INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI[®]

NOTE TO USERS

This reproduction is the best copy available.

UMI[®]

NECESSARY AND SUFFICIENT CONDITIONS FOR POTENTIALS
OF SINGLE AND DOUBLE LAYERS

By

George Alvin Garrett

A thesis submitted to the faculty of The Rice Institute
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy.

2-6-36 Author

Rice Institute
1935

36-443 ✓

UMI Number: 3079708

UMI Microform 3079708

Copyright 2003 by ProQuest Information and Learning Company.
All rights reserved. This microform edition is protected against
unauthorized copying under Title 17, United States Code.

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

NECESSARY AND SUFFICIENT CONDITIONS FOR POTENTIALS OF SINGLE AND DOUBLE LAYERS*

By George A. Garrett.

1. Introduction. We consider the problem of obtaining necessary and sufficient conditions that a harmonic function defined in a three-dimensional region T be such that it can be represented in the region as a potential function. Denoting the harmonic function by $v(M)$ or $U(M)$, we find conditions necessary and sufficient in order that the function be representable as a potential of a single-layer distribution,

$$v(M) = \int_{S_0} \frac{1}{MP} d\gamma(e_p),$$

and as the potential of a double-layer distribution,

$$U(M) = \int_{S_0} \frac{\cos \angle(MP, n_p)}{MP^2} d\gamma(e_p);$$

where S_0 is the boundary of T ; $\gamma(e)$ is the mass function, a completely additive function of point sets defined on S_0 ; P is a point of S_0 ; and n_p is the interior normal to S_0 at P .

These conditions are given in terms of "normal families" of surfaces.

2. Potentials and Normal Families of Surfaces.

Definition. We shall call a family $\{S\}$ of simple closed surfaces having a tangent plane at every point a normal family provided that they lie interior or exterior to the surface S_0 , which we shall consider as a member of the family, and

(a) for each surface of the family and for any two points A, B of the

* This paper is an extension of the corresponding treatment for the two-dimensional case as presented by Prof. G. O. Evans in a seminar conducted at The Rice Institute during the spring of 1933. Theorems 4.1 and 4.2 overlap somewhat a theorem published by Ch. J. De la Vallée Poussin in November of the same year. Cf. De la Vallée Poussin, "Propriétés des Fonctions Harmoniques dans un Domaine Ouvert Limité par des Surfaces à Courbure Bornée", Annali della R. Scuola Normale Superiore di Pisa, Serie II—Vol. II (1933—XI), pp. 167-199. A more complete reference to this theorem of De la Vallée Poussin's is given in the footnote to Theorem 4.2.

surface

$$|\angle(n_A, n_B)| < \gamma \overline{AB},$$

where γ is a constant, n_A is the normal at A, and n_B is the normal at B;

(b) for every two surfaces S_1 and S_2 of the family

$$|\angle(n_{A_1}, n_{A_2})| < \gamma \overline{A_1 A_2},$$

where n_{A_1} is the normal to S_1 at A_1 on S_1 and n_{A_2} is the normal to S_2 at A_2 on S_2 ;

(c) in any infinite subset of the family there is a subsequence $\{S_i\}$ which approaches a surface S of the family, approach being in the sense that the maximum normal distance from S approaches zero.

We take the positive directions of the normals to the surfaces to be toward the interiors of the surfaces. We denote by T^+ and T^- the regions interior and exterior respectively to S_0 .

Consider a surface S_0 having the property (a). Let P be a point of S_0 and M a point not on S_0 and such that the normal to S_0 at the point Q on S_0 passes through M . Let M' be the image of M in the tangent plane to S_0 at Q . Consider a normal family $\{S\}$ of surfaces inside or outside S_0 and let τ be the maximum normal distance of a member of the family from S_0 . We suppose that τ is small enough so that there is a unique 1:1 correspondence* of points A_0 of S_0 along the normals n_{A_0} with points A of an arbitrary surface of $\{S\}$. If M is on S , then we denote by n_M the normal to S at M . We suppose also that τ is so small that dS_Q/dS_M is bounded away from zero and infinity, where dS_Q and dS_M are the elements of area of S_0 at Q and S at M , respectively.

* That this is true for τ small enough is shown by G. C. Evans and E. R. C. Miles in "Potentials of General Masses in Single and Double Layers. The Relative Boundary Value Problems"; American Journal of Mathematics, Vol. 53 (1931), pp. 493-516. Cf. p. 497. Hereafter we refer to this article as (A).

We write

$$\begin{aligned} \varphi &= \angle(MP, \eta_P) & ; & \quad \varphi' = \angle(M'P, \eta_P) & ; \\ \Theta_M &= \angle(MP, \eta_M) & ; & \quad \Theta'_M = \angle(M'P, \eta_{M'}) & ; \\ \Theta_Q &= \angle(MP, \eta_Q) & ; & \quad \Theta'_Q = \angle(M'P, \eta_Q) & ; \\ \eta &= MP & ; & \quad \eta' = M'P. \end{aligned}$$

By the symbol Θ without subscripts we shall mean both Θ_M and Θ_Q ; and by Θ' , both Θ'_M and Θ'_Q . We consider the functions

$$(I). \quad U(M) = \int_{S_0} \frac{\cos \varphi}{r^2} d\mathcal{V}(e_P),$$

$$(II). \quad U_1(M) = \int_{S_0} \frac{\cos \Theta}{r^2} d\mathcal{V}(e_P),$$

$$(III). \quad U_2(M) = \int_{S_0} \frac{\sin \varphi}{r^2} d\mathcal{V}(e_P),$$

$$(IV). \quad U_3(M) = \int_{S_0} \frac{\sin \Theta}{r^2} d\mathcal{V}(e_P).$$

Let η_Q be the z -axis and the tangent plane to S_0 at Q be the xy -plane for a system of rectangular coordinates. Let s be the curve of intersection of S_0 and the plane determined by P and η_Q . We shall also denote by s the length of the arc QP measured along this curve. Take as the x -axis the tangent to s at Q . Let P have cylindrical coordinates (ρ, μ, z) . Let A be any point of S_0 . Then we denote by $S_0(\delta, A)$ the portion of S_0 containing A which is contained in the sphere of center A and radius δ ; and by $S(\delta, A)$ the portion of S nearest to $S_0(\delta, A)$ which is cut out by this sphere. We suppose that P is in $S_0(\delta, Q)$ and that δ is so small that

$$s < 2\rho$$

$$\varphi, \varphi' \leq \pi.$$

In the inequalities which follow we may suppose without loss of generality that $r \leq r'$. For P in $S_0(\delta, Q)$ we have

$$\begin{aligned}
 (2.1) \quad & \left\{ \begin{aligned}
 & \varrho_M = \varrho_{M'} \leq 2r \\
 & |\varphi - \vartheta_Q| \leq \frac{1}{2}(\pi_P, \pi_Q) ; \quad |\varphi' - \vartheta'_Q| \leq \frac{1}{2}(\pi_P, \pi_Q) ; \\
 & |\vartheta_M - \vartheta_Q| \leq \frac{1}{2}(\pi_M, \pi_Q) ; \quad |\vartheta'_M - \vartheta'_Q| \leq \frac{1}{2}(\pi_{M'}, \pi_Q) ; \\
 & |\cos \varphi - \cos \vartheta_Q| \leq 2 \left| \sin \frac{\varphi - \vartheta_Q}{2} \right| < 2\gamma P ; \\
 & |\cos \varphi' - \cos \vartheta'_Q| \leq 2 \left| \sin \frac{\varphi' - \vartheta'_Q}{2} \right| < 2\gamma P ; \\
 & |\sin \varphi - \sin \vartheta_Q| \leq 2 \left| \sin \frac{\varphi - \vartheta_Q}{2} \right| < 2\gamma P ; \\
 & |\sin \varphi' - \sin \vartheta'_Q| \leq 2 \left| \sin \frac{\varphi' - \vartheta'_Q}{2} \right| < 2\gamma P ; \\
 & |\sin \vartheta_Q - \sin \vartheta'_Q| \leq 2 \left| \cos \frac{\vartheta_Q + \vartheta'_Q}{2} \right| \leq 2 \left| \sin \frac{1}{2}(\rho, \rho_P) \right| < \frac{2}{\rho P} \int_0^{\frac{1}{2}} \gamma s ds < 4\gamma P ; \\
 & |\cos \vartheta_Q + \cos \vartheta'_Q| \leq 2 \left| \cos \frac{\vartheta_Q + \vartheta'_Q}{2} \right| < 4\gamma P ; \\
 & \left| \frac{1}{r^2} - \frac{1}{(r')^2} \right| = \left| \frac{\sin \vartheta_Q - \sin \vartheta'_Q}{\rho} \right| \left| \frac{1}{r} + \frac{1}{r'} \right| < 4\gamma \left(\frac{2}{r} \right) = \frac{8\gamma}{r} ; \\
 & |\sin \vartheta_M - \sin \vartheta_Q| \leq 2 \left| \sin \frac{\vartheta_M - \vartheta_Q}{2} \right| < \gamma \overline{\varrho_M} \leq 2\gamma r ; \\
 & |\cos \vartheta_M - \cos \vartheta_Q| \leq 2 \left| \sin \frac{\vartheta_M - \vartheta_Q}{2} \right| < 2\gamma r ; \\
 & |\sin \vartheta'_M - \sin \vartheta'_Q| \leq 2 \left| \sin \frac{\vartheta'_M - \vartheta'_Q}{2} \right| < \gamma \overline{\varrho_{M'}} \leq 2\gamma r ; \\
 & |\cos \vartheta'_M - \cos \vartheta'_Q| \leq 2 \left| \sin \frac{\vartheta'_M - \vartheta'_Q}{2} \right| < 2\gamma r .
 \end{aligned} \right.
 \end{aligned}$$

Since P is in $S_0(\delta, Q)$ if Q is in $S_0(\delta, P)$, the relations (2.1) subsist for Q in $S_0(\delta, P)$.

Now we wish to consider P as fixed and Q as a variable point in $S_0(\delta, P)$. Let π_P be the z' -axis and the tangent plane to S_0 at P be the $x'y'$ -plane. Take the x' -axis along the tangent to the curve s^* at P , where s^* is the curve of intersection of S_0 and the plane determined by π_P and Q . We also denote by s^* the length of the arc QP measured along this curve. Let Q have cylindrical coordinates (ρ', μ', z') . Let $C(\delta, P)$ be the circle $\rho' = \delta$ in the $x'y'$ -plane. Let δ be small enough so that the relations (2.1) hold and also small enough so that

$$\begin{aligned}
 s^* &< 2\rho' \\
 |ds^*| &< 2|d\rho'|
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 |ds_Q| &< 2\rho' d\rho' d\mu' \\
 \frac{1}{2} &< \frac{\rho}{\rho'} < 2.
 \end{aligned}$$

Lemma 1. The integrals

$$\begin{aligned} & \int_{S_0(\delta, P)} \left| \frac{\cos \theta}{r^2} + \frac{\cos \theta'}{(r')^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\sin \varphi}{r^2} - \frac{\sin \varphi'}{(r')^2} \right| dS_Q; \\ & \int_{S_0(\delta, P)} \left| \frac{\cos \varphi - \cos \theta}{r^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\cos \varphi' - \cos \theta'}{(r')^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\cos \theta_M - \cos \theta_Q}{r^2} \right| dS_Q; \\ & \int_{S_0(\delta, P)} \left| \frac{\cos \theta'_M - \cos \theta'_Q}{(r')^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\sin \theta_M - \sin \theta_Q}{r^2} \right| dS_Q; \quad \int_{S_0(\delta, P)} \left| \frac{\sin \theta'_M - \sin \theta'_Q}{(r')^2} \right| dS_Q; \end{aligned}$$

all approach zero with δ , independently of τ . This statement is true if dS_Q and $S_0(\delta, P)$ are replaced by dS_M and $S(\delta, P)$ respectively.

