of air, and $II = 3520$; the other condition on the constants is that Eqs. (12) and (13) represent (for this case) the data given in Table II.

If we substitute Eqs. (12) and (13) into Eqs. (7) and integrate, and if we then calculate several

TABLE I. σ^2 , R^2 vs. α_0 .

α_0	σ^2 , R^2	α_0	σ^2 , R
0.150	1.29	0.600	1.21
0.200	1.26	0.750	1.22
0.300	1.22	0.875	1.23
0.450	1.21	1.000	1.24

values of σ^2 , R^2 as a function of L (instead of α_0) as was done to obtain Table I, we find that

$$\sigma^2 = (1.27 \pm 0.05) R^2, \quad \left. \begin{array}{l} 2.5 \leq L < 4. \end{array} \right\} \quad (14)$$

Obviously the difference between Eqs. (11) on one hand and Eqs. (14) on the other hand, is essentially a small one relative to the difference between Eq. (9) and Eqs. (12)–(13).

V. EFFECT OF THE VARIATION OF PROTON'S CHARGE

All our previous considerations, now, have been based on the assumption that the charge on a proton is a constant $e = mK$ over the whole of the proton's path. It is known, however, that the charge of a proton traveling through a gas may be neutralized for short intervals of time if the kinetic energy of the proton is small. As Livingston and Bethe⁹ point out, only if a proton has a

TABLE II. Energy-range relation for low energy protons.

RANGE OF PROTON IN CM IN DRY AIR AT 15°C AND 760 MM OF HG	KINETIC ENERGY OF THE PROTON IN MEV
0.0	0.000
0.2	0.202
0.4	0.355
0.6	0.463
0.8	0.550
1.0	0.628

kinetic energy of 0.1 Mev or more, can one be reasonably safe in taking the charge to be the usual value, $e = 4.8 \times 10^{-10}$ e.s.u. Since the protons which we observed in the present experiment had energies less than 0.1 Mev over almost half their path lengths, we must make an allowance for their reduced charges at low energies if we intend to utilize their entire tracks for calculations. We shall derive a relation analogous to Eqs. (3) and (14) to allow for this variation of charge.

Let E be the kinetic energy in ergs of the proton when liberated by the disintegration of a deuteron; δ the ratio of the true charge on the proton to e (this, of course, varies as the proton traverses its path); i the kinetic energy in ergs of a proton which has a range r' ; $i = E$ when $r' = L'$. Now just as Eq. (2) was obtained, we may also find

$$i = (H\delta e \rho \sec \psi)^2 / 2m;$$

therefore,

$$v = 5 \times 10^{-15} K (II\delta \rho \sec \psi)^2. \quad (15)$$

Note that

$$\rho^2 = C_2 v / \delta^2, \quad (16)$$

where C_2 is a suitable constant. Hence, just as

⁹ M. S. Livingston and H. A. Bethe, reference 7, p. 262.

Eq. (3) was obtained,

$$W = 10^{-14} K (H \tau \sec \psi)^2, \quad (17)$$

where

$$\tau^2 = (1/W) \int_0^W \delta^2 \rho^2 dv. \quad (18)$$

Note that for $v \geq 0.18$, $\delta = 1$ very nearly and this last reduces to

$$\tau^2 = \frac{1}{2} \rho^2, \quad v \geq 0.18. \quad (19)$$

To evaluate this, we need to know essentially how δ varies with v and how ρ varies with v .

We can get an idea of how δ might be expected to vary with v from the data of Bartels¹⁰ on the variation of charge of positive rays of hydrogen passing through hydrogen. Bartels' data, as given for gas pressures of 1 mm Hg, are summarized in Table III, in which n_+/n_0 is the ratio of the number of protons to the number of neutral hydrogen atoms in a beam of such positive rays. The third column in this table is the ratio $n_+/(n_+ + n_0)$. These data are the circles in Fig. 2; the (dotted) curve for δ in Fig. 2 is taken, by analogy with the variations of α -particle charges, to pass near the point (0.1, 0.9); its equation will be of the form $\delta = \delta(v)$. This particular variation of δ with v , while of course indicative of the nature of the true variation, cannot be expected to give much more than the order of magnitude of the true variation under the very different conditions which existed in the cloud chamber. Here the pressure was 8 cm (80 times as great as the pressure for which the data in Table III are quoted); the gas mixture was only $\frac{1}{4}$ hydrogen (the remainder H_2O vapor), and the whole was in the presence of moderately intense γ -radiation. We shall see shortly how this variation of δ with v must be replaced by another law for the present experiment.

Continuing with the case (for simplicity) for

TABLE III. Bartels' data for n_+/n_0 as a function of v at 1 mm Hg pressure.

v	0.010	0.035	0.060
n_+/n_0	0.16	0.61	1.30
δ	0.14	0.38	0.57

¹⁰ H. Bartels, Ann. d. Physik 13, 373 (1932).

FIG. 2. Values of δ for various values of the kinetic energy v .

which $\psi = 0$ and for which τ is proportional to r :

$$\tau = C_3 r, \quad (20)$$

where $C_3 = 0.0713$ (this is appropriate to the example for $H = 3520$ and a stopping power of chamber gases of 0.075 that of air, which we have used in Section IV), we have

$$\rho = [g(r)]^{\frac{1}{2}} = 41 r^{\frac{1}{2}} \delta(v); \quad (21)$$

$C_2 = 1687$ from Eq. (16). Substituting this into Eqs. (7) and integrating, we get the path which a proton follows under these conditions. If from the path computed using the dotted-curve variation of δ with v (Fig. 2) we compute for several values of $(r=L)$ the values of R^2 , we find that for $2.5 \leq L \leq 4.0$, $R = 16.5 \pm 0.5$. But the values of R which were found in the present experiment were in the neighborhood of 12 cm. Since the values of R calculated from Eq. (21) are largely independent of C_3 , this means that the true variation of δ with v is not the one given by Table III for 1 mm Hg pressure.

The true variation of δ with v must obviously be such that at very small values of v , δ is appreciably greater than is indicated in Table III. Furthermore, the true variation of δ must be such that the values of R calculated from Eq. (21) by means of it, are somewhat smaller than 16.5 cm. An approximation to what must be the true variation of δ with v is given by the solid curve of Fig. 2. With this law for $\delta(v)$ in Eq. (21), the calculated values of R in the neighborhood of 13 cm, are more nearly equal to the observed values of about 12 cm. In view of the necessary experimental uncertainties in the observed values of R , this variation of $\delta(v)$ is probably sufficiently near to the true variation of $\delta(v)$ for the purposes of the present experiment. The values of τ^2/R^2 got

by this $\delta(v)$ are given in Table IV. Obviously we can take $\tau^2 = (1.11 \pm 0.09)R^2$ provided $2.5 \leq L \leq 4.0$. Allowing for deviations from Eq. (20) as was done in Section IV, we may take

$$\tau^2 = (1.15 \pm 0.09)R^2, \quad \left. \begin{array}{l} 2.5 \leq L \leq 4.0. \end{array} \right\} \quad (22)$$

Converting the bounds on L to bounds on W and substituting in Eq. (17), we thus get as our final expression for W ,

$$W = (1.15 \pm 0.09) \times 10^{-14} K (IR \sec \psi)^2, \quad \left. \begin{array}{l} 0.175 \leq W \leq 0.275. \end{array} \right\} \quad (23)$$

This expression for W is largely independent of uncertainties in the energy-range relation for protons of low energy, as are Eqs. (11) and (14).

VI. ERROR THEORY

Let ϵ_j , $j=1$ through 7, be the absolute errors which might be present in single determinations of R , ψ , W , V , θ , V_B , and in the mean value $\bar{V}_B$ of n determinations of V_B , respectively. Then it can be shown from Eq. (23), that

$$\epsilon_3 \leq 2W(\Delta k/2k + \epsilon_1/R + \epsilon_2 \tan \psi), \quad (24)$$

where k is the coefficient 1.15. In a similar manner,

$$\epsilon_6 \leq \epsilon_4 + 5 \times 10^4 \epsilon_3 + 2\epsilon_5. \quad (25)$$

Hence, substituting Eq. (24) into Eq. (25) and noting that in the mean, some of the errors will be reduced by a factor of $n^{1/2}$ (according to the usual theory of errors), we have

$$\epsilon_7 \leq \epsilon_4 + 2W(\Delta k/k) + [4W(\epsilon_1/R + \epsilon_2 \tan \psi) + 5 \times 10^4 \epsilon_3]n^{-1/2}; \quad (26)$$

here $\bar{\psi}$ is the mean of the n values of ψ .

TABLE IV. τ^2/R^2 vs. L .

L	2.5	3.0	3.5	4.0
τ^2/R^2	1.02	1.07	1.14	1.20

VII. OBSERVATIONS AND CALCULATED RESULTS

A low pressure (8 cm Hg) cloud chamber was used to observe the protons in this experiment. It was of a depth of about 2.5 cm and an inside diameter of 11 cm. A mixture of H_2O vapor and deuterium was used in the chamber, and a γ -ray

source (of Ra Th) was placed as close as possible to the glass face of the chamber. The γ -ray activity of the Ra Th was equivalent to about 5 mg of Ra; this source was in a 4-mm thick lead container. In spite of the extreme propinquity of the γ -ray source to the cloud chamber, no great trouble was experienced with the fogging of the chamber by the γ -radiation. This was probably due to the low average atomic weight of the gas in the chamber. The light source was a carbon arc with photographic-flame carbons at 100 amperes d.c. The photographs were taken with a stereoscopic camera having $f:1.9$ lenses. The magnetic field at the region of the cloud chamber was measured by the inductive method, with inductances, resistances, and standard cells which had been calibrated at the National Bureau of Standards. Under the conditions which prevailed when the photographs were made, H was found to be 3762 gauss to well within one percent.

