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GENERALIZATIONS TO SPACE OF THE CAUCHY AND
MORERA THEOREMS

Maxwell Reade

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GENERALIZATIONS TO SPACE OF THE CAUCHY AND MORERA THEOREMS

INTRODUCTION.

Consider the function

$$(1) \quad w = f(z) = x_1(u, v) + i x_2(u, v), \quad z = u + i v,$$

defined in a simply connected domain D . * A necessary and sufficient condition that $f(z)$ be analytic in D is that the Cauchy-Riemann equations be satisfied there:¹

$$\frac{\partial x_1}{\partial u} = \frac{\partial x_2}{\partial v}, \quad \frac{\partial x_1}{\partial v} = - \frac{\partial x_2}{\partial u};$$

that is

$$(2) \quad \lambda w = 0,$$

where

$$\lambda \equiv \frac{\partial}{\partial u} + i \frac{\partial}{\partial v}$$

is a differential operator. From (2) we obtain

$$(3) \quad \sum_{j=1}^2 (\lambda x_j)^2 = 0,$$

which can be written in the form

$$(4) \quad \sum_{j=1}^2 \left(\frac{\partial x_j}{\partial u} \right)^2 = \sum_{j=1}^2 \left(\frac{\partial x_j}{\partial v} \right)^2, \quad \sum_{j=1}^2 \frac{\partial x_j}{\partial u} \frac{\partial x_j}{\partial v} = 0.$$

Conversely, (3) implies either (2) or

$$\bar{\lambda} w = 0,$$

*We shall consider only finite simply connected domains.

1. See the theorem of Looman-Menchoff: S. Saks, Theory of the Integral, sec. rev. ed., (1937), p. 199.

where

$$\bar{\lambda} \equiv \frac{\partial}{\partial u} - i \frac{\partial}{\partial v}.$$

If (3) holds, then we say that $x_1(u, v)$ and $x_2(u, v)$ form a couple of conjugate harmonic functions.

The real functions

$$(5) \quad x_j = x_j(u, v), \quad j = 1, 2, 3,$$

defined and continuous in a simply connected domain

D , will be said to define a surface S . If the first partial derivatives of the functions (5) are continuous and satisfy

$$(6) \quad E(u, v) = G(u, v), \quad F(u, v) = 0,$$

where

$$(7) \quad E \equiv \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial u} \right)^2, \quad F \equiv \sum_{j=1}^3 \frac{\partial x_j}{\partial u} \frac{\partial x_j}{\partial v}, \quad G \equiv \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial v} \right)^2,$$

are the coefficients of the first fundamental differential quadratic form,²

$$ds^2 = E du^2 + 2F du dv + G dv^2$$

of the surface S , then the parameters u, v are said to be isothermic parameters,³ and the surface is said to be given in isothermic representation. The map of D on S is conformal except where $E = G = 0$. Now (6), which is a generalization of (4), can be written in the form

$$(8) \quad \sum_{j=1}^3 (\lambda x_j)^2 = 0,$$

2. W. C. Graustein, Differential Geometry, p. 82.

3. Graustein, op. cit., p. 131.

which is a generalization of (3).

The following is a combination of the Cauchy and Morera theorems.⁴

If the function $f(z)$ in (1) is continuous in a simply connected domain D , then a necessary and sufficient condition that $f(z)$ be analytic in D is that for each closed rectifiable Jordan curve γ lying in D ,

$$(9) \quad \int_{\gamma} f(z) dz = 0.$$

We may write (9), when we separate its left-hand member into its real and imaginary parts, in a form analogous to the Cauchy-Riemann equations:

$$(10) \quad \int_{\gamma} x_1(u,v) du = \int_{\gamma} x_2(u,v) dv, \quad \int_{\gamma} x_1(u,v) dv = - \int_{\gamma} x_2(u,v) du,$$

But the relations (10) imply

$$(11) \quad \sum_{j=1}^2 \left[\int_{\gamma} x_j(u,v) du \right]^2 = \sum_{j=1}^2 \left[\int_{\gamma} x_j(u,v) dv \right]^2, \quad \sum_{j=1}^2 \int_{\gamma} x_j(u,v) du \int_{\gamma} x_j(u,v) dv = 0,$$

which are analogous to the relations (4); and (11) may be written in the following form analogous to (3):

$$(12) \quad \sum_{j=1}^2 \left[\int_{\gamma} x_j(u,v) dz \right]^2 = 0$$

A generalization of (12), analogous to (8), is

$$(13) \quad \sum_{j=1}^2 \left[\int_{\gamma} x_j(u,v) dz \right]^2 = 0,$$

4. E. J. Townsend, Functions of a Complex Variable, p. 82.

where γ is an arbitrary closed rectifiable Jordan curve lying in D . In this paper we shall study (13) and

$$\sum_{j=1}^3 \left| \int_{C_\lambda} x_j(u, v) dz \right|^2 = \sigma(\lambda^\alpha), \quad 0 < \alpha,$$

where C_λ is the circle in D with variable radius λ and fixed center (u_0, v_0) , and where $\sigma(\lambda^\alpha)$ is a quantity (not always the same quantity) such that

$$\lim_{\lambda \rightarrow 0} \frac{\sigma(\lambda^\alpha)}{\lambda^\alpha} = 0.$$

If the functions (5) are harmonic and satisfy (6) in a simply connected domain D , they have been called a triple of conjugate harmonic functions.⁵ In terms of this definition, a theorem of Weierstrass⁶ may be stated as follows.

A necessary and sufficient condition that the functions (5), defined in a simply connected domain, be the coordinate functions of a minimal surface given in isothermic representation is that they form a triple of conjugate harmonic functions.

We recall the following theorem.⁷

Theorem A. A necessary and sufficient condition

5. E. F. Beckenbach and T. Radó, Subharmonic Functions and Minimal Surfaces, Trans. Am. Math. Soc., v. 35 (1933), pp. 648-661.

6. Loc. cit., p. 649.

7. Graustein, op. cit., p. 98.

that the functions (1), having continuous partial derivatives of the third order in a simply connected domain, be the coordinate functions of a spherical surface S is that there exist a constant α such that

$$e = \alpha E, \quad f = \alpha F, \quad g = \alpha G,$$

where

$$e \equiv \frac{|X_{1uu} X_{2u} X_{3u}|}{H}, \quad f \equiv \frac{|X_{1uv} X_{2u} X_{3u}|}{H}, \quad g \equiv \frac{|X_{1vv} X_{2u} X_{3u}|}{H},$$

are the coefficients of the second fundamental quadratic form⁹ of S , and where

$$H \equiv \sqrt{EG - F^2}, \quad X_{ju} \equiv \frac{\partial X_j}{\partial u} \text{ etc.}$$

Here we have used the notation

$$|a_1, b_1, c_1| \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

It follows immediately that a necessary and sufficient condition that the functions (5) be the coordinate functions of a spherical surface given in isothermic representation is that both (6) and

$$(14) \quad e = g, \quad f = 0$$

hold.

If the functions (5) have continuous partial derivatives of the m th order in a simply connected domain D , then about each point (u_0, v_0) of D we have a finite Taylor expansion for each function:

8. If $\alpha = 0$, then S is a plane surface; loc. cit., p. 97.

9. Loc. cit., pp. 93-94.

$$(15) \quad x_j(u, v) = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \left[\left(\cos \theta \frac{\partial}{\partial u} + \sin \theta \frac{\partial}{\partial v} \right)^m x_j \right] + \sigma(\rho^m), \quad j=1, 2, 3,$$

where

$$u - u_0 = \rho \cos \theta, \quad v - v_0 = \rho \sin \theta,$$

where

$$\cos \theta \frac{\partial}{\partial u} + \sin \theta \frac{\partial}{\partial v}$$

is a differential operator, and where the partial derivatives are evaluated at (u_0, v_0) .

We establish the following result for functions (5) with Taylor expansions (15).

If C_ρ is the circle in D with center at (u_0, v_0) and radius ρ , then

$$(16) \quad \sum_{j=1}^3 \left[\int_{C_\rho} x_j(u, v) dz \right]^2 = -\pi^2 \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{p=0}^k \rho^{2k+2p} B_{p,k} + \sigma(\rho^{m+2}),$$

where

$$B_{p,k} = \sum_{j=1}^3 \frac{C_{p+1}^{k+1} C_p^{k+1}}{2^{2k} [(k+1)!]^2} \Delta^p \lambda x_j \Delta^{k-p} \lambda x_j,$$

where

$$\Delta \equiv \lambda \bar{\lambda} \equiv \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}, \quad \Delta^0 \equiv 1, \quad \Delta^p \equiv \Delta(\Delta^{p-1}), \quad p \geq 1,$$

where

$$C_s = \frac{k!}{s!(k-s)!}, \quad C_0 \equiv 1,$$

and where $\lfloor (m-1)/2 \rfloor$ is the greatest integer not greater than $\{(m-1)/2\}$.

Proof. We make use of the following equalities.

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} \frac{\Gamma(\frac{m+1}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n+2}{2})}, \quad 10$$

$$(17) \quad \int_0^\pi \sin^m \theta \cos^n \theta d\theta = \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta + (-1)^m \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta,$$

$$\int_0^{2\pi} \sin^m \theta \cos^n \theta d\theta = [1 + (-1)^{m+n}] \int_0^\pi \sin^m \theta \cos^n \theta d\theta.$$

From (17) we obtain

$$(a) \quad \int_0^{2\pi} \sin^m \theta \cos^n \theta d\theta = R_{m,n},$$

(18)

$$(b) \quad = 0$$

where (a) holds for m and n both even and where (b) holds for m and n not both even, and where

$$R_{m,n} \equiv 2 \frac{\Gamma(\frac{m+1}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n+2}{2})}, \quad m, n = 0, 2, 4, \dots,$$

$$\equiv 2 \frac{\Gamma(p+\frac{1}{2}) \Gamma(q+\frac{1}{2})}{\Gamma(p+q+1)}, \quad p, q = 0, 1, 2, \dots,$$

where $m=2p$ and $n=2q$. But

$$\Gamma(s+1) = s!, \quad \Gamma(\frac{2s+1}{2}) = \frac{\sqrt{\pi}}{2^{2s}} \frac{(2s)!}{s!}, \quad s = 0, 1, 2, \dots; \quad 11$$

10. W. F. Osgood, Advanced Calculus, p. 485, ex. 2.

11. These results follow from the following properties of the Gamma function (loc. cit., pp. 482-483):

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha), \text{ for } \alpha > 0, \quad \Gamma(1) = 1 \quad \text{and} \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

therefore (18), (a), may be written as

$$(19) \quad \int_0^{2\pi} \sin^{2p}\theta \cos^{2q}\theta d\theta = \frac{\pi}{2^{2(p+q)+1}} \frac{(2p)!(2q)!}{p!q!(p+q)!}, \quad p, q = 0, 1, 2, \dots$$

Consider

$$\int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s} x_j \right] \cos\theta d\theta, \quad \begin{matrix} s = 0, 1, 2, \dots, \\ j = 1, 2, 3, \end{matrix}$$

where the partial derivatives are evaluated at (u_0, v_0) .

We obtain

$$(20) \quad \int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s} x_j \right] \cos\theta d\theta = \sum_{t=0}^{2s} \int_0^{2\pi} C_t \cos^{2s-t}\theta \sin^t\theta \frac{\partial^{2s} x_j}{\partial v^t \partial u^{2s-t}} d\theta = 0, \quad \begin{matrix} s = 0, 1, \dots, \\ j = 1, 2, 3, \end{matrix}$$

Here we have used (18), (b).

Similarly we obtain

$$(21) \quad \int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s} x_j \right] \sin\theta d\theta = 0, \quad \begin{matrix} s = 0, 1, \dots, \\ j = 1, 2, 3, \end{matrix}$$

Consider the expression

$$(22) \quad \int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s+1} x_j \right] \cos\theta d\theta, \quad \begin{matrix} s = 0, 1, \dots, \\ j = 1, 2, 3, \end{matrix}$$

Applying (18), (b), to (22) we obtain

$$\int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s+1} x_j \right] \cos\theta d\theta = \sum_{t=0}^s \int_0^{2\pi} \sin^{2t}\theta \cos^{2s-2t+1}\theta C_{2t} \frac{\partial^{2s+1} x_j}{\partial v^{2t} \partial u^{2s-2t+1}} d\theta,$$

which, by (19), may be written in the form

$$(23) \quad \begin{aligned} \int_0^{2\pi} \left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v} \right)^{2s+1} x_j \right] \cos\theta d\theta &= \frac{\pi}{2^{2s}} \sum_{t=0}^s C_{2t} \frac{(2s+1)!}{s!(s+1)!} \frac{\partial}{\partial u} \left[\frac{\partial^{2s} x_j}{\partial v^{2t} \partial u^{2s-2t}} \right] \\ &= \frac{\pi}{2^{2s}} \frac{(2s+1)!}{s!(s+1)!} \frac{\partial}{\partial u} \Delta^s x_j, \quad \begin{matrix} s = 0, 1, \dots, \\ j = 1, 2, 3. \end{matrix} \end{aligned}$$

Similarly we obtain

$$(24) \quad \int_0^{2\pi} \left(\cos \theta \frac{\partial}{\partial u} + \sin \theta \frac{\partial}{\partial v} \right)^{2s+1} \sin \theta d\theta = \frac{\pi}{2^{2s}} \frac{(2s+1)!}{s!(s+1)!} \frac{\partial}{\partial v} \Delta^s X_j, \quad s=0,1,\dots, \\ j=1,2,3,$$

From (20), (21), (23) and (24) we obtain, for the functions (15),

$$\int_{C_\lambda} X_j(u,v) dz = \pi \lambda^2 i \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\lambda^{2s}}{\lambda^{2s}} \frac{\Delta^s X_j}{s!(s+1)!} + O(\lambda^{m+1}), \quad j=1,2,3,$$

from which we obtain (16).

