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Poisson's Integral and Plurisegments on the
Hypersphere

A Thesis submitted by Nat Edmonson, Jr. to
the Mathematics Department of The Rice Institute
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy.

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Poisson's Integral and Plurisegments on the Hypersphere *

By Nat Edmonson

1. Introduction. In a recent paper†, Bray and Evans have obtained necessary and sufficient conditions that a function $u(M)$, harmonic at all interior points of the unit sphere S , be given by the formula

$$(1) \quad u(M) = \left(\frac{1}{4\pi}\right) \int_S \frac{1-r^2}{MP^3} dF(s_p)$$

In this formula, $F(s)$ is a bounded additive function of segments on the surface of the unit sphere S . The properties of such functions have been developed for two dimensional rectangular regions by Maria||, whose results hold without modification for segments and plurisegments either on the sphere or on the hypersphere. It is the purpose of the present paper to extend the results of Bray and Evans to n -dimensional space.

1.1. The rectangular equation of the hypersphere for n -dimensions is

$$x_1^2 + x_2^2 + \dots + x_n^2 = R^2;$$

a possible set of parametric coordinates are the following:

$$(2) \quad \left\{ \begin{array}{l} x_1 = R \cos \theta_1 \\ x_2 = R \sin \theta_1 \cos \theta_2 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ x_{n-1} = R \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} \\ x_n = R \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \sin \theta_{n-1} \end{array} \right.$$

* Presented to the American Mathematical Society, September, 1928.

† Bray and Evans: A Class of Functions Harmonic within the Sphere, American Journal of Mathematics, Vol. 49, No. 2, 1927, page 153.

|| Maria: Functions of Plurisegments, Transactions of the American Mathematical Society, Vol. 28, No. 2, 1926, page 448.

$$0 \leq \theta_1 \leq \pi$$

$$0 \leq \theta_2 \leq \pi$$

$$\text{---} \text{---} \text{---} \text{---} \text{---}$$

$$0 \leq \theta_{n-2} \leq \pi$$

$$0 \leq \theta_{n-1} \leq 2\pi$$

The measure of the surface of the hyper sphere is obtained by generalizing the ordinary three dimensional formula

$$A_3 = \iint \frac{R dx dy}{z}$$

to the form

$$A_n = \underbrace{\int \cdots \int}_{n-1} \frac{R dx_1 \cdots dx_{n-1}}{x_n}$$

the field of integration being defined by the inequality

$$R^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2 > 0;$$

x_n is given by the expression

$$x_n = \sqrt{R^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}$$

On changing to polar coordinates and integrating with respect to each variable between the corresponding limits of integration, the measure of the surface of the hypersphere is found to be

$$A_n = 2\pi R^{n-1} \int_0^\pi \cdots \int_0^\pi \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-1} d\theta_1 \cdots d\theta_{n-1}$$

$$(3) \quad = 2\pi^{\frac{n}{2}} R^{n-1} / \Gamma\left(\frac{n}{2}\right)$$

The element of this measure is

$$(4) \quad dA_n = R^{n-1} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1}$$

will be called a proper segment. In (7) and (8), $0 < \beta_k < \bar{\beta}_k < \pi$, $k = 1, \dots, (i-1)$; in (9) and (10), $0 < \beta_k < \bar{\beta}_k < \pi$, $k = 1, \dots, (n-2)$, while in (10), $\bar{\beta}_{n-1} - \beta_{n-1} \leq 2\pi$. In (7) and (8), β_i and $\bar{\beta}_i$ may be chosen close to zero and π , respectively, thus making the measure of the caps arbitrarily small. No configurations other than these caps and proper segments will be considered.

The boundaries of these caps and proper segments are $(n-2)$ -dimensional spreads in space of n -dimensions, the hypersphere itself being an $(n-1)$ -dimensional spread in n -dimensional space. Each one of these $(n-2)$ -dimensional coordinate spreads is defined by two equations:

$$(11) \quad R = 1, \quad \theta_k = \beta_k \text{ or } \bar{\beta}_k$$

The spreads which bound the caps are obtained by taking $k = 1, 2, \dots, (i-1)$, and adding to this set the equation

$$(12) \quad R = 1, \quad \theta_i = \beta_i$$

in the case of (7), or the equation

$$(13) \quad R = 1, \quad \theta_i = \bar{\beta}_i$$

in the case of (8). The boundary of a proper segment is obtained by taking $k = 1, 2, \dots, (n-2)$ in the case of (9), and taking $k = 1, 2, \dots, (n-1)$ in the case of (10).

It will be noticed that within and on the boundaries of the proper segments the correspondence between the rectangular coordinates and the polar coordinates is one-to-one, and that the bounding $(n-2)$ -dimensional spreads cut each other orthogonally. There are points within and on the boundaries of the caps where the one-to-one correspondence breaks down. Furthermore, the coordinate spreads may cut each other at angles oth-

ther than right angles in the interior of the caps. The purpose of introducing the caps is to remove such angles. Thus the hypersphere S is completely covered by caps and proper segments, and the caps and proper segments fit together to form an orthogonal system.