In all, about 2300 pictures were made, among which were found fifteen proton tracks which were suited to accurate measurement. The measurement of them was effected as follows:

(1) the tracks were projected on a screen by running the (well illuminated) film back through the camera and moving the screen until the stereoscopic images coincided;

(2) with a Brown and Sharp protractor, ψ and θ for each track was measured (see Table V)—the θ of a track was measured between the line from the center of the γ -ray source to the beginning of the track and the tangent to the track at its beginning point;

(3) the two ends and the mid-point N of each track were carefully recorded on small strips of paper with a very sharp pencil; this was done several times for each track;

(4) the chord (of length 2ω) P_2P_1' (see Fig. 1) was measured and its half-length recorded (see Table V);

(5) the mid-chord perpendicular (length h) from each track to its chord was measured and recorded (see Table V);

(6) from ω , d was calculated and recorded:

$$d = \omega \cos \psi.$$

This method, far from being inaccurate, affords a very reliable method of measuring such curved proton tracks. In fact, it is believed that

TABLE V. Data on proton tracks.

TRACK	ψ (°)	ϕ (°)	ω (cm)	h (cm)	d (cm)	V_B (Mev)	$\bar{V}_B - V_B$ (Mev)
1	37	137	1.52	0.086	1.215	2.220	0.05
2	30	173	1.25	0.062	1.082	2.197	0.03
3	17	63	1.59	0.096	1.520	2.153	0.02
4	18	134	1.76	0.125	1.681	2.145	0.02
5	35	97	1.61	0.089	1.319	2.166	0.00
6	2	47	1.35	0.069	1.350	2.117	0.05
7	3	38	1.51	0.097	1.540	2.198	0.03
8	42	159	1.33	0.064	0.975	2.261	0.09
9	0	76	1.45	0.086	1.450	2.167	0.00
10	27	131	1.30	0.062	1.160	2.201	0.03
11	12	97	2.00	0.174	1.955	2.198	0.03
12	23	86	2.03	0.151	1.869	2.201	0.03
13	9	0	1.79	0.118	1.769	2.118	0.05
14	7	94	1.20	0.057	1.191	2.123	0.05
15	21	93	1.62	0.112	1.512	2.144	0.03
Mean						2.174	0.03

the determinations of ω are accurate to within one percent and that those of h are accurate to within four percent.

In the seventh column of Table V are the values of V_B calculated by Eqs. (23) and (1) from the data in the earlier columns of the table.¹¹ In the eighth column are the deviations of the various values of V_B from the mean value $\bar{V}_B$. The mean value of V_B , as seen on the table, is 2.17 Mev. The average deviations of the several V_B 's from this mean is 0.03 Mev, corresponding to a "probable error" of 0.02 Mev.

The maximum error ϵ_7 to be expected in this value of $\bar{V}_B$ we get from Eq. (26) taking $\epsilon_1/R=0.05$, $\epsilon_2=0.025$, $\epsilon_3=0.005$, $\epsilon_4=0.02$, $\bar{\psi}=19^\circ$, $\Delta k=0.09$, and $n=15$. Substituting these values and taking $W \leq 0.2$ Mev, Eq. (26) gives $\epsilon_7=0.05$ Mev. Accordingly, we take as the final value given by this experiment

$$\bar{V}_B = 2.17 \pm 0.05 \text{ Mev.} \quad (27)$$

It is to be noted that the uncertainty 0.05 is a conservative value indicating the maximum error to be expected in this value of $\bar{V}_B$; as such, it may be expected to be of the order of two or three times as large as the "probable error" of $\bar{V}_B$. The precision of this experiment thus compares well

¹¹ With $V = 2.623 \pm 0.005$ Mev.

with the precisions of the previous determinations of the binding energy of the deuteron.

VIII. CONCLUSIONS

It will be noticed that the value $\bar{V}_B = 2.17$ Mev determined by this experiment is in excellent agreement with the value 2.17 found by Bethe³ on recalculation of the data of Chadwick, Feather, and Bretscher,² and with the value 2.18 got by Richardson and Emo⁴ from the disintegration of H^2 by the high energy γ -radiation from radioactive sodium. This concurrence of values tends to indicate that the presently accepted energy-range relation (used by Bethe and by Richardson and Emo) is correct for protons of energies in the neighborhood of 0.2 Mev. It also tends to fix the mass of the neutron at the presently accepted value of 1.0089 atomic mass units rather than at some higher value.¹² It also tends to indicate very definitely that the variation of protonic charge with energy as the proton traverses a gas such as the presently used H_2O-H_2 mixture, has $\delta(v)$ about twice as great (for $v < 0.1$) as the experimental data for pressures of 1 mm Hg would predict—note that this conclusion depends chiefly upon the agreement of calculated values of R with the observed values of R , and is only confirmed by the agreement of $\bar{V}_B$ with the other values of 2.17 and 2.18 Mev.*

Finally, we wish to acknowledge our indebtedness to Dr. H. A. Wilson for his having suggested to one of us that the binding energy of the deuteron be redetermined, and for his having provided us with every desired facility for carrying out the necessary experiment.

We are also grateful to Miss Eby Nell McElrath for assisting in the taking of the necessary photographs, and to Mr. Stanley Heaps for helping with the development of the negatives.

¹² See, e.g., E. Hudspeth and T. W. Bonner, Phys. Rev. **54**, 308 (1938).

* Note added in proof: In addition to the values of V_B quoted in the Introduction, G. Stetter and W. Jentschke (Zeits. f. Physik **110**, 214 (1938)) have more recently given the concordant value $V_B = 2.189 \pm 0.022$ Mev.

An Automatic Stabilizer Circuit

The circuit shown in Fig. 1 has been found to be a satisfactory modification of the circuit developed in 1934 by Wynn-Williams¹ for stabilizing the magnetic field produced by an electromagnet. It differs from that of Wynn-Williams in two respects: (1) it contains two separate photoelectric cells rather than one double-cathode photo-cell; and (2) it contains only parts which are readily obtainable in this country. Otherwise it is essentially the same as the circuit developed by Wynn-Williams. In fact, the binding posts *A* and *B* are meant to be connected to a relay system like that used by Wynn-Williams.

This circuit is such that when the light beam incident on the dividing mirror *M* is equally divided by *M*, both thyatron tubes are alight. The sensitivity of the circuit is controlled by the voltage dividers R_2 , R_2' , R_3 and R_3' , after R_1 and R_1' have been properly adjusted (by trial). R_1 and R_1' , when once "set," may be left unchanged for long periods of time.

This circuit, while developed specifically for magnetic-field stabilizing, may, of course, be used equally well for stabilizing other entities.

F. T. ROGERS, JR.

The Rice Institute,
Houston, Texas,
December 9, 1938.

¹ C. E. Wynn-Williams, Proc. Roy. Soc. A145, 250 (1934).

FIG. 1. Stabilizer circuit. The connections *A* and *B* are explained in the text. The parts are as follows:

- VT1 = G. E. type PJ-23 gas-filled photoelectric cell;
- VT2 = type 6N7 radio tube;
- VT3 = G. E. type FG-65 mercury-vapor thyatron tube;
- P = terminals for 110 v. 60-cycle a.c.;
- T1 = Thordarson type T-60R49 110 v to 560 (c. l.) v transformer; the filament winding is for 2.5 v.;
- T2 = Thordarson type T-61F85 transformer, using the 6.3 v terminals;
- R1, R1' = Electrad 10-watt 25,000-ohm resistor;
- R2, R2' = Yaxley type A-20M-P, wire-wound voltage divider;
- R3, R3' = Yaxley type A-10M-P, wire-wound voltage divider;
- R4 = 125-ohm (2-watt) resistor;
- R5 = 0.1-megohm (2-watt) resistors;
- R6 = 1.5-megohm (0.5-watt) resistors;
- R7 = 0.1-megohm (2-watt) resistors;
- C = 22-μ volt radio "C" batteries;
- D = 67 μ volts from radio "B" batteries.

Note on the Stopping Power of Hydrogen at Very Low Energies

An experiment has been reported recently¹ on the binding energy of the deuteron, as obtained from low pressure (H_2 — H_2O) cloud-chamber observations of proton tracks from photoelectrically disintegrated deuterium. It is desired here to record the information which this experiment yields about the stopping power of hydrogen for (protons of) very low energies.

It can be readily shown that the average stopping power of a mixture of n gases at a total pressure P (in cm Hg) is

$$S = \sum_{i=1}^n (p_i/76) S_i, \quad (1)$$

$$P = \sum_{i=1}^n p_i, \quad (2)$$

where S_i is the average stopping power of the i th gas at STP, and where p_i is the partial pressure of the i th gas. By "average" stopping power of a gas, we mean the average value of the stopping power over the path (in the gas) of a charged particle of a given initial energy. For convenience we refer these stopping powers to that of standard air, taken to be unity.

