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A THREE DIMENSIONAL TREATMENT OF GROUPS
OF LINEAR TRANSFORMATIONS

by

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Introduction. In a recent treatment of the theory of linear transformations* Professor L. R. Ford has introduced the concept of the isometric circle. Used in the theory of discontinuous groups it leads easily to the construction of a fundamental region of the group. The simplification of the treatment of the linear transformation by this concept suggested the use of the same notion to define a type of space transformation.

Although in many respects the properties of this type of space transformation are a natural generalization of those of the linear transformation, yet interesting differences appear, in particular, in the properties of the properly and the improperly discontinuous groups.

Poincaré was the first to introduce the notion of a space transformation corresponding to a linear transformation in the plane**. His idea was to replace the sequence of inversions in circles in the plane which is equivalent to a linear transformation by a sequence of inversions in spheres constructed on these circles as equatorial planes. He defines the space transformation corresponding to the linear transformation by means of a sequence of inversions in such spheres.

The chief difficulty with his generalization is that a theory of groups of space transformations based on it yields no additional information concerning groups of linear transformations. The concept of isometric circles and spheres can be applied to Poincaré's space transformation with a resulting simplification of treatment, but there is no

* On the Foundations of the Theory of Discontinuous Groups of Linear Transformations, L. R. Ford, Proceedings of the National Academy of Sciences, May, 1927. For a more complete treatment of this concept of the isometric circle see the same author's forthcoming book, Automorphic Functions, McGraw-Hill Book Co. 1929.

**Mémoire sur les groupes kleinéens, H. Poincaré, Acta Math., 3, 1883-1884, pp. 49-92.

geometrical restriction peculiar to the isometric spheres which makes all such spheres lie in a finite region. Such a restriction is imposed only by restricting the group of linear transformation.

In the following treatment the plane is first projected stereographically on a sphere Σ_0 . To each linear transformation there corresponds a single transformation of Σ_0 into itself. The concept of isometric circles on Σ_0 and isometric spheres through these circles orthogonal to Σ_0 leads to an interpretation of the space transformation as a sequence of inversions in spheres orthogonal to Σ_0 that are determined by the transformation and its isometric sphere.

A group of space transformations based on this generalization of the linear transformation has the peculiar properties that it contains a finite number of rotations of Σ_0 and is such that the radii of the isometric spheres are bounded. These two properties are sufficient to insure a fundamental region for any discontinuous group. This region in its properties resembles considerably that of the Fuchsian group in the plane and a study of it can be made in somewhat the same way.

The particular distinction then between this study of the plane and the space transformation theory is that both properly and improperly discontinuous groups in space have a fundamental region while only properly discontinuous groups in the plane possess such a region.

The following type of generalization of the linear transformation affords a means of obtaining a set of generating transformations for a properly discontinuous group where the methods used in the plane case fail.

The continuous and directly conformal Transformation of the Surface of a Sphere into Itself. If the surface of a sphere be mapped in a one-to-one and continuous manner on itself, there results, in general, a magnification of some parts of the surface and a diminution of other parts. If the mapping is conformal, we have the following

THEOREM 1. Let the surface of a sphere be transformed in a one-to-one and directly conformal manner on itself. Then, provided the transformation is not a rotation of the sphere about an axis, the locus of the points in the neighborhood of which lengths and areas are unaltered is a small circle.

Let the sphere Σ_0 be projected stereographically on the z -plane. Then, as is well known, the plane undergoes a linear transformation when the sphere is transformed into itself in a one-to-one and directly conformal manner. Let Σ_0 be the unit sphere with center at the origin in the z -plane. Denote by ds the length of an element of arc on the surface of Σ_0 and by ds' the length of the transformed element. The square of the linear element ds is given by

$$ds^2 = d\xi^2 + d\eta^2 + d\zeta^2,$$

where the point $P(\xi, \eta, \zeta)$ on Σ_0 is the initial point of the element ds .

The point P on Σ_0 and its stereographic projection $z = x + iy$ in the plane are connected by the relations*

$$\xi = \frac{2x}{z\bar{z} + 1}, \quad \eta = \frac{2y}{z\bar{z} + 1}, \quad \zeta = \frac{z\bar{z} - 1}{z\bar{z} + 1},$$

* The stereographic projection is made by projecting from the point $(0,0,1)$ upon the $\xi\eta$ -plane. The equator of Σ_0 in the $\xi\eta$ -plane remains fixed.

where bars indicate conjugate imaginaries. Using these, we can write ds^2 in the form

$$ds^2 = \frac{8(dx^2 + dy^2)}{(z\bar{z} + 1)^2}.$$

Similarly ds'^2 can be written

$$ds'^2 = \frac{8(dx'^2 + dy'^2)}{(z'\bar{z}' + 1)^2},$$

where z' is the transform of z by a linear transformation

$$z' = \frac{az + b}{cz + d},$$

with $ad - bc = 1$. Then since $dx'^2 + dy'^2 = dz'd\bar{z}'$ and

$$\frac{dz'}{dz} = \frac{1}{(cz + d)^2},$$

the condition that

$$ds'^2 = ds^2$$

is that

$$\frac{1}{(z'\bar{z}' + 1)^2 (cz + d)^2 (\bar{c}\bar{z} + \bar{d})^2} = \frac{1}{(z\bar{z} + 1)^2},$$

which reduces to

$$(1) \quad (1 - a\bar{a} - c\bar{c})z\bar{z} - (a\bar{b} + c\bar{d})z - (\bar{a}b + \bar{c}d)\bar{z} + (1 - b\bar{b} - d\bar{d}) = 0.$$

If $1 - a\bar{a} - c\bar{c} \neq 0$, equation (1) represents a circle C with center and radius respectively

$$p = \frac{\bar{a}b + \bar{c}d}{1 - a\bar{a} - c\bar{c}}, \quad r = \sqrt{\frac{a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2}{1 - a\bar{a} - c\bar{c}}}.$$

The expression under the radical sign is non-negative. Moreover, except

for rotations of the sphere Σ_0 into itself, it is positive. In fact,

$$a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2 = (a - \bar{d})(\bar{a} - d) + (b + \bar{c})(\bar{b} + c) \geq 0,$$

the equality sign holding if and only if $a = \bar{d}$, $b = -\bar{c}$, with $a\bar{a} + c\bar{c} = 1$. For these relations equation (1) is identically satisfied. They are the conditions for a rotation of Σ_0 about an axis.

If $1 - a\bar{a} - c\bar{c} = 0$ and $1 - b\bar{b} - d\bar{d} \neq 0$, equation (1) reduces to the equation of a straight line, not passing through the origin.

The projection of C on Σ_0 is a circle I_S . If, in particular, $1 - a\bar{a} - c\bar{c} = 0$ and $1 - b\bar{b} - d\bar{d} \neq 0$, the circle passes through the north pole. When a transformation S is made carrying the surface of Σ_0 into itself as described above, infinitesimal lengths in the neighborhood of I_S are unaltered; hence infinitesimal areas on the surface are unaltered also.

We call the circle I_S the isometric circle of the transformation S .

The circle I_S is the complete locus of points in the neighborhood of which lengths on Σ_0 are unaltered by the transformation S . For any point P on Σ_0 not on I_S has as projection in the plane a point $Q(z_0)$ which is not on C . This point, lying inside or outside C , has the coordinate z_0 such that the left member of (1) is definitely negative or positive. These possibilities lead respectively to the inequalities

$$ds'^2 > ds^2, \quad ds'^2 < ds^2.$$

Hence elements of length on Σ_0 are increased or decreased in the neighborhood of P and P is not a point of the required locus.

The isometric circle I_S cannot be a great circle of the sphere Σ_0 , for the distance of the plane of I_S from the center of Σ_0 is

greater than zero. The plane of I_s is given by

$$\xi(a\bar{b}+c\bar{d}+\bar{a}b+\bar{c}d)+i\eta(a\bar{b}+c\bar{d}-\bar{a}b-\bar{c}d)+\zeta(a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d})+(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}-2)=0.$$

The distance Δ of the center of Σ_0 from this plane is then

$$\Delta = \sqrt{\frac{a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}-2}{a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}+2}}$$

which is essentially positive, since the numerator vanishes only when $a=\bar{d}$, $b=-\bar{c}$, the conditions for a rotation of Σ_0 about an axis.

The transformation S carries I_s into a circle I'_s of the same size. I'_s is obviously the isometric circle of the inverse transformation S^{-1} .

The surface of the smaller polar cap of Σ_0 cut off by I_s we call the interior of I_s ; that of the larger polar cap we call the exterior of I_s .

THEOREM 2. Lengths and areas in the interior of I_s are increased in magnitude by the transformation S ; those in the exterior of I_s are decreased.

The transformation S carries I_s into a circle I'_s of the same size and transforms Σ_0 into itself. It is impossible, therefore, that all lengths not on I_s be increased (or decreased) by S . Furthermore, in the interior of I_s every element of length must be increased (or decreased). For let P, P' be two points inside I_s such that lengths near P are increased by S , near P' decreased. P and P' can be joined by a continuous curve not crossing I_s along which the ratio ds'/ds changes continuously from a value greater than 1 to a value less than 1. Hence at some point P_2 on PP' it has the value 1. This is impossible, however, for P_2 does not lie on I_s .

Areas inside I_s cannot be decreased by S , for the interior of I_s must go into the interior of I'_s with the same area, or into the

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exterior of I'_S with an increase in area. The former is impossible. If lengths inside I_S were decreased, then areas would be also. Hence lengths and areas inside I_S must be increased, and those outside I_S decreased by S .

In the course of the preceding proof we have established the following

THEOREM 3. The transformation S carries the interior and the exterior of I_S into the exterior and interior respectively of I'_S .

Relative position of I_S and I'_S . Let T be the linear transformation in the z -plane corresponding to S . We can state the following facts about the position of I_S with respect to I'_S for the non-loxodromic transformation:

THEOREM 4. If T is elliptic, I_S, I'_S intersect in the fixed points of S ; if T is hyperbolic, I_S, I'_S are external; if T is parabolic, I_S, I'_S are tangent at the fixed point of S .

If T is elliptic with fixed points ξ_1, ξ_2 , any fixed circle of T has one of these points in its interior, the other on its exterior. On Σ , therefore the corresponding fixed circle of S has the fixed points P_1, P_2 similarly located. Moreover, a fixed circle of T can be found with any arbitrary, previously assigned radius. Hence if P_1 does not lie on I_S , a fixed circle for S can be found so small that it lies entirely inside or outside I_S . But such a circle is either increased or decreased in magnitude and so cannot be carried into itself. P_1 therefore lies on I_S . It then also lies on I'_S .

If T is hyperbolic with fixed points ξ_1, ξ_2 , each circular arc joining ξ_1, ξ_2 is carried into itself. Hence on Σ each circular arc joining the fixed points P_1 and P_2 is carried into itself by S . One of these points is interior, one exterior to I_S . For unless P_1 and P_2 are separated by I_S , we can join them by an arc lying, except possibly for its ends, wholly within or without I_S . This arc then has its length altered by the transformation and cannot therefore be carried into itself.

Similarly we show that one fixed point is within, the other without I'_S .

Suppose I_S, I'_S have a point Q in common. Q lies on I_S and on a fixed arc K joining P_1 and P_2 . It is transformed by S into Q' on I'_S and on K . Hence I'_S , passing through Q and Q' and not coinciding with K , must include both P_1, P_2 or neither in its interior. Either case, however, has been shown impossible. Hence I_S, I'_S have no points in common.

If T is parabolic with fixed point ξ , the fixed circles are tangent at the fixed point. On Σ_0 the fixed circles are tangent at the fixed point P . By the same reasoning as that used in the elliptic case, P must lie on I_S and I'_S . Suppose I_S, I'_S have a second point Q in common. As in the hyperbolic case Q is transformed into Q' on I'_S and the fixed circle through Q . Hence I'_S coincides with the fixed circle since they have three points in common. This is possible only if I_S and I'_S coincide. But then, since lengths on I_S are unaltered, the point on I_S at the other end of a diameter through P is also fixed; which is contrary to the fact that there is a single point.