For $m=1,3,5$, (16) is displayed in (26), (72) and (58) respectively.

1. CHARACTERIZATION OF ISOTHERMIC MAPS.

Theorem 1. If the functions (1) have continuous partial derivatives of the first order in a simply connected domain D , then a necessary and sufficient condition that they map D isothermally on a surface S is that for each point (u_0, v_0) of D ,

$$(25) \quad \sum_{j=1}^3 \left| \int_{C_\lambda} X_j(u,v) dz \right|^2 = O(\lambda^4),$$

where C_λ is the circle in D with center at (u_0, v_0) and radius λ .

Proof. If the first partial derivatives of the functions (5) are continuous in D , then we obtain a finite Taylor expansion for each function about an arbitrary point (u_0, v_0) of D by setting $m=1$ in (15). If C_λ is the circle in D with center at (u_0, v_0) and

radius λ , then upon setting $m=1$ in (16) we obtain

$$(26) \quad \sum_{j=1}^3 \left| \int_{C_\lambda} x_j(u,v) dz \right|^2 = -\pi^2 \lambda^4 \sum_{j=1}^3 (\lambda x_j)^2 + o(\lambda^4).$$

From (26) it follows that a necessary and sufficient condition that (8) hold in D is that (25) hold.

This completes the proof of Theorem 1.

2. CHARACTERIZATION OF THOSE ISOTHERMIC SPHERICAL MAPS THAT DO NOT MAP CIRCLES ON CIRCLES.

Lemma 1. If the functions (5) are not identically constant in a simply connected domain D , and if they map D isothermically on a surface S that lies on a sphere $\mathcal{S}$ of finite radius a , such that circles are mapped on circles, then the functions (5) have the representation

$$x_1 = c_1 + c_2^2 \frac{R f(z)}{|f(z)|^2 + c_4^2},$$

$$x_2 = c_3 + c_2^2 \frac{I f(z)}{|f(z)|^2 + c_4^2},$$

$$x_3 = c_4 \left[1 - \frac{c_2^2}{|f(z)|^2 + c_4^2} \right]$$

where the c_j are real constants, where $f(z)$ has one of the following forms,

$$f(z) = \frac{\alpha z + \beta}{\delta z + \epsilon}, \quad \text{or} \quad f(z) = \frac{\alpha \bar{z} + \beta}{\delta \bar{z} + \epsilon}, \quad (\alpha \delta - \beta \epsilon) \neq 0,$$

and where $R f(z)$ is the real part, and $I f(z)$ the imaginary

part, of $f(z)$.

Proof. Since the functions (5) are not identically constant in D , it follows that $\mathcal{S}$ is not a point-sphere. Let the coordinates of the center Q of $\mathcal{S}$ be (a_1, a_2, a_3) . Since the functions (5) map D isothermally on S such that circles are mapped on circles, the stereographic projection of $\mathcal{S}$ on the s, t -plane, the coordinates of the pole of projection being (a_1, a_2, k) , where $k = a_3 \pm a$ and $|k|$ is the maximum of the two quantities $|a_3 + a|$ and $|a_3 - a|$, induces an isothermic map of D on a domain D^* of the s, t -plane whereby circles are mapped on circles.

Here the s, t -plane is so placed and oriented that the positive s - and t -axes coincide with the positive x_1 - and x_2 -axes respectively. The function mapping D on D^* must be linear,¹² and therefore of the form

$$(a) \quad F(z) = s + it = \frac{\alpha'z + \beta'}{\delta'z + \delta'}, \quad \alpha$$

$$(b) \quad F(z) = s + it = \frac{\alpha'\bar{z} + \beta'}{\delta'\bar{z} + \delta'}, \quad (\alpha'\delta' - \beta'\delta') \neq 0,$$

where (a) holds if the map of D on D^* is directly conformal and where (b) holds if the map of D on D^* is inversely conformal. The stereographic projection of D^* on $\mathcal{S}$ is given by

12. This follows immediately from Theorem 25, p. 32, of L. R. Ford, Automorphic Functions.

$$\begin{aligned}
 x_1 &= a_1 + \frac{2a|k|(s-a_1)}{(s-a_1)^2 + (t-a_2)^2 + k^2} \\
 (27) \quad x_2 &= a_2 + \frac{2a|k|(t-a_2)}{(s-a_1)^2 + (t-a_2)^2 + k^2}, \\
 x_3 &= k \left[1 - \frac{2a|k|}{(s-a_1)^2 + (t-a_2)^2 + k^2} \right].
 \end{aligned}$$

It follows that the map of D on S is given by

$$\begin{aligned}
 x_1 &= a_1 + \frac{2a|k| \operatorname{Re} f(z)}{|f(z)|^2 + k^2}, \\
 (28) \quad x_2 &= a_2 + \frac{2a|k| \operatorname{Im} f(z)}{|f(z)|^2 + k^2}, \\
 x_3 &= k \left[1 - \frac{2a|k|}{|f(z)|^2 + k^2} \right],
 \end{aligned}$$

where

$$f(z) = F(z) - (a_1 + i a_2),$$

and hence it follows that the functions (5) have the required form.

We note that if it is given only that one non-null circle in D is mapped on a circle on S by the isothermic functions (5), then, as in the above discussion, D is mapped isothermically on D^* such that one non-null circle in D is mapped on a circle in D^* . Therefore the mapping function is linear; hence it follows that the functions (5)

have the representation (28).

Lemma 2. If the functions (5) are not identically constant in a simply connected domain D , and if they map D isothermally on a surface S that lies on a sphere $\mathcal{S}$ of finite radius a , then a necessary and sufficient condition that they map circles on circles is that the first quadratic form of S have the representation

$$(29) \quad ds^2 = \frac{c^2(du^2 + dv^2)}{[(u-u_0)^2 + (v-v_0)^2 + b^2]^2}, \quad b > 0, c > 0,^{13}$$

Necessity. If the center Q of $\mathcal{S}$ is at (a_1, a_2, a_3) then, by Lemma 1, the functions (5) are given by (28). From (27), a computation shows that we obtain

$$\tilde{E} = \frac{4a^2k^2}{[(s-a_1)^2 + (t-a_2)^2 + k^2]^2},^{14}$$

where

$$\tilde{E} \equiv \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial s} \right)^2.$$

But

$$|f'(z)|^2 = \frac{|\alpha\delta - \beta t|^2}{|\delta z + \delta|^4}$$

and

$$(s-a_1)^2 + (t-a_2)^2 = |f(z)|^2 = \left| \frac{\alpha z + \beta}{\delta z + \delta} \right|^2;$$

therefore

13. Compare with L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, p. 109.

14. Compare with Ford, op. cit., p. 118.

$$E = |f'(z)|^2 \tilde{E} = \frac{4a^2 k^2 |\alpha\delta - \beta\gamma|^2}{[|\alpha z + \beta|^2 + k^2 |\gamma z + \delta|^2]^2},$$

which may be written as follows:

$$(30) \quad E = \frac{c^2}{[(u-u_0)^2 + (v-v_0)^2 + b^2]^2},$$

where

$$-Nu_0 = (\alpha_1\beta_1 + \alpha_2\beta_2) + k^2(\gamma_1\delta_1 + \gamma_2\delta_2),$$

$$-Nv_0 = (\alpha_1\beta_2 - \alpha_2\beta_1) + k^2(\gamma_1\delta_2 - \gamma_2\delta_1),$$

$$b^2 = \frac{|\beta|^2 + k^2|\delta|^2}{N} - (u_0^2 + v_0^2),$$

$$c^2 = \frac{4a^2 k^2 |\alpha\delta - \beta\gamma|^2}{N^2} > 0,$$

$$N = |\alpha|^2 + k^2|\gamma|^2 > 0,$$

where the α_i , β_i , γ_i and δ_i are real constants, such that

$$\alpha = \alpha_1 + i\alpha_2,$$

$$\beta = \beta_1 + i\beta_2,$$

$$\gamma = \gamma_1 + i\gamma_2,$$

$$\delta = \delta_1 + i\delta_2,$$

Here we have used the representation (a) on page 11 for $F(z)$ in order to derive the relation (30). Similarly, we can derive (30) when $F(z)$ has the form (b) on page 11, the values of the constants u_0 and b^2 in (30) being altered accordingly.

The Gaussian (total) curvature of S is given by¹⁵

$$(31) \quad K = -\frac{1}{2E} \Delta \log E,$$

which, with (30), yields

$$k = \frac{4b^2}{c^2}.$$

But $\mathcal{D}$ is a sphere of finite non-null radius; therefore $b^2 > 0$.

Under the hypotheses, u, v are isothermic parameters. From (6) and (30) it follows that the first quadratic form of S is precisely (29).

Sufficiency. Let the stereographic projection of D on the sphere $\mathcal{D}^*$ be S^* , where $\mathcal{D}^*$ is the sphere with center at $(u_0, v_0, b^{-c/2b})$ and radius $c/2b$, and where the pole of projection is at (u_0, v_0, b) . This projection is given by

$$\begin{aligned} x_1 &= u_0 + \frac{c(u-u_0)}{(u-u_0)^2 + (v-v_0)^2 + b^2}, \\ x_2 &= v_0 + \frac{c(v-v_0)}{(u-u_0)^2 + (v-v_0)^2 + b^2}, \\ x_3 &= b \left[1 - \frac{c}{(u-u_0)^2 + (v-v_0)^2 + b^2} \right]. \end{aligned} \quad (32)$$

The first quadratic form of S^* is found to be

$$ds^2 = \frac{c^2 (du^2 + dv^2)}{[(u-u_0)^2 + (v-v_0)^2 + b^2]^2}, \quad (33)$$

which is identical with that of S . It follows from (29), (31) and (33) that S and S^* lie on
tions and Surfaces of Negative Curvature, Trans. Am. Math. Soc., v. 35 (1933), pp. 662-674.

spheres with equal radii.

We shall show that to each circle in D there corresponds, by means of the functions (5), a circle on S . Consider the non-null circle C_0 in D , and let the interior of C_0 be denoted by R . The functions (32) map R isothermally on a surface Σ^* on S^* which is bounded by a non-null circle C^* , where C^* is the image of C_0 , and the functions (5) map R on a surface Σ on S which is bounded by a curve C . According to the reasoning in the proof of Lemma 1,¹⁶ it is sufficient that we take the non-null circle C_0 so small that the spherical cap Σ^* is contained on a hemisphere. There is a pole of C^* , say P_0^* , on Σ^* , and hence on S^* . Let P_1^* be a point of C^* , and let $\widehat{P_0^* P_1^*}$ be the arc of the geodesic¹⁷ passing through P_0^* and P_1^* and lying on Σ^* . Let the image of P_0^* in D be Q_0 , the image of P_1^* in D be Q_1 and the image of the arc $\widehat{P_0^* P_1^*}$ in D be the arc $\widehat{Q_0 Q_1}$. Let the image of Q_0 on S be P_0 , the image of Q_1 on S be P_1 ,

16. See the bottom of page 12.

17. The geodesics on spherical surfaces are arcs of great circles; Graustein, op. cit., p. 149. Σ^* is on a hemisphere; hence through each pair of points there passes exactly one geodesic that lies on Σ^* .

and the image of $\widehat{Q_0 Q_1}$ on S be $\widehat{P_0 P_1}$. Now $\widehat{P_0 P_1}$ is a geodesic.¹⁸ Keep P_0^* fixed and allow P_1^* to describe the circle C^* . The geodesic arc $\widehat{P_0^* P_1^*}$ is of fixed length ρ for all points P_1^* on C^* . As P_1^* describes C^* , P_1 describes C . From (29)

and (33) it follows that the geodesic arc $\widehat{P_0 P_1}$ also is of the same fixed length ρ for all points P_1 on C . Moreover, as P_1^* describes C^* exactly once around, P_1 describes C exactly once around. For, let the

arc $\widehat{Q_0 B_1}$ in D be the image of the geodesic arc $\widehat{P_0^* A_1^*}$ on S^* and let the image of $\widehat{Q_0 B_1}$ on S be the geodesic arc $\widehat{P_0 A_1}$, so that $\widehat{P_0^* A_1^*}$ and $\widehat{P_0 A_1}$

are corresponding geodesic arcs. Let $\widehat{P_0^* A_2^*}$ and $\widehat{P_0 A_2}$ be another pair of corresponding geodesic arcs. Here A_1^* and A_2^* are on C^* and A_1 and A_2 are on C . Since

18. $\widehat{P_0 P_1}$ is a geodesic arc. For, the geodesic arcs are arcs of shortest distance. If $\widehat{P_0 P_1}$ were not a geodesic arc on Σ , then there would be a

geodesic arc $\widehat{P_0 P_1}$ through P_0 and P_1 on Σ . The corresponding curve $\widehat{P_0^* P_1^*}$ through P_0^* and P_1^* on Σ^* would be of the same length. Since $\widehat{P_0 P_1}$ and $\widehat{P_0^* P_1^*}$ are of equal length and since $\widehat{P_0 P_1}$ is shorter than

the functions (5) and (32) are isothermic, it fol-

lows that the angle formed by $\widehat{P_0 A_1}$ and $\widehat{P_0 A_2}$

is equal to that between $\widehat{P_0^* A_1^*}$ and $\widehat{P_0^* A_2^*}$.

Hence there is a one-to-one correspondence between the points of C and C^* as P_1^* describes C^* once around.

Hence the curve C on S is a circle with center at P_0 . Hence we have shown that the non-null circle C_0 is mapped on a circle C on S by the functions (5). Hence the functions (5) map all circles in D isothermically on circles on S .