In the experiment mentioned above, $n=2$. Let $i=1, 2$ refer to deuterium and to heavy-water vapor, respectively; then $p_1 = P - p_2$. Also, $S_2 = S_1 + s$, where s is the average stopping power of $\frac{1}{2}\text{O}_2$. Thus Eqs. (1) and (2) yield

$$S_1 = (76S - p_2s)/P, \quad (3)$$

which is, of course, numerically equal to the average atomic

stopping power of deuterium relative to the average (atomic) stopping power of air. The experiment mentioned above used $P \leq 8.0$ cm Hg and $p_2 \leq 2.7$ cm Hg. Fifteen proton tracks (mean of the initial energies being 0.22 Mev) were observed, having an average length of 2.9 cm; these protons have a range of 0.228 cm in air;² therefore, $S = 0.228/2.9 = 0.079$. Substituting in Eq. (3), we thus find

$$S_1 \leq 0.35, \quad (4)$$

if we take $s = 1.2$.

This value 0.35 for S_1 for (proton) energies ≤ 0.22 Mev appears rather high; but in view of the fact that the stopping power of hydrogen increases³ from 0.20 at about 3 Mev to 0.25 at about 0.6 Mev, it is probably not much higher than would have been expected. This experimentally determined value of S_1 , through the (integral) relation between the energy and range of a particle and the stopping power of the gases through which the particle passes, constitutes a *bound* on the variation of the stopping power of hydrogen with energy for (proton) energies ≤ 0.22 Mev.

F. T. ROGERS, JR.

The Rice Institute,
 Houston, Texas,
 February 2, 1939.

¹ F. T. Rogers, Jr., and M. M. Rogers, Phys. Rev. 55, 263 (1939).

² H. A. Bethe, Phys. Rev. 53, 313 (1938).

³ M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. 9, 245 (1937)

**ON THE THEORY OF THE ELECTROSTATIC BETA-PARTICLE
ENERGY SPECTROGRAPH: II**

(A preliminary account of the results obtained in this research and given in this paper was presented at the Houston (Texas) Regional Meeting of The Texas Academy of Science, on 14 April 1939.)

ABSTRACT

This paper is a discussion of the effects, on the electrostatic energy spectrograph for charged particles, of the field distortions at the edges of the spectrograph plates and in between the plates. These effects are evaluated and shown not to be negligible in measurements of the highest precision. It is shown how, by properly locating the source and detector of particles, to eliminate these effects to within a first approximation.

The discussion is carried through ^{only} for _{one} form of spectrograph, but the modifications necessary to make it applicable to any other form of spectrograph are pointed out.

I. INTRODUCTION

For many investigations of charged particles¹ the electrostatic spectrograph may be used in the manner indicated by Fig. 1. The diagram is of a view parallel to the common axes of the cylindrical (radii r_1 and r_2) plates P_1 and P_2 between which an electrostatic field is maintained. The field between P_1 and P_2 , from E to D, is nearly radial (with center at O) in the two dimensions parallel to the plane of the diagram. S or S' is a source of particles; C or C' is a suitable detector of these particles.

In the ideal spectrograph the electrostatic field between P_1 and P_2 would be independent of ϕ from $\phi = 0$ to $\phi = \pi$ and would have the strength² $-km/re$, where e/m is the specific charge of a particle moving between P_1 and P_2 with position coordinates r and ϕ , and where k is a constant³ for fixed e/m , r_1 , r_2 , and potential difference (P) of P_1 and P_2 . Outside E and D the electric field

FIG. 1. Mid-plane cross-section plan of electrostatic spectrograph; see text. The x -direction is tangent at $\theta = \theta_0$ to the circle or radius r_0 .

would vanish. The distance l would be given⁴ by

$$l = (r_0/2\frac{1}{2})(\csc \sqrt{2}\frac{1}{2} - \cot \sqrt{2}\frac{1}{2}), \quad (1)$$

where $r_0 = \frac{1}{2}(r_1 + r_2)$, and many particles leaving S would be focussed to C . This ideal arrangement has been well discussed by Herzog⁵ and by Dempster.⁶

It is desired now to investigate the effects on the spectrograph of the situation (important in precise work) which occurs in practice, namely: the failure of the electrostatic field to vanish outside E and D and to be independent of ϕ in the interval $(0, \frac{1}{2})$. To make this investigation we consider (after the derivation in Section II of some necessary equations) first the case in which the electric field is ideal between the two regions A and B (see Fig. 1) and yet behaves in SA and BC as it actually should. Then in Section IV we consider the first-order effects produced by small deviations of the field in AB from the ideal. For simplicity in the derivations we shall restrict our considerations to the examination of particles

moving along SABC near the path of radius (in EABD)
 r_0 .

We now derive some useful equations for the departure of a particle's path from this "circular" path of radius r_0 . This departure will be caused by the addition to $-km/re$ of the small field strength $-F(\phi)m/e$.

II. EQUATIONS FOR DEPARTURE OF PATH FROM THE r_0 - CIRCULAR PATH

The equations of motion of a particle of constant mass m in a radial two-dimensional electrostatic field $-[k/r + F(\phi)]m/e$ reduce on division by m to

$$\left. \begin{aligned} \ddot{r} - r\dot{\phi}^2 + k/r + F(\phi) &= 0, \\ r^2\dot{\phi} &= N, \end{aligned} \right\} \quad (2)$$

where N is the (constant) angular momentum of the particle about the center of force. We shall consider only cases for which

$$|F(\phi)| \ll |k/r|. \quad (3)$$

To render Eqs. (2) independent of time, substitute the second of them into the first; we then have a differential equation defining r as a function of the variable ϕ :

$$r''/r - 2r'^2/r^5 - 1/r^3 + k/N^2 r + F(\phi)/N^2 = 0.$$

Then transform⁷ from r to u by the relation $r = 1/u$; this reduces to

$$(4)$$

$$u'' + u - k/N^2 u - F(\phi)/N^2 u^2 = 0. \quad (4)$$

Transforming again from u to ϵ by

$$u = k^{1/2}/N + \epsilon, \quad (5)$$

expanding expressions involving u by the binomial theorem and neglecting higher powers of ϵ , Eq. (4) becomes simply

$$\epsilon'' + 2\epsilon - F(\phi)/k = 0. \quad (6)$$

This differential equation defines an $\epsilon(\phi)$ which is a first approximation to the $\epsilon(\phi)$ of Eq. (5); this approximation cannot be expected to be any more nearly accurate than about $(100)\epsilon/(k^{1/2}/N)$ per cent. If $S(\phi)$ is a particular solution of Eq. (6), the general solution is of the form⁸

$$\epsilon(\phi) = C_1 \sin \sqrt{2}\phi + C_2 \cos \sqrt{2}\phi + S(\phi), \quad (7)$$

where C_1 and C_2 are arbitrary constants which are adjustable to the boundary conditions describing the motion of the particle at some given time.

A moderately convenient method of handling

this problem is, having expressed $F(\phi)$ as an $(n+1)$ -termed power series in ϕ about some constant value ϕ_0 of ϕ

$$F(\phi) = \sum_{i=0}^n a_i (\phi - \phi_0)^i, \quad (8)$$

to seek for $\epsilon(\phi)$ as another power series

$$\epsilon(\phi) = \sum_{j=0}^{\infty} b_j (\phi - \phi_0)^j. \quad (9)$$

Substituting Eq. (9) into Eqs. (6) and (8) and equating coefficients of powers of $(\phi - \phi_0)$ to zero, we find⁹ that this $\epsilon(\phi)$ may be a solution of Eqs. (6) and (8) only if

$$b_{j+2} = \frac{a_j/k - 2b_j}{(j+1)(j+2)}. \quad (10)$$

This formula of course allows two arbitrary constants, which may conveniently be taken to be b_0 and b_1 .

Other forms of solution, based on Eq. (7), of Eq. (6) may be available for other particular forms of $F(\phi)$ for which $S(\phi)$ may be determined simply.

The foregoing Eqs. (2)-(10) are the equations pertaining to the motion of a particle in a radial two-dimensional electrostatic field of strength $-[k/r + F(\phi)]m/e$, and $\epsilon(\phi)$ is a measure of the departure of the particle's path from a circular path having a radius $r_0 = N/k^{\frac{1}{2}}$.

For considerations of a particle's motion outside E and D, we examine the motion of a particle in an unidirectional field the strength of which varies with x only (see Fig. 1). Let this strength be $G(x)m/e$. Then the equations of motion of m in this field reduce to

$$\left. \begin{aligned} \ddot{x} &= 0, \\ \ddot{y} &= G(x). \end{aligned} \right\} \quad (11)$$

The solution of the first of these is just

$$x = C_3 t + C_4, \quad (12)$$

where C_3 and C_4 are arbitrary constants. Substituting this into the second of Eqs. (11) to eliminate x , we have for the y -coordinate of the par-

ticle the differential equation

$$\ddot{y} = G(C_3 t + C_4). \quad (13)$$

Provided $G(x)$ is known, y may then be obtained directly in terms of t by two successive integrations of Eq. (13). Elimination of t in this expression for y by means of Eq. (12) thus gives the equation of the path of the particle in the unidirectional field. Such an unidirectional field is often a good approximation (see Section III) to the true field along DC.

III. EDGE EFFECTS

As stated in the Introduction, we first consider the effects of the edges of the condenser plates at the regions EA and BD; see Fig. 1. We need to consider only the region near D, for that near E is just the same. We shall assume that the edges near D are constructed according to the heavy-lines plan in Fig. 2; this is only one possible arrangement at D, but since it is very easily approximated and likely to occur in practice, we shall consider only it. In Fig. 2 is also plotted the electrostatic field in the region BDC as got by the graphical method of curvilinear squares¹⁰. In Fig. 3 is a graph showing, as a function of the distance (s) from G along the curve FGC, the ratio of the true field strength to the ideal field strength $-k\eta/r_0 e$ in AB at $r=r_0$. C is the location the detector would have if the spectrograph were ideal, if $\phi = \pi/2$, and if we also assumed that $r_2 - r_1 = 0.0375r_0$. We shall trace the path of a particle moving initially along $r=r_0$ from A (with N and k

FIG. 2. Rough sketch of the field at the edges near D. F, on the r_0 -circle, is the point at which the field begins to depart appreciably from km/re. l is $0.351r_0$ by Eq. (1). The case for $r_2 - r_1 = 0.0375r_0$.