In order to examine the nature of the boundary of a proper segment s , the points of the boundary to be known as vertices of s will first be defined. Each vertex is defined by equations of the form

$$R = 1, \theta_{\kappa} = \bar{\beta}_{\kappa} \text{ or } \beta_{\kappa}, \kappa = 1, 2, \dots (n-1)$$

These equations show that through each vertex there pass $(n-1)$ mutually orthogonal one-parametered spreads or curves, each of which has equations of the form

$$R = 1, \theta_{\kappa} = \bar{\beta}_{i_{\kappa}} \text{ or } \beta_{i_{\kappa}}, \kappa = 1, 2, \dots (n-2)$$

A portion of each curve lies on the boundary of s . It is seen from these defining equations that each one-parametered curve can be regarded as the set of points common to $(n-2)$ orthogonal 2-parametered coordinate spreads, each of which has defining equations of the form

$$R = 1, \theta_{i_{\kappa}} = \bar{\beta}_{i_{\kappa}} \text{ or } \beta_{i_{\kappa}}, \kappa = 1, 2, \dots (n-3)$$

As a result of the continuation of this process, it is finally found that an $(n-3)$ -dimensional spread, defined by equations of the form

$$R = 1, \theta_{i_{\kappa}} = \bar{\beta}_{i_{\kappa}} \text{ or } \beta_{i_{\kappa}}, \kappa = 1, 2.$$

is the set of points common to two orthogonal $(n-2)$ -parametered

spreads of the type (11).

There is now sufficient material at hand for the definition of lattices on the hypersphere. By a lattice is meant a system of caps and proper segments covering S , the caps and segments thus being the meshes of the lattice. Each mesh is to be in diameter $< 2d\sqrt{n-1}$, where d is an arbitrarily chosen number. In constructing the lattice, the following system of caps will first be set up:

(1) 2 caps:

$$\begin{array}{ll} 0 \leq \theta_1 \leq \beta & \pi - \beta_1 \leq \theta_1 \leq \pi \\ \hline 0 \leq \theta_i \leq \pi & 0 \leq \theta_i \leq \pi \\ \hline 0 \leq \theta_{n-1} < 2\pi & 0 \leq \theta_{n-1} < 2\pi \end{array}$$

where β_1 is chosen so small that $|x_i| < d, i \geq 1$.

(2) A number of caps:

$$\begin{array}{ll} \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} & \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} \\ 0 \leq \theta_2 \leq \beta_2 & \pi - \beta_2 \leq \theta_2 \leq \pi \\ 0 \leq \theta_3 \leq \pi & 0 \leq \theta_3 \leq \pi \\ \vdots & \vdots \\ 0 \leq \theta_{n-2} \leq \pi & 0 \leq \theta_{n-2} \leq \pi \\ 0 \leq \theta_{n-1} < 2\pi & 0 \leq \theta_{n-1} < 2\pi \end{array}$$

where β_2 is chosen so small that $|x| < d, i \geq 2$; $(\theta_{1,p})$ is a set of values of θ_1 dividing the interval $(\beta_1, \pi - \beta_1)$ into sub-intervals such that the corresponding sub-intervals on the x_1 axis are each in length $< d$.

(3). A number of caps:

$$\begin{array}{ll} \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} & \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} \\ \theta_{2,p} \leq \theta_2 \leq \theta_{2,p+1} & \theta_{2,p} \leq \theta_2 \leq \theta_{2,p+1} \\ 0 \leq \theta_3 \leq \beta_3 & \pi - \beta_3 \leq \theta_3 \leq \pi \\ 0 \leq \theta_4 \leq \pi & 0 \leq \theta_4 \leq \pi \\ \vdots & \vdots \end{array}$$

$$\begin{array}{ll}
 0 \leq \theta_{n-2} \leq \pi & 0 \leq \theta_{n-2} \leq \pi \\
 0 \leq \theta_{n-1} < 2\pi & 0 \leq \theta_{n-1} < 2\pi
 \end{array}$$

where β_3 is chosen so small that $|x_i| < d$, $i > 3$; $(\theta_{1,p})$, $(\theta_{2,p})$ are sets of values of θ_1 , θ_2 respectively dividing the intervals $(\beta_1, \pi - \beta_1)$, $(\beta_2, \pi - \beta_2)$ into sub-intervals such that the corresponding sub-intervals on the x_1 , x_2 axes are each in length $< d$.

.....