Space Transformations. We wish to find a space transformation which transforms the surface of Σ_0 in exactly the same way that S transforms it. Consider the linear transformation T corresponding to S . It can be regarded in an infinite number of ways as the result of a sequence of an even number of inversions in circles of the plane. Let spheres be constructed through these circles orthogonal to the plane; a sequence of inversions in these spheres has the same effect on the plane as that of the sequence of inversions in the circles. Let the plane be projected stereographically on the sphere Σ_0 . Now this stereographic projection can be accomplished by inverting in a sphere K of radius $\sqrt{2}$ and center at the point $(0,0,1)$. By this inversion the plane is transformed into the surface of Σ_0 , and spheres orthog-

onal to it into spheres orthogonal to Σ_0 . An inversion in one of the spheres Σ' orthogonal to Σ_0 transforms Σ_0 into itself, for points on the plane inverse with respect to any one of the spheres Σ are transformed by inversion in K into points on Σ_0 inverse with respect to the transformed sphere Σ' , and Σ_0 , passing through these inverse points of Σ' , is transformed into itself by inversion in Σ' . Hence a transformation consisting of a sequence of an even number of inversions in these spheres orthogonal to Σ_0 transforms the surface of Σ_0 into itself. Since each inversion in a sphere is an inversely conformal transformation, the sequence of an even number of such is directly conformal. Hence a sequence of an even number of inversions in spheres orthogonal to Σ_0 transforms the surface of Σ_0 into itself as S does; it is then a space transformation such as we desired to find; that is, it possesses the same properties with respect to the sphere Σ_0 as does S , so that if the sequence can be chosen so as to carry certain points on the surface of Σ_0 into certain other points on Σ_0 , it will be equivalent to S .

To justify this definition we must show it to be unique.* The linear transformation T may be regarded in an infinite number of ways as an even number of inversions in circles of the plane. Hence the circles of inversion are not determined. But the space transformation corresponding to T , as a sequence of inversions in certain spheres orthogonal to Σ_0 , is a perfectly determined operation.

Suppose, in fact, that T can be regarded in one way as a sequence of n inversions in circles $C_1, C_2, \dots, C_n$; and in a second way as a sequence of p inversions in circles $C'_1, C'_2, \dots, C'_p$, with n and p even. Let spheres be constructed through these circles, or-

* H. Poincaré, Acta Math., 3, 1883-1884.

thogonal to the plane and let $\sigma_1, \sigma_2, \dots, \sigma_n$ and $\sigma'_1, \sigma'_2, \dots, \sigma'_p$ be their respective transforms by inversion in K , the equivalent of the stereographic projection. Let P be any point of space. We apply to P first the succession of inversions in the sequence $\sigma_1, \dots, \sigma_n$, which transforms it into a point Q ; and secondly, the sequence of inversions in $\sigma'_1, \dots, \sigma'_p$, which transforms it into Q' . Both of these sequences satisfy the definition of the space transformation. If Q coincides with Q' the transformation is unique.

Indeed, we can pass through P three spheres $\Sigma_1, \Sigma_2, \Sigma_3$ orthogonal to Σ_0 and having only two points in common, one inside, one outside Σ_0 , and cutting Σ_0 in the circles K_1, K_2, K_3 . The space transformation will carry these circles into three other circles, K'_1, K'_2, K'_3 , respectively, on Σ_0 . Let $\Sigma'_1, \Sigma'_2, \Sigma'_3$ be the spheres through these circles, orthogonal to Σ_0 . They also have just two points in common. Both the sequences of inversions carrying P into Q and P into Q' transform $\Sigma_1, \Sigma_2, \Sigma_3$ into $\Sigma'_1, \Sigma'_2, \Sigma'_3$ respectively. But the points P and P' lie on all three of these spheres and are both on the same side of Σ_0 . Hence Q and Q' are coincident. The definition is then unique.

We shall use S to represent the space transformation corresponding to the transformation of the surface of Σ_0 . No ambiguity is likely to arise. Consider the isometric circles I_s, I'_s of S and S^{-1} . S transforms I_s into I'_s and carries three distinct points on I_s into three distinct points on I'_s in a particular way. Let Σ_s, Σ'_s be spheres through these circles respectively, orthogonal to Σ_0 . S must also transform Σ_s into Σ'_s of the same size and three points on Σ_0 in a particular way.

Let S change a circle I on Σ_0 into a circle I' on Σ_0 of the same size with no increase or decrease in the length of any arc of I . Let $S(P_0) = P'_0$, P_0 on I , P'_0 on I' . Any other point Q on I goes into a point Q' on I' such that the distance P_0Q is equal

to the distance P_0Q' . If I is put down on I' so that P_0 coincides with P' when I is properly rotated about the axis through the centers of Σ_0 and I' , then all points of I will be brought into coincidence with the corresponding points of I' .

Let P_0 move around I in a counterclockwise direction with respect to the axis through the centers of I and Σ_0 . Suppose P' moves around in a counterclockwise direction also with respect to the axis through the centers of I' and Σ_0 . Then by a rotation of Σ_0 , I can be carried into I' so that all points occupy the proper position. But in this case all circles on Σ_0 are unchanged in size and there is no unique isometric circle.

Suppose now that S is not a rotation of Σ_0 . As P_0 moves around I_S in counterclockwise direction, P' moves around I'_S in clockwise direction. This change of sense can be accomplished by means of an even number of inversions in spheres. Let Π_S be the plane perpendicular bisector of the line of centers of Σ_S and Σ'_S . If Σ_S and Σ'_S coincide, we take Π_S as the plane perpendicular bisector of the line joining any point P on I_S with its transform P' on I'_S .

We first invert in Σ_S . This inversion leaves points on I_S unchanged but reverses the direction of angles on the surface of Σ_0 . We then reflect in Π_S so that the direction of lines on I_S is reversed and angles on Σ_0 are again changed. Finally we make a rotation of Σ'_S about the line of centers of Σ'_S and Σ_0 , so that the transform of P

by the inversion in the plane Π_S is brought into coincidence with $P' = S(P)$ and all other points of the inverse of I_S in Π_S into their proper positions on I'_S . Since the rotation is equivalent to two reflections in planes, the result of the three operations which we have done is to leave angles on Σ_0 unchanged and to transform points on I_S into points on I'_S in the same way that S transforms them.

We may state these results in the following form:

THEOREM 5. The space transformation S is equivalent to:

- (1) an inversion in Σ_S , followed by
- (2) a reflection in Π_S , followed by
- (3) a rotation through a suitable angle θ about the fixed axis OP_0 , through the centers of Σ'_S and Σ_0 .

Let U_Π denote the reflection in Π_S ; U_Σ , U'_Σ , the inversions in Σ_S , Σ'_S ; and R , R' , the rotations of Σ_S , Σ'_S respectively, the direction of rotation in R and R' being opposite. Then the transformations $R'U_\Pi U_\Sigma$, $U'_\Sigma U_\Pi R$, $U_\Pi U_\Sigma R$, $U_\Pi R U_\Sigma$, $U'_\Sigma R' U_\Pi$, and $R' U'_\Sigma U_\Pi$ are all equivalent since they transform three points in the same way. The sequence of the above theorem then may be varied for a given transformation.

Remark. The center of any isometric sphere Σ'_S is a transform of ∞ by S , where S^{-1} is the inverse of the transformation S .

Let Σ_S be the isometric sphere of S and Σ'_S that of S^{-1} . By Theorem 5 the inversion in Σ_S carries its center to ∞ and ∞ to its center; the reflection and rotation leave ∞ fixed and carry the transform of ∞ by the inversion in Σ_S into the center of Σ'_S ; that is, the center of Σ'_S is a transform of ∞ by S .

THEOREM 6. Lengths in the neighborhood of points inside Σ_S are increased in magnitude by the transformation S ; those outside Σ_S are decreased.

Of the geometric operations of Theorem 5 that accomplish the

transformation S only the inversion in Σ_s changes lengths. By inversion in this sphere its interior is carried into the exterior, the exterior into the interior, and the surface into itself. By the relations between the coordinates of a point P and those of its inverse with respect to Σ_s the distance ds from P to a point near P and its value ds' after the inversion have the relation

$$ds' = (r'/r) ds,$$

where r, r' are the distances from the point P and its inverse respectively from the center of Σ_s . If P is outside Σ_s , $r > r'$, so that $ds' < ds$. The opposite inequality holds for P inside. Hence lengths inside Σ_s are increased, those outside decreased by inversion in Σ_s . The same statement holds for the transformation S .

Remark. Any given sphere orthogonal to Σ_0 is the isometric sphere of an infinite number of transformations.

Let Σ be any given sphere orthogonal to Σ_0 . If it is the isometric sphere of a space transformation S , it must be transformed by S , into a sphere Σ' of the same size and orthogonal to Σ_0 . By Theorem 5 this transformation is equivalent to an inversion in Σ , a reflection in the plane perpendicular bisector of the line of centers of Σ and Σ' , and a rotation of Σ' about the line of centers of Σ' and Σ_0 . So with any sphere Σ' of the same size as Σ and orthogonal to Σ_0 we can accomplish one transformation by choosing an angle of rotation for Σ' . Then by different choices of the position of Σ' , the planes of reflection for the transformations are different and the transformations are distinct.

THEOREM 7. The distance of a point P from the center of a sphere Σ_s is unaltered by inversion in a sphere Σ orthogonal to Σ_s , if P lies on the surface of Σ_s or on that of Σ ; it is increased if P lies outside both Σ_s and Σ or inside both; it is decreased if P lies out-

side one and inside the other.

We shall choose the unit so that Σ has the radius 1 and so orient the axes that the center of Σ_1 lies on the ξ -axis.

We take Σ as the unit sphere with center at the origin and Σ_1 , orthogonal to Σ , with center on the ξ -axis and radius r_1 . If the center of Σ_1 has coordinates $(q, 0, 0)$, then $r_1^2 = q^2 - 1$, and the spheres are given by

$$\Sigma : \xi^2 + \eta^2 + \zeta^2 = 1 ; \quad \Sigma_1 : (\xi - q)^2 + \eta^2 + \zeta^2 = q^2 - 1.$$

Let $P(\xi, \eta, \zeta)$ be a point in space and $P'(\xi', \eta', \zeta')$ be its inverse with respect to the sphere Σ_1 . These points are such that

$$(\xi^2 + \eta^2 + \zeta^2)(\xi'^2 + \eta'^2 + \zeta'^2) = 1.$$

Let h, h' be the distance of P, P' respectively from the center of Σ_1 . Then

$$h^2 = (\xi - q)^2 + \eta^2 + \zeta^2,$$

$$h'^2 = (\xi' - q)^2 + \eta'^2 + \zeta'^2.$$

We form the difference $h^2 - h'^2$

$$\begin{aligned} h^2 - h'^2 &= (\xi - q)^2 - (\xi' - q)^2 + \eta^2 - \eta'^2 + \zeta^2 - \zeta'^2 \\ &= \xi^2 - \xi'^2 - 2q(\xi - \xi') + \eta^2 - \eta'^2 + \zeta^2 - \zeta'^2. \end{aligned}$$

With the relations $\xi' = \xi / (\xi^2 + \eta^2 + \zeta^2)$, $\eta' = \eta / (\xi^2 + \eta^2 + \zeta^2)$, $\zeta' = \zeta / (\xi^2 + \eta^2 + \zeta^2)$ this can be reduced to

$$h^2 - h'^2 = \frac{(\xi^2 + \eta^2 + \zeta^2 - 1) [(\xi - q)^2 + \eta^2 + \zeta^2 - q^2 + 1]}{(\xi^2 + \eta^2 + \zeta^2)}.$$

The first factor in the numerator is positive for points outside Σ , negative for points inside, and zero for points on the surface of Σ . The second factor is positive for points outside Σ_1 , negative for

points inside, and zero for points on the surface of Σ .

We conclude then that

- (1) If P is on either Σ or Σ_1 , $h^2 - h'^2 = 0$ and $h = h'$;
- (2) If P is outside Σ and inside Σ_1 , $h^2 - h'^2 < 0$ and $h < h'$,
If P is inside Σ and outside Σ_1 , $h^2 - h'^2 < 0$ and $h < h'$;
- (3) If P is outside both Σ and Σ_1 , $h^2 - h'^2 > 0$ and $h > h'$,
If P is inside both Σ and Σ_1 , $h^2 - h'^2 > 0$ and $h > h'$.

Now let P be any point of space. Consider the effect of a transformation S of the set of transformations on the distance of P from the center of Σ_0 . Of the three geometrical operations of S of Theorem 5 only the inversion in Σ_s can affect the distance of a point P from the center of Σ_0 , for the point O is a fixed point for the reflection and rotation, and neither operation changes lengths. Theorem 7 applied to the orthogonal spheres Σ_0 and Σ_s gives the following

THEOREM 8. The distance of any point P from the center of Σ_0 is unaltered by a transformation S of the set of space transformations if P lies on either Σ_0 or Σ_s ; it is increased if P lies outside or inside both; it is decreased if P lies outside one and inside the other.