Since dS_Q/dS_M is bounded away from zero and infinity, the second statement is a consequence of the first. Using θ_Q and θ'_Q for θ and θ' we have for the first integral

$$\begin{aligned} \int_{S_0(\delta, P)} \left| \frac{\cos \theta_Q}{r^2} + \frac{\cos \theta'_Q}{(r')^2} \right| dS_Q &= \int_{S_0(\delta, P)} \left| \frac{\cos \theta_Q + \cos \theta'_Q}{r^2} - \cos \theta'_Q \left(\frac{1}{r^2} - \frac{1}{(r')^2} \right) \right| dS_Q \\ &\leq \int_{S_0(\delta, P)} \left\{ \left| \frac{\cos \theta_Q + \cos \theta'_Q}{r^2} \right| + \left| \frac{1}{r^2} - \frac{1}{(r')^2} \right| \right\} dS_Q \\ &< 12\gamma \int_{S_0(\delta, P)} \frac{dS_Q}{r^2} \leq 48\gamma \int_{C(\delta, P)} d\rho' d\mu' = 96\pi\gamma\delta. \end{aligned}$$

This inequality proves the lemma for the first integral. The four succeeding integrals are treated similarly, using θ_Q and θ'_Q for θ and θ' . For the sixth integral we have

$$\int_{S_0(\delta, P)} \left| \frac{\cos \theta_M - \cos \theta_Q}{r^2} \right| dS_Q < 2\gamma \int_{S_0(\delta, P)} \frac{dS_Q}{r^2} \leq 16\pi\gamma\delta.$$

The three remaining integrals are treated similarly. For the first integral using θ_M and θ'_M for θ and θ' , we write

$$\begin{aligned} \int_{S_0(\delta, P)} \left| \frac{\cos \theta_M}{r^2} + \frac{\cos \theta'_M}{(r')^2} \right| dS_Q &\leq \int_{S_0(\delta, P)} \left| \frac{\cos \theta_M - \cos \theta_Q}{r^2} \right| dS_Q + \int_{S_0(\delta, P)} \left| \frac{\cos \theta'_M - \cos \theta'_Q}{(r')^2} \right| dS_Q \\ &\quad + \int_{S_0(\delta, P)} \left| \frac{\cos \theta_Q}{r^2} + \frac{\cos \theta'_Q}{(r')^2} \right| dS_Q. \end{aligned}$$

The lemma has been proved for these last three integrals. We treat similarly the four succeeding integrals, using θ_M and θ'_M for θ and θ' .

As a corollary to Lemma 1, we have

Lemma 2. The integrals

$$\begin{aligned} & \int_{S_0} \left| \frac{\cos \theta}{r^2} + \frac{\cos \theta'}{(r')^2} \right| dS_Q ; \int_{S_0} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q ; \int_{S_0} \left| \frac{\sin \varphi}{r^2} - \frac{\sin \varphi'}{(r')^2} \right| dS_Q ; \\ & \int_{S_0} \left| \frac{\cos \varphi - \cos \theta}{r^2} \right| dS_Q ; \int_{S_0} \left| \frac{\cos \varphi' - \cos \theta'}{(r')^2} \right| dS_Q ; \int_{S_0} \left| \frac{\cos \theta_M - \cos \theta_Q}{r^2} \right| dS_Q ; \\ & \int_{S_0} \left| \frac{\cos \theta'_M - \cos \theta'_Q}{(r')^2} \right| dS_Q ; \int_{S_0} \left| \frac{\sin \theta_M - \sin \theta_Q}{r^2} \right| dS_Q ; \int_{S_0} \left| \frac{\sin \theta'_M - \sin \theta'_Q}{(r')^2} \right| dS_Q ; \end{aligned}$$

are bounded independently of τ . This statement is true if dS_Q and S_0 are replaced by dS_M and S respectively.

We denote by E_S a finite number of non-overlapping regions on S and let the corresponding regions on S_0 be E_{S_0} . We denote the measures of these sets by $m(E_S)$ and $m(E_{S_0})$.

Theorem 2.1. The absolute continuity of the integrals

$$\begin{aligned} & \int_{E_{S_0}} |U_3(M) - U_3(M')| dS_Q ; \int_{E_{S_0}} |U_2(M) - U_2(M')| dS_Q ; \\ & \int_{E_S} |U_3(M) - U_3(M')| dS_M ; \int_{E_S} |U_2(M) - U_2(M')| dS_M ; \end{aligned}$$

is uniform independently of τ .

It suffices to prove the theorem for the first two integrals, since dS_Q/dS_M is bounded away from zero and infinity. We write

$$\begin{aligned} I &= \int_{E_{S_0}} |U_3(M) - U_3(M')| dS_Q \leq \int_{S_0} |d\gamma(e_p)| \int_{E_{S_0}} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q \\ &\leq \int_{S_0} |d\gamma(e_p)| \left\{ \int_{S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q + \int_{E_{S_0} - E_{S_0} \cap S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q \right\} \end{aligned}$$

We denote by $\bar{\gamma}(S_0)$ the total variation of $\gamma(e)$ over S_0 . Since $\gamma(e)$ is completely additive, $\bar{\gamma}(S_0)$ is bounded. By Lemma 1, given $\epsilon > 0$, we may

choose δ so small that the first of the inner integrals is less than $\frac{\epsilon}{2\bar{\gamma}(S_0)}$.

A positive constant K exists such that for Q in $E_{S_0} - E_{S_0} \cap S_0(\delta, P)$

$$\left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| < K$$

We now have

$$I < \frac{\epsilon}{2} + K_m(E_{S_0}) \bar{V}(S_0).$$

Hence, for all E_{S_0} such that $m(E_{S_0}) \leq \frac{\epsilon}{2K\bar{V}(S_0)}$, I is less than ϵ . This is what was to be shown. The second integral may be treated similarly.

Theorem 2.2. The integrals

$$\int_{S_0} |U_3(M) - U_3(M')| dS_Q ; \int_{S_0} |U_2(M) - U_2(M')| dS_Q ;$$

$$\int_S |U_3(M) - U_3(M')| dS_M ; \int_S |U_2(M) - U_2(M')| dS_M$$

remain bounded as τ approaches zero and approach zero uniformly with τ .

Again it suffices to prove the theorem for the first two integrals and again the same method of proof is used for both. That the integrals remain bounded as τ approaches zero follows immediately from the preceding theorem. For the first integral we have

$$\int_{S_0} |U_3(M) - U_3(M')| dS_Q \leq \int_{S_0} |d\bar{V}(e_P)| \left\{ \int_{S_0 - S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q + \int_{S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q \right\}$$

By Lemma 1 we may fix δ so small that, given any $\epsilon > 0$,

$$\int_{S_0(\delta, P)} \left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right| dS_Q < \frac{\epsilon}{2\bar{V}(S_0)}$$

independently of τ . Now for Q not in $S_0(\delta, P)$ the quantity $\left| \frac{\sin \theta}{r^2} - \frac{\sin \theta'}{(r')^2} \right|$ is bounded for each τ . Let $K_1(\tau)$ be the least upper bound of this quantity. We have then

$$\int_{S_0} |U_3(M) - U_3(M')| dS_Q < K_1(\tau) m(S_0) \bar{V}(S_0) + \frac{\epsilon}{2}$$

for each τ . But $K_1(\tau)$ obviously approaches zero uniformly with τ .

Hence the theorem is proved.

Theorem 2.3. The integrals

$$\int_S |U(M)| dS_M ; \int_S |U_1(M)| dS_M ; \int_{S_0} |U(M)| dS_Q ; \int_{S_0} |U_1(M)| dS_Q$$

remain bounded as τ approaches zero. The same holds if $U(M')$, $U_1(M')$,

~~are~~ substituted for $U(M)$, $U_1(M)$, ~~are~~ respectively.

It is sufficient to prove the theorem for the first two integrals.

In fact, we need only to prove the theorem for the second integral, for

$$\int_S |U(M)| - |U_1(M)| dS_M \leq \int_S |U(M) - U_1(M)| dS_M \leq \int_{S_0} |d\bar{V}(e_P)| \int_S \left| \frac{\cos \varphi - \cos \theta}{r^2} \right| dS_M$$

which is bounded by Lemma 2. Consider then

$$\int_S |U_1(M)| dS_M \leq \int_S dS_M \int_{S_0} \left| \frac{\cos \theta}{r^2} \right| |d\tau(e_P)|$$

Supposing for the moment that $\theta = \theta_M$, we have

$$\int_S \left| \frac{\cos \theta_M}{r^2} \right| dS_M = \int_S \left| \frac{\cos \angle(MP, n_M)}{r^2} \right| dS_M.$$

But S is a surface of "class Γ ". Hence a positive constant Γ exists so that $\int_S \left| \frac{\cos \angle(MP, n_M)}{r^2} \right| dS_M < \Gamma$. Hence we may interchange the order of integration above, obtaining

$$\int_S |U_1(M)| dS_M \leq \int_{S_0} |d\tau(e_P)| \int_S \left| \frac{\cos \theta_M}{r^2} \right| dS_M < \Gamma \bar{V}(S_0)$$

Therefore $\int_S |U_1(M)| dS_M$ is bounded for $\theta = \theta_M$. Moreover, for the purposes of this theorem it is immaterial whether we use θ_M or θ_Q for θ , since $\int_S \left| \frac{\cos \theta_M - \cos \theta_Q}{r^2} \right| dS_M$ is bounded, by Lemma 2.

For the second part of the theorem, we may write

$$|U(M')| \leq |U(M') + U(M)| + |U(M)| \quad ; \quad |U_1(M')| \leq |U(M')| + |U_1(M') - U(M')|$$

We have only to show, then, that $\int_S |U(M') + U(M)| dS_M$ and $\int_S |U_1(M') - U(M')| dS_M$ are bounded. We have

$$\int_S |U(M') + U(M)| dS_M \leq \int_{S_0} |d\tau(e_P)| \int_S \left| \frac{\cos \theta}{r^2} + \frac{\cos \theta'}{(r')^2} \right| dS_M$$

The inner integral on the right is bounded by Lemma 2. Hence the iterated integral is bounded. Similarly it may be shown that $\int_S |U_1(M') - U(M')| dS_M$ is bounded.

Let M, N on the surface S correspond to Q, P respectively on S_0 .

Lemma 3. The integrals

$$\int_{E_S} \frac{\cos \angle(MN, n_N)}{MN^2} dS_M \quad \text{and} \quad \int_{E_S} \frac{\cos \angle(MN, n_M)}{MN^2} dS_M$$

converge uniformly to the integrals

$$\int_{E_{S_0}} \frac{\cos \angle(QP, n_P)}{QP^2} dS_Q \quad \text{and} \quad \int_{E_{S_0}} \frac{\cos \angle(QP, n_Q)}{QP^2} dS_Q.$$

respectively as τ approaches zero.