(n-2). A number of caps:

$$\begin{array}{ll}
 \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} & \theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} \\
 \theta_{2,p} \leq \theta_2 \leq \theta_{2,p+1} & \theta_{2,p} \leq \theta_2 \leq \theta_{2,p+1} \\
 \vdots & \vdots \\
 \theta_{n-3,p} \leq \theta_{n-3} \leq \theta_{n-3,p+1} & \theta_{n-3,p} \leq \theta_{n-3} \leq \theta_{n-3,p+1} \\
 0 \leq \theta_{n-2} \leq \beta_{n-2} & \pi - \beta_{n-2} \leq \theta_{n-2} \leq \pi \\
 0 \leq \theta_{n-1} < 2\pi & 0 \leq \theta_{n-1} < 2\pi
 \end{array}$$

where $(\theta_{k,p})$, $k=1, 2, \dots, (n-3)$, have the same significance as in the previous cases.

The remainder of the hypersurface is covered by the proper segment:

$$\begin{array}{lll}
 \beta_1 \leq \theta_1 \leq \pi - \beta_1 & & \\
 \beta_2 \leq \theta_2 \leq \pi - \beta_2 & & \\
 \vdots & \vdots & \vdots \\
 \beta_{n-2} \leq \theta_{n-2} \leq \pi - \beta_{n-2} & & \\
 0 \leq \theta_{n-1} < 2\pi & &
 \end{array}$$

This may be divided into sub-segments, where the $(\theta_{k,p})$ have the same significance as before:

$$\begin{array}{ccc}
\theta_{1,p} \leq \theta_1 \leq \theta_{1,p+1} \\
\theta_{2,p} \leq \theta_2 \leq \theta_{2,p+1} \\
\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
\theta_{n-2,p} \leq \theta_{n-2} \leq \theta_{n-2,p+1} \\
\theta_{n-1,p} \leq \theta_{n-1} \leq \theta_{n-1,p+1}
\end{array}$$

If x and x' are any two points within or on the boundary of a cap or a proper segment of this system

$$|x'_i - x_i| < 2d \quad i = 1, 2, \dots, n$$

$$\therefore \sqrt{\sum (x'_i - x_i)^2} < 2d\sqrt{n-1}$$

Hence the diameter of any cap or segment of the system is in diameter

$$< 2d\sqrt{n-1}$$

In particular, the number d may be a member of a decreasing sequence having zero for its limit:

$$\lim_{m \rightarrow \infty} d_m = 0$$

With each d_m there is associated a lattice G_m of order m , and the meshes of G_{m+1} are obtained by sub-dividing the meshes of the lattices of G_m . The resulting set of lattices forms a net G on S for the given system of axes.

Definition: A plurisegment is a finite or denumerable infinity of segments no two of which have an interior point in common. If p_1 and p_2 are two plurisegments, p_1 is contained in p_2 if every segment of p_1 is contained in a finite number of segments of p_2 . The two are equivalent if each is contained in the other. If p_1 and p_2 have no interior points in common, $p_1 + p_2$ is the plurisegment composed of the segments of p_1 and

of p_2 .

1.3. A short resume of the properties of bounded additive functions of segments will now be given. $F(s)$ is a bounded additive function of segments s if it is defined for every finite plurisegment and if when p_1 and p_2 are two finite plurisegments which have no interior points in common,

$$F(p_1 + p_2) = F(p_1) + F(p_2)$$

The positive variation function $P(s)$ of $F(s)$ is defined as the upper bound of $F(s)$ for all finite plurisegments contained in s ; $P(s)$ is non-negative, additive, and bounded for finite plurisegments. Similarly, $N(s)$ is the upper bound of $-F(s)$, and

$$F(s) = P(s) - N(s)$$

$T(s)$, the total variation function, is defined by

$$T(s) = P(s) + N(s)$$

The extension of the definition of $P(s)$ and $N(s)$ to plurisegments composed of a denumerable infinity of segments is unique if it is made by means of the additive property. The details of this extension may be found in the paper of Bray and Evans, already referred to.* Consequently, $F(s)$ is uniquely defined for infinite plurisegments.

The non-negative function $F(s)$ is said to be without point values if $F(s)$ becomes infinitely small as the diameter of s approaches zero.

In particular, the foregoing discussion may be applied to the plurisegments of a net G and to functions which are defined,

* Bray and Evans, loc. cit., pages 156-157; Maria, loc. cit., page 449.

bounded, and additive only on those plurisegments.

1.31. The fundamental tool of the present investigation is the Stieltjes integral defined with respect to a bounded additive function of segments. This integral, which is the limit of a Riemann sum, may be defined in two ways, according to the hypotheses, given below, regarding $F(s)$:

$$(16) \quad \int_S h(P) dF(s_p) = \lim_{\delta \rightarrow 0} \sum_1^K h(P_i) F(s_i)$$

$$(17) \quad \int_S h(P) dF(s'_p) = \lim \sum_1^K h(P_i) F(s'_i)$$

where $S = \sum s_i, \sum s'_i$, diameter $s_i < \delta$, diameter $s'_i < \delta$, and P_i is a point of s_i , or s'_i , respectively. In case $F(s)$ is defined as a bounded additive function of segments for all caps and proper segments having a given system of rectangular axes, (16) and (17) are equivalent to each other, but if $F(s)$ is a bounded additive function of segments defined only on the meshes of a net G , (17) defines the integral.