Groups of Space Transformations. We have the following definitions:

A set of transformations is said to form a group if both the sequence of any two members of the set and the inverse of any member of the set are members of the set.

A group that contains infinitesimal transformations is said to be continuous; that is, if given $\epsilon > 0$, there exists a transformation S of the group such that the distance between any arbitrary point P on Σ_0 and its transform $P' = S(P)$ is less than ϵ .

A group not continuous is said to be discontinuous.

Discontinuous groups may be classified in the following way:

(1) Discontinuous groups such that in every neighborhood of any point P of Σ_0 there are transforms of P .

(2) Discontinuous groups such that in the neighborhood of some point P on Σ_0 there are none of its transforms by any transformation of the group.

Discontinuous groups of the first type are said to be improperly discontinuous; those of the second, properly discontinuous.

In the study of groups of space transformations we obtain first two fundamental properties that characterize the continuous and the discontinuous groups. For the derivation of these properties we need the following

LEMMA. Let L_1, L_2 be lines through two points Q_1, Q_2 respectively on Σ_0 , meeting Σ_0 again in the points Q'_1, Q'_2 respectively. If the distance Q_1Q_2 is less than ϵ and if the direction cosines of the lines L_1, L_2 differ respectively by an amount less than ϵ , then the distance $Q'_1Q'_2$ is less than $K\epsilon$, where $K = 17\sqrt{3}$.

Let l_1, m_1, n_1 be the direction cosines of L_1 and l_2, m_2, n_2 those of L_2 . Let $P(\xi, \eta, \zeta)$ be any point on L_1 . Then the equation of L_1 in parametric form may be written

$$\begin{aligned} \xi &= \xi_1 + l_1 s \\ \eta &= \eta_1 + m_1 s \\ \zeta &= \zeta_1 + n_1 s \end{aligned} \quad (1)$$

where (ξ_1, η_1, ζ_1) are the coordinates of Q_1 and s is the distance PQ_1 . The intersection of L_1 with Σ_0 determines Q'_1 . This point is given by the substitution of (1) into the equation of Σ_0 ,

$$\xi^2 + \eta^2 + \zeta^2 - 1 = 0,$$

which gives as roots of s the values

$$s = 0, \quad s = -2(l_1\xi_1 + m_1\eta_1 + n_1\zeta_1).$$

The value $s = 0$ gives the point $Q_1(\xi_1, \eta_1, \zeta_1)$; the second root gives $Q'_1(\xi'_1, \eta'_1, \zeta'_1)$

$$\begin{aligned}\xi'_1 &= \xi_1 - 2\pm_1(\pm_1\xi_1 + m_1\eta_1 + n_1\zeta_1) \\ \eta'_1 &= \eta_1 - 2m_1(\pm_1\xi_1 + m_1\eta_1 + n_1\zeta_1) \\ \zeta'_1 &= \zeta_1 - 2n_1(\pm_1\xi_1 + m_1\eta_1 + n_1\zeta_1)\end{aligned}$$

In a similar manner the coordinates of $Q'_2(\xi'_2, \eta'_2, \zeta'_2)$ are obtained by solving the parametric equations for L_2 with the equation of Σ_0 . They are

$$\begin{aligned}\xi'_2 &= \xi_2 - 2\pm_2(\pm_2\xi_2 + m_2\eta_2 + n_2\zeta_2) \\ \eta'_2 &= \eta_2 - 2m_2(\pm_2\xi_2 + m_2\eta_2 + n_2\zeta_2) \\ \zeta'_2 &= \zeta_2 - 2n_2(\pm_2\xi_2 + m_2\eta_2 + n_2\zeta_2)\end{aligned}$$

We form the difference

$$\xi'_1 - \xi'_2 = \xi_1 - \xi_2 - 2[(\pm_1\xi_1 - \pm_2\xi_2) + (\pm_1m_1\eta_1 - \pm_2m_2\eta_2) + (\pm_1n_1\zeta_1 - \pm_2n_2\zeta_2)]$$

By hypothesis each of the quantities $|\pm_1, -\pm_2|$, $|m_1, -m_2|$, $|n_1, -n_2|$, $|\xi_1 - \xi_2|$, $|\eta_1 - \eta_2|$, $|\zeta_1 - \zeta_2|$ is less than ε . We may write

$$\begin{aligned}\xi_2 &= \xi_1 + \varepsilon_1, & \pm_2 &= \pm_1 + \varepsilon_4, \\ \eta_2 &= \eta_1 + \varepsilon_2, & m_2 &= m_1 + \varepsilon_5, \\ \zeta_2 &= \zeta_1 + \varepsilon_3, & n_2 &= n_1 + \varepsilon_6,\end{aligned}$$

where each of the quantities $|\varepsilon_1|, \dots, |\varepsilon_6|$ is less than ε . With these relations the difference becomes

$$\begin{aligned}\xi'_1 - \xi'_2 &= \varepsilon_1 + 2(\pm_1 + \varepsilon_4)^2(\xi_1 + \varepsilon_1) - \pm_1^2\xi_1 - (\pm_1 + \varepsilon_4)(m_1 + \varepsilon_5)(\eta_1 + \varepsilon_2) + \pm_1 m_1 \eta_1 \\ &\quad - (\pm_1 + \varepsilon_4)(n_1 + \varepsilon_6)(\zeta_1 + \varepsilon_3) + \pm_1 n_1 \zeta_1,\end{aligned}$$

which may be written in the following form

$$\xi'_1 - \xi'_2 = \varepsilon_1 + 2\left[\xi_1(\pm_1 + \pm_1)\varepsilon_4 - \pm_1(\varepsilon_5 m_1 + \varepsilon_6 n_1 + \eta_1 \varepsilon_5 - \zeta_1 \varepsilon_6) + \varepsilon_4(m_1 \eta_1 + n_1 \zeta_1)\right]$$

Since the points Q_1 and Q_2 lie on Σ_0 and so $|\xi_i|, |\eta_i|, |\zeta_i|$ are each less than 1, and the direction cosines are less than 1, we have

$$|\xi_1' - \xi_2'| < 17 \varepsilon.$$

In like manner we have

$$|\eta_1' - \eta_2'| < 17 \varepsilon,$$

$$|\zeta_1' - \zeta_2'| < 17 \varepsilon.$$

Hence

$$Q_1'Q_2' = \sqrt{(\xi_1' - \xi_2')^2 + (\eta_1' - \eta_2')^2 + (\zeta_1' - \zeta_2')^2} < 17\sqrt{3} \varepsilon.$$

But since Q_1' and Q_2' lie on Σ_0 , we know that $Q_1'Q_2' \leq 2$ for any ε . Hence unless $\varepsilon \leq 2/17\sqrt{3}$, this upper bound for the distance is trivial.

We use this lemma to prove the two following theorems.

THEOREM 9. If a group contains an infinite number of rotations of the sphere Σ_0 into itself it is a continuous group.

A rotation of a sphere about an axis through an angle θ can be considered as equivalent to two reflections in planes that pass through the axis of rotation, and make an angle $\theta/2$ with each other.

To each rotation of Σ_0 there correspond an axis and an angle whose magnitude lies between 0 and 2π ; consequently there are an infinite number of equivalent planes of reflection, two to each axis. We pick out two planes for each rotation. Let these be Π_{i1}, Π_{i2} with direction cosines $\xi_{i1}, \eta_{i1}, \zeta_{i1}$ and $\xi_{i2}, \eta_{i2}, \zeta_{i2}$; these six quantities are bounded, less than 1, and infinite in number. Considered as a point in six dimensional space the infinite set of such points has at least one cluster point $(\xi_0, \eta_0, \zeta_0; \xi'_0, \eta'_0, \zeta'_0)$.

Let S and T be two rotations such that the direction cosines of the planes of reflection for S and T differ from those of the

cluster value by less than $\varepsilon/2$, ($\varepsilon > 0$), so that those of S and T differ by less than ε . Consider the effect of the transformation ST^{-1} on a point Q_1 on Σ_0 . This rotation is equivalent to a sequence of reflections in planes. In fact, the rotation T is equivalent to reflections in π_{T_1} and π_{T_2} , and S to reflections in π_{S_1} and π_{S_2} . Then ST^{-1} is the result of making the rotation T^{-1} and then the rotation S . T^{-1} is equivalent to

- (a) a reflection in π_{T_2} , carrying Q_1 to Q_2 ,
- (b) a reflection in π_{T_1} , carrying Q_2 to Q_3 .

Applying S to Q_3 we have

- (c) a reflection in π_{S_1} , carrying Q_3 to Q_4 ,
- (d) a reflection in π_{S_2} , carrying Q_4 to Q' .

Then $ST^{-1}(Q_1) = Q'$. The conditions for the above lemma are satisfied for the point Q_3 and its reflections in the planes π_{T_1} and π_{S_1} , since the lines $L_1 = Q_3Q_2$ and $L_2 = Q_3Q_4$ are perpendicular to the planes of reflection whose direction cosines differ by less than ε . Then $Q_2Q_4 < K\varepsilon$. Applying the lemma a second time to the reflections of Q_2 and Q_4 in π_{T_2} and π_{S_2} respectively we have $Q_2Q_4 < K^2\varepsilon$, that is, the distance between Q_1 and its transform by ST^{-1} is less than $K^2\varepsilon$.

We may choose S and T from those rotations whose equivalent planes of reflection have direction cosines which differ by less than ε' where $0 < \varepsilon' < \varepsilon$. Hence the distance Q_1Q' can be made arbitrarily small. The group thus contains infinitesimal transformations and so is continuous.

COROLLARY. A discontinuous group contains at most a finite number of rotations.

THEOREM 10. For a discontinuous group the number of isometric spheres with radii $> r > 0$ is finite.

Suppose on the contrary that the number is infinite.

Suppose first that the radii are bounded. To each transformation there correspond an isometric sphere with a radius r_k , a center $A_k(a_k, b_k, c_k)$, and the isometric sphere of its inverse transformation with center at $A'_k(a'_k, b'_k, c'_k)$, a plane of reflection with direction cosines l_k, m_k, n_k , which bisects the line of centers $A_k A'_k$ (if distinct; otherwise we accept the meaning given in Theorem 5), and an angle of rotation, for which we substitute two planes with direction cosines l_{k2}, m_{k2}, n_{k2} and l_{k3}, m_{k3}, n_{k3} respectively. That is, to each transformation there correspond fifteen quantities, not all independent, and since the direction cosines are less than 1, these quantities are bounded. The number of such sets of quantities is infinite, so that if each set is considered as representing a point in fifteen dimensional space, there is at least one cluster point of the set. Let $(a_0, b_0, c_0; a'_0, b'_0, c'_0; l_0, m_0, n_0; l'_0, m'_0, n'_0; l''_0, m''_0, n''_0)$ be one such point. Then a limit value r_0 for the radii is determined since the spheres with (a_0, b_0, c_0) as limit point of centers are orthogonal to Σ_0 .

Then for a given $\varepsilon > 0$ we can find two points in the fifteen dimensional space

$$(a_1, b_1, c_1; a'_1, b'_1, c'_1; l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3),$$

$$(a_2, b_2, c_2; a'_2, b'_2, c'_2; l_4, m_4, n_4; l_5, m_5, n_5; l_6, m_6, n_6),$$

such that the numerical difference of corresponding coordinates is less than ε . The radii r_1 and r_2 corresponding to these points differ by less than 2ε .

Let T and S be the transformations determined by the above two points with the corresponding values r_1 and r_2 as radii respectively. Consider the effect of ST^{-1} on a point Q_1 on Σ_0 . T is equivalent to two reflections in planes π_{T_3}, π_{T_2} , the operation which is substituted for the rotation of Σ_T , a reflection in the plane π_T , which is the perpendicular bisector of the line of centers of Σ_T and $\Sigma_{T^{-1}}$,

and an inversion in $\Sigma_{T^{-1}}$; S is equivalent to two reflections in planes Π_{S_3} , Π_{S_2} , a reflection in the plane Π_{S_1} , which bisects the line of centers of Σ_S and $\Sigma_{S^{-1}}$, and an inversion in $\Sigma_{S^{-1}}$. The transformation ST^{-1} is the result of performing T^{-1} and then S . T^{-1} is equivalent to

- (a) an inversion in $\Sigma_{T^{-1}}$, carrying Q_1 to Q_2 ,
- (b) a reflection in Π_{T_1} , carrying Q_2 to Q_3 ,
- (c) a reflection in Π_{T_2} , carrying Q_3 to Q_4 ,
- (d) a reflection in Π_{T_3} , carrying Q_4 to Q_{5-} .