Theorem 2. If the functions (5) are not identically constant in a simply connected domain D , and if they have continuous partial derivatives of the third order in D , then a necessary and sufficient condition that they map D isothermically on a surface S that lies on a sphere of finite radius, such that circles are not mapped on circles, is that for each point (u_0, v_0) in D ,

$$(34) \quad \sum_{j=1}^3 \left[\int_{C_n} x_j(u, v) dz \right]^2 = O(\lambda^4),$$

where C_n is the circle in D with center at (u_0, v_0) and radius n , and that there exist a point (u_1, v_1)

 $\widehat{P_0 P_1}$ it follows that $\widehat{P_0^* P_1^*}$ is not a geodesic arc.
 This is not so; hence $\widehat{P_0 P_1}$ is a geodesic arc.

in D such that

$$(35) \quad \sum_{j=1}^3 \left[\int_{C_n^*} x_j(u, v) dz \right]^2 \neq \sigma(1^8),$$

where C_n^* is the circle in D with center at (u, v) and radius n .

Necessity. Under the hypotheses, the functions (5) map D isothermally on a surface S that lies on a sphere of finite non-null radius. Hence (6) and (14) hold, and there exists a constant α such that

$$(36) \quad e = \alpha E, \quad f = \alpha F, \quad g = \alpha G, \quad \alpha \neq 0.$$

For the present representation of S , the formulas of Gauss¹⁹ become

$$(37) \quad \begin{aligned} x_{iuv} &= R u x_{ju} - R v x_{ju} + e \xi_j, \\ x_{iuv} &= R v x_{iu} + R u x_{iv}, \\ x_{iuv} &= -R u x_{iu} + R v x_{iv} + e \xi_j, \end{aligned} \quad j=1, 2, 3,$$

where

$$R = \frac{1}{2} \log E,$$

and where

$$(38) \quad \xi_j = \frac{x_{ku} x_{lv} - x_{kv} x_{lu}}{E}, \quad j, k, l = 1, 2, 3 \text{ in cyclic order,}$$

are the direction cosines of the normal to S .²⁰

We shall need the following relations.

19. Graustein, op. cit., pp. 135-137.

20. Loc. cit., p. 92.

$$(39) \quad \sum_{j=1}^3 \lambda x_j \Delta^2 \lambda x_j = -4\alpha^2 F \lambda^2 E,$$

$$(40) \quad \sum_{j=1}^3 (\Delta \lambda x_j)^2 = 4\alpha^2 (\lambda E)^2.$$

To obtain (39) and (40), we shall use a method which depends upon the existence and continuity of the partial derivatives of order higher than three of the functions (5). These functions map D isothermally on a spherical surface. Therefore, as in the proof of Lemma 1, we can consider an intermediate stereographic projection to show that the functions (5) have the representation (28), where either $f(z)$ or $\overline{f(z)}$ is analytic in D . From this representation, it follows that the functions (5) have continuous partial derivatives of all orders.

To obtain (39) and (40), we shall need the following equalities:

$$(41) \quad \Delta \lambda x_j = \alpha (e \lambda \xi_j + \xi_j \lambda e), \quad j=1,2,3,$$

$$(42) \quad \Delta^2 \lambda x_j = \alpha (e \Delta \lambda \xi_j + \alpha \lambda e \Delta \xi_j + 2 \Delta e \lambda \xi_j + \bar{\lambda} e \lambda^2 \xi_j + \lambda^2 e \bar{\lambda} \xi_j + \xi_j \Delta \lambda e), \quad j=1,2,3,$$

$$(43) \quad \sum_{j=1}^3 \xi_j \lambda x_j = 0,$$

$$(44) \quad \sum_{j=1}^3 \lambda x_j \lambda \xi_j = 0,$$

$$(45) \quad \sum_{j=1}^3 \lambda^2 \xi_j \lambda x_j = 0,$$

$$(46) \quad \sum_{j=1}^3 \lambda \kappa_j \Delta \zeta_j = 0,$$

$$(47) \quad \sum_{j=1}^3 \lambda \kappa_j \Delta \lambda \zeta_j = 0.$$

It follows from (37) that

$$\Delta \lambda \kappa_j = 2 \lambda (e \zeta_j), \quad j=1, 2, 3,$$

and hence it follows that (41) holds. Operating on (41) with the operator $\bar{\lambda}$ we obtain

$$(48) \quad \Delta^2 \kappa_j = 2(e \Delta \zeta_j + \bar{\lambda} e \lambda \zeta_j + \lambda e \bar{\lambda} \zeta_j + \zeta_j \Delta e), \quad j=1, 2, 3,$$

and operating on (48) with λ we obtain (42).

Recall the equalities:²¹

$$(49) \quad \sum_{j=1}^3 \zeta_j^2 = 1,$$

$$(50) \quad \sum_{j=1}^3 \zeta_j \lambda \zeta_j = 0,$$

$$(51) \quad \sum_{j=1}^3 \zeta_j \lambda \kappa_j = 0,$$

$$(52) \quad e = - \sum_{j=1}^3 \kappa_{ju} \zeta_{ju} = - \sum_{j=1}^3 \kappa_{ju} \zeta_{ju} = 1,$$

$$(53) \quad \bar{e} = - \sum_{j=1}^3 \kappa_{ju} \zeta_{ju} = - \sum_{j=1}^3 \kappa_{ju} \zeta_{ju} = 0.$$

We note that (43) and (51) are identical.

From (8) and (51) it follows that

$$\sum_{j=1}^3 \kappa_{ju} \kappa_{ju} = \sum_{j=1}^3 \kappa_{ju} \zeta_j = \sum_{j=1}^3 \kappa_{ju} \zeta_j = 0.$$

21. Graustein, op. cit., pp. 93-94.

Hence the vectors whose direction components are

$$X_{ju}, X_{jv}, \xi_j, \quad j=1,2,3,$$

respectively, are mutually perpendicular.²² From (50) and (53) we obtain

$$\sum_{j=1}^3 X_{jv} \xi_{ju} = \sum_{j=1}^3 \xi_j \xi_{ju} = 0,$$

and hence the vectors whose direction components are

$$\xi_{ju}, X_{jv}, \xi_j, \quad j=1,2,3,$$

respectively, are mutually perpendicular. Hence

$$(54) \quad \xi_{ju} = -\alpha X_{ju}, \quad j=1,2,3. \quad ^{23}$$

22. Loc. cit., p. 5.

23. ξ_{ju} and X_{ju} are proportional: $\xi_{ju} = p(u,v) X_{ju}$, $j=1,2,3$. Hence from (7), (36) and (52) we obtain $p(u,v) = -\alpha$. Here $E=0$ only at isolated points. For if not, then there would exist a sequence of points of D , $z_1, z_2, z_3, \dots$, and a point z_0 of D , with the following properties: $\lim_{n \rightarrow \infty} z_n = z_0$, $z_n = u_n + i v_n$ and $E(u_n, v_n) = 0$, $n=1,2,\dots$. But, as has been noted on page 20, the functions (5) have the representation (28), where either $f(z)$ or $\overline{f(z)}$ is analytic in D .

Then, as on page 14, we obtain

$$E = \frac{4a^2 k^2 |f'(z)|^2}{(|f(z)|^2 + k^2)^2}, \quad a, k \neq 0,$$

where a and k have the same meaning as in the proof of Lemma 2. Therefore $f'(z_n) = 0$, $n=1,2,\dots$, and hence $f'(z) = 0$. It follows that $f'(z) = 0$ in D ,

From (50) and (53) we obtain

$$\sum_{j=1}^3 \zeta_{jv} \zeta_j = \sum_{j=1}^3 \zeta_j x_{ju} = 0,$$

and therefore the vectors whose direction components are

$$\zeta_{jv}, x_{ju}, \zeta_j, \quad j=1, 2, 3,$$

respectively, are mutually perpendicular. Hence

$$(55) \quad \zeta_{jv} = -\alpha x_{ju}, \quad j=1, 2, 3.$$

We can write (54) and (55) in the following form:

$$(56) \quad \lambda \zeta_j = -\alpha \lambda x_j, \quad \alpha \neq 0, \quad j=1, 2, 3.$$

From (56) we obtain

$$\sum_{j=1}^3 \lambda x_j \lambda \zeta_j = -\alpha \sum_{j=1}^3 (\lambda x_j)^2,$$

which, with (8), yields (44).

From (44) we obtain

$$\lambda \sum_{j=1}^3 \lambda x_j \lambda \zeta_j = \sum_{j=1}^3 \lambda^2 x_j \lambda \zeta_j + \sum_{j=1}^3 \lambda x_j \lambda^2 \zeta_j = 0,$$

which, with (56), establishes (45).

From (44) we obtain

$$\lambda \sum_{j=1}^3 \lambda x_j \lambda \zeta_j = \sum_{j=1}^3 \Delta x_j \lambda \zeta_j + \sum_{j=1}^3 \lambda x_j \Delta \zeta_j = 0,$$

from which it follows that $f(x)$ is identically constant in D . But if $f(x)$ were identically constant in D , then it would follow from the representation (28) for the functions (5) that the functions (5) are identically constant in D . From this contradiction of the hypothesis, it follows that E can vanish only at isolated points of D .

24. See footnote 23.

which, with (56), establishes (46).

From (41), (43) and (44) we obtain

$$\sum_{j=1}^3 \lambda x_j \Delta \lambda x_j = 0,$$

which, with (56), yields (47).

From (42)-(47), inclusive, it follows that

$$(57) \quad \sum_{j=1}^3 \lambda x_j \Delta^2 \lambda x_j = 2 \lambda^2 e \sum_{j=1}^3 \lambda x_j \bar{\lambda} \zeta_j.$$

From (36), (56) and (57) we obtain (39).

From (8) and (56) we obtain

$$\sum_{j=1}^3 (\lambda \zeta_j)^2 = 0,$$

which combines with (36), (41), (49) and (50) to yield (40).

Since the functions (5) have continuous partial derivatives of all orders in $\mathcal{D}$, the following expression, obtained from (16) by setting $m=5$, is valid:

$$(58) \quad \sum_{j=1}^3 \left| \int_{C_n} x_j(u,v) dz \right|^2 = -\pi^2 \lambda^4 \sum_{j=1}^3 (\lambda x_j)^2 - \frac{\pi^2}{4} \lambda^6 \sum_{j=1}^3 \lambda x_j \Delta \lambda x_j - \\ \frac{\pi^2}{16} \lambda^8 \sum_{j=1}^3 \left[2 \lambda x_j \Delta^2 \lambda x_j + 3 (\Delta \lambda x_j)^2 \right] + O(\lambda^8),$$

where the partial derivatives are evaluated at (u_0, v_0) and where C_n is the circle in $\mathcal{D}$ with center at (u_0, v_0) and radius λ ; (u_0, v_0) is an arbitrary point in $\mathcal{D}$.

Applying (8) and the relation

$$\sum_{j=1}^3 \lambda x_j \Delta \lambda x_j = 0, \quad (25)$$

to the relation (58), we obtain

$$(59) \quad \sum_{j=1}^3 \left[\int_{C_n} x_j(u, v) dz \right]^2 = \frac{\pi^2}{192} \lambda^8 \sum_{j=1}^3 [2 \lambda x_j \Delta \lambda x_j + 3 (\Delta \lambda x_j)^2] + O(\lambda^6).$$

Therefore (34) holds.

Let us now suppose that (35) does not hold, i.e., for each point (u_0, v_0) in D ,

$$(60) \quad \sum_{j=1}^3 \left[\int_{C_\lambda} x_j(u, v) dz \right]^2 = O(\lambda^6),$$

where C_λ is the circle in D with center at (u_0, v_0) and radius λ . From (59) and (60) it follows that

$$(61) \quad \sum_{j=1}^3 [2 \lambda x_j \Delta^2 \lambda x_j + 3 (\Delta \lambda x_j)^2] = 0$$

holds. From (39), (40) and (61) we obtain

$$(62) \quad \alpha^2 [2 E \lambda^2 E - 3 (\lambda E)^2] = 0.$$

But, by (36), $\alpha \neq 0$, and therefore (62) yields

$$(63) \quad 2 E \lambda^2 E - 3 (\lambda E)^2 = 0.$$

From (63) we obtain

$$2 \frac{\lambda^2 E}{\lambda E} = 3 \frac{\lambda E}{E},$$

which may be written in the following form:

$$2 \lambda (\log \lambda E) = 3 \lambda (\log E).$$

25. This relation follows from (47) and (56).

Hence

$$(64) \quad \lambda E = E^{3/2} e^{Q(z)},$$

where

$$(65) \quad \lambda Q(z) = 0.$$

From the imaginary part of (63) we obtain

$$2 \frac{E_{uv}}{E_v} du = 3 \frac{E_u}{E} du, \quad 2 \frac{E_{uv}}{E_u} dv = 3 \frac{E_v}{E} dv,$$

which imply

$$(66) \quad E_u = E^{3/2} e^{\phi_1(u)}, \quad E_v = E^{3/2} e^{\phi_2(v)},$$

where $\phi_1(u)$ is a function of u and $\phi_2(v)$ is a function of v . From (64), (65) and (66) it follows that the function

$$(67) \quad e^{Q(z)} \equiv e^{\frac{\phi_1(u)}{1+i}} e^{\frac{\phi_2(v)}{1-i}}$$

is an analytic function of $z = u + i v$. From the Cauchy-Riemann equations for the function (67) it follows that

$$e^{\frac{\phi_1(u)}{1+i}} \equiv 2a_0 u + a_1, \quad e^{\frac{\phi_2(v)}{1-i}} \equiv 2a_0 v + a_2,$$

where a_0 , a_1 and a_2 are real constants. Hence (64) may be written in the form

$$-2\lambda E^{-1/2} = 2a_0(1+i)v + (a_1+i a_2),$$

which yields

$$(68) \quad \frac{E_u}{E^{3/2}} du = (2a_0 u + a_1) du, \quad \frac{E_v}{E^{3/2}} dv = (2a_0 v + a_2) dv,$$

From (68) we obtain

$$-2E^{-\frac{1}{2}} = a_0(u^2+v^2) + a_1u + a_2v + a_3,$$

where a_3 is a real constant. Hence

$$(69) \quad E = \frac{4}{[a_0(u^2+v^2) + a_1u + a_2v + a_3]^2}.$$

Substitution in (31) shows that

$$(70) \quad K = \frac{4a_0a_3 - a_1^2 - a_2^2}{4},$$

which, since S is on a sphere of non-null radius, implies that $a_0 \neq 0$. Hence (69) yields

$$E = \frac{1}{\frac{a_0^2}{4} \left[\left(u + \frac{a_1}{2a_0}\right)^2 + \left(v + \frac{a_2}{2a_0}\right)^2 + M \right]^2},$$

where

$$(71) \quad M = \frac{4a_0a_3 - a_1^2 - a_2^2}{4a_0^2}.$$

From (70) and (71) it follows that $K = a_0^2 M$. But since S is on a sphere of finite non-null radius, it follows that $M > 0$. Therefore the first quadratic form of S may be written as

$$ds^2 = \frac{\alpha^2 (du^2 + dv^2)}{\left[\left(u + \frac{a_1}{2a_0}\right)^2 + \left(v + \frac{a_2}{2a_0}\right)^2 + \beta^2 \right]^2},$$

where

$$\alpha = \left| \frac{2}{a_0} \right|, \quad \beta = \sqrt{M}.$$

Therefore the first quadratic form of S has the repre-

sentation (29). It follows that the functions (5) map all circles in D on circles on S ; here we have made use of Lemma 2. This is a contradiction of our hypothesis. Therefore (35) holds.