The integral just defined satisfies the properties (C), (A), (L), (M) of Daniell's S-integral*. The result of this is that its definition may be extended to any function $h(P)$, bounded on the hypersphere and measurable in the Borel sense, the fundamental unit of measure being the segment. If $h(P) = \lim_{\mathcal{A} \rightarrow \omega} h_{\mathcal{A}}(P)$, where the $h_{\mathcal{A}}(P)$ are bounded in their set and have S-integrals, the S-integral of $h(P)$ is the limit of that for $h_{\mathcal{A}}(P)$. In particular, if $h(P)$ is the limit of a sequence of continuous func-

* P. J. Daniell: A General Form of Integral, Annals of Mathematics, Vol.19, 1918, pp. 279-294.

tions bounded in their set, its S-integral is the limit of the integrals of these continuous functions.

1.32. The metric density of the points of a proper segment will now be considered at a point P. Various positions of P with respect to the segment will be considered. Through P there pass (n-1) one parameter curves mutually orthogonal to each other at P. This orthogonality follows from the character of the parametric equations of the hypersphere.

If θ_i be the parameter generating one of these curves, θ_i may vary in two directions with respect to P: in one direction such that the corresponding points on the curve lie on the boundary of the segment, the opposite direction being such that the corresponding points do not lie on the boundary of the segment. The first of these directions will be taken as the positive direction on the curve. Similar reasoning holds for all the curves passing through P.

A new system of rectangular axes is now selected in the following manner: the point P and the origin O determine the new x axis, the direction $\overrightarrow{OP}$ being the positive direction. The $x_1, x_2, \dots, x_n$ axes are taken parallel to the positive directions of the one parameter curves $\theta_1, \theta_2, \dots, \theta_{n-1}$ through P and intersect at the origin. The hypersphere is now referred to this new system of axes by means of the parameters $\psi_1, \psi_2, \dots, \psi_{n-1}$. There can be formed two regions on the hypersphere:

$$\sigma_1 : 0 \leq \psi_1 \leq \beta_1, -\epsilon \leq \psi_2 \leq \frac{\pi}{2} + \epsilon, \dots, -\epsilon \leq \psi_{n-1} \leq \frac{\pi}{2} + \epsilon$$

$$\sigma_2 : 0 \leq \psi_1 \leq \beta_1, \epsilon \leq \psi_2 \leq \frac{\pi}{2} - \epsilon, \dots, \epsilon \leq \psi_{n-1} \leq \frac{\pi}{2} - \epsilon$$

If β_1 is sufficiently small, all points of the segment in the neighborhood of P are contained in σ_1 and σ_2 is contained in the segment. The segment will be denoted by s. In fact, to each ϵ , there corresponds a β_1 such that this is true. Therefore a sequence $\{\epsilon_m\}$ such that $\lim_{m \rightarrow \infty} \epsilon_m = 0$ can be taken, and to each member of this sequence there corresponds a β_1 such that there exists the above relation between σ_1 , σ_2 , and s.

The measure of the cap of class (1) (see 1.2). pole at P, is

$$\int_0^\beta \sin^{n-2} \psi_1 d\psi_1 \int_0^\pi \sin^{n-3} \psi_2 d\psi_2 \cdots \int_0^\pi \sin \psi_{n-2} d\psi_{n-2} \int_0^{2\pi} d\psi_{n-1}$$

The metric density of the segment s at the point P will be the limit of the ratio

$$\frac{\int_0^\beta \sin^{n-2} \psi_1 d\psi_1 \int_{-\epsilon_m}^{\frac{\pi}{2} + \epsilon_m} \sin^{n-3} \psi_2 d\psi_2 \cdots \int_{-\epsilon_m}^{\frac{\pi}{2} + \epsilon_m} \sin \psi_{n-2} d\psi_{n-2} \int_{-\epsilon_m}^{\frac{\pi}{2} + \epsilon_m} d\psi_{n-1}}{\int_0^\beta \sin^{n-2} \psi_1 d\psi_1 \int_0^\pi \sin^{n-3} \psi_2 d\psi_2 \cdots \int_0^\pi \sin \psi_{n-2} d\psi_{n-2} \int_0^{2\pi} d\psi_{n-1}}$$

as m approaches infinity. This limit is found on evaluation to be $1/2^{n-1}$. In fact, the limit is the same if $-\epsilon_m$ is replaced by $+\epsilon_m$ and the metric density is contained between the two values, since

$$\lim_{m \rightarrow \infty} \int_{-\epsilon_m}^{\frac{\pi}{2} + \epsilon_m} \sin^k \theta d\theta = \lim_{m \rightarrow \infty} \int_{\epsilon_m}^{\frac{\pi}{2} - \epsilon_m} \sin^k \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^k \theta d\theta$$

Since 2^{n-1} segments may have P as a vertex, the sum of the metric densities is 1.