FIG. 3. $s=0$ corresponds to the point G. C is at $s=1=0.351r_0$. Note that to the right of G, $s \equiv x$. As in Fig. 2, l here is got by taking $\phi = \pi/2$ and $r_2 - r_1 = 0.0375r_0$ and by using Eq. (1).

such ¹¹that $N/k^{\frac{1}{2}} = r_0$) in such an apparatus.

It is to be noted that to within a very good degree of approximation, the field in BD near the path FG is very nearly radial even though its strength is decreasing. Therefore Eqs. (2)-(10) are valid for this region. Beyond D and near GC, note that the field is, to within quite a good approximation, unidirectional along the y-direction. In this region, therefore, we may apply Eqs. (11)-(13). Incidentally, it is obvious that these approximations cannot be expected to be valid unless

$$(r_2 - r_1) \ll r_0, \quad (14)$$

a condition which is generally satisfied if the spectrograph is designed for particles of moderately high energy. We carry out the necessary calculations in two parts, one for the region BD, another ofr the region DC.

A. In BD. In BD the field-change expression $F(\phi)/k$ which appears in Eq. (6) can well be represented as ¹²

$$\left. \begin{aligned} F(\phi)/k &= (\phi - \pi/2 + 0.025)^2 600/r_0, \\ (\pi/2 - 0.025) &\leq \phi \leq \pi/2, \end{aligned} \right\} \quad (15)$$

provided $|r_2 - r|$ and $|r_1 - r|$ are much greater than $|r_0 - r|$. Substituting this into Eq. (6), we note that a particular solution $S(\phi)$ can easily be got in the form

$$S(\phi) = d_0 + d_1(\phi - \pi/2 + 0.025)^2, \quad (16)$$

where d_0 and d_1 are suitable constants. Applying the boundary conditions $\epsilon = 0$ and $d\epsilon/d\phi = 0$ at $\phi = \pi/2 - 0.025$, the complete solution (Eqs. (7) and (16)) for this case can be got. This solution¹³ gives $\epsilon = -0.0001/r_0$ and $d\epsilon/d\phi = -0.0065/r_0$ at $\phi = \pi/2$. Introducing the new coordinate $\delta(\phi)$ defined by

$$r(\phi) = r_0 + \delta(\phi), \quad (17)$$

whence $\delta = 1/(k^{\frac{1}{2}}/N + \epsilon) - r_0$, we find that at $\phi = \pi/2$ $\delta = 0.0001r_0$ and¹⁴ $d\delta/d\phi = 0.0065r_0$.

B. In DC. In DC the field expression $G(x)m/e$ may adequately be represented by

$$G(x) = (0.625k/r_0)e^{-20x} \quad (18)$$

provided $|y| \ll |r_2 - r_0|$. Substituting this into Eq. (13), we have

$$\ddot{y} = (0.625k/r_0)e^{-20C_4}e^{-20C_3t}.$$

Two successive integrations of this yield

$$y = (0.625k/400C_3^2r_0)e^{-20x} + (C_5/C_3)(x - C_4) + C_6,$$

where C_5 and C_6 are arbitrary (with respect to these two integration processes) constants. The constants C_3 , C_4 , C_5 , and C_6 are obviously to be interpreted as follows:

$$\left. \begin{aligned} C_3 &= \dot{x}; \\ C_4 &= x|_{t=0}; \\ C_5 &= \dot{y}|_{t=0} + (0.625k/20C_3r_0)e^{-20C_4}; \\ C_6 &= y|_{t=0} - (0.625k/400C_3^2r_0)e^{-20C_4}. \end{aligned} \right\} \quad (19)$$

In our case, now, we take $C_4 = 0$. Also, ¹¹ $C_3 \approx k^{\frac{1}{2}}$. Also, $\dot{y}|_{t=0} \approx k^{\frac{1}{2}}(ds/ds)|_{t=0} = (k^{\frac{1}{2}}/r_0)(ds/d\phi)_{\phi=\pi/2}$,

$= 0.0065k^{\frac{1}{2}}$ in this case by the computations in Part A. Therefore for this case,

$$C_5 = 0.0065k^{\frac{1}{2}} + 0.0031k^{\frac{1}{2}}/r_0.$$

Also, $y|_{t=0} = \delta|_{\phi=\pi/2} = 0.0001r_0$ for this case by the computations in Part A; hence

$$C_6 = 0.0001r_0 - 0.0016/r_0.$$

Eq. (19) for the present case is thus

$$y = (0.0016/r_0)e^{-20x} + (0.0065 + 0.0031/r_0)x + 0.0001r_0 - 0.0016/r_0. \quad (21)$$

Since C is at a distance¹⁵ of $x = 1 \approx 0.35r_0$ from G, y at C must have the value, according to Eq. (21),

$$y_C = (0.0016/r_0)e^{-7r_0} + 0.0024r_0 + 0.0011 - 0.0016/r_0. \quad (22)$$

For reasonable values of r_0 this Eq.(22) gives fairly small values for y_C ; e. g., for $r_0 = 16$ cm., $y_C \approx 0.04$ cm., a value which is perhaps not as large as might have been expected. For such reasonable

values, say for $r_0 \gg 10$ cm., of r_0 , Eq. (22) has the very good approximation

$$y_C \approx 0.0011 + 0.0024r_0. \quad (23)$$

The quantity y_C is the amount by which the detector of particles must be shifted (to a new location C' , as indicated in Fig. 1) perpendicularly to the Gx direction at $x=1$ in order that particles traversing the "central path" of radius r_0 in AB may be detected at their focal point. For precise measurements with such an apparatus as this, y_C cannot in general be neglected.

It must be remembered that these particular results are limited in strictest applications only to the spectrograph having the "geometry" indicated in Figs. 1 and 2. For other arrangements, other somewhat similar equations must be derived in the same manner as we have done and must be used instead of Eqs. (22) and (23).

IV. CORRECTION FOR "NON-UNIFORMITY" OF ELECTRO- STATIC FIELD

Consider a particle moving between A and B with a velocity which is very nearly v_0 , along a path which deviates from the r_0 -path only slightly under the influence of the electrostatic field $-[k/r + F(\phi)]m/e$, where (as always) $|F(\phi)| \ll |k/r|$. Suppose that at A the particle had its $\xi = 0$ and $d\xi/d\phi = 0$.

Now, having represented $F(\phi)$ as in Eq. (8) and having got the solution $\epsilon(\phi)$ of Eq. (6) as indicated in Eqs. (9)-(10), we may calculate ϵ and $d\epsilon/d\phi$ for the particle in the region B. These values of ϵ and $d\epsilon/d\phi$ at B may then be taken as the boundary conditions to be used as in Section III,A, to get C_1 and C_2 . The remainder of the true path of the particle, from B to its C' , may then be calculated just as has been done in Section III for the case in which $\epsilon = d\epsilon/d\phi = 0$ at the B region. We may thus get another y_C which will be the resultant of

the effects of the "non-uniformities" of the field in AB (i. e., departures by $-F(\phi)m/e$ of the field from the ideal $-km/re$) and of the edge-effects in BC'.

In this manner, both non-ideal effects (those caused by the stray electric fields at the plate edges and those caused by variations in the field in AB from the ideal) can be allowed for to the extent of a first approximation. If the effects are relatively fairly small, these corrections will almost entirely eliminate them.

V. PRACTICAL ASPECTS

In applying the foregoing analysis to any spectrograph, it should be first ascertained that the physical arrangement or "geometry" of the spectrograph is the same as that indicated in Figs. 1 and 2. If the plate-shapes and plate-separations are not the same as those indicated in Figs. 1 and 2, the derivations in Section III must be carried out again using the correct stray-field formulae in place of Eqs. (15) and (18).

Second, $F(\phi)$ for the region from A to B must be determined. In view of the fact that the "corrections" we have considered are only of the first-order, it may be sufficient in many cases to get $F(\phi)$ simply by the measured departures of $r_2 - r_1$ from constancy. Thus, if the measurements indicate that we may express the separation of the spectrograph plates as a function of ϕ by an equation of the form

$$r_2 - r_1 = \overline{r_2 - r_1} + q(\phi), \quad (24)$$

where $\overline{r_2 - r_1}$ is the mean value (averaged over the whole range from 0 to $\pi/2$ or to $\frac{\pi}{2}$ of ϕ -values) and where $|q(\phi)| \ll |r_2 - r_1|$, one might well take¹⁶

$$F(\phi) \approx Pq(\phi)/(\overline{r_2 - r_1})^2. \quad (25)$$

This procedure, while only approximate, should be adequately valid and suited to the computation of first-order effects.