We may write

$$\left| \int_{E_{S_0}} \frac{\cos \angle(QP, \eta_P)}{QP^2} dS_Q - \int_{E_S} \frac{\cos \angle(MN, \eta_N)}{MN^2} dS_M \right| \leq J_1 + J_2$$

where

$$J_1 = \left| \int_{E_{S_0} - \frac{S_0(\delta, P)}{S_0(\delta, P)}} \frac{\cos \angle(QP, \eta_P)}{QP^2} dS_Q - \int_{E_S - \frac{S(\delta, P)}{S(\delta, P)}} \frac{\cos \angle(MN, \eta_N)}{MN^2} dS_M \right| ;$$

$$J_2 = \left| \int_{E_{S_0} \cdot \frac{S_0(\delta, P)}{S_0(\delta, P)}} \frac{\cos \angle(QP, \eta_P)}{QP^2} dS_Q - \int_{E_S \cdot \frac{S(\delta, P)}{S(\delta, P)}} \frac{\cos \angle(MN, \eta_N)}{MN^2} dS_M \right| .$$

Since these are surfaces of class Γ , the two integrals composing J_2 exist and approach zero uniformly with δ on account of the absolute continuity of the integral as a function of the set over which the integration is taken. Given $\epsilon > 0$ we may fix δ so that J_2 is less than $\frac{\epsilon}{2}$. Now let τ be less than δ so that N is in $S(\delta, P)$. The integrand of the second integral of J_1 converges uniformly to that of the first as τ approaches zero. Also dS_Q/dS_M approaches one uniformly as τ approaches zero. Hence for τ_1 sufficiently small we have $J_1 < \frac{\epsilon}{2}$ for all M, N, Q, P and all $\tau \leq \tau_1$, which proves the theorem. The other two integrals are treated similarly.

3. On the Solution of Integral Equations. Let S_0 be a surface of the kind described in the preceding section and let $\{S\}$ be a normal family of surfaces inside or outside S_0 and including S_0 . Let w be a regular closed curve* on S_0 and let σ be one part of S_0 enclosed by w . Denote by $q(P, w)$ the symmetric surface density of σ at the point P ; i.e.

$$\begin{aligned} q(P, w) &= 1 \quad \text{for } P \text{ inside } w, \\ &= 0 \quad \text{for } P \text{ outside } w, \\ &= \frac{\Psi}{2\pi} \quad \text{for } P \text{ on } w, \text{ where } \Psi \text{ is the angle between} \\ &\quad \text{the forward and backward tangents to } w \text{ at } P. \end{aligned}$$

Define the function of regular

* (A) p. 498.

curves with regular discontinuities

$$(3.1) \quad \gamma(w) = \int_{S_0} q(P, w) d\gamma(e_P)$$

Let $S(\tau)$ be a member of the family the least upper bound of whose normal distances from S_0 is τ , and let $w(\tau)$, $\sigma(\tau)$ be the sets of points on $S(\tau)$ corresponding to w , σ respectively on S_0 . Write

$$(3.2) \quad v(M) = \int_{S_0} \frac{1}{MP} d\gamma(e_P) ;$$

$$(3.3) \quad U(\tau, w) = \int_{\sigma(\tau)} U(M) dS_M ;$$

$$(3.4) \quad V(\tau, w) = \int_{\sigma(\tau)} \frac{dv}{dn_M} dS_M = \int_{\sigma(\tau)} U_1(M) dS_M, \quad [\Theta = \angle(MP, n_M)] .$$

Denoting approach to S_0 from T^+ and T^- by $\lim_{\tau \rightarrow 0^+}$ and $\lim_{\tau \rightarrow 0^-}$

respectively, we have*

$$(3.5) \quad \lim_{\tau \rightarrow 0^+} U(\tau, w) = -2\pi \gamma(w) + \int_{S_0} d\gamma(e_P) \int_{\sigma} \frac{\cos \angle(QP, n_P) dS_Q}{QP^2}$$

$$(3.6) \quad \lim_{\tau \rightarrow 0^+} V(\tau, w) = +2\pi \gamma(w) + \int_{S_0} d\gamma(e_P) \int_{\sigma} \frac{\cos \angle(QP, n_Q) dS_Q}{QP^2}$$

If $F(w)$ is a given completely additive function of regular curves on S_0 with regular discontinuities, then in order to obtain the mass function $\gamma(w)$ determining $F(w)$ by means of the relation $F(w) = \lim_{\tau \rightarrow 0^+} \int_{\sigma(\tau)} U(M) dS_M$, where $U(M)$ is given by (I) for M in T^+ , we have the equation

$$F(w) = -2\pi \gamma(w) + \int_{S_0} d\gamma(w_P) \int_{\sigma} \frac{\cos \angle(QP, n_P) dS_Q}{QP^2}$$

Putting

$$K(Q, P) = \frac{1}{2\pi} \frac{\cos \angle(QP, n_P)}{QP^2}$$

$$\Phi(w) = -\frac{1}{2\pi} F(w)$$

we obtain the equation

$$(3.7) \quad \gamma(w) = \Phi(w) + \int_{S_0} d\gamma(w_P) \int_{\sigma} K(Q, P) dS_Q .$$

Theorem 3.1. The equation (3.7) has the solution

$$(3.8) \quad \gamma(w) = \Phi(w) - \int_{S_0} d\Phi(w_p) \int_{\sigma} k(Q, P) dS_Q$$

where $k(Q, P)$ is the resolvent kernel for the Fredholm equation of the form

$$(3.9) \quad h(Q) = f(Q) + \int_{S_0} K(Q, P) h(P) dS_P$$

and where $f(Q)$ is given by (3.12) below. The function $\gamma(w)$ is a completely additive function of regular curves on S_0 with regular discontinuities and is uniquely determined.

The function $k(Q, P)$ is seen by its explicit expression, given by (3.17) below, to be continuous except when $Q = P$. Moreover, it is integrable with respect to Q and P over S_0 , as follows incidentally in the proof of Lemma 4 below.

We show first that if $\gamma(w)$ is a completely additive function of regular curves on S_0 with regular discontinuities and is a solution of (3.7) then $\gamma(w)$ satisfies (3.8). Put

$$(3.10) \quad \gamma(w) = \Phi(w) + H(w).$$

It follows immediately from (3.7) that $H(w)$ is absolutely continuous.

Hence the derivative of $H(w)$ exists almost everywhere and we may write

$$(3.11) \quad H(w) = \int_{\sigma} h(R) dS_R,$$

where $h(R)$ is the derivative almost everywhere of $H(w)$. Substituting (3.10) in (3.7) we have

$$H(w) = \int_{S_0} d\Phi(w_p) \int_{\sigma} K(Q, P) dS_Q + \int_{S_0} dH(w_p) \int_{\sigma} K(Q, P) dS_Q.$$

Hence

$$h(Q) = \int_{S_0} K(Q, P) d\Phi(w_p) + \int_{S_0} K(Q, P) h(P) dS_P.$$

Putting

$$(3.12) \quad f(Q) = \int_{S_0} K(Q, P) d\Phi(w_p)$$

we obtain the equation (3.9). But (3.9) has the unique* summable solution $h(Q)$

given by

$$(3.13) \quad h(Q) = f(Q) - \int_{S_0} k(Q,R)f(R)dS_R.$$

where*

$$(3.14) \quad k(Q,P) + K(Q,P) = \int_{S_0} k(Q,R)K(R,P)dS_R = \int_{S_0} K(Q,R)k(R,P)dS_R.$$

Substituting (3.13) in (3.11) we have

$$H(w) = \int_{\sigma} dS_Q \int_{S_0} K(Q,P)d\Phi(w_p) - \int_{\sigma} dS_Q \int_{S_0} k(Q,R)dS_R \int_{S_0} K(R,P)d\Phi(w_p)$$

By the first part of the lemma proved below we may change the order of integration to reduce this last equation to the form

$$H(w) = \int_{S_0} d\Phi(w_p) \int_{\sigma} dS_Q \left\{ K(Q,P) - \int_{S_0} K(R,P)k(Q,R)dS_R \right\}$$

Applying (3.14) we now have

$$H(w) = - \int_{S_0} d\Phi(w_p) \int_{\sigma} k(Q,P)dS_Q.$$

Substituting in (3.10), we obtain the equation (3.8). Similarly it may be shown that $\gamma(w)$ given by (3.8) is a completely additive function of curves and satisfies (3.7), which completes the proof of the theorem.

Lemma 4. The integrals

$$\int_{S_0} |k(Q,P)|dS_Q \quad \text{and} \quad \int_{S_0} |K(Q,P)|dS_P$$

are bounded for all Q,P on S_0 . If M,N on $S(\tau)$ correspond respectively to Q, P on S_0 , then

$$\int_{S(\tau)} |k(M,N)|dS_M \quad \text{and} \quad \int_{S(\tau)} |K(M,N)|dS_N$$

are bounded uniformly as τ approaches zero.

* See Appendix 1.

We define the iterated kernels $K_{\lambda}(Q,P)$ by the relation

$$K_{\lambda+1}(Q,P) = \int_{S_0} K(Q,R)K_{\lambda}(R,P)dS_R \text{ where } K_0(R,P) = K(R,P). \text{ The kernel } K(Q,P)$$

is not bounded. However $K_1(Q,P)$ is bounded and continuous except when $Q = P$, and $K_2(Q,P)$ is continuous* in Q and P . It may readily be shown that $K_1(M,N)$ is bounded uniformly as τ approaches zero[†]. Making use of this fact in conjunction with Lemma 3, we see that $K_{\lambda}(M,N)$, ($\lambda \geq 2$), changes continuously as τ approaches zero, the continuity being uniform in M and N . A proof of this last statement follows.

First, we prove the statement for $\lambda = 2$. Let R on S_0 and G on $S(\tau)$ be corresponding points. Then for any M, N, Q, P

$$\left| \int_{S_0} K(Q,R)K_1(R,P)dS_R - \int_{S(\tau)} K(M,G)K_1(G,N)dS_G \right| \leq J_1 + J_2$$

where

$$J_1 = \left| \int_{S_0 - S_0(\delta,Q) - S_0(\delta,P)} K(Q,R)K_1(R,P)dS_R - \int_{S(\tau) - S(\delta,Q) - S(\delta,P)} K(M,G)K_1(G,N)dS_G \right|$$

$$J_2 = \left| \int_{S_0(\delta,Q) + S_0(\delta,P)} K(Q,R)K_1(R,P)dS_R - \int_{S(\delta,Q) + S(\delta,P)} K(M,G)K_1(G,N)dS_G \right|$$

Due to the uniform boundedness of $K_1(M,N)$ we may fix δ so that J_2 is as small as we like. Then we let τ be less than δ so that M and N are in $S(\delta,Q)$ and $S(\delta,P)$ respectively. Now the integrand of the second integral in J_1 converges uniformly to that of the first as τ approaches zero and dS_R/dS_G approaches one uniformly, which proves the statement. The proof for $\lambda > 2$ is an obvious application of the method of mathematical induction.

Replacing $K(Q,P)$ by $\lambda K(Q,P)$ and substituting (3.9) into itself twice we obtain the equivalent[‡] equation

$$(3.15) \quad h(Q) = f_2(Q) + \lambda \int_{S_0} K(Q,P)h(P)dS_P$$

where

$$f_2(Q) = f(Q) + \lambda \int_{S_0} K(Q,P)f_1(P)dS_P$$

* (A) p. 507. See also Kellogg, Potential Theory, p. 301.

[†] From the nature of the proof that $K(Q,P)$ is bounded, using the proof suggested in the footnote given on p. 507 of (A), the uniform boundedness of $K(M,N)$ is seen immediately. See Appendix 2.