Sufficiency. By setting $m=3$ in (16) we obtain

$$(72) \quad \sum_{j=1}^3 \left| \int_{C_\lambda} x_j(u, v) dz \right|^2 = -\pi^2 \lambda^4 \sum_{j=1}^3 (\lambda x_j)^2 - \frac{\pi^2}{4} \lambda^6 \sum_{j=1}^3 \lambda x_j \Delta \lambda x_j + O(\lambda^6),$$

where C_λ is the circle in D with center at the arbitrary but fixed point (u_0, v_0) and radius λ , and where the partial derivatives are evaluated at (u_0, v_0) .

From (34) and (72) it follows that (8) and

$$(73) \quad \sum_{j=1}^3 \lambda x_j \Delta \lambda x_j = 0$$

must hold. Operating on (8) with $\bar{\lambda}$, we obtain

$$(74) \quad \sum_{j=1}^3 \lambda x_j \Delta x_j = 0.$$

Operating on (74) with the operator λ , and applying the relation (73) to the result, we obtain

$$(75) \quad \sum_{j=1}^3 \lambda^2 x_j \Delta x_j = 0.$$

The four real linear homogeneous equations in

$$\Delta x_j, \quad j=1, 2, 3,$$

implied by (74) and (75) are

$$(76) \quad \sum_{j=1}^3 x_{j,u} \Delta x_j = \sum_{j=1}^3 x_{j,v} \Delta x_j = \sum_{j=1}^3 (x_{j,uu} - x_{j,vv}) \Delta x_j = \sum_{j=1}^3 x_{j,uv} \Delta x_j = 0,$$

One solution of the system (76) is

$$(77) \quad \Delta \kappa_j \equiv 0, \quad j=1,2,3,$$

which, by (8) and the theorem of Weierstrass, implies that the functions (5) map D isothermally on a minimal surface. From (8), (16) and (77) it follows that

$$(78) \quad \sum_{j=1}^3 \left[\int_C \kappa_j(u,v) dz \right]^2 = 0$$

holds for all circles C in D . Therefore (60) holds. But this is a contradiction of (35), and therefore the functions (5) do not satisfy (77). It follows that the given functions, which satisfy (76), do not satisfy (77), and therefore the rank of the matrix

$$\begin{vmatrix} \kappa_{1u} & \kappa_{2u} & \kappa_{3u} \\ \kappa_{1v} & \kappa_{2v} & \kappa_{3v} \\ \kappa_{1uv} & \kappa_{2uv} & \kappa_{3uv} \\ \kappa_{1uu} - \kappa_{1vv} & \kappa_{2uu} - \kappa_{2vv} & \kappa_{3uu} - \kappa_{3vv} \end{vmatrix}$$

is less than three.²⁶ Hence, in particular, we obtain

$$|\kappa_{1u} \kappa_{2v} \kappa_{3uu} - \kappa_{3vv}| = |\kappa_{1u} \kappa_{2v} \kappa_{3uv}| = 0.$$

Therefore (14) holds. From (8), (14) and Theorem A it follows that the functions (5) map D isothermally on a surface S that lies on a sphere $\mathcal{S}$ of finite²⁷

26. M. Bocher, Introduction to Higher Algebra, p. 47.

27. If the functions (5) defined a plane surface in

non-null radius.

Suppose that the functions (5) map D isothermically on the spherical surface S such that circles are mapped on circles. Then the map of the circle C in D is a circle, C^* , on S . Project C^* stereographically on the plane P of C^* and let the map of S be D^* . It follows that D has been mapped isothermically on the plane surface D^* . By the theorem of Weierstrass, the mapping functions form a triple of conjugate harmonic functions:

$$u_j = u_j(u, v), \quad j=1, 2, 3,$$

where (6) and

$$\Delta u_j \equiv 0, \quad j=1, 2, 3,$$

hold. As above, it follows that

$$(79) \quad \sum_{j=1}^3 \left[\int_C u_j(u, v) dz \right]^2 = 0$$

holds for the arbitrary fixed circle C in D . But for (u, v) on C ,

$$K_j(u, v) = u_j(u, v), \quad j=1, 2, 3,$$

which with (79) implies that (78) holds. It follows that (60) holds. But this is a contradiction of (35). Hence the functions (5) map D isothermically on S such that circles are not mapped on circles.

isothermic representation, then (77) would hold. We have shown that (77) cannot hold; therefore the sphere is one of finite radius.

3. CHARACTERIZATION OF THOSE ISOTHERMIC SPHERICAL MAPS THAT MAP CIRCLES ON CIRCLES AND OF MINIMAL SURFACES IN ISOTHERMIC REPRESENTATION.

Theorem 3. If the functions (5) have continuous partial derivatives of the third order in a simply connected domain D , then a necessary and sufficient condition that they either map D isothermically on a surface that lies on a sphere of finite non-null radius such that circles are mapped on circles, or be the coordinate functions of a minimal surface given in isothermic representation, is that for each circle C in D ,

$$(80) \quad \sum_{j=1}^3 \left| \int_C x_j(u,v) dz \right|^2 = 0.$$

Necessity. We have already shown that if the functions (5) are the coordinate functions of a minimal surface in isothermic representation, then (80), i.e., (78), holds.²⁸ We have also shown that if the functions (5) map D isothermically on a surface S that lies on a sphere of finite non-null radius such that circles are mapped on circles, then (80) holds.²⁹

Sufficiency. If (80) holds, then (60) holds. It follows that (8) and (73) hold, and therefore, as in the proof of Theorem 2, we obtain (76). Then, as be-

28. See p. 29.

29. See p. 30.

fore, it follows that the functions either map D isothermally on a surface that lies on a sphere of finite non-null radius such that circles are mapped on circles, or map D isothermally on a minimal surface. Here we have used the results of the second part of the proof of Theorem 2. But in the first part of the proof of Theorem 2 we showed that (60) implies that the spherical surface S , of which the coordinate functions are the functions (5), has its first quadratic form in the representation (29); therefore, by Lemma 1, all circles in D are mapped on circles on S .³⁰

Corollary 1. If the functions (5) have continuous partial derivatives of the third order in a simply connected domain D , then a necessary and sufficient condition that

$$\sum_{j=1}^3 \left[\int_C x_j(u,v) dz \right]^2 = 0$$

hold for all circles C in D is that for each point (u_0, v_0) in D ,

$$\sum_{j=1}^3 \left[\int_{C_\lambda} x_j(u,v) dz \right]^2 = \sigma(\lambda^8),$$

where C_λ is the circle in D with center at (u_0, v_0) and radius λ .

30. If the radius of the sphere were either null or infinite, (5) would be a triple of conjugate harmonic functions.

Corollary 2. If the functions (5) have continuous partial derivatives of the third order in a simply connected domain D, then a necessary and sufficient condition that they either map D isothermally on a surface that lies on a sphere of finite non-null radius such that circles are mapped on circles, or map D isothermally on a minimal surface, is that for each point (u_0, v_0) in D,

$$\sum_{j=1}^3 \left| \int_{C_n} x_j(u, v) dz \right|^2 = \sigma(n^2),$$

where C_n is the circle in D with center at (u_0, v_0) and radius n .

4. CHARACTERIZATION OF ISOTHERMIC PLANE MAPS.

Lemma 3. Let Γ be a closed rectifiable Jordan curve; denote the simply connected domain inside Γ by D, and let R be a closed region that lies in D. If $f(z)$ is analytic in D and continuous in the closed region $D + \Gamma$, then for any arbitrary positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$, which is independent of z in R, for which

$$(80) \quad \left| \frac{f(z') - f(z'')}{z' - z''} - f'(z_0) \right| < \epsilon,$$

for all z' , z'' and z_0 in R that satisfy the inequalities

$$(81) \quad |z' - z_0| < \delta, \quad |z'' - z_0| < \delta, \quad z' \neq z'',$$

31. Clearly, by the uniform continuity of the

Proof. If z' and z'' are any two points of R , $z' \neq z''$, then by the Cauchy integral formula we obtain

$$(82) \quad \frac{f(z') - f(z'')}{z' - z''} - f'(z'') = \frac{z' - z''}{2\pi i} \int_{\Gamma} \frac{f(z) dz}{(z - z'')^2 (z - z')}.$$

If ϵ is an arbitrary pre-assigned positive number, then it follows from (82) that

$$(83) \quad \left| \frac{f(z') - f(z'')}{z' - z''} - f'(z'') \right| < \frac{\epsilon}{2},$$

if

$$(84) \quad |z' - z''| < \delta_1 = \pi d^3 \epsilon / LM,$$

where M is the maximum of $|f(z)|$ on Γ , L is the length of Γ and d is the minimum distance between R and Γ . Here $d > 0$.

The function $f'(z)$ is uniformly continuous in R ; hence there exists a positive number δ_2 such that in R

$$(85) \quad |f'(z'') - f'(z_0)| < \epsilon/2,$$

when

$$(86) \quad |z'' - z_0| < \delta_2.$$

If we choose

$$\delta = \min(\frac{1}{2}\delta_1, \delta_2),$$

then the inequalities (81) imply (84) and (86).

Hence, if (81) holds, then (83) and (85) hold.

From (83) and (85) we obtain (80).

function $f'(z)$ in R , this last restriction can be removed by defining

$$\frac{f(z'') - f(z')}{z'' - z'} = f'(z''), \quad z' = z''.$$

Lemma 4. Let the function $g_1(z)$ be schlicht,³² and the function $g_2(z)$ analytic, in the circle $|z| < \rho$. If δ is an arbitrary positive number, $0 < \delta < \rho$, then there exists a positive number $\epsilon = \epsilon(\delta)$ such that the function

$$(87) \quad f(z) = g_1(z) + \eta g_2(z)$$

is schlicht in the circle $|z| \leq \rho - \delta$, for all η such that $|\eta| < \epsilon$.

Proof. If the lemma does not hold, then there exists a triple of sequences,

$$(88) \quad [z'_m]_1^\infty = z'_1, z'_2, \dots, \quad \left\{ \begin{array}{l} z'_m \neq z''_m, \quad m=1, 2, \dots, \\ (89) \quad [z''_m]_1^\infty = z''_1, z''_2, \dots, \end{array} \right.$$

$$(90) \quad [\eta_m]_1^\infty = \eta_1, \eta_2, \dots,$$

with the properties

$$(91) \quad \lim_{m \rightarrow \infty} \eta_m = 0,$$

$$(92) \quad |z'_m| \leq \rho - \delta, \quad |z''_m| \leq \rho - \delta, \quad m=1, 2, \dots,$$

$$(93) \quad f_m(z'_m) = f_m(z''_m), \quad m=1, 2, \dots,$$

where

$$(94) \quad f_m(z) = g_1(z) + \eta_m g_2(z), \quad m=1, 2, \dots.$$

32. The analytic function $g_1(z)$ is said to be schlicht in $|z| < \rho$ if $g_1(z') = g_1(z'')$ implies $z' = z''$, $|z'| < \rho, |z''| < \rho$.