Applying S we have

- (e) a reflection in Π_{S_3} , carrying Q_{5-} to Q_6 ,
- (f) a reflection in Π_{S_2} , carrying Q_6 to Q_7 ,
- (g) a reflection in Π_{S_1} , carrying Q_7 to Q_8 ,
- (h) an inversion in $\Sigma_{S^{-1}}$, carrying Q_8 to Q_1 .

The sequence for T^{-1} is such that applying T after T^{-1} we should go back through Q_4 , Q_3 , Q_2 to Q_1 .

Applying the lemma to the reflections of Q_{5-} in planes Π_{T_3} and Π_{S_3} , we obtain $Q_4 Q_6 < K\varepsilon$; applying the lemma again to the reflections of Q_4 and Q_6 in Π_{T_2} and Π_{S_2} respectively, we have $Q_3 Q_7 < K^2\varepsilon$; and finally applying the lemma to the reflections of Q_3 and Q_7 we obtain $Q_2 Q_1 < K^3\varepsilon$.

We consider now the effect of the inversions in $\Sigma_{T^{-1}}$ and $\Sigma_{S^{-1}}$ on a point P on Σ_0 . An inversion in a sphere Σ orthogonal to Σ_0 carries P into P' on Σ_0 on the straight line through P and the center of Σ . Let $Q_1(\xi_1, \eta_1, \zeta_1)$ and $Q_2(\xi_2, \eta_2, \zeta_2)$ be two points on Σ_0 such that $Q_1 Q_2 < \varepsilon$. Let $Q'_1(\xi'_1, \eta'_1, \zeta'_1)$ and $Q'_2(\xi'_2, \eta'_2, \zeta'_2)$ be their inverses in $\Sigma_{T^{-1}}$ and $\Sigma_{S^{-1}}$, respectively. The direction cosines of the lines $Q_1 Q'_1$ and $Q_2 Q'_2$ are

$$l_1 = (a'_1 - \xi_1)/r_1, \quad m_1 = (b'_1 - \eta_1)/r_1, \quad n_1 = (c'_1 - \zeta_1)/r_1,$$

$$z_2 = (a'_2 - \xi_2)/r_2, \quad m_2 = (b'_2 - \eta_2)/r_2, \quad n_2 = (c'_2 - \zeta_2)/r_2,$$

where r_1 is the distance of Q_1 from the center (a'_1, b'_1, c'_1) of Σ_{T-1} , and r_2 that of Q_2 from the center (a'_2, b'_2, c'_2) of Σ_{S-1} . Then the difference

$$z_2 - z_1 = (a'_2 - \xi_2)/r_2 - (a'_1 - \xi_1)/r_1$$

can be shown less in numerical value than $4\varepsilon/\pm$ where $\pm$ is a constant independent of ε . In fact, by the choice we made of S and T we may write $a'_2 = a'_1 + \varepsilon'$, $\xi_2 = \xi_1 + \varepsilon''$, $r_2 = r_1 + \varepsilon'''$, where $|\varepsilon'|, |\varepsilon''| < \varepsilon$ and $|\varepsilon'''| < 2\varepsilon$, so that

$$|z_2 - z_1| = \left| \frac{\varepsilon' - \varepsilon'' - \varepsilon''' (a'_1 - \xi_1)/r_1}{r_2} \right| < \frac{4\varepsilon}{\pm}$$

since $(a'_1 - \xi_1)/r_1 < 1$, and $r_2 > \pm > 0$. Similarly

$$|m_2 - m_1| < 4\varepsilon/\pm, \quad |n_2 - n_1| < 4\varepsilon/\pm.$$

Hence the conditions of the lemma are satisfied where the ε of the lemma in the expression $K\varepsilon$ is replaced by $4\varepsilon/\pm$, so that the distance $Q_1 Q'_2$ is less than $4K\varepsilon/\pm$.

We apply this result now to the points Q_2 and Q_y . The point Q_1 is the inverse of Q_2 in Σ_{T-1} , and Q' that of Q_y in Σ_{S-1} . Then the distance $Q_1 Q'$ is less than $4K^4\varepsilon/\pm$; that is, the distance between the point Q_1 and its transform Q' by the transformation ST^{-1} is less than $4K^4\varepsilon/\pm$. Since Q_1 is an arbitrary point on Σ_0 , this relation holds for all points on Σ_0 .

But $Q_1 Q' < 4K^4\varepsilon'/\pm$ for any $\varepsilon' < \varepsilon$ since S and T may be chosen so that the quantities in the set that corresponds to S and those of T differ by less than $\varepsilon' < \varepsilon$. Hence ST^{-1} can be chosen to shift the position of every point of Σ_0 by an arbitrarily small distance. However, this is contrary to the hypothesis that the given group is dis-

continuous.

Consider secondly the case where the radii are not bounded.

To each transformation S_k in this case there correspond two lines u_k, u'_k instead of the coordinates of the center of Σ_{s_k} and of $\Sigma_{s_k'}$ as in the first case. The lines u_k, u'_k join the centers of Σ_0 and Σ_{s_k} , and of Σ_0 and $\Sigma_{s_k'}$. The plane of reflection and the planes that correspond to the rotation of the sphere Σ_{s_k} are considered as in the above case. Let the direction cosines of u_k, u'_k be l_k, m_k, n_k and l'_k, m'_k, n'_k and those of π_{k_1}, π_{k_2} , and π_{k_3} be $l_{k_1}, m_{k_1}, n_{k_1}; l_{k_2}, m_{k_2}, n_{k_2}; l_{k_3}, m_{k_3}, n_{k_3}$ respectively. Each of these quantities is less than 1. Since there is an infinite number there is at least one cluster value, where these quantities are considered as in the first case as representing points in fifteen dimensional space. Let one cluster point be $(l_0, m_0, n_0; l'_0, m'_0, n'_0; l_{0_1}, m_{0_1}, n_{0_1}; l_{0_2}, m_{0_2}, n_{0_2}; l_{0_3}, m_{0_3}, n_{0_3})$.

Let r_1 be the distance of a point Q , on Σ_0 , from the center of an isometric sphere of radius r . We take r and r_1 greater than $1/\epsilon$. Then we can find two distinct transformations T and S such that the direction cosines of the lines and planes corresponding differ by less than ϵ respectively. Since r can be made as large as we please, ϵ can be made arbitrarily small.

Consider as before the effect of ST^{-1} on a point Q on Σ_0 , and use the same sequence of geometric operations to accomplish ST^{-1} as that used in the case for the radii bounded. The reflections in the planes perpendicular to the line of centers of Σ_s and Σ_{s-1} , and of Σ_T and Σ_{T-1} , and the reflections in the planes ~~in the planes~~ that replace the rotations have the same effect on Q as before. We need consider then only the effect of the inversions in Σ_{T-1} and Σ_{s-1} .

Let (a, b, c) ^{$+(a_1, b_1, c_1)$} be the centers of Σ_{T-1} and Σ_{s-1} respectively and p_1 and p_2 the distances of the points from the origin. Then

$$|a_1/p_1 - \xi_1| < 2\varepsilon, \quad |b_1/p_1 - \eta_1| < 2\varepsilon, \quad |c_1/p_1 - \xi_1| < 2\varepsilon, \\ |a_2/p_2 - \xi_2| < 2\varepsilon, \quad |b_2/p_2 - \eta_2| < 2\varepsilon, \quad |c_2/p_2 - \xi_2| < 2\varepsilon.$$

Let $Q_1(\xi_1, \eta_1, \xi_1)$ and $Q_2(\xi_2, \eta_2, \xi_2)$ be any two points on Σ_0 such that $Q_1 Q_2 < \varepsilon$, and let $Q'_1(\xi'_1, \eta'_1, \xi'_1)$ and $Q'_2(\xi'_2, \eta'_2, \xi'_2)$ be their inverses in Σ_{T-} and Σ_{S-} , respectively. The direction cosines of Q, Q'_1 and $Q_2 Q'_2$ differ then from those of the lines of centers of Σ_{T-} and Σ_0 and Σ_{S-} and Σ_0 respectively by less than 2ε . In fact, if r_1, r_2 represent the distances of Q_1, Q_2 from the centers of Σ_{T-} and Σ_{S-} , we have

$$p_1 - 1 < r_1 < p_1 + 1, \\ \text{or} \quad r_1 = p_1 + t, \quad \text{where} \quad |t| < 1.$$

The direction cosines of Q, Q'_1 are $(a_1 - \xi_1)/r_1, (b_1 - \eta_1)/r_1, (c_1 - \xi_1)/r_1$, so that

$$\left| \frac{a_1 - \xi_1}{p_1 + t} - \frac{a_1}{p_1} \right| = \left| \frac{\xi_1 + (a_1/p_1)t}{r_1} \right| < 2\varepsilon,$$

since the point Q_1 is on Σ_0 and $a_1/p_1 < 1$. Similar relations hold for $(b_1 - \eta_1)/r_1$ and $(c_1 - \xi_1)/r_1$, and for the direction cosines of $Q_2 Q'_2$. The direction cosines of Q, Q'_1 and $Q_2 Q'_2$ differ therefore by less than 5ε , since

$$\left| \frac{a_1 - \xi_1}{p_1 + t} - \frac{a_2 - \xi_2}{p_2 + t} \right| < \left| \frac{a_1 - \xi_1}{p_1 + t} - \frac{a_1}{p_1} \right| + \left| \frac{a_1}{p_1} - \frac{a_2}{p_2} \right| + \left| \frac{a_2}{p_2} - \frac{a_2 - \xi_2}{p_2 + t} \right| < 5\varepsilon,$$

By the above lemma, therefore, the distance Q'_1 from Q'_2 is less than $5K\varepsilon$; that is, the distance between Q_1 and its transform by inversion in Σ_{T-} and Σ_{S-} is less than $5K\varepsilon$.

Applying this result to Q_2 and its inverse Q'_2 in Σ_{T-} , and to Q'_1 and its inverse Q_1 in Σ_{S-} , we conclude, since $Q_2 Q'_2 < K^3\varepsilon$, that $Q_1 Q'_1 < 5K^4\varepsilon$. Since $\varepsilon > 0$ is arbitrary this distance can be made suitably

small; that is, ST^{-1} is an infinitesimal transformation. This is contrary to hypothesis, however, for the group is discontinuous.

Transformation of the Group. We wish to find a transformation S which when applied to a discontinuous group of transformations gives STS^{-1} , where T is the given group, that contains no rotations of Σ_0 into itself but transforms Σ_0 into itself in a one-to-one and directly conformal manner.

Let S be a transformation which transforms Σ_0 into itself in a one-to-one and directly conformal manner, and which possesses an isometric sphere with center at P_0 . Let the center of the isometric sphere of S be P'_0 . Then

$$S^{-1}(\infty) = P'_0,$$

$$T(P'_0) = P',$$

$$S(P') = P'',$$

so that

$$STS^{-1}(\infty) = P''.$$

If STS^{-1} is to contain no rotations of Σ_0 into itself, it must be such that

$$STS^{-1}(\infty) \neq \infty,$$

since for rotations the point at infinity is fixed. We choose S then so that $P'' \neq \infty$; that is to say, so that P'_0 and P' are not coincident. In particular, we may take $S^{-1} = S$, with $P'_0 = P_0$. S must be such that P_0 is not a fixed point of any transformation of the group, otherwise if $T(P_0) = P_0$, then $STS(\infty) = \infty$. It is sufficient if we take S so that P_0 lies outside a sphere concentric with Σ_0 and of radius $M = 1 + 2\rho$. In fact, we may take ρ as the greatest of the radii of the isometric spheres of the group. This is possible by Theorem 10,

for there is only a finite number of isometric spheres with radii exceeding a certain finite number. Let ρ be the greatest of these.

We have proved then the following

THEOREM 11. Let S be a transformation that transforms the surface of Σ_0 into itself in a one-to-one and directly conformal manner. If $S = S^{-1}$ is such that it possesses an isometric sphere whose center is not a fixed point of any transformation of the discontinuous group T , then the transformed group STS contains no rotations of Σ_0 into itself.

Limit Points of a Discontinuous Group.

Let G be an infinite discontinuous group of space transformations containing no rotations of Σ_0 into itself. In all the following work we shall consider only such groups unless otherwise specifically stated.

Let $r_1, r_2, \dots$ be the radii of the isometric spheres of the transformations. Then any sequence of radii of distinct spheres must be such that $\lim_{i \rightarrow \infty} r_i = 0$, by Theorem 10.