Third, the path of a particle which would follow the r_0 -path in AB if $q(\phi)$ were identically zero must be found, for precise calculations based on observations with a spectrograph are generally made for particles focussed at the focal point of the r_0 -path. Thus, for particles traversing a path that would be an r_0 -path if $q(\phi)$ were zero we should have $mv_0^2/r_0 = X_0 e$, where X_0 is the field strength k/r_0 at $r = r_0$; hence

$$mv_0^2/e = X_0 r_0, \quad (26)$$

in which the property mv_0^2/e of the particles is given in terms of the observable r_0 and X_0 . The

determination of the complete path must, then, be done in three steps:

- (1) the location S' of the source must be so made that a particle of velocity v_0 at A (see Fig. 1) is on the r_0 -circular path¹⁷ at A; the necessary calculations are obviously just the converse of the calculations given in Section III;
- (2) the determination of the path in AB, as done in Section IV, together with the values of ϵ and $d\epsilon/d\phi$ at B; and
- (3) the determination of the remainder of the path in BC⁰ and the location of C', as directed in Sections III and IV.

A detector placed at the O^0 so located would then detect the particles which would have, had $q(\phi)$ been identically zero and had the edge-effects at E and D been "ideal", appeared at C; for these particles Eq. (26) would hold if r_0 were taken to be the mean value (over all ϕ) of the radial distance to the middle of the space separating the plates P_1 and P_2 . In this manner the edge effects

at E and D and the "non-uniformity" effects in AB may be, to a first approximation, eliminated by simply compensating for them by the proper location of S' and C'.

APPENDIX I: DERIVATION OF EQ. (6)

Beginning with the equations

$$\left. \begin{aligned} \ddot{r} - r\dot{\phi}^2 + k/r + F(\phi) &= 0 \\ \text{and} \quad r^2\dot{\phi} &= N, \end{aligned} \right\} \quad (A)$$

substitute the second into the first; we get

$$\ddot{r} - rN^2/r^4 + k/r + F(\phi) = 0. \quad (B)$$

Then since $dr/dt = (dr/d\phi)(d\phi/dt)$, we have $dr/dt = (N/r^2)(dr/d\phi)$; similarly $d^2r/dt^2 = (N^2/r^4)(d^2r/d\phi^2) - (2N^2/r^5)(dr/d\phi)^2$; thus Eq. (B) becomes

$$(N^2/r^4)(d^2r/d\phi^2) - (2N^2/r^5)(dr/d\phi)^2 - N^2/r^3 + k/r + F(\phi) = 0,$$

which we shall call Eq. (C).

Now, letting $r = u^{-1}$, since $dr/d\phi = -(du/d\phi)/u^2$ and $d^2r/d\phi^2 = (2u^3)(du/d\phi)^2 - (1/u^2)(d^2u/d\phi^2)$, the substitution of this into Eq. (C) yields

$$u'' + u = k/N^2u + F(\phi)/N^2u^2, \quad (D)$$

where $u'' \equiv d^2u/d\phi^2$.

Now let

$$u(\phi) = k^{\frac{1}{2}}/N + \epsilon(o); \quad (E)$$

since $1/u = 1/(k^{\frac{1}{2}}/N + \epsilon) = (k^{\frac{1}{2}}/N + \epsilon)^{-1}$

$$= N/k^{\frac{1}{2}} - \epsilon N^2/k + \epsilon^2 N^3/k^{3/2} - \dots,$$

and $1/u^2 = (k^{\frac{1}{2}}/N + \epsilon)^{-2}$

$$= N^2/k - 2\epsilon N^3/k^{3/2} + 3\epsilon^2 N^4/k^2 - \dots,$$

we have instead of Eq. (D)

$$\epsilon'' + 2\epsilon = (N/k^{\frac{1}{2}})(\epsilon^2 - \epsilon^3 N/k^{\frac{1}{2}} + \epsilon^4 N^2/k - \dots)$$

$$+ F/k - (FN/k^{3/2})(2\epsilon - 3\epsilon^2 N/k^{\frac{1}{2}} + \dots). \quad (F)$$

Now we consider only the cases for which

$$|F(\phi)| < |k/r|$$

and seek for an $\epsilon(\phi)$ such that

$$|\epsilon(\phi)| < |k^{\frac{1}{2}}/N|.$$

We shall thus require that $2\epsilon/(k^{\frac{1}{2}}/N) \ll 1$, whence $\epsilon^2/(k^{\frac{1}{2}}/N) \ll 2\epsilon$. Eq. (F) then reduces to the approximation which is Eq. (6):

$$\epsilon'' + 2\epsilon - F(\phi)/k = 0. \quad (G)$$

APPENDIX II: DERIVATION OF EQ. (10)

Let
$$F(\phi) = \sum_{i=0}^n a_i (\phi - \phi_0)^i, \quad (H)$$

and substitute it into Eq. (6). We then seek a solution

$$e(\phi) = \sum_{j=0}^{\infty} b_j (\phi - \phi_0)^j \quad (I)$$

of the resulting equation.

Since
$$de/d\phi = \sum_{j=1}^{\infty} j b_j (\phi - \phi_0)^{j-1}$$

and since
$$d^2e/d\phi^2 = \sum_{j=2}^{\infty} j(j-1) b_j (\phi - \phi_0)^{j-2},$$

we shall therefore need the constants b_j chosen so that

$$\sum_{j=2}^{\infty} j(j-1) b_j (\phi - \phi_0)^{j-2} + 2 \sum_{j=0}^{\infty} b_j (\phi - \phi_0)^j = (1/k) \sum_{i=0}^n a_i (\phi - \phi_0)^i.$$

This must hold for all values of $(\phi - \phi_0)$, so we must have

$$b_{j+2} = \frac{a_j/k - 2b_j}{(j+1)(j+2)}, \quad (J)$$

which is Eq. (10).

Note that for $j > n$, all a_i are zero. Hence all b_m for which $m \geq n+3$ will be given by an Eq. (J) having no a_i -term.

APPENDIX III. COMPUTATIONS FOR SECTION III

Using the expression for $F(\phi)/k$ as given by Eq. (15) and substituting it into Eq. (6), we seek a solution of the form of Eq. (16). It is obviously

$$S(\phi) = -300/r_0 + 300(\phi - \pi/2 + 0.025)^2/r_0. \quad (K)$$

Substituting this into Eq. (7), we have

$$\epsilon(\phi) = C_1 \sin \sqrt{2}\phi + C_2 \cos \sqrt{2}\phi - 300/r_0 + 300(\phi - \pi/2 + 0.025)^2/r_0,$$

which we call Eq. (L).

To determine C_1 and C_2 , apply the boundary conditions that $\epsilon = d\epsilon/d\phi = 0$ at $\phi = \pi/2 - 0.025 = (1/2^\pi) \times 125^\circ 15' 12.2''$. Using seven-place tables, we find that $C_1 = 244.9822$ and $C_2 = -173.1581/r_0$. Substituting these into Eq. (L) and carrying out the computations for $\phi = \pi/2 = (1/2^\pi) \times 127^\circ 16' 49.5''$, we find that at this ϕ , $\epsilon = -0.0001/r_0$ and $d\epsilon/d\phi = -0.0065/r_0$.

Thus at $\phi = \pi/2$, $\delta = 0.0001r_0$ and $d\delta/d\phi = 0.0065r_0$.

REFERENCES

- ¹See, e. g., S. K. Allison, L. S. Skaggs, and N. M. Smith, Jr., Phys. Rev., 54, 171 (1938), and A. J. Dempster, Phys. Rev., 51, 67 (1937).
- ²The negative sign indicates that the action of the field is directed toward 0.
- ³It can readily be shown that $k = P/\ln(r_2/r_1)$, where P is the potential difference of P_1 and P_2 .
- ⁴See, e. g., A. J. Dempster, Phys. Rev., 51, 67 (1937).
- ⁵R. Herzog, Ann. d. Phys., 33-5, 89 (1938), and other papers referred to in this one.
- ⁶See the first part of the paper in reference 4.
- ⁷See Appendix I.
- ⁸See L. R. Ford, Differential Equations (McGraw-Hill, 1933), p. 65.
- ⁹See Appendix II.

¹⁰See, e. g., A. D. Moore, Fundamentals of Electrical Design (McGraw-Hill, 1927), Chapters IV-VII.

¹¹For such an r_0 -orbit in AB, the speed v_0 of the particle must be such that $mv_0^2/r_0 = (ke/r_0)e$, whence $v_0 = k^{1/2}$ and $N = v_0 r_0 = r_0 k^{1/2}$.

¹²This expression vanishes at $s = -0.025r_0$, and at $s = 0$ has the value $-0.375/r_0$.

¹³See Appendix III. The computations were made with seven-place tables.

¹⁴Since $|E| \ll |U|$, $|S|$ must be $\ll r_0$; thus $ds/d\phi = -r_0^2 (d\epsilon/d\phi)$, the approximation being good to about 2.00 %/r₀ per cent.

¹⁵See Fig. 2.

¹⁶Since a change q in $r_2 - r_1$ produces a change of approximately $-Pqr_1/r(r_2 - r_1)^2$ in the field $P/r \ln(r_2/r_1)$, we have Eq. (25). See ref. 3.

¹⁷Cf. references 3 and 11.

NUMBER ELEVEN

A BETA-PARTICLE ELECTROSTATIC SPECTROGRAPH

A BETA-PARTICLE ELECTROSTATIC SPECTROGRAPH

(Preliminary Report)

(ABSTRACT)

An account is given of the design and construction of an electrostatic spectrograph (using counter recording) suited to use for observations on particles with energies as great as 500,000 electron-volts. A discussion of the performance and operating details is given. With appropriate care, the instrument can be readily adapted to precision observations.