[‡] That (3.9) and (3.15) are equivalent if λ is not a characteristic value is shown in Goursat, Cours d'Analyse Mathématique, Vol. 3, pp. 355-356.

and

$$f_1(Q) = f(Q) + \lambda \int_{S_0} K(Q,P) f(P) dS_P$$

The solution of (3.15) may be written

$$(3.16) \quad h(Q) = f_2(Q) - \lambda^3 \int_{S_0} k_2(Q,R;\lambda^3) f_2(R) dS_R$$

where $k_2(Q,R;\lambda^3)$ is continuous in Q and R , providing λ^3 is not a characteristic value for $k_2(Q,R;\lambda^3)$. Denoting by $k(Q,P;\lambda)$ the resolvent kernel for (3.9)

where $K(Q,P)$ has been replaced by $\lambda K(Q,P)$ we have, from (3.13),

$$(3.17) \quad k(Q,R;\lambda) = -K(Q,R) - \lambda K_1(Q,R) + \lambda^2 k_2(Q,R;\lambda) + \lambda^3 \int_{S_0} k_2(Q,P;\lambda^3) K(P,R) dS_P + \lambda^4 \int_{S_0} k_2(Q,P;\lambda^3) K_1(P,R) dS_P.$$

Since $\lambda = 1$ is not a characteristic value for $K(Q,P)$ and the characteristic values of $K(Q,P)$ are real, $\lambda^3 = 1$ is not a characteristic value for $K_2(Q,P)$, and $k_2(Q,R;1)$ is a continuous function of Q and R . From the known expressions for $k_2(Q,P;\lambda^3)$ in terms of the iterated kernels of index at least as great as two it is evident that $k_2(M,N;1)$ converges uniformly to $k_2(Q,P;1)$ as τ approaches zero. That $k(Q,P)$ and $k(M,N)$ have the properties stated in the lemma follows from (3.17) and the fact that $K(Q,P)$ and $K(M,N)$ have those properties.

From the solution (3.8) of the equation (3.7) we may obtain an explicit formula for the potential given by (I) in terms of $F(w)$. Substituting (3.8) in (I), we have

$$(3.18) \quad U(M) = \frac{1}{4\pi} \int_{S_0} g_n(M,P) dF(w_p) = \frac{1}{4\pi} \int_{S_0} g_n(M,P) dF(e_p)$$

where

$$(3.19) \quad g_n(M,P) = 2 \int_{S_0} k(R,P) \frac{\cos \angle(MR,n_R)}{MR^2} dS_R - 2 \frac{\cos \angle(MP,n_P)}{MP^2}$$

The function of point sets $\mathcal{V}(e)$ and the function of regular curves $\mathcal{V}(w)$ are connected by the relations

$$(3.20) \quad \begin{aligned} \mathcal{V}(w) &= \int q(P,w) d\mathcal{V}(e_p) = \int q(P,w) d\mathcal{V}(w_p) \\ \mathcal{V}(e) &= \int_{S_0} \phi(P,e) d\mathcal{V}(w_p) = \int_{S_0} \phi(P,e) d\mathcal{V}(e_p) \end{aligned}$$

where $\phi(P,e)$ is the characteristic function of the set e . Similar relations

exist* between $F(e)$ and $F(w)$ and also between the positive and negative variation functions $F_{\pm}(e)$, $F_{\pm}(w)$ and $\gamma_{\pm}(e)$, $\gamma_{\pm}(w)$. If the function of point sets is completely additive, the corresponding function of regular curves is completely additive and conversely.

Theorem 3.2. If $U(M)$ is given by (I) for M in T^+ , a necessary and sufficient condition that $\gamma(e)$ be absolutely continuous is that $F(e)$ be absolutely continuous. Further, if in this situation $\gamma'(P)$ is continuous, then $F'(P)$ is continuous and conversely, where $\gamma'(P)$ and $F'(P)$ are the derivatives respectively of $\gamma(e)$ and $F(e)$.

From the equations corresponding to (3.20) for the positive and negative variation functions for $\gamma(e)$ and $F(e)$ and $\gamma(w)$ and $F(w)$ we have, denoting by $\bar{\sigma}$ the set of points composing w ,

$$\begin{aligned}\gamma_{\pm}(\sigma) &\leq \gamma_{\pm}(w) \leq \gamma_{\pm}(\sigma + \bar{\sigma}) ; \\ F_{\pm}(\sigma) &\leq F_{\pm}(w) \leq F_{\pm}(\sigma + \bar{\sigma}) .\end{aligned}$$

Suppose $\gamma(e)$ is absolutely continuous. Then $\gamma_{\pm}(e)$ are absolutely continuous.

Since $\gamma_{\pm}(e)$ are also additive, we have

$$\gamma_{\pm}(\sigma + \bar{\sigma}) = \gamma_{\pm}(\sigma)$$

since $\bar{\sigma}$ is of zero superficial measure. Hence $\gamma_{\pm}(w) = \gamma_{\pm}(\sigma)$. From (3.7) we may write

$$F_{\pm}(\sigma) \leq F_{\pm}(w) \leq 2\pi\gamma_{+}(\sigma) + 2\pi\gamma_{-}(\sigma) + \int_{S_0} |d\gamma(e_p)| \int_{\sigma} \frac{|\cos \angle(OP, n_p)|}{OP^2} dS_p$$

where each term on the right is absolutely continuous. Therefore $F_{\pm}(e)$, and hence $F(e)$, are absolutely continuous. For the sufficiency of the condition stated in the theorem we use the same method of proof, but use equation (3.8) instead of (3.7).

We note that if $U(M)$ is given by (I) where $\gamma(e) = \int_{S_0} \gamma'(P) dS_p$ and if $\gamma'(P)$ is continuous on S_0 , then $U(M)$ takes on continuously its boundary values $F'(P)$. In fact

$$U(M) = \int_{S_0} \frac{\cos \angle(MP, n_p)}{MP^2} \gamma'(P) dS_p$$

* See Bray and Evans, "A Class of Functions Harmonic Within the Sphere", American Journal of Mathematics, Vol. 49 (1927), pp. 158-159. The proofs given for functions of segments on the sphere apply equally well to bounded additive functions of regular curves on S_0 .

and the statement follows as in the case of regular surfaces*.

Theorem 3.3. If $U(M)$ is given by (I), it may be written in T^+ in the form (3.18) where $g_n(M, P)$ is continuous in P and harmonic in M , for M in T^+ , and is not negative; and

$$F(w) = \lim_{\tau \rightarrow 0^+} U(\tau, w) = \lim_{\tau \rightarrow 0^+} \int_{\sigma(\tau)} U(M) dS_M$$

where M is on the surface $S(\tau)$ of the normal family inside S_0 .

That $g_n(M, P)$ is harmonic in M and continuous in P for M in T^+ follows immediately[†] from the equation (3.19). We have only to show that $g_n(M, P)$ is not negative for M in T^+ and P on S_0 . Suppose the contrary. Then for some point M_1 in T^+ and some point P_1 of S_0 , $g_n(M, P)$ is negative. From the continuity of $g_n(M, P)$ in P there is a neighborhood ω_δ on S_0 containing P_1 such that $g_n(M, P) < 0$ for P in ω_δ . Suppose $F(w) = \int_{\sigma} F'(P) dS_P$ where $F'(P)$ is continuous on S_0 , positive for P in ω_δ , and zero otherwise. Then

$$U(M_1) = \frac{1}{4\pi} \int_{\omega_\delta} g_n(M_1, P) F'(P) dS_P < 0.$$

But $U(M)$ cannot be negative anywhere in T^+ since it is the solution of the interior Dirichlet problem for continuous not-negative boundary values. This contradiction establishes the theorem.

Theorem 3.4. The function $U(M)$ given by (I) for M in T^+ is the difference of two not-negative functions harmonic in T^+ .

This theorem follows immediately from the equation (3.18) and the preceding theorem, together with the fact that the completely additive function $F(w)$ may be expressed as the difference of two not-negative functions.

Theorem 3.5. If $U(M)$ is given by (I) for M in T^+ , a necessary and sufficient condition that $\gamma(e)$ be absolutely continuous is that the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M) dS_M$ be uniform as τ

* Kellogg, Potential Theory, pp.167-168.

† See Appendix 3.

approaches zero.

We prove first the necessity of the condition, supposing $\gamma(e)$ to be absolutely continuous. $F(e)$ is absolutely continuous by ^{Theorem 3.2.} ~~the preceding~~ theorem. It is sufficient to consider the case where $F(e)$, and therefore $U(M)$, is not negative. But

$$F(\sigma) = \lim_{\tau \rightarrow \sigma^+} \int_{\sigma(\tau)} U(M) dS_M.$$

Since $F(\sigma)$ is an absolutely continuous function of point sets, it has a derivative $F'(P)$ almost everywhere, and

$$(3.21) \quad \begin{cases} F(\sigma) = \int_{\sigma} F'(Q) dS_Q \\ F'(Q) = \lim_{\tau \rightarrow Q^+} U(M) \end{cases}$$

almost everywhere*. Suppose the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M) dS_M$ is not uniform. Then there exists a denumerable sequence $\{S_i\}$ of surfaces of the family for which the absolute continuity of the integrals $\int_{\sigma_i} U(M) dS_M$ is not uniform, where σ_i denotes the region on S_i corresponding to σ on S_0 . By the property (c) of normal families the sequence $\{S_i\}$ converges to some surface S^* of the family. We denote by $U_i(M^*)$ the function $U(M) dS_M / dS_{M^*}$ where M^* on S^* corresponds to M on S_i . Hence $\int_{\sigma_i} U(M) dS_M = \int_{\sigma^*} U_i(M^*) dS_{M^*}$, where σ^* on S^* corresponds to σ_i on S_i . By De la Vallée Poussin's converse to Vitali's Theorem, either $\left\{ \lim_{i \rightarrow \infty} U_i(M^*) \right\}$ is not summable or $\lim_{i \rightarrow \infty} \int_{\sigma^*} U_i(M^*) dS_{M^*}$ is not equal to $\int_{\sigma^*} \left\{ \lim_{i \rightarrow \infty} U_i(M^*) \right\} dS_{M^*}$. If S^* is S_0 , neither of these conditions is possible as is shown by (3.21). But if S^* is not S_0 , then $U(M)$ is continuous, and again neither condition can subsist.

* Due to the absolute continuity of F and γ , we may compute the derivatives of both members of (3.7), obtaining

$$\gamma'(Q) = -\frac{1}{2\pi} F'(Q) + \int_{S_0} K(Q, P) \gamma'(P) dS_P$$

where $\gamma'(Q)$ and $F'(Q)$ are the derivatives almost everywhere of $\gamma(w)$ and $F(w)$. Transposing, we obtain

$$F'(Q) = -2\pi \gamma'(Q) + 2\pi \int_{S_0} K(Q, P) \gamma'(P) dS_P = \lim_{\substack{\tau \rightarrow Q^+ \\ (M \rightarrow Q)}} U(M),$$

as is shown in ~~the preceding~~ Potential Theory, p. 226--227, (A), p. 511.

To prove the sufficiency, we note that if the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M) dS_M$ is uniform, then $F'(Q)$ is summable by Vitali's Theorem, and $\lim_{\tau \rightarrow 0+} \int_{\sigma(\tau)} U(M) dS_M = \int_{\sigma} F'(Q) dS_Q$. Hence $F(e)$ is absolutely continuous and therefore $\mathcal{V}(e)$ is absolutely continuous by Theorem 3.2.

4. Necessary and Sufficient Conditions for Representation as Potentials.

Theorem 4.1. A necessary and sufficient condition that $U(M)$ harmonic in T^+ be representable in the form (I) is that $\int_{S(\tau)} |U(M)| dS_M$ remain bounded over a normal family $\{S(\tau)\}$ within S_0 as τ approaches zero.