Definition. A limit point of the group is a cluster point of centers of isometric spheres.

Any other point not a limit point of the group is called an ordinary point.

THEOREM 12. All the limit points of a discontinuous group lie on Σ_0 .

This is immediate, for since the radii of the isometric spheres of the transformations whose centers approach the limit point approach zero, the centers must approach the surface of Σ_0 since the isometric spheres are orthogonal to Σ_0 .

Definition. Two points P_1 and P_2 are said to be congruent with respect to a transformation T of the group if $T(P_1) = P_2$, where T is not the identical transformation. Two figures F_1 and F_2 are

said to be congruent with respect to a transformation $T \neq 1$ of a group if $T(F_1) = F_2$.

THEOREM 13. The set of limit points is transformed into itself by every transformation of the group.

Let P_0 be a limit point of the group, and $P_1, P_2, \dots$ the centers of the isometric spheres which cluster about P_0 . Make the transformation T . Let $T(P_0) = P'_0$, $T(P_1) = P'_1$, $T(P_2) = P'_2$, $\dots$. If $P_1, P_2, \dots$ lie arbitrarily near to P_0 , then in any neighborhood of P'_0 there is at least one of the points P'_i distinct from P'_0 . Hence P'_0 is a cluster point of the points $P'_1, P'_2, \dots$.

The points $P_1, P_2, \dots$ are congruent to ∞ by the transformations $T_1^{-1}, T_2^{-1}, \dots$ since they are the centers of the isometric spheres respectively of these transformations. Then $P'_1, P'_2, \dots$ are also congruent to ∞ by the transformations $TT_1^{-1}, TT_2^{-1}, \dots$; but these transformations, as well as their inverses, belong to the group and have isometric spheres. Each TT_i^{-1} transforms ∞ into the center Q_i of the isometric sphere of its inverse. Then $Q_i = P'_i$, for $TT_i^{-1}(\infty) = P'_i$. Hence P'_0 is a limit point of the group.

THEOREM 14. In the neighborhood of any point P_0 of a discontinuous group of space transformations lie an infinite number of distinct points congruent to any point of space with the possible exception of P_0 itself and one other limit point.

Let P_0 be a limit point of the group; it must lie on Σ_0 . Let σ be a sphere with P_0 as center and radius r . In the neighborhood of P_0 , hence in σ , lie an infinite number of distinct isometric spheres whose centers cluster about P_0 . Let $\Sigma_1', \Sigma_2', \dots$ be isometric spheres whose centers have the single cluster point P_0 . Then the isometric spheres $\Sigma_1, \Sigma_2, \dots$ congruent to these respectively by the transformations $T_1, T_2, \dots$ are also infinite in number. Their centers have a cluster point P'_0 on Σ_0 .

Suppose P'_0 is distinct from P_0 . Take σ sufficiently small

so as to exclude P'_0 . We can choose n so large that Σ'_n lies inside σ . The point P'_0 may be such that it is fixed for all transformations of those whose isometric spheres have P_0 as limit point; in that case P'_0 does not have any transforms inside σ . Also the point P_0 itself may be fixed for all these transformations; in that case the transforms of P_0 are not distinct.

We wish to show then that any point Q in space other than P_0 and P'_0 has an infinite number of transforms in the neighborhood of P_0 . We shall show first that it has a transform distinct from P_0 in σ , then by choosing $\sigma_1, \sigma_2, \dots$ inside and concentric with σ , we conclude that it has an infinite number of such transforms near P_0 .

Let Q be any point in space distinct from P_0 and P'_0 . We take σ_1 inside σ so small that Q and P'_0 are on its exterior, and choose n such that Σ'_n lies wholly within, Σ_n lies wholly without σ_1 and has Q on its exterior. The transformation T_n carries Q into Q' inside Σ'_n , hence inside σ_1 . If $T_n(Q) = P_0$, then $T_n^2(Q) = Q''$ which is not coincident with P_0 , and lies inside Σ'_n .

In fact, if $T_n(Q) = P_0$, and $T_n^2(Q) = T_n(P_0) = P_0$, then $T_n^{-1}(P_0) = P_0$ at the same time as $T_n^{-1}(P_0) = Q$. This is impossible unless P_0 coincides with Q which is contrary to hypothesis. Since Q is outside Σ_n , Q' is inside Σ'_n , within σ_1 . If $Q' = P_0$, then $Q'' = T_n(P_0)$ is also within σ_1 , for P_0 is outside Σ_n and so is transformed into a point inside Σ'_n , that is, into a point in σ_1 .

So if $T_n(Q) = P_0$ we make the transformation T_n twice and get a point congruent to Q within σ_1 and distinct from P_0 .

Taking $\sigma_2, \sigma_3, \dots$ within σ_1 and concentric with it, we show an infinite number of distinct transforms of Q lie near P_0 .

Suppose secondly that P_0 and P'_0 are coincident. Let Q, Q' be any two distinct points different from P_0 . Take σ with P_0 as center and radius r sufficiently small that Q, Q' are exterior to σ . There exists a number n such that Σ_n, Σ'_n lie entirely inside σ .

The transformation T_n carries Q into Q_n , Q' into Q'_n , both within Σ_n and so in σ . Q_n and Q'_n cannot both coincide with P_0 , for Q and Q' are distinct and T_n is a one-to-one transformation. Hence at least one point Q_n , or Q'_n , has a transform in σ , distinct from P_0 .

Take a second sphere σ_1 with center P_0 , such that the point or points Q_n , Q'_n , distinct from P_0 are on its exterior. Let T_{n_1} be such that Σ_{n_1} , Σ'_{n_1} lie inside σ_1 . Both $T_{n_1}(Q)$ and $T_{n_1}(Q')$ lie inside σ_1 . Both cannot coincide with P_0 . Continuing in this way we obtain in the neighborhood of P_0 an infinite number of points congruent to one at least of Q , Q' .

There cannot be more than one point in space distinct from P_0 and not having an infinite number of transforms in the neighborhood of P_0 , for otherwise we could show, as above, by choosing σ small enough to exclude both these points, that at least one of them has an infinite number of transforms in the neighborhood of P_0 .

In the first part of the theorem the exceptional point P'_0 for P_0 is a limit point; in the second, the one of the points Q , Q' which is exceptional point for P_0 is transformed into P_0 and so by Theorem 13 is also a limit point. Both these points lie on Σ_0 . Hence every point in space not on Σ_0 has an infinite number of distinct transforms in the neighborhood of a limit point.

THEOREM 15. If the set A of limit points contains more than two points, the set is perfect.

Let H_A be the set of cluster points of the limit points A . We wish to show that

$$H_A \subset A \quad \text{and} \quad A \subset H_A.$$

Let P be a cluster point of limit points $P_1, P_2, \dots$. In any neighborhood of P there is a limit point P_n which is a cluster point of centers of isometric spheres. Hence in any neighborhood of P there

is at least one point, the center of an isometric sphere, distinct from P . Then P is a point of A .

Suppose P is a point of A . By hypothesis there are two other limit points Q, Q' distinct from P . By Theorem 14 in the neighborhood of P there are an infinite number of distinct transforms of one at least of Q, Q' ; by Theorem 13 these are all limit points. Hence P belongs to H_A .

An important application of this theorem is to prove the following

PROPOSITION. If the set of limit points of a discontinuous group consists of more than two points, there are no exceptional points to Theorem 14.

Suppose the theorem is not true. If there are more than two limit points of the group, the set is perfect. Let P_0 be a limit point of the group. Then near P_0 lie an infinite number of distinct limit points of the group. If every limit point has only itself as exceptional point, then near one, P say, there are limit points near which there are transforms of P ; hence near P there are transforms of itself, contrary to hypothesis. Hence at least one limit point P_0 exists which has an exceptional point P_1 distinct from itself. Let P_2 be any other limit point distinct from P_0 and P_1 . There is such a point by hypothesis.

If we can find a limit point P_2 distinct from P_0 and P_1 , which does not have P_1 as exceptional point, there is again a contradiction, for near P_2 there are transforms of P_1 , and near P_0 transforms of P_2 ; so near P_0 there are transforms of P_1 , contrary to hypothesis.

Suppose such a limit point does not exist. Then all the limit points distinct from P_1 have P_1 as exceptional point. Moreover, P_1 alone is exceptional for itself. In fact, suppose P_1 has a point P'_1 exceptional for itself, P_1 and P'_1 distinct. Then near some limit

point $P_m (P_m \neq P_1)$ are transforms of P_1 and near P_1 transforms of P_m ; hence near P_1 are transforms of P_1 . If P_1 has any exceptional point, therefore, it must be P_1 itself. It must necessarily have itself as exceptional point, for otherwise P_1 would not be exceptional for P_0 since in any neighborhood of P_1 there would be transforms of itself, which are limit points and not exceptional for P_0 .

This point P_1 is transformed into itself by every transformation of the group, for if it were carried into some limit point P_n by a transformation, then since P_n is not exceptional for any point except possibly P_n , near some limit point distinct from P_1 , there would be transforms of P_1 , contrary to hypothesis.

Let $\Sigma_1, \Sigma_2, \dots$ be the isometric spheres of the transformations $T_1, T_2, \dots$ respectively, whose centers have the single cluster point P_0 .

Let $\Sigma'_1, \Sigma'_2, \dots$ be the isometric spheres of $T_1^{-1}, T_2^{-1}, \dots$ respectively.

We can choose n sufficiently large that Σ_n has P_1 on its exterior; that is, choose n such that the radius of Σ_n is less than $1/2 |P_1 - P_0|$. Then P_1 must be interior to Σ'_n , otherwise, being exterior to Σ_n , it would be carried into the interior of Σ'_n and so could not be fixed.

If all the isometric spheres $\Sigma'_1, \Sigma'_2, \dots$ do not have centers that approach the point P_1 as limit point, then we can select a subsequence $\Sigma'_{n_1}, \Sigma'_{n_2}, \dots$ whose centers approach a point Q , distinct from P_1 . By choosing n_i sufficiently large we can make Σ'_{n_i} of radius less than $1/2 |P_1 - Q|$ and so have P_1 exterior to both Σ_{n_i} and Σ'_{n_i} . Hence P_1 is the only cluster point for the sequence $\Sigma_n, \Sigma'_{n_i}, \dots$.

Now let P_2 be any other limit point distinct from both P_0 and P_1 . Let a sequence of isometric spheres whose centers approach P_2 , distinct from both P_0 and P_1 , be $\Lambda_1, \Lambda_2, \dots$ of the transforma-

tions $S_1, S_2, \dots$ respectively.

We can take n , and m , such that the transformations T_n , and S_m , have isometric spheres, Σ_n , exterior to Σ_n' , and Λ_m , exterior to Λ_m' . Also we may suppose Σ_n and Λ_m are exterior to one another (since we may take n , and m , large enough that the radius of each of Σ_n , Λ_m , is less than $1/2 |P_0 - P_2|$).

The linear transformation L_n , corresponding to T_n , cannot be elliptic, for by Theorem 4 the isometric circles I_n, I_n' , on Σ_0 , through which Σ_n, Σ_n' respectively are constructed, would intersect and hence Σ_n, Σ_n' would intersect, contrary to the fact that these spheres are external. Also L_n cannot be parabolic, for then I_n, I_n' would be tangent, hence also Σ_n, Σ_n' . A similar statement holds for the linear transformation K_m , corresponding to the space transformation S_m .

The linear transformations L_n, K_m , therefore, are either hyperbolic or loxodromic. The projection of P_1 on the plane is a common fixed point of L_n and K_m ; it lies inside the isometric circles of L_n^{-1} and K_m^{-1} . The other fixed point for L_n , must by linear transformation theory lie inside the isometric circle of L_n , and the other for K_m , inside that of K_m . These fixed points are distinct, for otherwise their projections P_0 and P_2 on Σ_0 would be coincident, which would contradict the fact that P_0 and P_2 are distinct.

But by a proposition already proved*, if a group of linear transformations contains two hyperbolic or loxodromic transformations which have one fixed point in common but the other fixed points distinct, the group contains infinitesimal transformations. Hence the linear transformations corresponding to the given group of space transformations considered in the preceding work form a continuous group. Then by stereographic projection the space transformations form also a continuous group. This is a contradiction, for the group is discontinuous.

* The limit points of a group, L. R. Ford.

The Region R of the Group. A fundamental region R_0 of a discontinuous group is a region such that

- (1) No two points of the region are congruent;
- (2) In the neighborhood of any point on the boundary there are points congruent to points of the region.

THEOREM 16. If no two points of a region are congruent, the transforms of the region by two distinct transformations of the group do not overlap ; that is, do not have an interior point in common.