I. INTRODUCTION

For making many types of investigations of the properties of charged particles, such as momentum, the usual method has been to use the magnetic spectrograph in some form or other¹. The distinguishing feature of the magnetic spectrograph is its magnetic field, which is usually made as nearly uniform as possible. In this magnetic field, the charged particles under investigation traverse paths (which can be determined by suitable observations) which are partly characterized (in a known manner) by the properties being investigated².

For making other types of investigations, however, some form of the electrostatic spectrograph is preferable³. The distinguishing feature of the electrostatic spectrograph is its radial electrostatic field; this field may be radial in two³ or three⁴ dimensions. Such a radial electrostatic field has been shown^{4,5} to possess, in common with (but in a somewhat different manner from that of) an uniform magnetic field⁶, focus-

sing properties -- i. e., if like charged particles of the same speed emanate from a suitably located fixed source (S) (see Fig. 1) in not-too-widely divergent directions, many of their paths will (because of the influence of the electrostatic field) converge in the neighborhood of some other fixed point (C). In this electrostatic field the paths of the particles (which can be determined by suitable observations) are partly characterized (in a known manner⁷) by the properties of the particles being investigated.

In the following sections of this paper we present a detailed account of the design, construction, and operation of an electrostatic spectrograph employing an electrostatic field which is radial in two dimensions. This instrument has been found to be well suited to precision observations on particles having energies as high as $5 \cdot 10^5$ electron-volts.

II. DESIGN AND CONSTRUCTION OF THE SPECTROGRAPH

Since it was desired to construct without excessive difficulty an electrostatic spectrograph capable of precision performance, a two-dimensional radial field was used, following the practice of Allison, Skaggs, and Smith³. This radial field was to be produced between the plates (radii r_1 and r_2) of a cylindrical condenser; see Fig. 1. The mean radius $r_0 = \frac{1}{2}(r_1 + r_2)$ was taken to be about 16 cm., chiefly because only a 15" lathe was available for shaping the condenser plates. A potential difference (P) of about 207 es. u. was available; since it was planned ultimately to use this instrument with particles (electrons) having energies of about 10^6 electron-volts, r_1 and r_2 were taken⁸ to be such that $r_2/r_1 \cong 1.038$.

The condenser plates were made of two pieces of sheet brass, $\frac{1}{8}$ " x 3" x 12" each. Each of these was clamped in a large vise and bent roughly into the desired shape (as indicated by wooden templates for r_0 and for $r_0 - \frac{1}{8}$ ") with a 36" pipe wrench. It

Fig. 1. Mid-plane cross-section^{plan} of electrostatic spectrometer; see text. The x-direction is tangent at $\phi = \phi_0$ to the circle of radius r_0 .

It was found that this bending to shape could readily be done if the brass plates had been previously annealed by heating to a cherry-red color and quenching in cold water. It was not difficult to bend the two plates so that the outer face of the inner plate departed by not more than 2 mm. from the inner face of the outer plate.

To the "backs" of the two roughly shaped plates were then brazed a number of blocks of brass, $3/4" \times 3/4" \times 3"$ in size. Three of these blocks were each of attached to the plates, their long dimensions being parallel to the axis of the cylinders of which the plates were segments; this is illustrated in Fig. 2. After this, the edges of the plates were filed to near flatness and $1/4"$ -diameter holes were drilled through the back-blocks along their long dimensions, and then were countersunk. By flat-head iron bolts recessed well into the countersinks, the two plates were firmly bolted to the bed of a milling machine. By taking light "cuts" with a 3"-wide cutter, the plates' edges were planed down so that the plates

FIG. 1.

had the same width (within 0.001"), which was about 7.16 cm. Since the separation of the plates was to be about 6 mm., this width was adequate to remove sensibly all "edge effects" at the center of the plates. It should be explained that the taking of light cuts in these milling operations was necessitated by the use of only $\frac{1}{4}$ "-diameter bolts to hold the plates to the bed of the milling machine. After this milling operation, the $\frac{1}{4}$ " holes in the back-blocks were enlarged to $\frac{3}{8}$ " diameter. It was necessary to dull the cutting edges of the $\frac{3}{8}$ " drill in order to prevent them from "grabbing" excessively.

The milled plates were then mounted by $\frac{3}{8}$ " bolts directly on the face plate of a 15" lathe in preparation for final shaping. An angle which was accurately 90° was laid off of the center of the face plate and extended to the edges of the plate; in this manner the milled condenser plates were marked so that, after their ends had been sawed off and carefully dressed to these marks by filing, they intercepted an angle $\frac{\pi}{2}$ of nearly 90° at their axis. Final measurements of $\frac{\pi}{2}$ gave for it the

value $89^{\circ} 51^{\prime} \pm 05'$.

The final shaping process was, of course, that of turning the faces of the condenser plates to cylindrical shapes. Since $3/8"$ bolts formed only a quite light mounting for the plates on the face plate, it was necessary to take only light cuts on the surfaces of the plates. An ordinary lathe tool mounted in an extended holder was used for the entire shaping operation. For the first cutting, the lathe was run at a moderate speed; for the final cutting, which was done with $0.0005"$ cuts, the lathe was turned by hand. After the final cuts on the plates, the plates were lightly polished with an oiled bit of very fine emery cloth. The inner surface of the outer plate was done first; then the outer face of the inner plate was finished; in each operation, the other plate was removed. No trouble of any kind was experienced as long as the plates were tightly bolted to the face plate, as long as the tool was kept very sharp, and as long as sufficiently slow cutting speeds were used.

The measurements of the plates' separation was done with the finished plates both clamped to the face plate. A standard 3/16"-diameter ball-bearing was soldered to the end of a brass rod 6" long and 1/8" in diameter. This piece was mounted in the tool holder on the lathe, the ball-bearing being between the condenser plates, and was insulated electrically from the lathe body. A 4½-volt battery and a voltmeter in series were connected to the lathe body and to the 1/8" rod; in this manner it was easy to ascertain just when the ball-bearing was in contact with either of the two condenser plates. Since the tool holder was mounted on a slide operated by a screw calibrated in 1/1000th's of an inch, it was thus possible to find the difference between the ball-bearing's diameter and the plates' separation. In this manner it was found that $r_2 - r_1 = 0.5965 \pm 0.0045$ cm., the tolerance 0.0045 being the maximum deviation of $r_2 - r_1$ from 0.5965 cm.; except for only very small region, the tolerance was 0.0025 cm.

From an iron center-piece which was mounted on the face plate and turned as accurately as the lathe would allow, it was found that over the width (7.16 cm.) of the condenser plates, the radial tapering caused by misalignment of the lathe amounted to less than 0.0001". From the center of this piece, measurements of r_2 and r_1 were made which gave $r_0 = 16.05 \pm 0.007$ cm.; the larger tolerance expressed for this value of r_0 than for the above value of $r_2 - r_1$ results only from less precise measurements, not from any structural uncertainties. Where ever possible, all measurements were checked with standard Brown and Sharp gauges.

The finished plates were mounted with brass bolts to an insulating framework, as indicated in Fig. 2. The main piece of insulator, the base, was 7/8" thick; it was of a commercially available product consisting of many layers of cloth bound together with an impregnating binding of a phenolic plastic. The brass "bolts" used for mounting the plates ~~for~~ the insulators were of 3/8" round brass

rod, threaded on each end, and 0.005" under-size. Holes were drilled part-way through the base in the same pattern as the holes for mounting in the face plate (the face plate was actually used as a pattern for drilling these holes) and were threaded to take the bolts. 1/8"-thick washers of hard rubber (see Fig. 2) were used in several places for spacing. The tops of the bolts were held at the proper separations by short bars of a paper and ~~methanolic~~^{insulator} plastic. All holes for passage of the bolts were made several thousandths of an inch over-size. Standard hex nuts were used on the upper ends of the bolts to clamp the assembly in place. Fig. 2 shows the details of the assembly adequately.

After the plates were assembled on the insulating base, three brass rods two inches long and 0.5965 ± 0.0015 cm. in diameter were turned on a bench lathe. Using these rods as spacers, the two condenser plates were adjusted on the insulating framework to have the correct separation. It was to al-

low for this small adjustment that the brass bolts were made under-size and the holes through which they passed were made over-size. Final checking of the separation, made with one of the brass gauges and with calibrated shims, several days later showed that no observed changes had taken place and that the separation departed nowhere more than 0.005 cm. from the mean value of the separation. It is our opinion, based on the experience gained with this particular instrument, that this maximum departure of 0.005 cm. from the mean separation could be halved without excessive labor simply by the use of more care in the processes of making the condenser plates and mounting them.

Since the present spectrograph was to be used in an investigation of β -particles from a radium active deposit, the source holder (Fig. 2 at S) was a needle holder (Central Scientific Company's catalog number 7005). This needle holder was cut off at a suitable length and its lower $\frac{1}{8}$ " was threaded. It was decided to make the distance from the source

to the entering edges of the plates equal to the distance from the exit edges to the detector mechanism at C; these distances come out to be 5.62 cm. each⁹. At a distance of 5.62 cm., then, from the entering edges a hole was drilled in the insulating base and tapped out to take the needle holder. The top of the needle holder was 3.26 cm. from the base, allowing the insertion of a wire source $\frac{1}{8}$ " long.

The detector was chosen to be a simple Geiger counter¹⁰. A slit about $\frac{1}{2}$ mm. wide and $\frac{1}{8}$ " long was arranged along an element of the cylindrical wall of the counter, and was covered with sheet copper 0.001" thick. The counter was then mounted on the base plate in a manner similar to the source holder, the slit being at a distance of 5.62 cm. from the exit edges of the condenser plates. In this arrangement, only the central regions of the electrostatic field are used, thus eliminating the plate-edge effects on the field traversed by the particles.