The necessity of the condition is given by Theorem 2.3. To prove the sufficiency we note that by hypothesis the functions of point sets $F(\sigma(\tau))$, given by

$$F(\sigma(\tau)) = \int_{\sigma(\tau)} U(M) dS_M,$$

and therefore the corresponding functions of regular curves $F(w(\tau))$ defined on the surfaces $S(\tau)$, are of uniformly bounded total variation. Moreover $U(M)$ is uniformly continuous inside and on $S(\tau)$. From the equations corresponding to (3.8) formed for the surfaces of the family it follows that the functions $\mathcal{V}(w(\tau))$ are of uniformly bounded total variation, and by Theorem 3.2 we may write $\mathcal{V}(w(\tau)) = \int_{\sigma(\tau)} \mathcal{V}'(M) dS_M$ where $\mathcal{V}'(M)$ is continuous on $S(\tau)$. Consequently we may pick out a subsequence $\{S_k\}$ of surfaces of the family for which the functions $\mathcal{V}_k(w')$ converge in the weak sense* to a completely additive function $\mathcal{V}(w)$ defined on S_0 , where the $\mathcal{V}_k(w')$ represent the solutions of the equations corresponding to (3.7) for the surfaces S_k . Denoting by P_k a variable point on S_k , we have, for k sufficiently large,

$$U(M) = \int_{S_k} \frac{\cos \angle(MP_k, \nu_{P_k})}{MP_k^2} d\mathcal{V}_k(w'_{P_k}).$$

In fact, the $\mathcal{V}_k(e)$ is the integral of a continuous $\mathcal{V}'(P_k)$ and the right-hand

* J. Radon, "Über die Randwertaufgaben beim Logarithmischen Potential", Wiener Akademie Sitzungsberichte, Vol. 128 (1919), IIa, p. 1153. The methods employed by Radon for a certain class of curves may be extended readily to apply to normal families of surfaces. See Appendix 4.

member is harmonic within S_k and takes on continuously the boundary values $U(M_k)$ as M tends to M_k on S_k . But there can be only one such function. Consequently that function is $U(M)$. From the weak convergence of the functions $\gamma_k(w')$ to $\gamma(w)$ defined on S_0 , we now have

$$U(M) = \int_{S_0} \frac{\cos \angle(M, P, \gamma_p)}{MP^2} d\gamma(w_p) = \int_{S_0} \frac{\cos \angle(M, P, \gamma_p)}{MP^2} d\gamma(e_p)$$

which is what was to be shown.

Lemma 5. The function $g_n(M, P)$ given by (3.14) is positive for M in T^+ and P on S_0 .

By Theorem 3.3, $g_n(M, P)$ is not negative. Suppose there exists a point M_1 of T^+ and a point P_1 of S_0 such that $g_n(M_1, P_1)$ is zero. Then $g_n(M, P_1)$ is identically zero, since $g_n(M, P)$ is harmonic in M and not-negative. For any given completely additive function of regular curves with regular discontinuities, $F(w)$, we may solve the equation (3.7) for $\gamma(w)$ so that $U(M)$ is given by (3.18) and

$$F(w) = \lim_{\tau \rightarrow 0^+} \int_{\sigma(\tau)} U(M) dS_M.$$

Define $F(e)$ so that

$$F(e) = 1 \quad \text{if } e \text{ contains } P_1.$$

$$F(e) = 0 \quad \text{otherwise.}$$

From (3.17) we now have $U(M) \equiv 0$. Hence

$$F(w) = \lim_{\tau \rightarrow 0^+} \int_{\sigma(\tau)} U(M) dS_M = 0$$

for all w , and therefore $F(e) = 0$ for all e , which contradicts the assumption that $F(e) = 1$ for all e containing P_1 .

Theorem 4.2. A necessary and sufficient condition* that $U(M)$ harmonic in T^+ be given by (I) is that $U(M)$ be the difference of two not-negative functions harmonic in T^+ .

The necessity of the condition is given by Theorem 3.4. For the

* De la Vallée Poussin (loc. cit. p. 199) proves that a necessary and sufficient condition that the harmonic function U defined in T^+ be the difference of two positive harmonic functions is that $\int_{\sigma} |U| dS$ be bounded over a normal family of surfaces in T^+ and arbitrarily near S_0 . This theorem follows directly from Theorems 4.1 and 4.2.

sufficiency we need only prove the theorem for $U(M)$ not-negative and harmonic in T^+ . If $U(M)$ vanishes anywhere in T^+ , then it vanishes identically, and the theorem is trivial. Hence, we shall suppose that $U(M)$ is positive. Consider a fixed point $\bar{M}$ of T^+ and a normal family $S(\tau)$ of surfaces inside S_0 , where $\tau_1 > \tau > 0$ and τ_1 is small enough so that $\bar{M}$ is inside all the surfaces. We form the functions $g_{\tau}(\bar{M}, N)$ corresponding to $g_{\tau}(\bar{M}, P)$ where N is a point of $S(\tau)$. From the form of $g_{\tau}(\bar{M}, P)$ given by (3.19) and from Lemma 4 it is easily seen that $g_{\tau}(\bar{M}, N)$ converges uniformly* to $g_{\tau}(\bar{M}, P)$ as N on $S(\tau)$ approaches P on S_0 along n_p and that the convergence is uniform in τ and P . There exists a number $\epsilon > 0$ such that

$$g_{\tau}(\bar{M}, N) > \epsilon$$

for all N of $S(\tau)$ and all τ ; for, supposing the contrary, we can find a sequence of points N_i on surfaces of the family, having a limit point $\bar{N}$ on one of the surfaces such that $g_{\tau}(\bar{M}, \bar{N})$ equals zero. This is impossible by Lemma 5. Defining the functions $F(w(\tau))$ on $S(\tau)$ by the relation

$$F(w(\tau)) = \int_{S(\tau)} U(M) dS_M$$

we have for every τ in $\tau_1 > \tau > 0$

$$U(\bar{M}) = \frac{1}{4\pi} \int_{S(\tau)} g_{\tau}(\bar{M}, N) dF(w(\tau_N)).$$

In fact $F(w(\tau))$ is the integral of a continuous function $F'(N)$, and $F'(N) = \lim_{M \rightarrow N} U(M)$.

That $U(\bar{M})$ may be written in this form follows from Theorem 3.3 and the fact that inside $S(\tau)$, $U(M)$ is given by

$$U(M) = \int_{S(\tau)} \frac{\cos \angle(MN, n_N)}{MN^2} d\tau(w(\tau_N))$$

as was noted in the proof of Theorem 4.1. Now $F(w(\tau))$ is positive since $U(M)$ is positive. Hence

$$U(\bar{M}) \geq \frac{\epsilon}{4\pi} F(S(\tau)).$$

Therefore the functions $F(S(\tau))$ remain bounded uniformly over the normal family inside S_0 . The theorem now follows immediately from Theorem 4.1.

*.See Appendix 5.

Theorem 4.3. A necessary and sufficient condition that $v(M)$ harmonic in T^- , except at infinity, be representable in T^- by (3.2) is that

- (a) $v(M)$ approach zero continuously at infinity and
 (b) $\int_{S(\tau)} |U_1(M)| dS_M = \int_{S(\tau)} \left| \frac{dv}{dn_M} \right| dS_M$ remain bounded over a normal family $S(\tau)$ outside S_0 as τ approaches zero.

The necessity of (a) is immediate. The necessity of (b) is given by Theorem 2.3. Before proving the sufficiency we consider (3.6) in the form

$$G(w) = 2\pi \gamma(w) + \int_{S_0} d\gamma(e_p) \int_{\sigma} \frac{\cos \angle(QP, n_Q)}{QP^2} dS_Q;$$

or

$$(4.1) \quad \gamma(w) = \Psi(w) + \int_{S_0} d\gamma(e_p) \int_{\sigma} K(P, Q) dS_Q$$

where

$$\Psi(w) = \frac{1}{2\pi} G(w) = \lim_{\tau \rightarrow 0} \frac{1}{2\pi} \int_{S(\tau)} \frac{dv}{dn_M} dS_M$$

We may solve this equation in a manner similar to the one used in solving

(3.7). We may write the solution in the form

$$(4.2) \quad \gamma(w) = \Psi(w) - \int_{S_0} d\gamma(e_p) \int_{\sigma} k(P, R) dS_R.$$

Given $v(M)$, in order to prove the sufficiency of the conditions (a) and (b), we form the functions $\Psi(w(\tau)) = \int_{S(\tau)} U_1(M) dS_M$. By hypothesis, these functions are of uniformly bounded total variation. From the equations corresponding to (4.2) formed for the surfaces $S(\tau)$ it is seen that the functions $\gamma(w(\tau))$ have the same property. Hence, as in the proof of Theorem 4.1, we may pick out subsequences $\gamma_k(w')$ defined on surfaces S_k of the family outside S_0 converging in the weak sense to a completely additive function $\gamma(w)$ defined on S_0 . From (a) it follows that $v(M)$ is regular* at infinity. Since $U_1(M)$ is continuous outside S_k and takes on continuous boundary values on S_k , it follows from (4.2) that $\gamma_k(e)$ is the integral of a continuous function. Hence for k sufficiently large, we may write, as in the case of regular surfaces†,

$$v(M) = \int_{S_k} \frac{1}{MP_k} d\gamma_k(w'_k)$$

* Appendix 6.

† Kellogg, Potential Theory, p. 311.

for the right-hand member is harmonic outside S_k and is regular at infinity and its normal derivative takes on continuously* the boundary values $U_1(M_k)$ as M tends to M_k on S_k . But there can be only one such function†. Consequently that function is $v(M)$. From the weak convergence, we now have

$$v(M) = \int_{S_0} \frac{1}{MP} d\gamma(w_P) = \int_{S_0} \frac{1}{MP} d\gamma(e_P)$$

which was to be shown.

Theorem 4.4. A necessary and sufficient condition that $v(M)$ harmonic in T^+ be representable in T^+ in the form (3.2) is that $\int_{S(\tau)} |U_1(M)| dS_M$ remain bounded over a normal family $S(\tau)$ inside S_0 as τ approaches zero.

The necessity of the condition is given by Theorem 2.3. Incidentally, for $v(M)$ harmonic in T^+ we have $G(S(\tau)) = \int_{S(\tau)} \frac{dv}{dn_M} dS_M = 0$. Before proving the sufficiency we first consider the equation corresponding to (3.6). This equation is

$$(4.3) \quad \gamma(w) = \Psi(w) - \int_{S_0} d\gamma(e_P) \int_{\sigma} K(P, Q) dS_Q$$

where

$$\Psi(w) = -\frac{1}{2\pi} \lim_{\tau \rightarrow 0^+} \int_{S(\tau)} \frac{dv}{dn_M} dS_M.$$

The value $\lambda = -1$ is a characteristic value for the kernels $\lambda K(Q, P)$ and $\lambda K(P, Q)$, and the equation (4.3) can be satisfied if and only if $\Psi(S_0)$

is zero. The mass function $\gamma(w)$ is then determined except for an arbitrary additive term of the form $C \int_{\sigma} \phi_2(P) dS_P$, where C is a constant and $\phi_2(P)$ is a solution of the homogeneous equation‡

$$\phi(P) = - \int_{S_0} \phi(Q) K(Q, P) dS_Q$$

From the resolvent kernel $k(P, Q)$ we can find a particular solution of (4.3).