If P is a point of a one parameter curve determined by (n-2) (n-2)-dimensional bounding spreads, the segment s may be split into two segments having P as a common vertex. The metric

density of each segment at P is $1/2^{n-1}$, and, as a result, the metric density of s at P is $1/2^{n-2}$. The continuation of this process to the various types of bounding spreads yields a series of values $0, 1/2^{n-1}, 1/2^{n-2}, \dots, 1/2, 1$, according as P is outside s , a vertex, on a one parameter spread, on a two parameter spread, $\dots$, on an $(n-2)$ parameter spread, interior to s . Thus the metric density of s may be regarded as a point function which takes on a finite number of values. It will be denoted by $q(P, s)$, and will be shown to be the limit of a sequence of continuous functions, bounded in their set (see section 2). Therefore its S -integral exists, and is the limit of a sequence of integrals of continuous functions.

1.33. If $F(s)$ is a bounded additive function of segments, the quantity

$$(18) \quad F'_1(s) = \int_s q(P, s) dF(s_P)$$

is also a bounded additive function of segments on S . If the identity

$$(18') \quad F(s) = \int_s q(P, s) dF(s_P)$$

is valid for all segments on S , the possible discontinuities of $F(s)$ are said to be regular. If $F(s)$ is defined only on the segments of a net G , its discontinuities will still be said to be regular if $(18')$ holds for every segment of the net. The word segment is used to refer to both caps and proper segments, and will be used in this sense from now on.

It is useful at this point to have a definition of a segment of discontinuity.

Definition: A segment s is said to be a segment of contin-

uity for $F(s)$, a bounded additive function of segments, if for every sequence $\{s_n\}$ of segments such that $\lim_{n \rightarrow \infty} s_n = s$,

$$\lim_{n \rightarrow \infty} T(s_n) = T(s)$$

where $T(s)$ is the total variation associated with $F(s)$.

Since only a denumerable infinity of the values of θ_i , $i=1,2,\dots,(n-1)$, can give rise to segments of discontinuity, it is possible to form a net G , all segments of which are segments of continuity for $F(s)$. Moreover, on a segment of continuity for $F(s)$, $F_1(s) = F(s)$.

Theorem: If $F(s')$ is bounded and additive on a net G and if $F_1(s)$ is given by the definition

$$(19) \quad F_1(s) = \int_S q(P, s) dF(s'_P)$$

then, $h(P)$ being any bounded function measurable Borel,

$$(20) \quad \int_S h(P) dF_1(s_P) = \int_S h(P) dF(s'_P)$$

and $F_1(s)$ has regular discontinuities. In particular, the system of rectangular axes which corresponds to the segmets s need not be the same as that for the net segments s' .

To prove this theorem, let $h(P)$ be continuous on S , and divide S into segments s with respect to any system of rectangular axes. This system of axes can be the system with respect to which the net G is defined, but this is not necessarily the case. The diameter of each segment of the sub-division is taken to be less than δ , and the function

$$h_\delta(P) = \sum_i h(P_i) q(P, s_i)$$

is set up, P_i being a point of s_i . It now follows from the definition of $F_1(s)$ that

$$\sum_i h(P_i) F_i(s_i) = \int_S h_s(P) dF(s'_p)$$

Since $\lim_{s \rightarrow 0} h_s(P) = h(P)$ uniformly, the limit on the right side is unique, and the limit of the Riemann-Stieltjes sum on the left is uniquely defined. It satisfies the postulates (C), (A), (L), (M) of the S-integrals*, and consequently,

$$\int_S h(P) dF_i(s_p) = \int_S h(P) dF(s'_p)$$

for all functions measurable Borel and bounded..In particular, $h(P)$ may be taken equal to $q(P, s)$:

$$\int_S q(P, s) dF_i(s_p) = \int_S q(P, s) dF(s'_p)$$

that is,

$$(21) \quad F_i(s) = \int_S q(P, s) dF_i(s_p)$$

Therefore $F_i(s)$ has regular discontinuities.

Theorem: If $F(s)$ has regular discontinuities, its positive and negative variation functions, and hence its total variation function, have regular discontinuities on any segment on S .

The proof of this theorem is similar to the proof in the three dimensional case, and therefore will be omitted*.

Definition: A set of functions $F_k(s)$, additive and bounded, are of uniformly limited variation if their total variation functions are bounded in their set. A sequence of bounded additive functions $F_k(s)$ approaches $F(s)$ on the net G if $\lim F_k(s_{im}) = F(s_{im})$ for every mesh s_{im} of the net.

* P. J. Daniell, loc. cit..