Suppose on the contrary that two transforms R_S, R_T of the given region by two distinct transformations S, T of the group have an interior point in common. Let this point be Q . Then

$$S^{-1}(Q) = Q', \quad \text{and} \quad T^{-1}(Q) = Q'', \quad \text{so} \quad T^{-1}S(Q') = Q'';$$

that is, Q' and Q'' are congruent unless $T^{-1}S = 1$, or $T = S$. But S and T are distinct. Hence Q' and Q'' are congruent, which is contrary to hypothesis.

COROLLARY. No two transforms of a fundamental region of a group by two distinct transformations of the group overlap.

This follows immediately from the definition of a fundamental region.

Definition. The region R of a group is the space exterior to all isometric spheres of the group. More specifically, a point P belongs to R if a region about P exists no point of which is interior to an isometric sphere of the group.

THEOREM 17. No two points of R are congruent by any transformation of the group.

This is immediate, for all points exterior to the isometric sphere of a transformation of the group are carried into the interior of the isometric sphere of its inverse, and so outside R . Since every point of R is outside all isometric spheres, no transform of a point of R is a point of R .

THEOREM 18. R and the regions congruent to R form a set of regions which extend into the neighborhood of every point of space.

Suppose the theorem is not true. Let P_0 be a point in whose neighborhood there are no points of R or points congruent to points of R . Then with P_0 as center and with radius r sufficiently small a sphere S can be described in which there are no transforms of R or points of R itself.

The interior points of S are ordinary points, for if any one were a limit point, it would have in its neighborhood by Theorem 14 an infinite number of transforms of points of R , contrary to hypothesis. Every point in the interior of S is within or on an isometric sphere, for otherwise it would lie outside all isometric spheres and so belong to R , or have an infinite number of spheres in its neighborhood and so be a limit point, contrary to hypothesis. This is true for all points within each of the regions congruent to S . In fact, if in any region S' congruent to S there were a point outside all isometric spheres, and so belonging to R , the transformation that carries S' into S would transform a point of R into a point of S . This is impossible, for S contains no transforms of R .

Let Σ_T be an isometric sphere having P_0 as interior or boundary point. Consider the effect of the transformation T on the sphere S .

Suppose first that P_0 lies on Σ_T of radius r_T . Then $r_T > r$, for otherwise the center p_0 of Σ_T would lie in S which is contrary to hypothesis since p_0 is congruent to ∞ , a point of R . Of the geometrical operations in the transformation T the inversion in Σ_T alone affects the size of S . Let $T(S) = S_1$, of radius r_1 . Then r_1 is given by

$$r_1 = \frac{1}{2} \left[\frac{r_T^2}{r_T - r} - \frac{r_T^2}{r_T + r} \right] = \frac{r r_T^2}{r_T^2 - r^2} = \frac{r}{1 - (r/r_T)^2}.$$

But since $r_T < M$, a constant, and $r_T > r$ we have

$$r_1 > \frac{r}{1 - (r/M)^2} = k r, \quad \text{where } k = \frac{1}{1 - (r/M)^2}.$$

If P_0 lies inside Σ_T , the radius r_1 is larger than it is for P_0 on Σ_T , and the inequality still holds.

Let P_1 be the center of S_1 . It is, as was P_0 , inside or on the boundary of some isometric sphere, say Σ_{T_1} . T_1 transforms S_1 into S_2 , with center P_2 and radius r_2 such that

$$r_2 > \frac{r_1}{1 - (r_1/M)^2} > \frac{r_1}{1 - (r/M)^2} > k r_1 > k^2 r.$$

Another transformation T_2 carries S_2 into S_3 with center P_3 and radius r_3 such that

$$r_3 > k^3 r,$$

and P_3 is in or on an isometric sphere.

For the n -th transformation S_{n-1} is transformed into S_n with radius r_n such that

$$r_n > k^n r, \quad k > 1.$$

For n sufficiently large r_n can be made as large as we please, hence greater than M . Then S_n must contain points of R . Since S_n is congruent to S , then S contains points congruent to points of R , contrary to hypothesis.

Hence no point of space exists in the neighborhood of which there are neither points of R nor transforms of points of R .

THEOREM 19. R constitutes a fundamental region of the group.

The first condition of a fundamental region of the group is satisfied by R by application of Theorem 17.

Let P be a point on the boundary. In any given neighborhood of

P there is a point Q within an isometric sphere and hence not on the boundary of R . By Theorem 18 in the neighborhood of Q there are points congruent to points of R . The second condition for a fundamental region is therefore satisfied.

THEOREM 20. Any closed region not containing limit points of the group can be filled by a finite number of transforms of R , including possibly R itself. These regions fit together without lacunary spaces.

Let B be a closed region not containing limit points. There is a finite number of isometric spheres containing points of B . Otherwise we could select an infinite sequence such that the limit of their radii is zero. The centers of the spheres then have a cluster point which must be in B since B is closed. But this is a contradiction.

Every point of R is transformed by a transformation T of the group into the interior of the isometric sphere Σ'_T of T^{-1} . So if Σ'_T contains no point of B , then certainly $R_T = T(R)$ contains no point of B .

The number of regions R_T containing points of B is not greater than the number of isometric spheres containing points of B . This number is finite.

Since R and its transforms extend into the neighborhood of every point of space, there can be no lacunary spaces of B not covered by transforms of R .

THEOREM 21. Within any region enclosing a limit point there lie an infinite number of distinct transforms of the entire region R .

Let P_0 be a limit point and σ be a small sphere with P_0 as center. Let Σ'_T be an isometric sphere lying entirely inside σ . A transformation T of the group carries R into R_T which lies entirely within Σ'_T .

But in σ there are an infinite number of isometric spheres of transformations of the group; hence inside σ of arbitrarily small

radius; that is, in the neighborhood of P_0 , there lie an infinite number of transforms of R . The theorem is then proved.

Groups with more than Two Limit Points. If there are more than two limit points of a group, the set of such points is perfect. Since they all lie on Σ_0 , the discontinuous groups have the following classifications:

- (1) Groups whose set of limit points consists of all points on Σ_0 .
- (2) Groups whose set of limit points is not everywhere dense on Σ_0 .

The distinction between groups of the two types may be made clearer by means of regions about a point P of Σ_0 . Let U_P be any spherical cap on Σ_0 enclosing P . If every U_P , arbitrarily small, contains limit points of the group for all P on Σ_0 , then the set of limit points of the group is everywhere dense on Σ_0 ; if for some P a neighborhood U_P of P can be found which contains no limit point, then the set of limit points is not everywhere dense on Σ_0 .

In the neighborhood of every limit point P_0 of Σ_0 there are an infinite number of transforms of every point in space, hence in particular of P_0 itself. A discontinuous group of the first type is therefore an improperly discontinuous group.

For a group of the second type there exists at least one point P_1 of Σ_0 which is not a limit point, and so a neighborhood U_{P_1} of P_1 in which there are no limit points. Every transform of P_1 is within or on an isometric sphere. Since there is only a finite number of isometric spheres at most inside U_{P_1} (otherwise U_{P_1} would contain a limit point), there can be at most only a finite number of transforms of P_1 inside U_{P_1} . Moreover, not every point of U_{P_1} is a fixed point of some transformation of the group; for otherwise, since a fixed point of a transformation must lie within or on an isometric sphere, U_{P_1} would

contain an infinite number of isometric spheres in its interior and so contain a limit point. Hence a group of the second type is properly discontinuous, for a neighborhood U_P of some point P , exists which contains no transforms of itself by any transformation of the group, except the identical transformation.

THEOREM 22. A necessary and sufficient condition that a group be properly discontinuous is that the region R contain in its interior a point of Σ_0 .

Since R contains no limit point in its interior, if P on Σ_0 belongs to the interior of R , P must be an ordinary point; that is, some neighborhood U_P of P exists which contains no limit point of the group. Since P belongs to R it is not a fixed point for any transformation of the group. The group then is properly discontinuous.

Conversely, if the group is properly discontinuous, there exists at least one point P on Σ_0 which is an ordinary point. We wish to show that a point on Σ_0 exists which belongs to the interior of R . There exists about P a sphere σ in which there are no limit points of the group. σ can be covered by a finite number of transforms of R . Let R_m be one of these transforms. R_m has a finite number of bounding spherical surfaces in σ , otherwise σ would contain a limit point. Since each face of R_m is orthogonal to Σ_0 , there exists at least one point P' in R_m on Σ_0 . The transformation $T_m^{-1}(P') = P$, transforms P' into P , in R and lying on Σ_0 . Hence the theorem is proved.

For an improperly discontinuous group, therefore, no point of the sphere Σ_0 belongs to the interior of R . Hence R is divided into two parts having in common at most only boundary points which are limit points.

THEOREM 23. The region R is transformed into itself by an in-

version in Σ_0 .

An inversion in Σ_0 transforms any sphere orthogonal to Σ_0 into itself. In particular, then, the isometric spheres are carried into themselves by an inversion in Σ_0 . Consequently R is transformed into itself.

The inversion in Σ_0 transforms the region about the origin into the region about ∞ . Hence a portion of R surrounds the origin, a part includes ∞ in its interior.

Let R_0 be the portion of R interior to Σ_0 . A study of the region R_0 gives information concerning the part of R outside Σ_0 by inversion in Σ_0 .

THEOREM 24. The region R_0 is simply connected.

A region in space is said to be connected if any two points of the region can be joined by a curve lying entirely within this region. The region is said to be simply connected if any closed surface entirely within the region can be shrunk to a point without crossing the boundary of the region.

In the first place, R_0 is connected, for a curve can be drawn from any point of R_0 to the origin, lying entirely within R_0 , namely, a radius. For if a radius of Σ_0 cuts an isometric sphere in a point inside Σ_0 , its second point of intersection is the inverse of the first point with respect to Σ_0 , hence lies outside Σ_0 .

Now let S be any closed surface in the region R_0 . It subtends a solid angle at the center of Σ_0 and the boundary of this angle is a cone. By sliding the surface S along the interior boundary of the cone we can shrink S to a point, the center of Σ_0 , without crossing the boundary.

The Boundary of R_0 . The points on the boundary of R_0 are either limit points or ordinary points. If a point P on the boundary of R is an ordinary point, it lies on just a finite number of isometric spheres.

A boundary point P belongs to one of the following classes:

(α) P a limit point.

(β) P an ordinary point on just one isometric sphere.

(γ) P an ordinary point on two or more isometric spheres having a circle in common.

(δ) P an ordinary point not on Σ_0 and on spheres not having a circle in common.

(ϵ) P an ordinary point on Σ_0 and on more than one isometric sphere.

If P is a fixed point of a transformation $S = S^{-1}$ and P on no other isometric sphere we shall regard it as belonging to class (γ).

A point P of class (β) is said to lie on a face; of class (γ), to lie on an edge; of class (δ), to be a vertex.

Points of class (α). Let P belong to class (α). The theorems on limit points proved above apply to this class of points.

Points of class (β). Let P belong to class (β). Through P there is just one isometric sphere Σ_S of the transformation S . The transform P' of P by S lies on Σ_S' and since P is of class (β), P and P' are distinct.

The point P' is also on the boundary of R . For suppose P' is inside Σ_0 , U a transformation of the group and not the identical transformation. Consider the effect of US on P .

$S(P) = P'$ with lengths near P unchanged;

$U(P') = P''$ with lengths near P' increased;

therefore

$US(P) = P''$ with lengths near P increased.

Hence P lies inside Σ_0 , which is contrary to hypothesis that P is a boundary point of R .

Suppose P' lies on Σ_0 , different from Σ_s' , $U \neq 1$. Then

$S(P) = P'$ with lengths near P unchanged;

$U(P') = P''$ with lengths near P' unchanged.

Hence

$US(P) = P''$ with lengths near P unchanged.

Therefore P lies on Σ_0 also, which is contrary to hypothesis, unless $US = S$, that is, $U = 1$, the identical transformation. This is impossible, for U is not the identical transformation.

Hence P' is on Σ_s' and is a boundary point of R .

In the neighborhood of P points on Σ_s are ordinary points also, for otherwise P would be a limit point of limit points, hence itself a limit point.

Part of the boundary of R is therefore a portion of the surface of Σ_s and a congruent portion of Σ_s' . The two congruent parts are equal in area, for lengths and consequently areas on Σ_0 are unchanged by S . Σ_s is called a face of the boundary of R . By Theorem 8 congruent points on the two faces are equidistant from the center of Σ_0 .