The whole assembly, plates, counter, source

support, and insulating framework, was placed in a large air-tight box made of $\frac{1}{4}$ "-thick sheet iron, one end of which was removable. In use, the source was put in its holder, and the box was closed and vacuum-sealed with a rosin-and-beeswax mixture while a vacuum was being obtained. A Cenco Megavac pump was used to obtain this vacuum. For the β -particle investigations for which this instrument was designed, a vacuum of 10^{-3} mm. Hg was sufficient and was obtainable with only an hour's pumping. The leads for the necessary high-voltages were brought through the walls of the vacuum chamber in heavy hard rubber insulators. Since the electrical center of the high-potential circuit was grounded, as was the vacuum chamber, each of the lead-in insulators needed to withstand only one-half of the potential difference actually appearing across the condenser plates.

AUXILIARY APPARATUS

The auxiliary apparatus used with the spectrograph plates just described consists of: (1) a Geiger-Mueller counter, the associated amplifier and mechanical recorder; and (2) a source of high steady potential difference.

The amplifier and recorder were of the simplest type, which has been adequately described elsewhere¹¹. The amplifier output could, however, be fed either into the input of the mechanical recorder, or into headphones for auditory recording.

The high-potential circuit, a diagram of which is given in Fig. 2, was quite simple and dependable. The ^{motor-}generator set was run by a 110-volt direct-current line from a bank of Edison storage cells. The generator alternating current output was of a frequency of 500 cycles per second. The transformer for converting the

590-cycle a. c. to a high potential was a 3 k. v. a. transformer rated at 100,000 volts at 60 cycles. In use, the output voltage of this transformer was controlled by a finely adjustable resistance network in the primary circuit. The filter and rectifier system were of the simplest kind.

The rectifier tubes were "kenotrons" manufactured by the Machlett Laboratories and rated to operate at 75,000 volts each. Their filaments were heated by "home-made" transformers which were insulated to withstand reliably (by calculation!) 30,000 volts each -- these filament transformers have, to date, not been tested to beyond 12,500 volts each. The filter network was simply a pair of 0.2 microfarad condensers in series, these being then connected across the rectified transformer output. These condensers, supplied from stock by the Cornell-Dubilier corporation, were rated to withstand 60,000 volts each as a d. c. working voltage.

The output voltage of this circuit was measured by a ~~calibrated~~ galvanometer in the electrical center of a bank of 60 megohms of wire-wound resistance. The 60-megohm resistance was connected directly across the output of the high potential circuit. This resistance consisted of sixty one-megohm resistors connected in series and enclosed in corona-reducing metal cases. This unit was supplied by the Challenor Company and is a stock item. For precision work, this resistance was always checked for its accurate value with a large high-precision Leeds and Northrup Wheatstone's bridge.

It will be noted that the electrical centers of the transformer secondary, of the filter condensers, and of the voltage-measuring resistors were connected to ground.

No stabilizer was provided for this high potential. It was found that the very rare small quick fluctuations in the input of the motor-generator were well "smoothed out" by the ~~motor~~ ~~generator~~

energy of the rotating armature. Larger and/or longer-period fluctuations were compensated for by hand adjustments of the control resistor.

PREFORMANCE

The assembly and testing of the entire electrostatic spectrograph circuit was effected in several steps. First, the G-M counter was installed and tested. The chief difficulty encountered here was an electrical instability of sorts: the operation of the counter was not reliable. Continued adjustments of the R and C (resistance and capacity) values in the circuits eventually brought about satisfactory counting. It was found that the resolving time of the entire circuit in good operation was of the order of 0.09 seconds.

Second, the high voltage circuit and insulators were tested. In view of the fact that first experiments scheduled to be done with this electrostatic spectrograph did not require a total potential difference of more than 25,000 volts, the testing (to date) has not been carried

beyond this voltage. No difficulties of any kind were experienced with the circuit in any detail. The stability of the output voltage was all that could be desired.

Third, the vacuum inside the box enclosing the deflecting plates had to be obtained and maintained. The greatest distance between conductors at different potentials (one-half the potential difference of the deflecting plates) was about 30 cm. Since one-half the potential across the deflecting plates was at first to be not over 12,500 volts, a pressure of 0.001 mm. Hg was sufficiently low to provide adequate vacuum insulation. A great deal of difficulty¹² was encountered with the attaining of this vacuum however -- it was found to be very difficult to clamp the top on the top of the iron box (using a single 1/16" rubber gasket), to seal the joint adequately against leaks, and to pump the pressure down to 10^{-3} mm. Hg, all before the source of β -particles being studied decayed. It was

ultimately found best to liberally paint the whole joint with hot, pure beeswax. In this manner, it was found possible quite often to get well below a pressure of 10^{-3} mm. Hg in much less than an hour's time.

It was hoped to be possible to include in the present edition of this paper some typical observations and data got with the instrument just described. At the moment of writing, however, such data are not available. It is now expected that they will be available shortly.

CONCLUSION

The electrostatic spectrograph described in this paper is, by all indications, to be a highly satisfactory instrument for particle energies as high as 500,000 electron-volts. Its component parts operate separately, excellently, and it is expected that the ensemble of them all will do as well.

For adapting this instrument to work of the highest precision, it will be necessary to make the allowances for stray fields and for "non-uniformities" of field, which are discussed in the preceding paper.¹³

The development and entire construction of this electrostatic spectrograph would not have been possible without the very generous provision through Dr. H. A. Wilson for every desired facility for doing the necessary shop work, and without the necessary equipment for

the high-voltage source, which was also provided through Dr. Wilson. Mr. A. W. McReynolds, in highly effective collaboration, provided indispensable help. Finally, Mr. J. D. van der Henst made many fruitful technical suggestions and gave much reliable technical advice in connection with the necessary shop work.

REFERENCES

- ¹See, e. g., Rutherford, Chadwick, and Ellis, Radiations from Radioactive Substances (Cambridge Press, 1930), pp. 341-347, for typical investigations.
- ²For ion paths in an uniform magnetic field, see J. J. Thomson, Conduction of Electricity through Gases (Cambridge Press, 1928), Chap. V.
- ³See, e. g., S.K. Allison, L. S. Skaggs, and N. M. Smith, Jr., Phys. Rev., 54, 171(1938).
- ⁴E. M. Purcell, Phys. Rev., 54, 818 (1938).
- ⁵For the two-dimensional field, see, e. g., A. J. Dempster, Phys. Rev., 51, 67 (1937). This paper contains further references.
- ⁶See reference 1, p. 344.
- ⁷See first part of reference 5.
- ⁸Many equations used in the design of this instrument are given in reference 7 and in F. T. Rogers, Jr., Rev. Sci. Inst., 8, 22 (1936).

⁹Using Equations in reference 5 for l' and l'' in terms of f and g .

¹⁰See A. K. Brewer and A. Bramley, J. App. Phys., 9, 778 (1938).

¹¹See, e. g., H. V. Neher's chapter (VII) in Procedures in Experimental Physics (Prentice-Hall, 1938), edited by John Strong.

¹²An excellent set of data on sparking potentials in gases at reduced pressures is given by

¹³See the preceding article in this volume.

In connection with the design of insulators for high voltages, a very good reference is Alfred Still's Elements of Electrical Design (McGraw-Hill, 1932), Chap. VI.

THE MASSES AND VELOCITIES OF SOME RADIUM B

BETA-PARTICLES

(Preliminary Report)

(Abstract)

This paper is an account of an experiment being carried on ^{to} ~~the~~ determine how the masses of the β -particles from a Ra A+B+C active deposit are related to the velocities of these particles. A discussion of the method being used is given, together with the necessary theory. It is concluded that the experiment is entirely adequate to determine beyond all reasonable doubt whether the β -particles conform to the Abraham or to the Lorentz model of the electron.

INTRODUCTION

As has been well discussed by Zahn and Spees¹, a highly precise experiment on the masses of β -particles at high velocities has not as yet been performed. Many experiments have been done recently and earlier. The earliest experiments²⁻⁴ suffered from a lack of sufficiently good "resolving power" to be of great significance. The later experiments⁵⁻⁹, while much more precise than the earlier, still leave a reasonable amount of doubt as to whether β -particles "obey" the Abraham¹⁰ model or the Lorentz model¹¹. These later experiments adequately confirm the increase of mass with velocity approximately as required by the theories; by "approximately" it is meant to imply agreement with theory to within a few per cent. The available evidence indicates very strongly that β -particles conform the more nearly with the Lorentz model, in spite of the uncertainties in the measurements. The most recently reported experiment¹², that

of Lahaye, is of still better precision and tends definitely to favor the Lorentz model, though the final results are subject to an unpleasant amount of uncertainty.

It is thus believed at present to be most likely that the masses of β -particles (particularly of those β -particles which constitute the "discrete spectrum" of the Ra A+B+C active deposit) vary with velocity (for velocities up to $\beta=0.888c$, where c is the velocity of light) as predicted by the Lorentz theory of the electron.

All the data that are available have been taken by observations on the motions of β -particles in suitable combinations of electrostatic and magnetostatic fields. The observations at the higher velocities have been made, as previously pointed out, on the β -particles of the discrete spectra of Ra A+B+C. The observations on the lower velocities have been made in some cases with β -particles of known energies from cathode-ray sources.