From (92) it follows that (88) and (89) have cluster points in the circle $|z| \leq \rho - \delta$. Hence there exist subsequences,

$$\{z'_{m_p}\}_{p=1}^{\infty} \quad \text{and} \quad \{z''_{m_p}\}_{p=1}^{\infty},$$

such that

$$(95) \quad \lim_{p \rightarrow \infty} z'_{m_p} = z'_0,$$

$$(96) \quad \lim_{p \rightarrow \infty} z''_{m_p} = z''_0,$$

where z'_0 and z''_0 are cluster points of (88) and (89) respectively. Then

$$(97) \quad |z'_0| \leq \rho - \delta, \quad |z''_0| \leq \rho - \delta,$$

and from (91) it follows that

$$(98) \quad \lim_{p \rightarrow \infty} \eta_{m_p} = 0$$

From (93)-(96) and (98) it follows that

$$(99) \quad g_1(z'_0) = g_1(z''_0)$$

holds. Since $g_1(z)$ is schlicht in the circle $|z| \leq \rho - \delta$, it follows that (97) and (99) imply

$$(100) \quad z' = z'' = z_0.$$

From (93)-(100) and Lemma 3 we obtain

$$(101) \quad \lim_{p \rightarrow \infty} \frac{f_{m_p}(z'_{m_p}) - f_{m_p}(z''_{m_p})}{z'_{m_p} - z''_{m_p}} = \left. \frac{d}{dz} g_1(z) \right|_{z=z_0} = 0.$$

Since $|z_0| \leq \rho - \delta$, (101) implies that the function $g_1(z)$ is not schlicht in the circle $|z| < \rho$. From this

contradiction of our hypothesis, it follows that Lemma 4 must hold.

Theorem 4. If the functions (5) have continuous partial derivatives of the third order in a simply connected domain D , then a necessary and sufficient condition that they map D isothermally on a plane surface is that for all closed rectifiable Jordan curves δ lying in D ,

$$(102) \quad \sum_{j=1}^3 \left[\int_{\delta} \chi_j(u, v) dz \right]^2 = 0.$$

Necessity. If the functions (5) map D isothermally on a surface S that lies on a plane ρ , then we make a rigid transformation of the x_1, x_2, x_3 -space such that ρ coincides with the plane $x_3' = 0$ and the positive normal at the image of an arbitrary point (u_0, v_0) of D coincides with the positive x_3' -axis. Let this transformation be given by

$$(103) \quad x_j' = \sum_{k=1}^3 \lambda_{kj} x_k + a_j, \quad j=1,2,3,$$

where

$$(104) \quad \sum_{k=1}^3 \lambda_{sk} \lambda_{jk} = 0, \quad s \neq j \\ = 1, \quad s=j, \quad s, j=1,2,3,$$

and where the a_j and λ_{jk} are real constants.³³ Therefore the functions

33. C. H. Sisam, Analytic Geometry, p. 267.

$$(105) \quad \chi'_j(u, v) = \sum_{k=1}^3 \lambda_{kj} [\chi_k(u, v) - \chi_k(u_0, v_0)], \quad j = 1, 2, 3,$$

$$\chi'_3(u, v) \equiv 0,$$

map D isothermally on a plane surface; hence, by the theorem of Weierstrass, they form a triple of conjugate harmonic functions. From (3), (6), (7), (104) and (105) it follows that the functions $\chi'_1(u, v)$ and $\chi'_2(u, v)$ are a couple of conjugate harmonic functions. Hence by Cauchy's theorem, analogous to (12), we obtain

$$\sum_{j=1}^2 \left[\int_{\gamma} \chi'_j(u, v) dz \right]^2 = 0,$$

which, with (105), implies

$$(106) \quad \sum_{j=1}^3 \left[\int_{\gamma} \chi'_j(u, v) dz \right]^2 = 0.$$

From (9), (103) and (104) we obtain

$$(107) \quad \sum_{j=1}^3 \left[\int_{\gamma} \chi'_j(u, v) dz \right]^2 = \sum_{j=1}^3 \left[\int_{\gamma} \chi_j(u, v) dz \right]^2.$$

From (106) and (107) we obtain (102).

Sufficiency. If (102) holds for all closed rectifiable Jordan curves γ lying in D , then, by Theorem 3, the functions (5) either are the coordinate functions of a minimal surface given in isothermic representation, or they map D isothermally on a surface that lies on a sphere of finite non-null radius such that circles are mapped on circles.

Part I. We first consider the case when the func-

tions (5) are the coordinate functions of a minimal surface given in isothermic representation. Let (u_0, v_0) be an arbitrary point in D and let the circle C_0 in D have its center at (u_0, v_0) . If γ is an arbitrary closed rectifiable Jordan curve lying in D , which contains (u_0, v_0) in its interior, then the interior of γ can be mapped conformally on the interior of the circle C : $s^2 + t^2 = \rho^2$ in the s, t -plane such that the image of (u_0, v_0) is the center of C . The expansion of the inverse of this mapping function has the representation

$$(108) \quad z = h(w) = z_0 + \sum_{n=1}^{\infty} b_n w^n,$$

where

$$w = s + it, \quad z_0 = u_0 + i v_0.$$

Since the functions (5) are the isothermic coordinate functions of a minimal surface, it follows, from the theorem of Weierstrass, that these functions may be written in the form

$$(109) \quad x_j(u, v) = \frac{1}{2} \left[\phi_j(z) + \overline{\phi_j(z)} \right], \quad j=1, 2, 3,$$

where $\phi_j(z)$ is a function that is analytic in D and $\overline{\phi_j(z)}$ its conjugate function. We may write

$$\phi_j(z) = \sum_{n=0}^{\infty} a_{j,n} (z - z_0)^n, \quad j=1, 2, 3;$$

then

$$\overline{\phi_j(z)} = \sum_{m=0}^{\infty} \overline{a_{j,m}} (\overline{z-z_0})^m, \quad j=1,2,3, \quad {}^{34}$$

where the series for $\phi_j(z)$ and $\overline{\phi_j(z)}$ are absolutely convergent in the interior of and on C_0 , and where $\overline{a_{j,m}}$ and $(\overline{z-z_0})^m$ are conjugates of $a_{j,m}$ and $(z-z_0)^m$ respectively. Therefore

$$(110) \quad \overline{\phi_j(z)} = \overline{a_{j,0}} + \sum_{m=1}^{\infty} \overline{a_{j,m}} \sum_{n=1}^{\infty} C_{m,n} \overline{w}^n, \quad j=1,2,3, \quad {}^{35}$$

since

34. This is a Taylor expansion for $\overline{\phi_j(z)}$ and hence it converges absolutely in and on C_0 ; likewise for $\phi_j(z)$.

35. Since (108) is absolutely convergent for $|w| < \rho$, the series for $(z-z_0)^m$ is absolutely convergent in the circle $|w| < \rho$, $m=1,2,\dots$. See W. L. Ferrar, Convergence, p. 75. Moreover, the Taylor series for $\phi_j(z)$ is uniformly convergent in C_0 , and therefore it is uniformly convergent in and on γ . Hence the series

$\sum_{m=1}^{\infty} a_{j,m} [h(w)-z_0]^m$ is uniformly convergent, $|w| \leq \rho'$, for all ρ' , $0 \leq \rho' < \rho$. Hence by the Weierstrass double series theorem (K. Knopp, Funktionentheorie, I, p. 89), the series $\sum_{m=1}^{\infty} a_{j,m} [h(w)-z_0]^m$ is uniformly convergent to

the analytic function $\phi_j[h(w)] = \phi_j(z)$, $j=1,2,3$. Since a similar discussion is valid for the right-hand member of (110), it follows that (110) must hold.

$$\overline{(z - z_0)}^n = \sum_{m=1}^{\infty} C_{n,m} \bar{w}^m, \quad n = 1, 2, \dots,$$

where

$$(111) \quad \begin{aligned} C_{n,m} &= \sum_{k_j} \bar{b}_{k_1} \dots \bar{b}_{k_m}, \\ k_1 + \dots + k_m &= m, \quad k_j \geq 1, \quad j = 1, 2, \dots, m, \\ C_{n,m} &= 0, \quad m > n. \end{aligned}$$

If C_R is the circle concentric with C and of radius $R < \rho$, then the function (108) maps the interior of C_R conformally on the interior of a closed rectifiable Jordan curve δ_R that lies in the interior of δ ; then from (9), (108) and (109) we obtain

$$(112) \quad \int_{\delta_R} x_j(u, v) dz = \frac{1}{2} \int_{C_R} \overline{\phi_j[h(w)]} h'(w) dw, \quad j = 1, 2, 3.$$

From (108) we obtain

$$(113) \quad h'(w) = \sum_{m=1}^{\infty} m b_m w^{m-1}.$$

After setting $w = Re^{i\varphi}$ in (108), (110), (112) and (113), it follows, since the resulting series are uniformly convergent, that

$$(114) \quad \int_{\delta_R} x_j(u, v) dz = \pi i \sum_{n=1}^{\infty} \bar{a}_{j,n} \sum_{m=1}^{\infty} m b_m C_{n,m} R^{2m}, \quad j = 1, 2, 3,$$

must hold. From (114) it follows that

$$(115) \quad \sum_{j=1}^3 \left[\int_{\delta_R} x_j(u, v) dz \right]^2 = -\pi^2 \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \sum_{j=1}^3 \bar{a}_{j,n} \bar{a}_{j,s} \sum_{m=1}^{\infty} \sum_{t=1}^{\infty} b_{n,m} b_{s,t} R^{2m+2t},$$

holds, where

$$(116) \quad B_{n,m} = m b_m C_{n,m}, \quad n, m = 1, 2, \dots.$$

Note that

$$(117) \quad B_{\kappa, \lambda} = 0, \quad \kappa > \lambda, \quad ^{36}$$

Therefore it follows from (102), (115) and (117) that

$$(118) \quad \sum_{t=2}^{\infty} \sum_{m=1}^{t-1} \sum_{M=1}^m \sum_{s=1}^{t-m} A_{m,s} B_{m,m} B_{s,t-m} R^{2t} = 0,$$

where

$$(119) \quad A_{m,s} = A_{s,m} = \sum_{j=1}^3 \bar{a}_{j,m} \bar{a}_{j,s}, \quad m, s = 1, 2, \dots.$$

But recall that the circle C_R was an arbitrary circle concentric with and interior to the circle C . Therefore the relation (118) is independent of R , i.e., (118) holds for all R , $0 \leq R < \rho$. Hence

$$(120) \quad \sum_{m=1}^{t-1} \sum_{s=1}^{t-m} \sum_{M=1}^m A_{m,s} B_{m,m} B_{s,t-m} = 0, \quad t = 2, 3, \dots. \quad ^{37}$$

From (120) we obtain, by an induction,

$$(121) \quad A_{m,s} = 0, \quad m, s = 1, 2, \dots,$$

For, when $t=2$ in (120) we obtain

$$(122) \quad A_{1,1} B_{1,1}^2 = 0.$$

From (111), (116) and (122) we obtain

$$(123) \quad A_{1,1} b_1^2 \bar{b}_1^2 = 0.$$

36. This follows from (111) and (116).

37. See theorem on p. 170 of K. Knopp, Theorie und Anwendung der Unendlichen Reihen, 2nd ed..

But the function whose inverse is (108) maps the interior of γ on the interior of C in a one-to-one manner.³⁸ Therefore

$$(124) \quad b_1 \neq 0,$$

From (123) and (124) we obtain

$$(125) \quad A_{1,1} = 0.$$

Now suppose that

$$(126) \quad A_{m,s} = 0, \quad 2 \leq m+s \leq p-1,$$

where m , s and p are positive integers. We shall show that

$$(127) \quad A_{m,s} = 0, \quad m+s = p,$$

holds. For $t=p$, (120) yields

$$(128) \quad \sum_{m=1}^{p-1} \sum_{s=1}^{p-m} \sum_{n=1}^m A_{m,s} B_{m,n} B_{s,p-m} = 0,$$

From (126) and (128),

$$(129) \quad \sum_{k=1}^{p-1} A_{k,p-k} B_{k,k} B_{p-k,p-k} = 0.$$

But (111) and (116) yield

$$B_{s,s} = s b_s \bar{b}_1^s, \quad s \geq 1,$$

and hence (129) may be written in the form

38. A consequence of Darboux's theorem. See

W. F. Osgood, Functions of a Complex Variable, p. 167.

$$(130) \quad \bar{b}_1^p \sum_{k=1}^{p-1} k(p-k) b_k b_{p-k} A_{k,p-k} = 0.$$

From (124) and (130) we obtain

$$(131) \quad \sum_{k=1}^{p-1} k(p-k) b_k b_{p-k} A_{k,p-k} = 0.$$

By Lemma 4, for a fixed R , $0 < R < \rho$, the functions

$$(132) \quad G_g(w) = h(w) + \eta_g w^g = z_0 + \sum_{m=1}^{\infty} b_m w^m + \eta_g w^g, \quad g=1,2,\dots,p-1,$$

are schlicht in the circle $|w| \leq R$ provided

$$|\eta_g| < \epsilon_{1,g}, \quad g=1,2,\dots,p-1,$$

where $\epsilon_{1,g}$ is a positive constant whose existence was established in Lemma 4. We further restrict η_g to satisfy the condition

$$R^g |\eta_g| < \delta,$$

for $g=1,2,\dots,p-1$, where δ is the distance between γ_R and γ , in order that the map $\gamma_{g,R}$ of C_R in C_0 by (132) shall lie inside γ . Moreover, (u_0, v_0) is inside $\gamma_{g,R}$. Hence, if

$$(133) \quad 0 < |\eta_g| < d_g = \min[\epsilon_{1,g}, \delta/R^g], \quad g=1,2,\dots,p-1,$$

then we obtain the following result which is analogous to (131) for the functions (132):

$$(134) \quad (a) \sum_{k=1}^{p-1} k(p-k) b_k b_{p-k} A_{k,p-k} + 2g(p-g) b_{p-g} A_{g,p-g} \eta_g = 0,$$

$$(134) \quad (b) \sum_{k=1}^{p-1} k(p-k) b_k b_{p-k} A_{k,p-k} + \delta^2 (\eta_\delta^2 + 2b_\delta \eta_\delta) A_{\delta,\delta} = 0,$$

where (a) holds for all δ , $1 \leq \delta \leq p-1$, except $\delta = p/2$, and where (b) holds for $\delta = p/2$.³⁹ From (131) and (134) we obtain

$$(135) \quad (a) \quad 2\delta(p-\delta) b_{p-\delta} A_{\delta,p-\delta} \eta_\delta = 0,$$

$$(b) \quad \delta^2 (\eta_\delta^2 + 2b_\delta \eta_\delta) A_{\delta,\delta} = 0,$$

where (a) holds for all δ , $1 \leq \delta \leq p-1$, except $\delta = p/2$, and where (b) holds for $\delta = p/2$.