These results may be stated in the following theorem:

THEOREM 25. The boundary points of R of class (β) form sets of spherical faces which are congruent in pairs. The congruent faces are equal in area and congruent points on the faces are equidistant from the center of Σ_0 .

Points of class (γ) . Let P be a point of class (γ) . Let

C , be the circle common to the two or more isometric spheres, and E_1 the arc (or arcs) of C , which consists entirely of ordinary points and belongs to the boundary of R . We call this arc an edge of the boundary of R . Let Σ_{T_0} , Σ_T , form the boundary of R at E_1 . The following figure shows the boundary of R at a point P of class (γ) .

The transformation T_1 transforms E_1 into E_2 on Σ_T' with no change in lengths in any part of E_1 . As in case (β) points of E_2 are also on the boundary of R . E_2 is also an edge, of the same length and on the same number of isometric spheres as is E_1 . In the first place, E_2 is of the same length as E_1 , for E_1 is on Σ_{T_0} , the isometric sphere of T_1 . Moreover, points on K_0 , the polar cap of Σ_T , inside the isometric sphere Σ_{T_0} at the edge E_1 , are transformed by T_1 into the polar cap K_0' of the same area on Σ_T' at E_2 . The points of K_0' are inside an isometric sphere also, for each being on Σ_T' cannot belong to the interior of R , and if any one were on the boundary of R it would be transformed by T_1^{-1} into a point on K_0 on the boundary also, which is contrary to hypothesis. The distance of every point Q on E_2 from the center of Σ_0 is the same as that of its corresponding point $T_1^{-1}(Q)$ on E_1 .

The edges E_1 and E_2 are on the same number of isometric spheres. In fact, suppose E_1 lies on Σ_T , Σ_S , Σ_U , Σ_V , ... all distinct; then E_2 lies on $\Sigma_{ST^{-1}}$, $\Sigma_{UT^{-1}}$, $\Sigma_{VT^{-1}}$, ... as well as on Σ_T' . All these spheres are distinct, for if $ST^{-1} = UT^{-1}$, then $S = U$.

Consider the effect of ST^{-1} on E_2 .

$$T^{-1}(E_2) = E_1, \quad \text{with lengths near } E_2 \text{ unchanged;}$$

$S(E_1) = E'_1$ with lengths near E_1 unchanged, for E_1 is on Σ_S .

Hence

$ST^{-1}(E_2) = E'_1$ with no change in lengths near E_2 ;

that is, E_2 is on $\Sigma_{ST^{-1}}$.

So for each isometric sphere through E_1 there is at least one through E_2 . Similar reasoning holds for E_1 and E_2 interchanged. Then E_1 and E_2 lie on the same number of isometric spheres.

Since E_2 is an edge and on Σ_{T_1}' , which forms the boundary of R near E_2 , there is another isometric sphere through E_2 and forming the boundary of R near E_2 . Let this sphere be Σ_{T_2} . As for T_1 , we have $T_2(E_2) = E_3$, where E_3 is also an edge of the same length as E_2 . E_3 is on Σ_{T_2}' . Let Σ_{T_3} form the other face at E_3 . Then $T_3(E_3) = E_4$, which is also an edge on Σ_{T_3}' . Continuing in this way, we get congruent edges $E_1, E_2, E_3, \dots$ all of equal length and each point of E_{i+1} at the same distance from the center of Σ_0 as the corresponding point of E_i .

There is then a finite number n such that

$$T_n(E_n) = E_1 \quad \text{and} \quad T_n(\Sigma_n) = \Sigma_0.$$

Indeed, if no such number exists, there is either a transformation which carries an edge E_n into an arc E'_n not an edge, or there is an infinite number of distinct congruent edges $E_1, E_2, \dots$. The former alternative is impossible as has been proved above. The transformations $T_1, T_2, \dots$ corresponding to the edges $E_1, E_2, \dots$ are also infinite in number and have radii $r_1, r_2, \dots$ such that $\lim_{i \rightarrow \infty} r_i = 0$ by the result of Theorem 10. But all the edges are arcs of circles of non-zero radius on the isometric spheres $\Sigma_{T_1}, \Sigma_{T_2}, \dots$ and do not decrease in size. The second alternative, therefore, is also not possible. Hence a number n exists such that

$$T_n(E_n) = E_1.$$

Moreover, the isometric sphere of any transformation T_n that carries E_n into E_1 of the same size with no increase or decrease of any part must pass through E_n , for if E_n were entirely inside or outside Σ_{T_n} , its length would be increased or decreased by T_n ; if it were partly within, partly without, some elements of length of E_n would be changed by T_n . Either case is contrary to the condition that E_n goes into E_1 of the same length and with no change of any element of length of E_n .

The transformation T_n , therefore, whose isometric sphere Σ_{T_n} passes through E_n and forms the boundary of R near E_n , carries E_n into E_1 , and Σ_{T_n} into Σ'_{T_n} at E_1 . The edge E_1 then lies on Σ'_{T_n} and Σ_{T_0} , both forming the boundary of R near E_1 . Now Σ'_{T_n} cannot coincide with Σ_{T_0} since $T_n \neq T_0$. Hence Σ'_{T_n} and Σ_{T_0} coincide.

These results give the following theorem:

THEOREM 26. The boundary points of class (Y) form sets of congruent circular arcs, or edges. All congruent edges are equal in length.

Points of class (S). Let P_1 be a vertex not on Σ_0 . P_1 is transformed into another vertex P_2 by a transformation T_1 , whose isometric sphere Σ_{T_1} passes through P_1 . For let F_1 be the bounding face of the vertex P_1 on Σ_{T_1} . T_1 carries F_1 into a bounding face F'_1 of the same area and shape on Σ'_{T_1} . (See the accompanying figure). Then

at least two other isometric spheres Σ_{T_2} and Σ_{T_3} must pass through P_2 , each forming part of the boundary of R .

The corresponding transformations T_2, T_3 , and others possibly whose isometric spheres pass through P_2 , transform P_2 into $P_3, P_4, \dots$ (at most a finite number of such points), at each of which at least two other isometric spheres form parts of the boundary of R , for each of the points P_i is a vertex. Continuing in this way we get vertices $P_1, P_2, P_3, P_4, \dots$ congruent to the vertex P_1 .

Now for each vertex P_i congruent to P_1 by a transformation U there is an isometric sphere through P_i , for P_i cannot be inside Σ_0' since it is by hypothesis on the boundary and it cannot be outside Σ_0' since U^{-1} would transform P_i into P_1 inside Σ_0 and so not on the boundary of R . Then the number of vertices congruent to P_1 is finite, for P_1 is an ordinary point.

By Theorem 8 all the congruent vertices are equidistant from the center of Σ_0 .

Hence we have proved

THEOREM 27. The boundary points of class (δ) form finite sets of congruent vertices. All vertices of a set are equidistant from the center of Σ_0 .

Points of class (ϵ) . A point P of this class may be a point on two or more isometric spheres which are tangent at P , or a point on three or more spheres which intersect in a point on Σ_0 .

Let P be an ordinary point of the boundary of R and a point of tangency of two or more (but necessarily a finite number) of isometric spheres. If P is a point of internal tangency of the spheres we have the following theorem:

THEOREM 28. If an ordinary point P on the boundary of R is a point of internal tangency of two or more isometric spheres of which Σ_{T_1} of the transformation T_1 is the largest, its transform P' by T_1 is on the boundary of R and is a point of external tangency of two isometric spheres.

Let Σ_r , form the boundary of R near P , and let Σ_u of the transformation U be tangent internally to Σ_r , at P . $T_1(P) = P'$ is on the boundary of R by the same reasoning as that used in case (β) . Let Q be a point outside Σ_r , and near P' on the radial line of Σ_r , through P' . Denote by η the distance $P'Q$. Consider the effect of the transformation UT_1^{-1} on the length of η . Let the length η be constructed at P on Σ_r , outward along a radius extended. The inversion in Σ_r , will have the same effect on η at P as that in Σ_r' at P' . Suppose ξ is the length of η after inversion in Σ_r , and ζ the length of ξ after inversion in Σ_u . The relation between ζ and η is given by

$$\zeta - \eta = \frac{\xi^2(r - r')}{(r' - \xi)(r - \xi)} > 0,$$

where $r > r'$, r and r' being the radii respectively of Σ_r , Σ_u . Then $\zeta > \eta$; that is, lengths near P' on Σ_r' and on a radius of Σ_r' , extended beyond P' are increased by UT_1^{-1} . Hence points on this line lie inside the isometric sphere $\Sigma_{u,r'}$ of UT_1^{-1} . Also all points on Σ_r , near P' are boundary points of R . Hence $\Sigma_{u,r'}$ and Σ_r' are tangent externally at P' .

This reasoning holds for all transformations whose isometric spheres pass through P and are tangent internally to Σ_r , there. Hence P' lies on the isometric spheres of $U_1T_1^{-1}$, ..., $U_kT_1^{-1}$. Of all these there is one $U_jT_1^{-1}$ with the greatest radius. Then $\Sigma_{u,r'}$ forms part of the boundary of R near P' , so that P' is on the boundary of R and is a point of external tangency of Σ_r' and $\Sigma_{u,r'}$, where U_j is one of the isometric spheres tangent internally to Σ_r , at P .

Conversely, if P is an ordinary point of the boundary of R and is a point of external tangency of two isometric spheres Σ_r, Σ_u which form the boundary of R near P , the transform P' of P by T

is a point on the boundary of R and is a point of internal tangency of Σ'_T and Σ_{O_T-1} . However, some other transformation may exist whose isometric sphere is tangent externally to Σ'_T at P' .

Let P_1 be a point of the boundary of R of the type mentioned in Theorem 28, and let Σ_{T_1} form the boundary of R near P_1 . $T_1(P_1) = P_2$ is a point of external tangency of Σ'_T and the isometric sphere Σ_{T_2} of some other transformation T_2 . $T_2(P_2) = P_3$ is a point on the boundary of R and a point of external or internal tangency of two spheres Σ'_T and Σ_{T_3} ; if P_3 is a point of external tangency of Σ_{T_2} and Σ_{T_3} , we have $T_3(P_3) = P_4$ which is again a point of the boundary of R and a point of external or internal tangency. We can continue in this way as long as we have congruent points $P_2, P_3, \dots$ on the boundary of R which are points of external tangency.

We come finally to a transformation T_n such that $T_n(P_n) = P'_n$ is a point of the boundary of R and a point of internal tangency of Σ'_T and of some other isometric sphere Σ_{T_n} forming the boundary of R at P'_n . For suppose on the contrary we should have an infinite number of distinct congruent points $P_1, P_2, P_3, \dots$ all except P_1 being points of external tangency, congruent respectively by $T_1, T_2, \dots$. Each point P_i on the boundary of R congruent to P_1 by a transformation U has an isometric sphere Σ_U through P_i , for P_1 cannot be inside Σ_U , since it is by hypothesis on the boundary, and it cannot be outside Σ_U , since U^{-1} would transform P_1 into P_i inside Σ'_U , and so not on the boundary of R . Then through P_1 there is an infinity of distinct isometric spheres. But this is impossible, for P_1 is an ordinary point.

Hence there exists a finite number n such that $T_n(P_n) = P'_n$ which is a point of internal tangency of Σ'_T and some other isometric sphere or spheres. All points $P_1, \dots, P_n, P'_n$ are on Σ_0 , for P_1 is on Σ_0 .

These results constitute the theorem

THEOREM 29. The points of class $(\mathcal{E})$ that are points of tangency of two or more isometric spheres form sets of congruent points $P_1, P_2, \dots, P_n, P'_n$ all of which are points of external tangency except two, which are points of internal tangency. All points of a set are on Σ_0 .

For points of class $(\mathcal{E})$ which are vertices the same result holds as for a vertex not on Σ_0 .

Angles at Edges and Vertices of Sets of Congruent Edges and Vertices. We consider first the dihedral angles at the edges and at the points of tangency of class $(\mathcal{Y})$. We can state the following

THEOREM 30. The sum of the dihedral angles at the edges of a set of congruent edges is 2π , or a sub-multiple of 2π .

Let $E_1, E_2, \dots, E_n$ be a set of congruent edges, each being transformed into the one following by the transformations $T_1, T_2, \dots, T_n$ respectively, with $T_n(E_n) = E_1$. Let the faces of R at E_1 be F_0, F_1 ; at E_2 , F'_1, F_2 ; ...; at E_n , F'_{n-1}, F_n , with $T_n(F_n) = F_0$.