We first reduce (4.3) to the Fredholm form. We put $\gamma(w) = \Psi(w) - H(w)$.

From (4.3) it is seen that $H(w)$ is absolutely continuous, and hence we may write

* Kellogg, Potential Theory, p. 164.

† Ibid, p. 213.

‡ (A) p. 505. (An error in printing occurs here. The λ should be replaced by $-\lambda$.)

$$H(w) = \int_{\sigma} h(R) dS_R$$

where $h(R)$ is the derivative almost everywhere of $H(w)$. Proceeding as in the proof of Theorem 3.1 we obtain the Fredholm equation

$$(4.4) \quad h(Q) = f(Q) - \int_{S_0} K(P, Q) h(P) dS_P$$

where

$$f(Q) = \int_{S_0} K(P, Q) d\Psi(w_P).$$

We write

$$K(P, Q; \lambda) = \frac{A(P, Q)}{\lambda + 1} + B(P, Q; \lambda)$$

where $A(P, Q)$ is the residue at the pole $\lambda = -1$ and is continuous*.

A particular solution of (4.4) is now given by

$$h(Q) = f(Q) + \int_{S_0} B(P, Q; -1) f(P) dS_P$$

or

$$(4.5) \quad h(Q) = f(Q) + \int_{S_0} B(P, Q; -1) f(P) dS_P - \int_{S_0} A(P, Q) f(P) dS_P$$

since

$$\int_{S_0} A(P, Q) f(P) dS_P = 0.^\dagger$$

From (4.5) we have

$$H(w) = \int_{\sigma} dS_Q \int_{S_0} K(P, Q) d\Psi(w_P) + \int_{\sigma} dS_R \int_{S_0} B(R, Q; -1) dS_R \int_{S_0} K(P, R) d\Psi(w_P) - \int_{\sigma} dS_Q \int_{S_0} A(R, Q) dS_R \int_{S_0} K(P, R) d\Psi(w_P).$$

The first part of the lemma proved below permits us to change the order of integration in this equation so that we may write

$$H(w) = \int_{S_0} d\Psi(w_P) \int_{\sigma} dS_Q \left\{ K(P, Q) B(R, Q; -1) dS_R - \int_{S_0} K(P, R) A(R, Q) dS_R \right\}.$$

From (3.14) we obtain the relation

$$K(P, Q) + B(P, Q; \lambda) = \int_{S_0} K(P, R) A(R, Q) dS_R - \int_{S_0} K(P, R) B(R, Q; \lambda) dS_R.$$

Substituting in the previous equation we now have

$$H(w) = - \int_{S_0} d\Psi(w_P) \int_{\sigma} B(P, Q; -1) dS_Q$$

or

*Kellogg, Potential Theory, p. 308.

†Ibid, p. 298.

$$(4.6) \quad \gamma(w) = \psi(w) + \int_{S_0} d\psi(w_p) \int_{\sigma} B(P, Q; -1) dS_Q$$

as a particular solution of (4.5).

Lemma 6. The integrals $\int_{S_0} |B(P, Q; -1)| dS_P$ and $\int_{S_0} |B(P, Q; -1)| dS_Q$ are bounded for all Q, P . The integrals $\int_{S(\tau)} |B(N, M; -1)| dS_N$ and $\int_{S(\tau)} |B(N, M; -1)| dS_M$ are bounded uniformly as τ approaches zero.

We may write

$$k_2(Q, P; \lambda) = \frac{A_2(Q, P)}{\lambda^3 + 1} + B_2(Q, P; \lambda^3)$$

where $A_2(Q, P)$ is continuous in Q, P and $B_2(Q, P; \lambda^3)$ is a power series in $(\lambda^3 + 1)$ uniformly convergent in the neighborhood of $\lambda = -1$ and with coefficients which are continuous* in Q and P . Moreover, $B_2(M, N; -1)$ and $A_2(M, N)$ are bounded as τ approaches zero, since $k_2(M, N; \lambda^3)$ changes continuously with τ . Substituting in (3.17) and equating like powers of $(\lambda + 1)$, we obtain for $\lambda = -1$

$$(4.7) \quad \begin{aligned} B(Q, R; -1) = & -K(Q, R) + K_1(Q, R) - \frac{1}{3}A_2(Q, R) + B_2(Q, R; -1) \\ & + \frac{2}{3} \int_{S_0} A_2(Q, P) K(P, R) dS_P - \int_{S_0} A_2(Q, P) K_1(P, R) dS_P \\ & - \int_{S_0} B_2(Q, P; -1) K(P, R) dS_P + \int_{S_0} B_2(Q, P; -1) K_1(P, R) dS_P. \end{aligned}$$

Since

$$A_2(Q, R) = - \int_{S_0} K_1(Q, P) A_2(P, R) dS_P = - \int_{S_0} K(Q, P) A_2(P, R) dS_P = + \int_{S_0} K_1(Q, P) A_2(P, R) dS_P$$

we reduce (4.7) to the form

$$(4.8) \quad \begin{aligned} B(Q, R; -1) = & -K(Q, R) + K_1(Q, R) + B_2(Q, R; -1) - 2A_2(Q, R) \\ & - \int_{S_0} B_2(Q, P; -1) K(P, R) dS_P + \int_{S_0} B_2(Q, P; -1) K_1(P, R) dS_P. \end{aligned}$$

The lemma is proved by treating (4.8) in precisely the same manner as we treated (3.17) in proving Lemma 4.

We continue now with the proof of Theorem 4.4. By hypothesis the functions $\psi(w(\tau)) = -\frac{1}{2\pi} \int_{\sigma(\tau)} d\tau dS_M$ are of uniformly bounded total variation. Since $\psi(S(\tau)) = 0$, the equations corresponding to (4.3) formed for the surfaces $S(\tau)$ may be solved for the mass functions $\gamma(w(\tau))$

* Kellogg, loc. cit., p. 294.

defining on $S(\tau)$ the functions* $\psi(w(\tau))$. The functions $\bar{\psi}(w(\tau))$, particular solutions of the equations corresponding to (4.3), given by the equations of the form (4.6) are of uniformly bounded total variation. Hence we may select a subsequence $\bar{\psi}_k(w')$ defined on the surfaces S_k and converging in the weak sense to $\bar{\psi}(w)$ defined on S_0 . These functions $\bar{\psi}_k(w')$ satisfy the equations of the form (4.5), but so also do the functions

$$\begin{aligned}\gamma_k(w') &= \bar{\gamma}_k(w') + C_k \int_{\sigma} \phi_2(P_k) dS_{P_k} \\ &= \bar{\gamma}_k(w') + \gamma_k^*(w')\end{aligned}$$

where w', σ, P_k on S_k correspond respectively to w, σ, P on S_0 ; C_k is an arbitrary constant; and $\phi_2(P)$ is a solution of the homogeneous equation $\phi(P) = - \int_{S_k} \phi(M) K(M, P_k) dS_M$. The potential of the form (3.2) due to the mass function $\gamma_k^*(w')$ reduces inside S_k to a constant†, where

$$K_k = \int_{S_k} \frac{1}{MP_k} d\gamma_k^*(w'_{P_k}) = C_k \int_{S_k} \frac{1}{MP_k} \phi_2(P_k) dS_{P_k}.$$

By properly choosing the constants K_k , we have

$$v(M) - K_k = \int_{S_k} \frac{1}{MP_k} d\gamma_k(w'_{P_k})$$

for M inside S_k ; for, as in the previous theorem, $\gamma_k(w')$ is the integral of a continuous function and the normal derivative of the right-hand member takes on the same boundary values as does the normal derivative of $v(M)$ and therefore the right-hand member differs from $v(M)$ by a constant‡. Since $v(M)$ is given and the functions $\bar{\gamma}_k(w')$ converge in the weak sense to $\bar{\gamma}(w)$ defined on S_0 , we have for M in T^+

$$v(M) = \int_{S_0} \frac{1}{MP} d\bar{\gamma}(w_P) + K$$

where $K = \lim_{k \rightarrow \infty} K_k$. But inside T^+ we may write

$$K = B \int_{S_0} \frac{1}{MP} \phi_2(P) dS_P,$$

where B is a constant, due to the fact that

$$v_0(M) = \int_{S_0} \frac{1}{MP} \phi_2(P) dS_P \neq 0$$

(for otherwise, $v_0(M)$ being a conductor potential, we have $\phi_2(P) \equiv 0$,

which is impossible since $\phi_2(P)$ is a non-zero solution of the homogeneous

* (A) p. 505.

† (A) p. 509 (footnote).

‡ Kellogg, Potential Theory, p. 213.

equation). Defining $\mathcal{V}(w)$ by the relation

$$\mathcal{V}(w) = \overline{\mathcal{V}}(w) + B \int_{\sigma^-} \phi_2(P) dS_P$$

we now have

$$v(M) = \int_{S_0} \frac{1}{MP} d\mathcal{V}(w_P) = \int_{S_0} \frac{1}{MP} d\mathcal{V}(e_P) ,$$

which establishes the theorem.

Theorem 4.5. A necessary and sufficient condition that $U(M)$ harmonic in T^- except at infinity be representable in T^- in the form (I) is that

- (a) $U(M)$ approach zero continuously at infinity and $\int_{\Omega} \frac{dU}{dn} dS = 0$, where Ω is a sphere of radius $r > r_0$ sufficiently large, and
 (b) that $\int_{S(\tau)} |U(M)| dS_M$ remain bounded uniformly over a normal family $S(\tau)$ outside S_0 as τ approaches zero.

The necessity of (a) is well known. The necessity of (b) is given by Theorem 2.3. To prove the sufficiency we must be able to solve the integral equations corresponding to (3.5) for the surfaces $S(\tau)$. These equations take the form

$$(4.9) \quad \mathcal{V}(w(\tau)) = \Phi(w(\tau)) - \int_{S_0} d\mathcal{V}(e_N) \int_{S(\tau)} K(M, N) dS_M$$

where

$$\Phi(w(\tau)) = \frac{1}{2\pi} \int_{S(\tau)} U(M) dS_M .$$

A sufficient condition that these equations have solutions is that

$$\int_{S(\tau)} \phi_2(M) d\Phi(e_M) = \frac{1}{2\pi} \int_{S(\tau)} \phi_2(M) U(M) dS_M = 0 ,$$

where $\phi_2(M)$ is a solution of the homogeneous equation

$$\phi(N) = - \int_{S(\tau)} \phi(M) K(M, N) dS_M .$$

But the potential of the form (3.2) for $\mathcal{V}(w(\tau)) = \int_{S(\tau)} \phi_2(M) dS_M$ may be expressed as a conductor potential, and hence we may take $\phi_2(M)$ as the density of the distribution producing the conductor potential, which we denote by $V(M)$. Now let Γ be a sphere of sufficiently large radius, a , and fixed center. We have

$$\int_{S(\tau)+\Gamma} U \frac{dV}{dn^-} dS - \int_{S(\tau)+\Gamma} V \frac{dU}{dn^-} dS = 0.$$

Since $U(M)$ vanishes at infinity like $\frac{1}{n^2}$ and $\frac{dU}{dn}$ vanishes like $\frac{1}{n^3}$ *,

we may let a become infinite, obtaining

$$\int_{S(\tau)} U \frac{dV}{dn^-} dS - \int_{S(\tau)} V \frac{dU}{dn^-} dS = 0.$$

But on $S(\tau)$, $\frac{dV}{dn^-}$ vanishes identically. Moreover

$$\frac{dV}{dn_m^-} - \frac{dV}{dn_m^+} = 4\pi \phi_2(M).$$

Hence we have, since $V = 1$ on $S(\tau)$ and $\int_{S(\tau)} \frac{dU}{dn_m^-} dS_m = 0$,

$$\int_{S(\tau)} U \frac{dV}{dn_m^-} dS_m = 4\pi \int_{S(\tau)} \phi_2(M) U(M) dS_m = 0.$$

Therefore the equations (4.9) may be solved. Consequently we may write $U(M)$ outside $S(\tau)$ in the form

$$U(M) = \int_{S(\tau)} \frac{\cos \angle(MN, \eta_N)}{MN^2} d\gamma(w(\tau)_N),$$

where $\gamma(w(\tau))$ is the solution of (4.9); for the right-hand member is harmonic outside $S(\tau)$, vanishes at infinity, and takes on the same values on $S(\tau)$ as $U(M)$, and is therefore identically equal to $U(M)$. The solutions of (4.9) contain arbitrary additive constants, but we may take these constants to be zero, since they add nothing to the potentials given in the form (I) for M in T^- . To complete the proof we apply the weak convergence analysis as in the preceding theorems.