Since η_δ is an arbitrary constant which is subject only to the restriction (133), it follows from the relation (135), (b), that

$$(136) \quad A_{\delta,\delta} = 0, \quad \delta = p/2,$$

must hold, provided p is even. If δ_1 is a fixed positive integer, $1 \leq \delta_1 \leq p-1$, and $\delta_1 \neq p/2$, and if $b_{p-\delta_1} \neq 0$, then it follows from (135), (a), that

39. The statement has meaning only when p is an even positive integer because δ is a positive integer. Hence when we state "except for $\delta = p/2$ " we mean that if p is an even integer, then δ does not take on the integral value $p/2$. In a similar manner, we say that (134), (b), holds for $\delta = p/2$.

$$(137) \quad A_{g_1, p-g_1} = 0.$$

If g_2 is a fixed positive integer, $1 \leq g_2 \leq p-1$, $g_2 \neq p/2$, and if $b_{p-g_2} = 0$, then we consider the function

$$(138) \quad h_1(w) \equiv h(w) + \eta'_{p-g_2} w^{p-g_2},$$

where η'_{p-g_2} is a fixed constant,

$$0 < |\eta'_{p-g_2}| < d_{p-g_2}.$$

The function (138) is schlicht in the circle $|w| \leq R$,⁴⁰ and maps C_R on a closed rectifiable Jordan curve δ'_R which lies inside γ . Moreover, (u_0, v_0) is inside δ'_R . From (108) and (138) we obtain

$$(139) \quad h_1(w) = z_0 + \sum_{m=1}^{\infty} b_{1,m} w^m,$$

where

$$b_{1,m} = b_m, \quad m \neq p-g_2,$$

$$b_{1,p-g_2} = \eta'_{p-g_2}.$$

If we apply (139) as we have applied (108) in the earlier part of this proof of Theorem 4, then we obtain the following result, which is analogous to the relation (135), (a):

$$(140) \quad 2g_2(p-g_2)A_{g_2, p-g_2} \eta'_{p-g_2} \eta''_{g_2} = 0,$$

40. This holds because $|\eta'_{p-g_2}| < d_{p-g_2}$, where d_{p-g_2} was defined in (133).

where η''_{δ_2} is an arbitrary constant,

$$(141) \quad 0 < |\eta''_{\delta_2}| < d'_{\delta_2}.$$

Here the constant d'_{δ_2} is in the same relation to $h(\omega)$ as d_{δ_2} is to $h(\omega)$. Since $\eta'_{p-\delta_2} \neq 0$ and $\delta_2 \neq p/2$, we obtain the following result from (140) and (141):

$$(142) \quad A_{\delta_2, p-\delta_2} = 0.$$

From (120), (136), (137) and (142) it follows that (127) holds provided (126) holds. Since (125) holds, our induction is now complete.

From (119), (121) and the substitution

$$\beta_{j,0} = 0, \quad \alpha_{j,m} = \alpha_{j,m} - i \beta_{j,m}, \quad m=0,1,\dots, \quad j=1,2,3,$$

we obtain the following relations

$$(143) \quad \sum_{j=1}^3 (\alpha_{j,m} \alpha_{j,s} - \beta_{j,m} \beta_{j,s}) = 0,$$

$$\sum_{j=1}^3 (\alpha_{j,m} \beta_{j,s} + \alpha_{j,s} \beta_{j,m}) = 0, \quad m, s \geq 1, \quad 4'$$

where the $\alpha_{j,0}$, $\alpha_{j,m}$ and $\beta_{j,m}$ are the Fourier coefficients in the expansion of $x_j(u,v)$, $j=1,2,3$, about the point (u_0, v_0) . These expansions are

$$(144) \quad x_j(u,v) = \alpha_{j,0} + \sum_{m=1}^{\infty} r^m (\alpha_{j,m} \cos m\theta + \beta_{j,m} \sin m\theta), \quad j=1,2,3.$$

41. Compare with J. W. Hahn and E. F. Beckenbach,
Triples of Conjugate Harmonic Functions and Minimal Sur-

We transform the axes in the x_1, x_2, x_3 -space such that the new origin is at the image of (u_0, v_0) , the plane $x_3^1 = 0$ is tangent to the surface there, and the positive normal to the surface there coincides with the positive x_3^1 -axis. This transformation is given by the equations in (103), where

$$a_j = - \sum_{k=1}^3 \lambda_{kj} x_k(u_0, v_0), \quad j=1, 2, 3,$$

and where (104) holds. Then the relations (143) are invariant under this rigid transformation. For, since the functions (5) form a triple of conjugate harmonic functions, it follows that the new coordinate functions

$$(145) \quad x_j^1(u, v) = \sum_{k=1}^3 \lambda_{kj} [x_k(u, v) - x_k(u_0, v_0)], \quad j=1, 2, 3,$$

form a triple of conjugate harmonic functions.⁴² Hence we may write

faces, Duke Math. Journ., v. 2 (1936), Lemma 1, p. 699, and footnote on p. 700.

42. Since the functions (5) are harmonic in D , it follows that the functions (145) are harmonic in D .

From (104) and (145) we obtain $\sum_{j=1}^3 \left(\frac{\partial x_j^1}{\partial u} \right)^2 = \sum_{j=1}^3 \left(\frac{\partial x_j^1}{\partial v} \right)^2$,

$$\sum_{j=1}^3 \frac{\partial x_j^1}{\partial u} \frac{\partial x_j^1}{\partial v} = \sum_{j=1}^3 \frac{\partial x_j^1}{\partial u} \frac{\partial x_j^1}{\partial v} \quad \text{and} \quad \sum_{j=1}^3 \left(\frac{\partial x_j^1}{\partial v} \right)^2 = \sum_{j=1}^3 \left(\frac{\partial x_j^1}{\partial u} \right)^2.$$

Hence the functions (145) are isothermic.

$$(146) \quad x_j^1(u, v) = \sum_{m=0}^{\infty} R^m (\alpha_{j,m}^1 \cos m\theta + \beta_{j,m}^1 \sin m\theta), \quad \beta_{j,0}^1 = 0, \quad j=1,2,3.$$

Since the $x_j^1(u, v)$ are equal to zero, it follows from the relation (146) that

$$(147) \quad \alpha_{j,0}^1 = 0, \quad j=1,2,3.$$

From (144), (145) and (146) we obtain

$$\alpha_{j,m}^1 = \sum_{k=1}^3 \lambda_{kj} \alpha_{k,m}, \quad \beta_{j,m}^1 = \sum_{k=1}^3 \lambda_{kj} \beta_{k,m}, \quad \begin{matrix} m=1,2,\dots \\ j=1,2,3 \end{matrix}$$

which, with (104) and (143), imply

$$(148) \quad \begin{aligned} \sum_{j=1}^3 (\alpha_{j,s}^1 \alpha_{j,m}^1 - \beta_{j,s}^1 \beta_{j,m}^1) &= 0, \\ \sum_{j=1}^3 (\alpha_{j,s}^1 \beta_{j,m}^1 + \alpha_{j,m}^1 \beta_{j,s}^1) &= 0, \quad m, s \geq 1, \end{aligned}$$

where m and s are positive integers. Therefore the relations (143) are invariant under the rigid transformation defined by (103) and (104).

To prove the functions (5) define a plane surface it is sufficient that we show that

$$(149) \quad \alpha_{3,0}^1 = \alpha_{3,m}^1 = \beta_{3,m}^1 = 0, \quad m=1,2,\dots, \quad 43$$

holds. Let ϵ be the positive integer for which

$$(150) \quad \begin{aligned} (a) \quad \sum_{j=1}^3 \alpha_{j,m}^{1,2} &= 0, \quad m=1,2,\dots, \epsilon-1, \\ (b) \quad \sum_{j=1}^3 \alpha_{j,\epsilon}^{1,2} &\neq 0. \quad 44 \end{aligned}$$

43. If (149) holds, then $x_3^1(u, v) \equiv 0$ and S is a plane surface.

44. If $\sum_{j=1}^3 \alpha_{j,m}^{1,2}$ for all m , then it follows from the relation in (148), for $m=s=m$, that $\sum_{j=1}^3 \beta_{j,m}^1 = 0$ for all m . Hence (149) holds. If $\sum_{j=1}^3 \alpha_{j,m}^{1,2}$ does not vanish for

From (146), using formulas analogous to (38), we obtain

$$\begin{aligned}
 (a) \quad & \alpha'_{2,\tau} \beta'_{3,\tau} - \alpha'_{3,\tau} \beta'_{2,\tau} = 0, \\
 (151) \quad (b) \quad & \alpha'_{3,\tau} \beta'_{1,\tau} - \alpha'_{1,\tau} \beta'_{3,\tau} = 0, \\
 (c) \quad & \alpha'_{1,\tau} + \alpha'_{2,\tau} + \alpha'_{3,\tau} = \alpha'_{1,\tau} \beta'_{2,\tau} - \alpha'_{2,\tau} \beta'_{1,\tau}, \quad 45
 \end{aligned}$$

since the positive normal to S at the point $(0,0,0)$ coincides with the z -axis, then there is a least positive integer τ for which (150), (b), holds; therefore (150) holds.

45. Since the functions (145) are harmonic, we may

write $x'_j(u,v) = R \sum_{m=0}^{\infty} (\alpha'_{j,m} - i \beta'_{j,m}) (z-z_0)^m$, $j=1,2,3$, from which we obtain

$$\begin{aligned}
 \frac{\partial x'_j}{\partial u} &= R \sum_{m=0}^{\infty} (m+1) (\alpha'_{j,m+1} - i \beta'_{j,m+1}) (z-z_0)^m, \\
 \frac{\partial x'_j}{\partial v} &= R \sum_{m=0}^{\infty} (m+1) (\beta'_{j,m+1} + i \alpha'_{j,m+1}) (z-z_0)^m.
 \end{aligned}$$

Whenever $\sum_{j=1}^3 \left(\frac{\partial x'_j}{\partial u} \right)^2 \neq 0$, then

$\zeta'_j = \left[(x'_{ku} x'_{lv} - x'_{kv} x'_{lu}) / \sum_{j=1}^3 \left(\frac{\partial x'_j}{\partial u} \right)^2 \right]^{1/2}$, $j,k,l=1,2,3$, in cyclic order, are the direction cosines of the normal to S . Since the functions (145) are harmonic, then

$\sum_{j=1}^3 \left(\frac{\partial x'_j}{\partial u} \right)^2 = 0$ near (u_0, v_0) only if these functions are identically constant in D . If the functions (145) are identically constant in D , then (149) holds, since the functions have expansions (146). If the functions (145) are not identically constant in D , then for λ sufficiently small

$$\sum_{j=1}^3 \left(\frac{\partial x'_j}{\partial u} \right)^2 = \sum_{j=1}^3 \left[\sum_{m=0}^{\infty} (m+1) \lambda^m (\alpha'_{j,m+1} \cos m\theta + \beta'_{j,m+1} \sin m\theta) \right]^2 \neq 0 \quad \text{where}$$

$z-z_0 = \lambda (\cos \theta + i \sin \theta)$. Therefore $\zeta'_j = \left[(\alpha'_{k,\tau} \beta'_{l,\tau} - \alpha'_{l,\tau} \beta'_{k,\tau}) / \sum_{j=1}^3 \alpha'^2_{j,\tau} \right] + O(\lambda)$, where $O(\lambda)$ denotes a quantity (not always the same quan-

cides with the positive X_3^1 -axis. Equations (151), (a) and (b), are linear and homogeneous in $\alpha'_{3,t}$ and $\beta'_{3,t}$; since by (150), (b), and (151), (c), their determinant of coefficients is not zero, it follows that

$$(152) \quad \alpha'_{3,t} = \beta'_{3,t} = 0.$$

From (148), for $m \geq t$, (150), (b), and (152) we obtain

$$(153) \quad \alpha'^2_{1,t} + \alpha'^2_{2,t} = \beta'^2_{1,t} + \beta'^2_{2,t} \neq 0,$$

$$(154) \quad \alpha'_{1,t} \beta'_{1,t} + \alpha'_{2,t} \beta'_{2,t} = 0.$$

From (151), (c), (152), (153) and (154) we obtain

$$(155) \quad \alpha'_{1,t} = \beta'_{2,t}, \quad \alpha'_{2,t} = -\beta'_{1,t}.$$

From (148), for $s = t \leq m$, (152) and (155) it follows

$$(156) \quad \begin{aligned} &\alpha'_{1,m} \alpha'_{1,t} + \alpha'_{2,m} \alpha'_{2,t} + \alpha'_{1,t} \beta'_{1,m} - \alpha'_{1,t} \beta'_{2,m} = 0, \\ &-\alpha'_{1,m} \alpha'_{2,t} + \alpha'_{2,m} \alpha'_{1,t} + \alpha'_{1,t} \beta'_{2,m} + \alpha'_{1,t} \beta'_{1,m} = 0, \quad m \geq t, \end{aligned}$$

hold. The elimination of $\alpha'_{2,m}$ from the relations (156) yields.

$$(157) \quad (\alpha'^2_{1,t} + \alpha'^2_{2,t})(\beta'_{2,m} - \alpha'_{1,m}) = 0, \quad m \geq t,$$

and the elimination of $\beta'_{2,m}$ from (156) yields

$$(158) \quad (\alpha'^2_{1,t} + \alpha'^2_{2,t})(\beta'_{1,m} + \alpha'_{2,m}) = 0, \quad m \geq t.$$

 tity) $\varphi(\lambda)$ such that $|\varphi(\lambda)/\lambda|$ is bounded for λ sufficiently small. Then (151) follows, since $\zeta'_1 = 0$, $\zeta'_2 = 0$, $\zeta'_3 = 1$ at $(0, 0, 0)$. Note that ζ'_j remains continuous at $\lambda = 0$, thus insuring the existence of a normal to S even at points where $\sum_{j=1}^3 \left(\frac{\partial X_j^1}{\partial u}\right)^2 = 0$ when $\sum_{j=1}^3 \left(\frac{\partial X_j^1}{\partial u}\right)^2 \neq 0$. See Hahn and

Beckenbach, loc. cit., pp. 701-703.