Consider the transformation

$$U = T_n T_{n-1} T_{n-2} \dots T_2 T_1;$$

this transformation has E_1 as fixed edge. Let

$$U_n = T_n, \quad U_{n-1} = T_n T_{n-1}, \quad \dots, \quad U_2 = T_n \dots T_2, \quad U_1 = T_n \dots T_1.$$

U_n transforms E_n into E_1 and the dihedral angle in R between F'_{n-1} and F_n at a point P_n on E_n into an equal angle at P' on E_1 , and R into R_n joining onto R along F_0 ; U_{n-1} carries E_{n-1} into E_1 , and the dihedral angle at a point P_{n-1} on E_{n-1} into an equal angle at the point P' on E_1 and R into R_{n-1} fitting onto R_n along the open face of R_n through E_1 ;; finally U_1 carries E_1 into E_1 and the dihedral angle at the point P on E_1 into an equal angle at P

on E_1 and R into R_1 , fitting onto R_2 along the open face of R_2 through E_1 .

If R_1 coincides with R , then U is the identical transformation and the sum of the dihedral angles about congruent points on a set of congruent edges of class (γ) is 2π .

If R_1 does not coincide with R , then a finite number of repetitions of U will carry R into itself. In this case the sum of the dihedral angles at congruent edges of a set of congruent edges is $2\pi/k$, where k is a finite integer.

Suppose on the contrary that a finite number of repetitions of U will not fill the dihedral angle about a point P of E_1 . We make $U, U^2, U^3, U^4, \dots$. If an infinite number of repetitions of U fill the angle, it is filled completely with no overlapping. All these transformations have E_1 as fixed edge. Being infinite in number they have isometric spheres whose radii are such that $\lim_{i \rightarrow \infty} r_i = 0$. This is impossible since E_1 , a circle on the isometric spheres of all the transformations, has a definite radius.

It is impossible, therefore, to have an infinite number of distinct transformations with E_1 as fixed edge; hence a finite number k exists such that $U, U^2, U^3, \dots, U^k$ are all those derived by repetition from U which have E_1 as fixed edge. This number k is such that U^k completely fills the dihedral angle at P on E_1 , for otherwise we could make $UU^k = U^{k+1}$, with E_1 as fixed edge, $U^k \neq 1$, and fill more of it without overlapping, which contradicts the fact that $U, U^2, \dots, U^k$ are all the distinct transformations derived from U and having E_1 as fixed edge.

THEOREM 31. The sum of the dihedral angles at points of a set of congruent points of tangency of class $(\mathcal{E})$ is 2π .

In a set of congruent points of this type, $P_1, P_2, \dots, P_n$, two of the points are points of internal tangency; so the dihedral angle

at each of those points is π . The other points of the set are points of external tangency; at such points the dihedral angle is zero. Hence the sum of the dihedral angles at the set of points of this type is 2π .

We investigate secondly the solid angles at the vertices of classes (8) and (9).

THEOREM 32. The set of transformations that leave a vertex invariant forms a finite group.

Consider the set of all transformations which leave a vertex P_1 invariant. If U_1, U_2 are two such transformations the product $U_1 U_2$ leaves P_1 fixed. Moreover, if U_1 leaves P_1 unchanged, so also does its inverse U_1^{-1} . Hence if transformations exist which leave the vertex P_1 invariant, the set of all such transformations forms a group.

Suppose U is a transformation that has the vertex P_1 as invariant point. Then Σ_0 passes through P_1 , otherwise by Theorem 8 the distance of P_1 from the origin would be increased or decreased by U and so could not be carried into itself. Since P_1 is an ordinary point it lies on just a finite number of isometric spheres. Hence the set of transformations that leave a vertex invariant forms a finite group.

THEOREM 33. The sum of the solid angles at the vertices of a set of congruent vertices of class (8) is $4\pi/m$, m a positive integer.

It is a property of conformal space transformations that solid angles are preserved.

Let the congruent set of vertices be $P_1, P_2, \dots, P_n$. We construct equal spheres $\sigma_1, \sigma_2, \dots, \sigma_n$ about the vertices $P_1, P_2, \dots, P_n$ respectively, orthogonal to the isometric spheres through the vertices and so small that they intersect only those isometric spheres which pass through the respective vertices. A transformation T_i carrying P_i into P_j transforms σ_i into σ_j , for σ_i being orthogonal to Σ_{T_i} through P_i , is carried into σ_i' of the same size, orthogonal to the isometric spheres about P_j ; Hence it must coincide with σ_j since

there is only one sphere of this size about P_j orthogonal to all the isometric spheres that pass through P_j .

Since P_1 is an ordinary point the space about P_1 inside σ_1 can be filled completely by a finite number of transforms of R . The boundaries of these transforms are spherical faces which we may take as having P_1 for a common vertex (otherwise we could decrease the radius of the spheres σ_i and exclude such transforms of R that do not have this property). The surface of σ_1 is then cut up into a finite number of spherical polygons.

Consider now the finite group of transformations that leave the vertex P_1 invariant. Such a set must exist since R and any transform of R about P_1 in σ_1 have P_1 as common vertex, which fact can be possible only if both the isometric sphere of the transformation and of its inverse pass through P_1 . But no transformations, excepting the identical one, other than those formed from the set by which $P_1, P_2, \dots, P_n$ are congruent can transform P_1 into itself, for otherwise the set $P_1, \dots, P_n$ would not form the complete set of congruent vertices.

Let $\tau_1, \tau_2, \dots, \tau_n$ be the spherical polygons cut out on the spheres $\sigma_1, \sigma_2, \dots, \sigma_n$ by the faces of R at the vertices $P_1, P_2, \dots, P_n$ respectively. There exists a transformation S_i carrying P_i into P_1 and τ_i into τ_i' on σ_1 . Such a transformation exists for each of the vertices $P_2, P_3, \dots, P_n$ carrying $\tau_2, \tau_3, \dots, \tau_n$ into $\tau_2', \tau_3', \tau_4', \dots, \tau_n'$ respectively on σ_1 . Let Ω be the region consisting of $\tau_1, \tau_2', \dots, \tau_n'$ on σ_1 , none of which overlap since each τ_i' is a transform of points of R and no two transforms of R overlap. The solid angles at the vertices are preserved by the transformations S_i .

Consider all the transformations $T_0, T_1, \dots, T_{m-1}$ ($T_0 = 1$) of the group that leaves P_1 invariant and transforms σ_1 into itself.

Let

$$\Omega_0, \Omega_1, \dots, \Omega_{m-1}$$

be the transforms of Ω by this group. They do not overlap. Moreover, they fill up completely the surface of σ_i . Suppose there is a region ω on σ_i , not covered by Ω or one of its transforms. Since ω does not belong to R , otherwise it would be contained in τ_i , it or a part ω' of ω has a congruent region in R . Let S carry ω or part of ω into the interior of R . Then since the sphere σ_i is outside all isometric spheres except those which pass through P_i , the same is true of ω . Hence Σ_j passes through P_i , so that $S(P_i)$ is some P_k of the set of congruent vertices and $S(\omega') = \omega''$ is a part of τ_k on σ_k . We make then the transformation S_i ; so that

$$S_i S(\omega') = \omega, \quad \text{on } \tau_i'.$$

$S_i S$ carries P_i into P_i and σ_i into σ_i ; that is, $S_i S$ belongs to the group $T_0, T_1, \dots, T_{m-1}$. The inverse transformation $S^{-1}S_i^{-1}$ also belongs to the group, so that $\omega' = S^{-1}S_i^{-1}(\omega)$, where ω is a part of τ_i' , is a transform of part of τ_i' ; that is, ω' is a part of some Ω_k of the set $\Omega_0, \Omega_1, \dots, \Omega_{m-1}$. Then the surface of σ_i is filled completely by Ω and its congruent regions $\Omega_1, \dots, \Omega_{m-1}$.

The sum of the solid angles at the vertices of the set is repeated $(m-1)$ times, and fills up the total solid angle 4π about the vertex P_i . So the sum of the solid angles at the vertices of the set is $4\pi/m$. The theorem is thus proved.

The Possible Finite Groups. If any set of transformations exists which transform a vertex into itself it is necessarily either the elliptic cyclic group or one of the polyhedral groups. In the latter case it may be the tetrahedral group with 12 transformations, the octohedral group with 24 transformations, the icosohedral group with 60 trans-

formations, or the n -group with $2n$ transformations.

If the group is the tetrahedral group, $m = 12$ and by Theorem 33 the sum of the solid angles at the vertices of the set of congruent vertices is $\psi = \pi/3$; if the octohedral group, $\psi = \pi/6$; if the icosohedral group, $\psi = \pi/15$; and if the n -group, $\psi = 2\pi/n$.

Generating Transformations of the Group.

Definition. A set of transformations $T_1, T_2, \dots$ is said to generate a group if every transformation of the group is a combination of members of the set $T_1, T_2, \dots$.

Any region interior to Σ_0 can be filled by a finite number of transforms of R_0 and therefore by a finite number of transforms of R_0 since the interior of Σ_0 is carried into itself by every transformation of the group.

THEOREM 34. The set of transformations which connect the faces of R_0 form a set of generating transformations of the group.

Let $T_1, T_2, T_3, \dots$ be the transformations connecting the faces of R_0 . Let S be any transformation of the group and $S(R_0) = R_3$. Any point Q of R_3 can be joined to a point of R_0 by a curve L every point of which is an ordinary point. Then about each point of L there is a region which consists only of ordinary points. A finite number of such regions cover L completely so that L is interior to a region A consisting only of ordinary points.

The region A can be filled by a finite number of transforms of R_0 by the transformations $T_1, T_2, T_3, \dots$. In fact, A is divided into two regions by certain of the faces of R_0 , say $F_1, F_2, \dots, F_k$ on the isometric spheres of $T_1, T_2, \dots, T_k$ respectively, for A contains a part of R_0 in its interior. We can fill that part of A not in R_0 by transforms of R_0 . The only face that fits exactly onto F_i in A is F_i^1 on $\Sigma_{T_i}^1$ of the transformation T_i^1 . But F_i^1 is on the boundary of R_0 and T_i^1 belongs to the set $T_1, T_2, \dots$. By T_i^1 ,

R_0 is carried into R_i inside Σ_{T_n} , bounded by certain spheres, which are transforms by T_n of the isometric spheres forming the boundary of R_0 . Let one sphere of the boundary of R_i be σ which is the transform of the isometric sphere Σ_{T_m} , say, by T_i . We make then the transformation $T_i T_m$ which adds onto σ in its interior another transform $R_{i,m}$ of R_0 . In turn each of its bounding faces can be added onto so that the interior of Σ_{T_i} which belongs to A is completely filled by transforms of R_0 . The number of such transforms is finite and each is a transform of R_0 by one of $T_1, T_2, \dots$.

In a similar way each face of R_0 in A can be added onto, so that A is completely filled by a transform of R_0 by a combination of members of the set $T_1, T_2, T_3, \dots$.

But A contains a portion of R_S . Thus R_S and a transform of R_0 by a combination of $T_1, T_2, T_3, \dots$ can overlap if and only if S is identical with this combination. The theorem is thus proved.

Application to Properly Discontinuous Groups in the Plane.

Let G be a properly discontinuous group of linear transformations. A sufficient condition for finding a set of generating transformations for the group is that it be possible to join every ordinary point of the plane to a point of R by a curve not passing through a limit point; in such a case, the transformations connecting the sides of R form a set of generating transformations for the group.

Suppose this sufficient condition is not satisfied. In this case we are able by considering the group of space transformations corresponding to G to obtain a set of generating transformations for G where the isometric circles in the usual linear theory do not necessarily enable us to do so.

Project the plane stereographically on Σ_0 and form the corresponding group of space transformations. The fundamental region R for

the space-group is found by constructing the isometric spheres for the group. R must contain in its interior one or more parts of Σ_0 , for otherwise by Theorem 22 the group would be improperly discontinuous; the same would be true of the linear group G contrary to hypothesis.

By projecting Σ_0 back on the plane those parts of Σ_0 belonging to R form a fundamental region (possibly disconnected) for the group in the plane, since the definition for such a region is satisfied.

The region R of the space transformations furnishes us a set of generating transformations for the space-group; hence the corresponding set of linear transformations forms a generating set for the linear group.

For example, suppose the boundary of R for a properly discontinuous group in the plane consists entirely of limit points. In this case any ordinary point of the plane obviously cannot be joined to a point of R by a curve not passing through a limit point, since any curve joining an ordinary point exterior to R to a point of R must cross the boundary.

We then form the space-group corresponding to the linear group and by the method given in the above paragraphs we are able to find a set of generating transformations for the linear group.

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