Appendix

1.

We derive the equation (3.14) by making use of the formula for $k(Q, P)$ in terms of K , K_1 , and k_2 ; namely, the equation (3.17). For $\lambda = 1$ the equation (3.17) becomes

$$(1.1') \quad k(Q, R) = -K(Q, R) - K_1(Q, R) + k_2(Q, R) + \int_{S_0} k_2(Q, x) \{K(x, R) + K_1(x, R)\} dS_x$$

Since $K_2(Q, P)$ is continuous in Q and P , $K_2(Q, P)$ satisfies the Volterra relations

$$(1.2') \quad \int_{S_0} k_2(Q, R) K_2(R, P) dS_R = \int_{S_0} k_2(R, P) K_2(Q, R) dS_R = k_2(Q, P) + K_2(Q, P)$$

From (1.1') we have

$$\begin{aligned} \int_{S_0} k_2(Q, R) K(R, P) dS_R &= - \int_{S_0} K(Q, R) K(R, P) dS_R - \int_{S_0} K_1(Q, R) K(R, P) dS_R \\ &\quad + \int_{S_0} k_2(Q, R) K(R, P) dS_R + \int_{S_0} k_2(Q, x) dS_x \int_{S_0} \{K(x, R) K(R, P) + K_1(x, R) K(R, P)\} dS_R \\ &= -K_1(Q, P) - K_2(Q, P) + \int_{S_0} k_2(Q, x) dS_x \{K(x, P) + K_1(x, P)\} \\ &\quad + \int_{S_0} k_2(Q, x) K_2(x, P) dS_x \end{aligned}$$

Replacing the last integral by its value given by (1.2'), we have

$$\begin{aligned} \int_{S_0} k_2(Q, R) K(R, P) dS_R &= -K_1(Q, P) + k_2(Q, P) + \int_{S_0} k_2(Q, x) \{K(x, P) + K_1(x, P)\} dS_x \\ (1.3') \quad &= k_2(Q, P) + K(Q, P) \end{aligned}$$

by (1.1').

Since

$$-k_2(Q, R) = K_2(Q, R) + K_3(Q, R) + K_4(Q, R) + \dots,$$

this series being uniformly convergent for Q, R on S_0 ; and since

$$\int_{S_0} K_i(Q, x) K_j(x, R) dS_x = \int_{S_0} K_j(Q, x) K_i(x, R) dS_x = K_{i+j}(Q, R)$$

it follows that

$$(1.4') \quad \int_{S_0} k_2(Q, x) K_i(x, R) dS_x = \int_{S_0} K_i(Q, x) k_2(x, R) dS_x \quad (i = 0, 1, \dots).$$

By (1.1')

$$k(R,P) = -K(R,P) - K_1(R,P) + k_2(R,P) + \int_{S_0} k_2(R,x) \{K(x,P) + K_1(x,P)\} dS_x$$

Making use of (1.4') we write this equation in the equivalent form

$$(1.5') \quad k(R,P) = -K(R,P) - K_1(R,P) + k_2(R,P) + \int_{S_0} k_2(x,P) \{K(R,x) + K_1(R,x)\} dS_x.$$

Hence we have

$$\begin{aligned} \int_{S_0} k(R,P) K(Q,R) dS_R &= -K_1(Q,P) - K_2(Q,P) + \int_{S_0} k_2(x,P) K(Q,x) dS_x \\ &\quad + \int_{S_0} K(Q,R) dS_R \int_{S_0} k_2(x,P) \{K(R,x) + K_1(R,x)\} dS_x \\ &= -K_1(Q,P) - K_2(Q,P) + \int_{S_0} k_2(x,P) \{K(Q,x) + K_1(Q,x)\} dS_x \\ &\quad + \int_{S_0} k_2(x,P) K_2(Q,x) dS_x. \end{aligned}$$

By (1.2') this last equation may be written

$$(1.6') \quad \begin{aligned} \int_{S_0} k(R,P) K(Q,R) dS_R &= -K_1(Q,P) + k_2(Q,P) + \int_{S_0} k_2(x,P) \{K(Q,x) + K_1(Q,x)\} dS_x \\ &= k(Q,P) + K(Q,P) \end{aligned}$$

from (1.5'). Combining (1.3') and (1.6') we obtain the Volterra relations (3.14), which was to be shown

2.

Lemma 1a. $K_1(Q,P)$ is bounded.

We suppose δ to be a positive constant, and consider two cases.

Case 1. $\overline{QP} \geq 2\delta > 0$. We may write

$$4\pi^2 K_1(Q,P) = 4\pi^2 \int_{S_0 - S_0(\delta,Q) - S_0(\delta,P)} K(Q,R) K(R,P) dS_R + 4\pi^2 \int_{S_0(\delta,Q)} K(Q,R) K(R,P) dS_R + 4\pi^2 \int_{S_0(\delta,P)} K(Q,R) K(R,P) dS_R.$$

We denote the three terms on the right by I_1 , I_2 , and I_3 respectively.

We now have

$$\begin{aligned} |I_1| &\leq \frac{1}{\delta^4} m(S_0) \\ |I_2| &\leq \frac{1}{\delta^2} \int_{S_0(\delta,Q)} \frac{|\cos \angle(QR, n_R)|}{QR^2} dS_R \\ |I_3| &\leq \frac{1}{\delta^2} \int_{S_0(\delta,P)} \frac{|\cos \angle(RP, n_P)|}{RP^2} dS_R \end{aligned}$$

It is obvious that $|I_1|$ is bounded. $|I_2|$ and $|I_3|$ are bounded since these are surfaces of class Γ .

Case 2. $\overline{QP} < 2\delta$. We let $\delta' = 4\delta$ and write

$$4\pi^2 K(Q, P) = 4\pi^2 \int_{S_0 - S_0(\delta', P)} K(Q, R) K(R, P) dS_R + 4\pi^2 \int_{S_0(\delta', P)}$$

We denote the terms on the right by J_1 and J_2 respectively. We now have

$$|J_1| \leq \frac{1}{64\delta^4} m(S_0)$$

$$|J_2| \leq \int_{S_0(\delta', P)} \left| \frac{\cos \angle(QR, \eta_R)}{\overline{QR}^2} \cdot \frac{\cos \angle(RP, \eta_P)}{\overline{RP}^2} \right| dS_R$$

Since $|J_1|$ is obviously bounded, we have only to prove that $|J_2|$ is bounded.

Let arc PR measured along the curve of intersection of S_0 and the plane determined by η_P and R be s . Let η_P be the z -axis and the tangent plane to S_0 at P be the xy -plane. Let R have cylindrical coordinates (ρ, μ, z) . In a similar manner set up a system of cylindrical coordinates at Q. Let (ρ', μ', z') be the coordinates of R in this system. Denote by ρ' the projection of QR in the tangent plane at P. Let $\bar{s}$ be the arc QR measured along the curve of intersection of S_0 and the plane determined by η_Q and R. Let δ be small enough so that

$$(2.1') \quad \begin{cases} |\angle(QR, \eta_R)| \leq \pi; |\angle(QR, \eta_Q)| \leq \pi, |\angle(\eta_Q, \eta_R)| \leq \pi \\ \frac{\rho'}{\rho} < 2 \\ s < 2\rho \\ ds < 2d\rho \\ \bar{s} < 2\rho' \\ d\bar{s} < 2d\rho' \end{cases}$$

Now we have

$$|\cos \angle(RP, \eta_P)| = \frac{z}{RP} < 4YP, \quad (\text{see the 9th of the inequalities (2.1)})$$

$$|\cos \angle(QR, \eta_Q)| = \frac{z'}{RQ} < 4YP'$$

$$|\angle(QR, \eta_R) - \angle(QR, \eta_Q)| \leq |\angle(\eta_Q, \eta_R)|$$

$$|\cos \angle(QR, \eta_R) - \cos \angle(QR, \eta_Q)| \leq |\angle(\eta_Q, \eta_R)| < Y\overline{QR}.$$

Hence

$$|\cos \angle(QR, \eta_R)| < |\cos \angle(QR, \eta_Q)| + Y\overline{QR} \\ \leq 4YP' + Y\overline{QR}$$

Therefore

$$\left| \frac{\cos \angle(RP, \eta_P)}{RP^2} \right| < \frac{4YP}{RP^2} \leq \frac{4Y}{P}$$

$$\left| \frac{\cos \angle(QR, \eta_R)}{\overline{QR}^2} \right| < \frac{4YP'}{\overline{QR}^2} + \frac{Y}{\overline{QR}} \leq \frac{5Y}{P'} \leq \frac{10Y}{P'}$$

We have now

$$(2.2') \quad |J_L| < 40Y^2 \int_{S_0(\delta', P)} \frac{dS_R}{PP'}$$

Let $O(\delta', P)$ be the circle with center P and radius δ' and lying in the tangent plane to S_0 at P . We have $|dS_R| < 2dS'$ where dS' is the projection of dS_R on the tangent plane at P . We have finally

$$|J_L| < 80Y^2 \int_{C(\delta', P)} \frac{dS'}{PP'}$$

This last integral is known to be bounded*.

Lemma 2a. $K_1(M, N)$ is bounded uniformly as τ approaches zero.

We suppose M, N, G on $S(\tau)$ correspond respectively to Q, P, R on S_0 . For simplicity of notation we denote $S(\tau)$ by S . Since MN converges uniformly to QP as τ approaches zero, given a positive number δ , then τ_1 exists so that for all $\tau \leq \tau_1$, either $MN \geq 2\delta$ uniformly or $MN \leq 2\delta$ uniformly. Hence, we consider two cases as before.

Case 1. $MN \geq 2\delta > 0$. In this case we write

$$4\pi^2 K_1(M, N) = 4\pi^2 \int_{S-S(\delta, M)-S(\delta, N)} K(M, G)K(G, N) dS_G + 4\pi^2 \int_{S(\delta, M)} + 4\pi^2 \int_{S(\delta, N)}$$

Denoting the terms on the right by I_1^1 , I_2^1 , and I_3^1 respectively, we have

$$|I_1^1| \leq \frac{1}{\delta^4} m(S)$$

$$|I_2^1| \leq \frac{1}{\delta^2} \int_{S(\delta, M)} \frac{|\cos 4(MG, n_G)|}{MG^2} dS_G$$

$$|I_3^1| \leq \frac{1}{\delta^2} \int_{S(\delta, N)} \frac{|\cos 4(GN, n_N)|}{GN^2} dS_G$$

Obviously $|I_1^1|$ is uniformly bounded as τ approaches zero. From

Lemma 3 it follows that $|I_2^1|$ and $|I_3^1|$ are uniformly bounded.