From (153), (157) and (158) it follows that

$$(159) \quad \alpha'_{1,m} = \beta'_{2,m}, \quad \alpha'_{2,m} = -\beta'_{1,m}, \quad m \geq t.$$

From (148), for $m \geq t$, and (159) we obtain

$$\alpha'_{3,m} \beta'_{3,m} = 0, \quad \alpha'^2_{3,m} = \beta'^2_{3,m}, \quad m \geq t,$$

from which it follows that

$$(160) \quad \alpha'_{3,m} = \beta'_{3,m} = 0, \quad m \geq t$$

From (148), for $m = s < t$, (150), (a), and (160) we obtain (149). This completes the proof for Part I.

Corollary. If the functions (5) are harmonic in a simply connected domain D such that their Fourier expansions about the point (u_0, v_0) of D are given by (144), and if (143) holds, then the functions (5) are isothermic coordinate functions of a plane surface.

Part II: We now consider the case when the functions (5) map D isothermically on the surface S which lies on the sphere $\mathcal{S}$, whose radius a is finite and non-null, such that circles are mapped on circles. Since the functions (5) have the representation (28), it follows that the functions (5) may be continued analytically to map the entire closed u, v -plane isothermically on $\mathcal{S}$ such that circles are mapped on circles. Let the point P on S correspond to the point $z = \infty$, and let $\mathcal{P}$ be the equatorial plane corresponding to P as a pole. Then the stereographic projection of $\mathcal{S}$ on $\mathcal{P}$, with P as pole, induces an isothermic map of the u, v -plane on $\mathcal{P}$ such that circles are mapped on circles and such that the point at infinity-

ty in the u, v -plane corresponds to the point at infinity in the plane P . Here we have taken the center

Q of J as the origin in a system of coordinates on P , such that the positive S_1 -axis, the positive t_1 -axis and the ray QP , in that order, have the same disposition as the coordinate axes (x_1, x_2, x_3) . The mapping function must have one of the following representations:⁴⁶

$$(161) \quad \begin{aligned} (a) \quad w_1 &= f(z) = \alpha_1 z + \beta_1, \\ (b) \quad w_1 &= f(z) = \alpha_1 \bar{z} + \beta_1, \quad w_1 = s_1 + i t_1, \end{aligned} \quad 47$$

where (a) holds if the map of the u, v -plane on P is directly conformal and where (b) holds if the map is inversely conformal; α_1 and β_1 are constants, $\alpha_1 \neq 0$.

Consider the system of axes (x'_1, x'_2, x'_3) whose origin is at Q , where the positive x'_1 -, x'_2 - and x'_3 -axes have the directions of the positive S_1 -axis, the positive t_1 -axis and the ray QP respectively. There exists a rigid transformation of the x_1, x_2, x_3 -space which carries the origin into the point Q , such that the positive x_1 -, x_2 - and x_3 -axes coincide with the positive x'_1 -, x'_2 - and x'_3 -axes respectively. This transformation has the representation (103), where (104) holds. If the new coordinate functions of S are

46. See L. R. Ford, loc. cit., p. 32, Theorem 25.

47. When we refer to (161) we shall refer to that part of (161) that effects the mapping noted in the proof.

$$(162) \quad x_j' = x_j'(u, v), \quad j=1, 2, 3,$$

then it follows from (9), (103) and (104) that (107) holds, where γ is an arbitrary closed rectifiable Jordan curve lying in D .

Since the functions (5) map circles on circles, it follows that the functions (162) map circles on circles. Hence it follows from Lemma 1 that the functions (162) have the representation (28); moreover, in (28), the function $f(z)$ is given by (161). Therefore

$$(163) \quad x_1'(u, v) = \frac{2a^2 s_1}{a^2 + s_1^2 + t_1^2}, \quad x_2'(u, v) = \frac{2a^2 t_1}{a^2 + s_1^2 + t_1^2}, \quad x_3'(u, v) = a \left[1 - \frac{2a^2}{a^2 + s_1^2 + t_1^2} \right].$$

From (9), (107) and (163) we obtain

$$(164) \quad \sum_{j=1}^3 \left[\int_{\gamma} x_j'(u, v) dz \right]^2 = \frac{4a^4}{a_1^2} \sum_{j=1}^3 \left[\int_{\gamma^*} \varphi_j(s_1, t_1) d\omega_1 \right]^2, \quad 48$$

where γ^* is the map of γ on the plane $\mathbb{P}^2$ by (161), and where

$$(165) \quad \varphi_1(s_1, t_1) = \frac{s_1}{a^2 + s_1^2 + t_1^2}, \quad \varphi_2(s_1, t_1) = \frac{t_1}{a^2 + s_1^2 + t_1^2}, \quad \varphi_3(s_1, t_1) = \frac{1}{a^2 + s_1^2 + t_1^2}.$$

Since $a \neq 0$, it follows from (102) and (164) that

$$(166) \quad \sum_{j=1}^3 \left[\int_{\gamma^*} \varphi_j(s_1, t_1) d\omega_1 \right]^2 = 0.$$

48. Since the functions (5) are not identically constant in D , it follows that $a_1 \neq 0$ in (161).

Without any loss of generality we may assume that the line $s_1 = 0$ passes through D^* , where D^* is the map of D on ρ by (161).⁴⁹ Therefore there exist a curve Γ with the following properties;

1) the vertices of Γ have the following polar coordinates,

$$A_1: (\rho_1, \tau_1), \quad A_2: (\rho_1, -\tau_1),$$

$$A_3: (\rho_1 + \omega_1, -\tau_1), \quad A_4: (\rho_1 + \omega_1, \tau_1), \quad 0 < \tau_1 < \pi/4, \quad 0 < \rho_1, \quad 0 < \omega_1;$$

2) Γ is composed of two arcs of circles and two straight-line segments: $\widehat{A_1 A_2}$ is an arc of the circle $s_1^2 + t_1^2 = \rho_1^2$, $\widehat{A_3 A_4}$ is an arc of the circle $s_1^2 + t_1^2 = (\rho_1 + \omega_1)^2$, each arc subtending an angle $2\tau_1$ at the origin, and $\widehat{A_2 A_3}$ and $\widehat{A_4 A_1}$ are on rays through the origin.

For the closed rectifiable Jordan curve Γ we obtain

$$\sum_{j=1}^3 \left[\int_{\Gamma} \varphi_j(s_1, t_1) d\omega_1 \right]^2 = -\beta_1 \sin \beta_1 k_1(\rho_1, \omega_1) - 4 \sin^2 \frac{\beta_1}{2} k_2(\rho_1, \omega_1), \quad \beta_1 = 2\tau_1,$$

where

49. For, consider the rotation defined by $w_2 = s_2 + it_2 = e^{i\theta} w_1$, where θ has been so chosen that the line $s_2 = 0$ passes through D . If $\varphi_1 = \frac{s_2}{\rho^2 + s_2^2 + t_2^2}$, $\varphi_2 = \frac{t_2}{\rho^2 + s_2^2 + t_2^2}$, and $\varphi_3 = \frac{1}{\rho^2 + s_2^2 + t_2^2}$, then

$$\sum_{j=1}^3 \left[\int_{\Gamma^*} \varphi_j(s_2, t_2) d\omega_2 \right]^2 = e^{-2i\theta} \sum_{j=1}^3 \left[\int_{\Gamma} \varphi_j(s_1, t_1) d\omega_1 \right]^2.$$

$$K_1(\lambda_1, \omega_1) = \left[\frac{(\lambda_1 + \omega_1)^2}{a^2 + (\lambda_1 + \omega_1)^2} - \frac{\lambda_1^2}{a^2 + \lambda_1^2} \right] \left[\frac{(\lambda_1 + \omega_1)^2}{a^2 + (\lambda_1 + \omega_1)^2} - \frac{\lambda_1^2}{a^2 + \lambda_1^2} + \log \frac{a^2 + \lambda_1^2}{a^2 + (\lambda_1 + \omega_1)^2} \right],$$

$$K_2(\lambda_1, \omega_1) = \left[\frac{a(\lambda_1 + \omega_1)}{a^2 + (\lambda_1 + \omega_1)^2} - \frac{a\lambda_1}{a^2 + \lambda_1^2} + \arctan \frac{\lambda_1}{a} - \arctan \frac{\lambda_1 + \omega_1}{a} \right]^2,$$

and where the ψ_j were obtained from (165). Since

$$\frac{\partial}{\partial \omega_1} [K_2(\lambda_1, \omega_1)]^{\frac{1}{2}} = \frac{-2(\lambda_1 + \omega_1)^2}{[a^2 + (\lambda_1 + \omega_1)^2]^{\frac{3}{2}}} \neq 0,$$

it follows that

$$K_2(\lambda_1, \omega_1) \neq 0.$$

Fix λ_1 and ω_1 so that

$$K_2(\lambda_1, \omega_1) \neq 0.$$

Since $\beta \sin \beta_1$ and $4 \sin^2 \frac{\beta_1}{2}$ are not proportional,

$$\beta \sin \beta_1 = \beta_1^2 - \frac{\beta_1^4}{6} + \dots,$$

$$4 \sin^2 \frac{\beta_1}{2} = \beta_1^2 - \frac{\beta_1^4}{12} + \dots,$$

there exists a β_1 such that

$$(1.67) \quad \sum_{j=1}^3 \left[\int_r \psi_j(s, t_1) d\omega_1 \right]^2 \neq 0.$$

Now (1.67) is a contradiction of (1.66). Therefore the functions (5) do not map D isothermally on a surface that lies on a sphere of finite non-null radius. This completes the proof of the theorem.

5. MEAN-VALUE SURFACES.

5.1. Mean-value surfaces. If the coordinate functions

$$(168) \quad x_j = x_j(u, v)$$

of a surface S are continuous in a domain D , then the circular averages

$$(169) \quad A_{j,\rho}(u, v) = \frac{1}{\pi\rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x_j(u + \xi, v + \eta) d\xi d\eta, \quad j=1, 2, 3,$$

where ρ is a positive constant, will be said to define a mean-value surface S_ρ associated with S . We define

$$A_{j,0}(u, v) \equiv x_j(u, v), \quad j=1, 2, 3.$$

We note that the functions (169) are defined and have continuous partial derivatives of the first order in an open set of points D_ρ which is interior to D .⁵⁰

Theorem 5.1.1. If the functions (168) are continuous in a simply connected domain D , then a necessary and sufficient condition that

$$(170) \quad \sum_{j=1}^3 \left[\int_C x_j(u, v) dz \right]^2 = 0$$

⁵⁰. Since D_ρ is not necessarily a connected set of points, S_ρ may consist of several pieces.

hold for each circle C in D is that all mean-value surfaces S_p associated with S be given in isothermic representation by (169).

Proof. The first partial derivatives of the functions (169) are given by the relations

$$(171) \quad \frac{\partial A_j}{\partial u} = \frac{1}{\pi p} \int_{\xi^2 + \eta^2 = p^2} x_j(u + \xi, v + \eta) d\eta, \quad \frac{\partial A_j}{\partial v} = -\frac{1}{\pi p} \int_{\xi^2 + \eta^2 = p^2} x_j(u + \xi, v + \eta) d\xi, \quad j=1,2,3, \quad 51$$

which are valid for points of D_p . From (171) we obtain

$$\sum_{j=1}^3 \left| \int_{\xi^2 + \eta^2 = p^2} x_j(u + \xi, v + \eta) (d\xi + i d\eta) \right|^2 = \pi^2 p^4 \sum_{j=1}^3 (A_{j,p})^2,$$

which, with (8) and (170), yields our theorem.

From Theorems 3 and 6.1.1 we obtain the following result.

Theorem 5.1.2. If the functions (168) have continuous partial derivatives of the third order in a simply connected domain D , then a necessary and sufficient condition that they map D isothermically either on a surface that lies on a sphere of finite non-null radius such that circles are mapped on circles, or on a minimal surface, is that all mean-value surfaces S_p associated with the surface S

51. See T. Radó, Subharmonic Functions, p.11.

defined by the functions (168) be given in isothermic representation by (169).