* Bray and Evans, loc. cit., page 160

The Helly-Bray theorem will now be stated: If the sequence $F_k(s)$, additive and of uniformly limited variation for all k , approaches $F(s)$, bounded and additive, on a net G on S , then

$$(2.2) \quad \lim_{\pi \rightarrow \infty} \int_S h(P) dF_\pi(s'_p) = \int_S h(P) dF(s'_p)$$

where $h(P)$ is a continuous function of P . Moreover, if $h_k(P)$ approaches $h(P)$ uniformly,

$$(2.2') \quad \lim_{\substack{\pi \rightarrow \infty \\ m \rightarrow \infty}} \int_S h_m(P) dF_\pi(s'_p) = \int_S h(P) dF(s'_p)$$

Again the proof presents nothing new.

2. Necessary and Sufficient conditions. Introductory

Theorem: Given the function u , harmonic within the unit hypersphere S , a set of necessary and sufficient conditions that there exists on the surface of the unit hypersphere S a bounded additive function $F(s)$ of segments such that

$$(I) \quad u(M) = \frac{1}{A'_n} \int_S \frac{1-r^2}{MP^n} dF(s_p)$$

is the following:

(a) There exists a sequence of values of r , $r_1 < r_2 < r_3 \dots$, $\lim_{i \rightarrow \infty} r_i = 1$, such that for any segment s , $F(r_i, s)$, defined by

$$F(r_i, s) = \int_{\sigma(r_i, s)} u(M) dM$$

approaches a limiting value as r_i approaches 1; here $\sigma(r_i, s)$ represents the projection of s on S , a hypersphere of radius r_i , and center O .

(b) $F(r_i, s)$ is of uniformly limited variation as a function of segments for all i .

The function $F(r_i, s)$ is evidently a bounded additive function of segments, since it is defined by an integral which certainly exists. In order to prove the necessity of condition (a),

consider the point function

$$p(P; r_i, s) = \frac{1}{A'_n} \int_{G(r_i, s)} \frac{1-r_i^2}{MP^n} dM$$

which is continuous in P for every r and is bounded as a function of P , s , and r_i . Its boundedness with respect to r_i , as r_i approaches 1, is demonstrated as follows: For a given value of r_i , the integrand of the above integral is positive. Hence the following inequality is true:

$$\begin{aligned} \frac{1}{A'_n} \int_{G(r_i, s)} \frac{1-r_i^2}{MP^n} dM &\leq \frac{1}{A'_n} \int_{S_{r_i}} \frac{1-r_i^2}{MP^n} dM = \\ &= \frac{r_i^{n-1} [1-r_i^2]}{A'_n} \int_0^\pi \frac{\sin^{n-1} \theta_1 d\theta_1}{[1+r_i^2-2r_i \cos \theta_1]^{\frac{n}{2}}} \int_0^\pi \sin^{n-2} \theta_2 d\theta_2 \dots \\ &\dots \int_0^\pi \sin \theta_{n-2} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \leq 1 \end{aligned}$$

This result shows that $p(P; r_i, s)$ is bounded for all values of r .

The point P is now taken as a vertex of a proper segment s . A new system of rectangular axes is set up precisely in the same way as in the definition of the metric density, and the hypersphere is referred to this new system of axes by means of the parameters ψ_i , $i=1, 2, \dots, (n-1)$. The integral defining $p(P; r_i, s)$ is divided into two parts, one part being taken over an arbitrarily small part of the segment s in the vicinity of P . If this arbitrarily small region be denoted by $G'(r_i, s)$, the remainder of the integral is taken over the region $[G(r_i, s) - G'(r_i, s)]$. This latter part disappears as r approaches unity. Therefore

$$\lim_{i \rightarrow \infty} p(P; r_i, s) = \lim_{i \rightarrow \infty} \frac{1}{A'_n} \int \frac{1 - r_i^L}{M P^n} dM$$

$$G'(r_i, s)$$

The following series of inequalities is seen to be valid:

$$\frac{[1 - r_i^L] \Gamma(\frac{n}{2})}{2 \pi^{\frac{n}{2}}} \int_0^\beta \frac{\sin^{n-2} \theta_1 d\theta_1}{[1 + r_i^L - 2r_i \cos \theta_1]^{\frac{n}{2}}} \int_{-\epsilon}^{\frac{\pi}{2} + \epsilon} \sin^{n-3} \theta_2 d\theta_2 \cdots \int_{-\epsilon}^{\frac{\pi}{2} + \epsilon} \sin \theta_{n-2} d\theta_{n-2} \int_{-\epsilon}^{\frac{\pi}{2} + \epsilon} d\theta_{n-1}$$

$$\geq \frac{1}{A'_n} \int \frac{1 - r_i^L}{M P^n} dM \geq \frac{[1 - r_i^L] \Gamma(\frac{n}{2})}{2 \pi^{\frac{n}{2}}} \int_0^\beta \frac{\sin^{n-2} \theta_1 d\theta_1}{[1 + r_i^L - 2r_i \cos \theta_1]^{\frac{n}{2}}}$$