In view of the recent high-precision determinations¹³⁻¹⁵ of the absolute (rather than relative) value of the $H\beta$ of the most intense line in the Ra B β -particle spectrum (which thus allows the highly precise absolute specification of the $H\beta$ values of the remaining lines in the Ra B+C spectra from the precise relative values determined by Ellis and Skinner¹⁶ and by Ellis^{17,18}) and in view of the recent advances in the uses of the electrostatic spectrograph¹⁹⁻²¹, it was considered desirable to make further observations on the masses and velocities of the Ra B+C β -particles. It was hoped that one or more of the several following goals could be attained:

- (1) to obtain precise^{absolute} values for the masses and velocities of several of the discrete-spectra β -particles of Ra B+C;
- (2) to have sufficient accuracy and precision to allow a decision which would be beyond all reasonable doubt, as to what model to Ra B+C β -particles most nearly conform to;
- (3) to obtain a good value of the rest mass of the Ra B+C β -particles by a relatively

direct method;

(4) to extend beyond $\beta = 0.888c$ the range of β for which data have been obtained;

(5) to accomplish one or more of these goals by a method which, though related to the previous methods of study of masses and velocities, is practically a new and independent method.

In the following parts of this paper we present an account of the details of such an experiment.

THEORY

Let m be the mass of a β -particle whose velocity is V and whose charge is e . If this particle move in an uniform magnetic field H perpendicular to the direction of H , the radius of curvature ρ of its path in the field is given by the well known equation

$$mV^2/\rho = HeV.$$

Thus $H\rho = mV/e.$ (1)

$H\rho$ is observable and by Eq. (1) thus yields the product mV/e .

Now let this particle move in a radial (in two dimensions) electric field, as in the instrument diagrammed in Fig. 1, along a circular path of radius R (see Fig. 1). This can be effected if the field strength X at $r = R$ is so chosen that

$$mV^2/R = Xe.$$

Thus $XR = mV^2/e.$ (2)

FIG. 1.

XR is observable and by this Eq. (2) yields the product mV^2/e .

Therefore, from Eqs. (1) and (2) we see that V and m/e are given in terms of $H\rho$ and XR by the equations

$$V = XR/H\rho, \quad (3)$$

and
$$m/e = (H\rho)^2/XR. \quad (4)$$

Hence the determination of XR and $H\rho$ for a particle is sufficient to determine the V and m/e of the particle.

If ϵ_3 and ϵ_4 are the maximum relative errors which might be present in $H\rho$ and XR respectively, then the maximum possible errors (relative) ϵ_1 and ϵ_2 which can be present in V and m/e respectively as got by Eqs. (3) and (4) are

$$\epsilon_1 = \epsilon_3 + \epsilon_4, \quad (5)$$

and
$$\epsilon_2 = 2\epsilon_3 + \epsilon_4. \quad (6)$$

For future reference in the analysis of

the data obtained in this experiment, we record the well-known^{theoretical} formulae for the transverse mass of the β -particle as a function of the velocity of the particle. If the particle follow the Abraham model, its transverse mass m_A is given by the equation

$$\frac{m_A}{e} = \frac{3}{8} \frac{m_0/e}{\beta^3} \left[(1+\beta^2) \ln \frac{1+\beta}{1-\beta} - 2\beta \right], \quad (7)$$

where m_0 is the rest mass of the electron. If the particle follow the Lorentz model, its transverse mass m_L is given by

$$m_L/e = (m_0/e)/(1-\beta^2)^{3/2}. \quad (8)$$

These equations are adequately worked out in several other places^{10,11} and their derivations do not form a part of this problem; for the necessary mathematical analyses, see the several references which contain them^{10,11}.

In his book²², Lorentz gives a graph showing m/m_0 for the two electron models for $\beta \leq 0.5$. At $\beta = 0.5$, the "difference" in^{the} masses predicted by the two theories is shown to be about three

per cent. This "difference" D between the masses predicted by the two theories, expressed in percent., is shown in Table I for some typical values of $\beta \geq 0.5$; these values of D were computed directly from Eqs. (7) and (8). They are plotted graphically in Fig. 2.

TABLE I.

β	D
0.50	3
0.60	5
0.70	8
0.80	13
0.85	17
0.90	23

FIG. 2.

EXPERIMENTAL DETAILS

The determination of the behaviour of the masses of the β -particles from the Ra A + B + C active deposit requires, by virtue of Eqs. (3) and (4), the product XR for the several kinds of β -particles. Therefore, the particles were deflected into a circular, observable path of known radius in a known electrostatic field. The instrument used was the electrostatic spectrograph.

This spectrograph ~~is~~ described in the preceding paper²³ (q. v.). It was so constructed that²⁴

$$R = 16.05 \pm 0.007 \text{ cm.}$$

If P is the potential difference of the plates of the spectrograph, and if r_1 and r_2 are the radii of the inner and outer surfaces of these plates, then²⁵

$$X = P/R \ln(r_2/r_1).$$

In this instrument $r_2 - r_1 = 0.5965 \pm 0.0045 \text{ cm.}$

Thus $XR = P / \ln(r_2/r_1)$

becomes simply

$$XR = 26.92 P, \quad (9)$$

which (aside from possible errors which might be present in P) is good to within 0.9 percent.

The potential P was measured by a calibrated galvanometer which indicated the current flowing in a high resistance (48.74 megohms to within 50,000 ohms) across the output of the source of high voltage applied to the spectrograph plates. In practice the particles from the active deposit were deflected by the spectrograph until they were incident on a G-M counter at the observing end of the spectrograph, when the value of P was read. (It was hoped to have some final data available for incorporation in this report, but at the time of writing only preliminary data are at hand. It is expected that all the final data will be taken shortly.) In all cases, the values of P for the several β -particles are believed to be certainly reliable to within a maximum of 0.5 %.

DISCUSSION

The method of analyzing the data to be got in this experiment is to compare the observed transverse masses with the transverse mass M at the value of β for the β -particles of least observed energy. Since the behaviour of m/M can be predicted for both theories of the electron from Eqs. (7) and (8), it will thus be possible to ascertain the relation between the true mass-velocity law for these β -particles and the two theoretically predicted laws.

Since F in all cases is good to within 0.5 % and since $r_2 - r_1$ is accurate to within 0.9 %, we may expect the values of XR got from this experiment by Eq. (9) to be uncertain by not more than 0.9 % + 0.5 %, or 1.5%. Since the $H\beta$ -values for the several β -particles have been got to within maximum uncertainties of at most 1 part in 2,500 for Ra B¹⁴ and to within about 1 part in 1,000 for Ra C¹⁶, we may (by Eqs. (5) and (6)) take

as very definite upper limits to the uncertainties possibly present in m/e and V

$$\epsilon_1 = \epsilon_2 = 0.02. \quad (10)$$

These are the maximum relative errors which can be present (aside from any undetected systematic errors) in the values of m/e and V got by this experiment. In view of the magnitudes of the D values given in Table I, it will thus be possible to distinguish beyond all reasonable doubt for the values of β being studied (above $\beta = 0.6$) what is the true β -particle^{mass} behaviour

In conclusion, it should be mentioned that this experiment could not have been even started without the provision by Dr. H. A. Wilson of the necessary apparatus and of all desired facilities for working. This work was done with the indispensable collaboration of Mr. A. W. McReynolds.

REFERENCES

- ¹C. T. Zahn and A. H. Spees, Phys. Rev., 53, 511 (1938).
- ²W. KAUFMANN, Ann. d. Phys., 19, 487 (1906).
- ³A. H. Bucherer, Ann. d. Phys., 28, 513 (1908).
- ⁴G. Neumann, Ann. d. Phys., 45, 529 (1914).
- ⁵E. Hupka, Ann. d. Phys., 31, 176 (1910).
- ⁶Guye, Ratnowski, and Lavanchy; see Handb. d. Phys., vol. 22 (1926), p. 76.
- ⁷R. A. Tricker, Proc. Roy. Soc., A109, 384 (1925).
- ⁸H. Starke, Verh. d. Deut. Phys. Gesel., 5, 241 (1903).
- ⁹H. Strake and M. Nacken, Ann. d. Phys., 21, 67 (1934).
- ¹⁰H. A. Lorentz, Lectures on Theoretical Physics, vol. 3 (MacMillan, 1931), p. 266.

- ¹¹ Lorentz, loc. cit., chapter 3.
- ¹² H. Lahaye, Ann. d. Phys., 34, 60 (1938).
- ¹³ F. A. Scott, Phys. Rev., 46, 633 (1934).
- ¹⁴ F. T. Rogers, Jr., Phys. Rev., 50, 515 (1936).
- ¹⁵ Norman Roberts, private communication of results just obtained at the University of Sydney.
- ¹⁶ Ellis and Skinner, Proc. Roy. Soc., A105, 165 (1924).
- ¹⁷ Ellis,
- ¹⁸ Rutherford, Chadwick, and Ellis, Radiations from Radioactive Substances (Cambridge Press, 1930), p. .
- ¹⁹ A. J. Dempster, Phys. Rev., 51, 61 (1937)
- ²⁰ Allison, Skaggs, and Smith, Phys. Rev., 54, 171 (1938).
- ²¹ F. T. Rogers, Jr., Rev. Sci. Inst., 8, 22 (1937).

²²Lorentz, loc. cit., chapter VII.

²³F. T. Rogers, Jr., A Beta-Particle Electrostatic Spectrograph.

²⁴See page 8 in reference 23.

²⁵Cf. F. T. Rogers, Jr., reference 21.

²⁶The results for the relative values of the $H\rho$'s being corrected to the proper absolute values by the work of Scott, Rogers, and Roberts.