5.2. Mean-value surfaces and transformations of axes. In §5.3 we shall make use of the following observations.

5.2.1. S_p is invariant under rigid transformations in the x_1, x_2, x_3 -space: if

$$x'_j = \sum_{k=1}^3 \lambda_{kj} x_k + a_j, \quad j=1,2,3,$$

where the a_j are constants and where (104) holds, then

$$A'_{j,p} = \sum_{k=1}^3 \lambda_{kj} A_{k,p} + a_j, \quad j=1,2,3,$$

where

$$A'_{j,p}(u,v) \equiv \frac{1}{\pi p^2} \int \int_{\xi^2 + \eta^2 \leq p^2} x'_j(u+\xi, v+\eta) d\xi d\eta, \quad j=1,2,3.$$

5.2.2. S_p is invariant under each of the reflections,

$$u' = u, \quad v' = -v \quad \text{and} \quad u' = -u, \quad v' = v.$$

If, for example,

$$x'_j(u', v') \equiv x'_j(u', -v'), \quad j=1,2,3,$$

then

$$A'_{j,p}(u', v') \equiv A'_{j,p}(u', -v'),$$

where

$$(172) \quad A'_{j,p}(u,v') \equiv \frac{1}{\pi \rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x'_j(u' + \xi, v' + \eta) d\xi d\eta, \quad j=1,2,3.$$

5.2.3. Similarly, S_p is invariant under the rigid transformations in the u, v -plane.

5.2.4. Under the transformation

$$(173) \quad u' = \alpha u, \quad v' = \alpha v, \quad \alpha > 0,$$

the mean-value surface S_p is transformed into the mean-value surface $S_{\alpha p}$ whose coordinate functions are given by (172), where

$$x'_j(u', v') \equiv x_j\left(\frac{u'}{\alpha}, \frac{v'}{\alpha}\right), \quad j=1,2,3.$$

Hence the family of mean-value surfaces S_p associated with a given surface S is invariant under the transformation (173).

5.3. Conformal mean-value surfaces. Since, by Theorem 5.1.2, the only surfaces in isothermic representation for which all associated mean-value surfaces are given in isothermic representation by (169) are spherical maps, in representation whereby circles are mapped on circles, and minimal surfaces, the question arises as to the nature of the mean-value surfaces in these two cases.

Theorem 5.3.1. If the functions (168) map a simply connected domain D isothermally on a minimal surface S , then each mean-value surface S_p associated

with S is a minimal surface given in isothermic representation by (169) and coinciding with S for (u, v) in D_p .

Proof. By the theorem of Weierstrass, the functions (168) are harmonic in D ; consequently, as is well known, the functions (169) coincide with the functions (168),

$$A_{j,p}(u, v) \equiv X_j'(u, v), \quad j=1, 2, 3,$$

in the open set D_p . Hence all mean-value surfaces associated with a given minimal surface are themselves minimal surfaces given in isothermic representation by (169); moreover, S_p coincides with S for (u, v) in D .

Theorem 5.3.2. If the functions (168) a simply connected domain D isothermically on a surface S that lies on a sphere $\mathcal{S}$ of finite non-null radius a , such that circles are mapped on circles, then each mean-value surface S_p associated with S lies on a surface of revolution T_p and is given in isothermic representation by (169). Further, for $p > 0$, T_p is not a sphere.

Proof. As has been shown before,⁵² the functions defining S , (168), may be continued analytically to map the entire closed u, v -plane isothermically on the

52. See the proof of Theorem 4, Part II, p. 52.

whole of $\mathcal{S}$. Further, the functions (168) then have the representation (28) where $f(z)$ is given by (161).

From 5.2.1-5.2.3 it follows that we may assume

$$a_1 = a_2 = a_3 = 0, \quad k = a,$$

and

$$(174) \quad f(z) = \alpha z, \quad \alpha > 0,$$

in (28). This is equivalent to assuming that the point P on $\mathcal{S}$ corresponding to the point $z = \infty$ is the point $(0, 0, a)$, that $z = 0$ corresponds to the point P' on $\mathcal{S}$ that is diametrically opposite P , and that the z -point corresponding to

$P'(a, 0, 0)$ is real and positive. Since we are investigating all mean-value surfaces associated with it follows from 5.2.4 that we may take $\alpha = 1$ in (174), in which case the functions (168), as given by (28), have the following familiar representation:

$$(175) \quad \begin{aligned} x_1 &= \frac{2a^2 u}{u^2 + v^2 + a^2} \\ x_2 &= \frac{2a^2 v}{u^2 + v^2 + a^2}, \\ x_3 &= a \left(1 - \frac{2a^2}{u^2 + v^2 + a^2} \right). \end{aligned}$$

If C_1 is the circle $u^2 + v^2 = R^2$, and if $z = u + iv = e^{i\theta}$, then (169) and (175) yield

$$A_{3,\rho}(u,v) = A_{3,\rho}(n,o),$$

$$(176) \quad A_{1,\rho}(u,v) + i A_{2,\rho}(u,v) = e^{i\theta} [A_{1,\rho}(n,o) + i A_{2,\rho}(n,o)].$$

From (176) it follows that the map $C_{\lambda'}^*$ of C_{λ} on the mean-value surface T_{ρ} associated with the sphere $\mathcal{S}$, defined by (175), is a circle in a plane perpendicular to the $A_{3,\rho}$ -axis. The center of $C_{\lambda'}^*$ is on the $A_{3,\rho}$ -axis and its radius is λ' :

$$\lambda' = |A_{1,\rho}(n,o) + i A_{2,\rho}(n,o)|.$$

Since $\mathcal{S}$ lies on T_{ρ} , it follows that the functions (169) map D_{ρ} isothermally on a surface that lies on a surface of revolution, namely T_{ρ} .

In § 5.4, it will appear that T_{ρ} is not a sphere for $0 < \rho < \infty$. We shall call T_{ρ} a mean-value sphere.

5.4. Mean-value spheres. The mean-value sphere T_{ρ} is a surface of revolution about the $A_{3,\rho}$ -axis. Accordingly, to investigate T_{ρ} it is sufficient to study the intersection $\tilde{T}_{\rho}$ of T_{ρ} with the plane $A_{2,\rho} = 0$. Since, by (175),

$$X_2(u,-v) = -X_2(u,v),$$

it follows that

$$A_{2,\rho}(u,o) = 0;$$

hence, by (176), the intersection of T_p with the plane $A_{2,p}=0$ can be obtained from (169) by setting $v=0$ in (175). A computation yields the following results:

$$\begin{aligned} A_{2,p}(u,0) &= \frac{1}{\pi p^2} \iint_{\xi^2 + \eta^2 \leq p^2} \frac{2a^2(u+\xi)}{(u+\xi)^2 + \eta^2 + a^2} d\xi d\eta, \\ &= \frac{1}{\pi p^2} \int_0^p \int_0^{2\pi} \frac{2a^2(u+r\cos\theta) r dr d\theta}{(a^2+u^2+r^2) + 2ur\cos\theta}, \\ &= \frac{a^2}{u^2} \left[a^2 + u^2 + p^2 - \sqrt{(a^2 - u^2 + p^2)^2 + 4a^2u^2} \right], \end{aligned}$$

and

$$\begin{aligned} A_{3,p}(u,0) &= \frac{1}{\pi p^2} \iint_{\xi^2 + \eta^2 \leq p^2} a \left(1 - \frac{2a^2}{(u+\xi)^2 + \eta^2 + a^2} \right) d\xi d\eta, \\ &= a - \frac{1}{\pi p^2} \int_0^p \int_0^{2\pi} \frac{2a^2 r dr d\theta}{(a^2+u^2+r^2) + 2ur\cos\theta}, \\ &= a - \frac{2a^3}{p^2} \log \left[\frac{a^2 - u^2 + p^2 + \sqrt{(a^2 - u^2 + p^2)^2 + 4a^2u^2}}{2a^2} \right], \end{aligned}$$

from which it follows that the coordinate functions of $\tilde{T}_p$ are

$$\begin{aligned} A_{1,p} &= \frac{a^2}{u^2} \left[a^2 + u^2 + p^2 - \sqrt{(a^2 - u^2 + p^2)^2 + 4a^2u^2} \right], \\ (177) \quad A_{3,p} &= a - \frac{2a^3}{p^2} \log \left[\frac{a^2 - u^2 + p^2 + \sqrt{(a^2 - u^2 + p^2)^2 + 4a^2u^2}}{2a^2} \right]. \end{aligned}$$

We make the following observations.

5.4.1. The curve $\tilde{T}_p$ has horizontal tangents at the points

$$A_{1,p} = 0, \quad A_{2,p} = a,$$

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$$A_{1,p} = 0, \quad A_{2,p} = a - \frac{4a^3}{p^2} \log \frac{\sqrt{a^2 + p^2}}{a},$$

and vertical tangents at the points

$$A_{1,p} = \pm \frac{2a^2}{p^2} [\sqrt{a^2 + p^2} - a], \quad A_{2,p} = a - \frac{2a^3}{p^2} \log \frac{\sqrt{a^2 + p^2}}{a}.$$

Here we have made use of the following expression for the slope of $\tilde{T}_p$:

$$m = \frac{2au}{a^2 - u^2 + p^2}.$$

The horizontal tangents occur for $u = 0, \infty$ and the vertical tangents occur when $u = \pm \sqrt{a^2 + p^2}$. Moreover,

$$A_{1,p}(u, 0) = A_{1,p}\left(\frac{a^2 + p^2}{u}, 0\right),$$

from which we can obtain the result that the curve $\tilde{T}_p$ is symmetric about the line

$$A_{2,p} = a - \frac{2a^3}{p^2} \log \frac{\sqrt{a^2 + p^2}}{a}.$$

In addition, from the representation (177) for $\tilde{T}_p$, it follows that the curve $\tilde{T}_p$ is also symmetric about the $A_{1,p}$ -axis. From these symmetries, and from the following relations for the slope m of

$$m = \frac{2au}{a^2 - u^2 + p^2}, \quad \frac{dm}{du} = \frac{2a(a^2 + u^2 + p^2)}{(a^2 - u^2 + p^2)^2},$$

it follows that $\tilde{T}_p$ is convex.

5.4.2. The height h of $\tilde{T}_p$ is

$$h = \frac{2a^3}{p^2} \log \frac{a^2 + p^2}{a^2}$$

and the width l is

$$l = \frac{4a^2}{p^2} [\sqrt{a^2 + p^2} - a].$$

Therefore, for $0 < p < \infty$, the curve $\tilde{T}_p$ is not a circle. Moreover,

$$\lim_{p \rightarrow \infty} \frac{l}{h} = \infty;$$

so that $\tilde{T}_p$ "flattens out" while approaching the point $(0, a)$, as p becomes infinite.

5.4.3. Each member of the family $[\tilde{T}_p]$ passes through the point $(0, a)$ and is tangent there to every other member of the family. Further, for $p=0$, $\tilde{T}_p$ is the circle

$$A_{1,p}^2 + A_{3,p}^2 = a^2,$$

and for $p=0$, $\tilde{T}_p$ is the point $(0, a)$.⁵³

From 5.4.1-5.4.3 it follows that the family of surfaces of revolution $[T_p]$ consists of convex surfaces each of which passes through the point $(0, 0, a)$ in the $A_{1,p}, A_{2,p}, A_{3,p}$ -space and is tangent there to every other member of the family. Since the ratio of width to height becomes infinite with p , it follows that T_p "flattens out" as p becomes infinite. For $p=0$, T_p is the sphere with radius a and center at the origin, and for $p=0$, T_p is the point $(0, 0, a)$. From (177) we obtain the following

53. For $a=1$ and the values $p=0, 1, 2\sqrt{6}$, there is a picture of the curve $\tilde{T}_p$ on the page following page 68.

representation for $T\rho$:

$$A_{1,\rho} = \frac{a^2 u}{\rho^2 (a^2 + v^2)} \left[a^2 + u^2 + v^2 + \rho^2 - \sqrt{(a^2 - u^2 - v^2 + \rho^2)^2 + 4a^2(u^2 + v^2)} \right],$$

$$A_{2,\rho} = \frac{a^2 v}{\rho^2 (a^2 + v^2)} \left[a^2 + u^2 + v^2 + \rho^2 - \sqrt{(a^2 - u^2 - v^2 + \rho^2)^2 + 4a^2(u^2 + v^2)} \right],$$

$$A_{3,\rho} = a - \frac{2a^3}{\rho^2} \log \left(\frac{a^2 - u^2 - v^2 + \rho^2 + \sqrt{(a^2 - u^2 - v^2 + \rho^2)^2 + 4a^2(u^2 + v^2)}}{2a^2} \right).$$

5.5. Homothetic mean-value surfaces. If the functions (168) are analytic in the u, v -plane, a computation shows that the circular averages (169) have the representation

$$(178) \quad A_{j,\rho}(u,v) = \sum_{m=0}^{\infty} \rho^{2m} \frac{\Delta^m x_j}{2^{2m} m! (m+1)!}, \quad j=1,2,3.$$

Two surfaces S and S' , whose coordinate functions are

$$x_j'(u,v) \quad \text{and} \quad y_j'(u,v), \quad j=1,2,3,$$

respectively, will be said to be homothetic provided there exists a constant $k \neq 0$ such that

$$y_j'(u,v) = k x_j'(u,v), \quad j=1,2,3.$$

By (178) it follows that the problem of determining analytic surfaces $S : x_j'(u,v)$, $j=1,2,3$, such

that all corresponding mean-value surfaces are homothetic to S is equivalent to determining solutions of

$$(179) \quad \sum_{m=0}^{\infty} \rho^{2m} \frac{\Delta^m x_j}{2^{2m} m! (m+1)!} = c_p x_j, \quad j=1,2,3.$$

In particular, all solutions of

$$(180) \quad \Delta x_j = c_p x_j, \quad j=1,2,3,$$

satisfy (179).

For $c_p = 0$, we see that all mean-value surfaces associated with harmonic surfaces S , (surfaces whose coordinate functions are harmonic), are homothetic to S , and, in fact, coincide with S .

The functions

$$(181) \quad \begin{aligned} x_1 &= a \sin u \sin v, \\ x_2 &= a \sin u \cos v, \\ x_3 &= a \cos u, \end{aligned}$$

which map the u, v -plane on an infinite sheeted sphere, also satisfy (179), with

$$c_p = -1.$$

We note that the surface defined by the functions (181) is not given in isothermic representation by (181).