$$\cdot \int_{\epsilon}^{\frac{\pi}{2} - \epsilon} \sin^{n-3} \theta_2 d\theta_2 \cdots \int_{\epsilon}^{\frac{\pi}{2} - \epsilon} \sin \theta_{n-2} d\theta_{n-2} \int_{\epsilon}^{\frac{\pi}{2} - \epsilon} d\theta_{n-1}$$

$$\frac{[1 - r_i^L] \Gamma(\frac{n}{2})}{2 \pi^{\frac{n}{2}}} \int_0^\beta \frac{\sin^{n-2} \theta_1 d\theta_1}{[1 + r_i^L - 2r_i \cos \theta_1]^{\frac{n}{2}}} \int_0^{\frac{\pi}{2}} \sin^{n-3} \theta_2 d\theta_2 \cdots \int_0^{\frac{\pi}{2}} \sin \theta_{n-2} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1}$$

$$+ H \geq \frac{1}{A'_n} \int \frac{1 - r_i^L}{M P^n} dM \geq \quad [\text{see next page}]$$

$$G'(r_i, s)$$

$$\frac{[1-r_i^2] \Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_0^\beta \frac{\sin^{n-2} \theta_1 d\theta_1}{[1+r_i^2-2r_i \cos \theta_1]^{\frac{n}{2}}} \int_0^{\frac{\pi}{2}} \sin^{n-3} \theta_2 d\theta_2 - \dots -$$

$$\dots - \int_0^{\frac{\pi}{2}} \sin \theta_{n-2} d\theta_{n-2} \int_0^{\frac{\pi}{2}} d\theta_{n-1} + \dots$$

Here H and L are quantities which arise from the presence of ϵ in the limits of integration and can be made arbitrarily small by taking β small enough. It is to be remembered that the first inequalities given above are valid for sufficiently small values of β . Consequently, by a direct calculation of the value of the integral*, if P is a vertex of a proper segment,

$$\lim_{r_i \rightarrow 1} p(P; r_i, s) = 1/2^{n-1}$$

If P lies on a one parameter curve of the boundary, on a two parameter spread of the boundary, and so on, a series of values $1/2^{n-1}, 1/2^{n-2}, \dots, 1/4, 1/2, 1$, is obtained. The value 1 corresponds to a position of P interior to the segment s . If P is exterior to the segment s , $\lim p(P; r_i, s)$ is zero. Therefore $p(P; r_i, s)$ has as its limit as r_i approaches 1 the metric density function $q(P, s)$. Thus $q(P, s)$ is the limit of a sequence of continuous functions, bounded in their set (see 1.32).

The application of the Helly-Bray theorem now yields the following result:

$$\lim_{r_i \rightarrow 1} F(r_i, s) = \lim_{r_i \rightarrow 1} \int_{G(r_i, s)} u(M) dM =$$

*The computation of the integral $\int_0^\beta \frac{\sin^{n-2} \theta_1}{[1+r_i^2-2r_i \cos \theta_1]^{\frac{n}{2}}} d\theta_1$ involves merely a finite number of integrations by parts.

$$= \lim_{r_i \rightarrow 1} \int_{G(r_i, S)} dM \int_S \frac{1 - r_i^2}{|MP|^n} dF(s_P)$$

$$= \lim \int_S p(P; r_i, S) dF(s_P)$$

$$= \int_S q(P, S) dF(s_P)$$

$$= F_1(S)$$

In this formula $F_1(s)$ is the bounded additive function of segments associated with $F(s)$ and coincident with $F(s)$ on all segments of continuity for $F(s)$. Moreover, $F_1(s)$ possesses regular discontinuities, as has already been seen.

The necessity of condition (b) follows directly from the properties of the Stieltjes integral:

$$\begin{aligned} \sum_j |F(r_i, S)| &\leq \frac{1}{A_n} \int_S dM \int_S \frac{1 - r_i^2}{|MP|^n} dT(s_P) \\ &\leq T(S) \end{aligned}$$

2.1. The sufficiency of conditions (a) and (b) is demonstrated by considering an arbitrary fixed point M_0 interior to S and to S_{r_i} and expressing the harmonic function $u(M_0)$ by means of the

Poisson's integral over S_{r_i} . There exists the identity

$$\begin{aligned}
 (23) \quad u(M_0) &= \frac{1}{r_i A'_n} \int_{S_{r_i}} \frac{r_i^L - r_0^L}{M_0 M^n} u(M) dM \\
 &= \frac{1}{r_i A'_n} \int_{S_{r_i}} \frac{r_i^L - r_0^L}{M_0 M^n} dF(r_i, S_M) \\
 &= \lim_{i \rightarrow \infty} \frac{1}{r_i A'_n} \int_{S_{r_i}} \frac{r_i^L - r_0^L}{M_0 M^n} dF(r_i, S_M)
 \end{aligned}$$

where the segments s are arbitrary segments having a given system of rectangular axes.