Case 2. $MN \leq 2\delta$. Let $\delta' = 4\delta$. Now we have

$$4\pi^2 K_1(M, N) = 4\pi^2 \int_{S-S(\delta', N)} K(M, G)K(G, N) dS_G + 4\pi^2 \int_{S(\delta', N)}$$

Denoting the terms on the right by J_1^1 and J_2^1 respectively, we have at once that

* Kellogg, Potential Theory, pp. 302 - 303.

$$|J_1| \leq \frac{1}{64\delta^4} m(S),$$

and hence $|J_1|$ is uniformly bounded. Let $O(\delta', N)$ be the circle with center N and radius δ' and lying in the tangent plane to S at N . Let

ρ_I, ρ_I' be the projections of NG and MG respectively in the tangent plane at N , and let dS'' be the projection of dS on the tangent plane at N .

Without affecting the validity of the proof we may suppose δ small enough

so that the inequalities corresponding to (2.1') hold uniformly as τ approaches zero. From the inequality (2.2') we have evidently

$$|J_2| < 80Y^2 \int_{C(\delta', N)} \frac{dS''}{\rho_I \rho_I'}$$

But since $\frac{dS''}{dS} \cdot \frac{\rho_I}{\rho_I'}$, $\frac{\rho_I'}{\rho_I}$ all approach 1 uniformly as τ approaches zero, $\int_{C(\delta', N)} \frac{dS''}{\rho_I \rho_I'}$ approaches $\int_{C(\delta', P)} \frac{dS'}{\rho_I \rho_I'}$ uniformly as τ approaches zero. We have seen that $\int_{C(\delta', P)} \frac{dS'}{\rho_I \rho_I'}$ is bounded, which completes the proof of the lemma.

3.

Lemma 3a. $g_n(M, P)$ is continuous in P for P on S_0 and M in T^+ .

Let M be a fixed point of T^+ with minimum distance D from points of S_0 . It is evident that the second term on the right in equation (3.19) is continuous in P . For the first term we have

$$\left| \int_{S_0} k(R, P) \frac{\cos \angle(MR, n_R)}{MR^2} dS_R - \int_{S_0} k(R, P_1) \frac{\cos \angle(MR, n_R)}{MR^2} dS_R \right| \leq I + J;$$

where

$$I = \frac{1}{D^2} \int_{S_0 \subset \delta, P} |k(R, P) - k(R, P_1)| dS_R$$

$$J = \frac{1}{D^2} \int_{S_0 - S_0 \subset \delta, P} |k(R, P) - k(R, P_1)| dS_R$$

We may fix δ so that I is as small as we like. Suppose $\overline{PP_1} < \frac{\delta}{2}$. To

complete the proof we have only to show that $k(R, P_1)$ approaches $k(R, P)$

as P_1 approaches P , with $RP > \frac{\delta}{2}$ and $RP_1 > \frac{\delta}{2}$. But with this restriction

on R , it is seen that each term in $k(R, P_1)$ as given by (3.17) approaches the corresponding term in $k(R, P)$ as P_1 approaches P . Hence the lemma is true.

4.

We shall make use of the following theorem due to J. Radon*:

Theorem. Let $f_1, f_2, \dots$ be a sequence of completely additive functions of point sets whose total variation over J is uniformly bounded and let E_0 be a closed set in J . Then we can pick out a subsequence $\{f_{n^*}\}$ of the functions f_n and a completely additive function of point sets, $f(e)$, so that for each function $\phi(P)$ continuous on E_0 .

$$(4.1') \quad \lim_{n^* \rightarrow \infty} \int_{E_0} \phi(P) df_{n^*}(e_p) = \int_{E_0} \phi(P) df(e_p).$$

It is readily seen that the function $f(e)$ and the sequence f_{n^*}

so chosen will also satisfy the relation

$$(4.2') \quad \lim_{n^* \rightarrow \infty} \int_{E_0} \phi(P) df_{n^*}(e_p) = \int_{E_0} \phi(P) df(e_p),$$

where ϕ_{n^*} is continuous and ϕ_{n^*} converges uniformly to ϕ on E_0 .

This relation is derived from the inequality

$$\left| \int_{E_0} \phi_{n^*} df_{n^*} - \int_{E_0} \phi df \right| \leq \left| \int_{E_0} \phi_{n^*} df_{n^*} - \int_{E_0} \phi_{n^*} df \right| + \left| \int_{E_0} \phi_{n^*} df - \int_{E_0} \phi df \right|$$

The first term on the right approaches zero as n^* becomes infinite, by the theorem above. The second term approaches zero as n^* becomes infinite on account of the uniform convergence of ϕ_{n^*} to ϕ . Hence we have the relation (4.2'). The subsequence $f_{n^*}(e)$ satisfying (4.1') is said to converge to the function $f(e)$ in the weak sense.

Extension for Normal Families of Surfaces. If $\{f_n(e)\}$ is a sequence of completely additive functions of point sets defined on the subsequence $\{S_n\}$ of a normal family of surfaces and whose total variations are uniformly bounded, and if the sequence $\{S_n\}$ converges to the surface S_0 of the normal family as n becomes infinite, then we may pick out a subsequence f_{n^*} of the f_n 's and a completely additive function of point

* J. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen", Wiener Akademie Sitzungsberichte, 122.2, II(a), (1913), p. 1337.

sets $f(e)$ defined on S_0 such that for each $\phi(P_n)$, defined for P_n on S_n and continuous in P_n and converging ^{uniformly} to $\phi(P)$ defined on S_0 .

$$\lim_{n^* \rightarrow \infty} \int_{S_{n^*}} \phi(P_{n^*}) df_{n^*}(e_{P_{n^*}}) = \int_{S_0} \phi(P) df(e_P).$$

This extension is immediate on account of the 1 : 1 correspondence between points of S_0 and points of S_{n^*} for n^* sufficiently large. We define $f_{n^*}(e)$ on S_0 equal to $f_{n^*}(e')$, where e' is the set of points on S_{n^*} corresponding to e on S_0 , and then apply (4.2'). We say that the functions f_{n^*} on S_{n^*} converge in the weak sense to $f(e)$ on S_0 .

5.

Lemma 42. The functions $g_n(M, N)$ converge uniformly to $g_n(M, P)$ as τ approaches zero, M being a fixed point in T^+ [where N on $S(\tau)$ corresponds to P on S_0].

For convenience, we denote $S(\tau)$ by S . Let τ be small enough so that M is interior to S as well as S_0 . In the formula for $g_n(M, N)$ corresponding to (3.19) it is obvious that the second term converges uniformly to the corresponding term in the formula (3.19) as τ approaches zero. For the first term we have, letting G on S correspond to R on S_0 ,

$$(5.1') \quad \left| \int_{S_0} k(R, P) \frac{\cos \angle(MR, \eta_R)}{MR^2} dS_R - \int_S k(G, N) \frac{\cos \angle(MG, \eta_G)}{MG^2} dS_G \right| \leq I + J,$$

where

$$I = \left| \int_{S_0(\delta, P)} k(R, P) \frac{\cos \angle(MR, \eta_R)}{MR^2} dS_R - \int_{S(\delta, P)} k(G, N) \frac{\cos \angle(MG, \eta_G)}{MG^2} dS_G \right|$$

$$J = \left| \int_{S_0 - S_0(\delta, P)} k(R, P) \frac{\cos \angle(MR, \eta_R)}{MR^2} dS_R - \int_{S - S(\delta, P)} k(G, N) \frac{\cos \angle(MG, \eta_G)}{MG^2} dS_G \right|$$

Given $\epsilon > 0$ we may fix δ so that $I < \frac{\epsilon}{2}$, since both integrals of (5.1') converge. Now we let τ_1 be less than δ so that N is in $S(\delta, P)$. The integrand of the second integral of J converges uniformly to the integrand of the first integral as τ approaches zero. Also dS_R/dS_G approaches 1 uniformly as τ approaches zero. Hence τ_1 exists

so that

$$J < \frac{\epsilon}{2}$$

for all $\tau \leq \tau_1$. Hence the lemma is proved.

6.

Theorem. If $v(x,y,z)$ is harmonic outside a sphere of radius a and approaches zero continuously at infinity, then $v(x,y,z)$ approaches zero canonically at infinity.

It follows immediately from a theorem due to Poincaré * that we may write

$$v(x,y,z) = K + \frac{H_0}{R} + \frac{H_1}{R^2} + \frac{H_2}{R^3} + \dots + \frac{H_n}{R^{2n+1}} + \dots$$

valid outside a sphere of sufficiently large radius; where K is a constant;

H_i is a spherical harmonic of degree i in x, y, z ; and $R = +\sqrt{x^2+y^2+z^2}$

We note at once that $K = 0$, and hence we have

$$v(x,y,z) = \frac{H_0}{R} + \frac{H_1}{R^2} + \frac{H_2}{R^3} + \dots + \frac{H_n}{R^{2n+1}} + \dots$$

which proves the theorem.

7.

Theorem. If $U(M)$ is harmonic outside a bounded region S , and $U(M)$ approaches zero continuously at infinity and $\int_{\Omega} \frac{dU}{dn} dS = 0$, where Ω is a sphere of radius $r > r_0$, sufficiently large, then $U(M)$ vanishes at infinity like $\frac{1}{r^2}$ and $\frac{dU}{dn}$ vanishes like $\frac{1}{r^3}$.

By Appendix 6 we may write

$$(7.1') \quad U(M) = U(x,y,z) = \frac{H_0}{R} + \frac{H_1}{R^2} + \dots + \frac{H_n}{R^{2n+1}} + \dots$$

uniformly convergent outside a sphere of radius sufficiently large, where

$R = +\sqrt{x^2+y^2+z^2}$. We may suppose r_0 large enough so that this series is

uniformly convergent outside the sphere of radius r_0 . On Ω we have

* Poincaré, Théorie du Potentiel Newtonien, p. 210.

$$U(M) = \frac{H_0}{r} + \frac{H_1}{r^3} + \dots + \frac{H_n}{r^{2n+1}} + \dots$$

and

$$\frac{dU}{dr} = -\frac{\partial U}{\partial r} = -\frac{\partial U}{\partial x} \frac{x}{r} - \frac{\partial U}{\partial y} \frac{y}{r} - \frac{\partial U}{\partial z} \frac{z}{r}.$$

But

$$\frac{\partial}{\partial r} \left(\frac{H_n}{r^{2n+1}} \right) = \frac{r^{2n+1} \left(\frac{H_n''}{r} \right) - (2n+1) H_n r^{2n}}{r^{4n+2}} = -\frac{H_n' (x, y, z)}{r^{2n+2}}$$

where H'' and H' are spherical harmonics of order n . Hence

$$\frac{dU}{dr} = \frac{H_0}{r^2} + \frac{H_1'}{r^4} + \dots + \frac{H_n'}{r^{2n+2}} + \dots$$

Therefore

$$\lim_{r \rightarrow \infty} \int_{\Omega} \frac{dU}{dr} dS = 4\pi H_0.$$

But it follows immediately from our hypothesis that

$$\lim_{r \rightarrow \infty} \int_{\Omega} \frac{dU}{dr} dS = 0$$

Hence $H_0 = 0$. That $U(M)$ and $\frac{dU}{dr}$ vanish at infinity like $\frac{1}{r^2}$ and $\frac{1}{r^3}$ respectively now follows immediately from (7.1').