Let $F(s) = \lim F(r_i, s)$. Then $F(s)$ is an additive function of segments, by definition. It is bounded, for if $p = \sum_k s_k$ is any plurisegment

$$|F(p)| = \lim_{i \rightarrow \infty} \left| \sum_k F(r_i, s_k) \right| \leq C, \text{ a constant,}$$

by (b). The application of the Helly-Bray theorem now gives the result

$$\lim_{i \rightarrow \infty} \frac{1}{r_i A'_n} \int_{S_{r_i}} \frac{r_i^L - r_0^L}{M_0 M^n} dF(r_i, S_M) = \frac{1}{A'_n} \int_S \frac{1 - r_0^L}{M_0 P^n} dF(s_P)$$

and the theorem is proved.

It is to be noticed that the above results are valid regardless of the manner in which r approaches 1.

2.2. It will be shown that condition (b) of the introductory theorem implies condition (a). By definition

$$F(r_i, s) = \int_{\sigma(r_i, s)} u(M) dM$$

and by hypothesis,

$$T(r_i, s) = \int_{\sigma(r_i, s)} |u(M)| dM < K, \text{ a positive constant}$$

1. e., the functions $F(r_i, s)$ are of uniformly limited variation
In particular, the last inequality holds if $s = S$.
 (see 1.32). The following auxiliary theorem will first be proved:

Theorem: The hypothesis that the set of functions $F(r_i, s)$
are of uniformly limited variation with respect to r_i , $0 < r_i < 1$,
is sufficient for the existence of a sequence $\{r_i'\}$, $r_i' < r_{i+1}'$,
 $\lim_{i \rightarrow \infty} r_i' = 1$, such that $F(r_i', s)$ approaches a definite limit $F(s)$ on
all the meshes of a given net G ; $F(s)$ is a bounded additive func-
tion of segments of the net.

The hypothesis of the theorem is expressed analytically as follows:

$$(2.4) \quad \sum_{j=1}^{\nu} |F(r_i, s_j)| \leq T(r_i, S) < K$$

in which K is a positive constant, and $T(r_i, S)$ is the total variation function of $F(r_i, S)$. This condition holds for any mode of dividing S into ν caps and proper segments.

A sequence $\{r_i'\}$, $r_i' < r_{i+1}'$, $\lim_{i \rightarrow \infty} r_i' = 1$, may be selected from the interval $(0, 1)$. The corresponding sequence of numbers $F(r_i', s)$ possesses both an upper limit $\overline{F(s)}$ and a lower limit $\underline{F(s)}$, because of (2.4), s_i being any cap or proper segment on S . If $\overline{F(s)} = \underline{F(s)}$, the sequence $\{F(r_i', s)\}$ is convergent and no further work is necessary so far as s_i is concerned. If $\overline{F(s)} \neq \underline{F(s)}$, a

sub-sequence $\{r_i\}$ of $\{r_i\}$ can be so selected that

$$\lim_{i \rightarrow \infty} F(r_i, s) = \overline{F(s)}$$

A second s_2 is taken on S . The number sequence $\{F(r_i, s_2)\}$ has, as before, an upper limit $\overline{F(s_2)}$ and a lower limit $\underline{F(s_2)}$. A sub-sequence $\{r_{i_2}\}$ is so selected ~~that~~ from $\{r_{i_1}\}$ that

$$\lim_{i \rightarrow \infty} F(r_{i_2}, s_2) = \overline{F(s_2)}, \quad r_{i_2} = r_{i_1}.$$

In general, to the segment s_m there corresponds the sub-sequence $\{r_{i_m}\}$ selected from $\{r_{i_{m-1}}\}$, so that

$$\lim_{i \rightarrow \infty} F(r_{i_m}, s_m) = \overline{F(s_m)} \quad r_{i_m} < r_{i_{m+1}}$$

and also so that

$$r_{1,m} = r_{1,m-1} = \dots = r_{1,2} = r_{1,1}.$$

$$r_{2,m} = r_{2,m-1} = \dots = r_{2,2}$$

$$r_{3,m} = r_{3,m-1} = \dots = r_{3,3}$$

$$\dots$$

$$r_{m-1,m} = r_{m-1,m-1}.$$

Thus there is found for any denumerable infinity of segments s_m a sequence $\{r'_i\}$, where $r'_i = r_{i_i}$, such that $r'_{i+1} > r'_i$, $\lim_{i \rightarrow \infty} r'_i = 1$, and

$$(25) \quad \lim_{i \rightarrow \infty} F(r'_i, s_m) = \overline{F(s_m)}$$

In particular, the segments may constitute the meshes of the successive lattices G_m of a given net G . Moreover, the limit functions are bounded additive functions of segments, since the functions $F(r'_i, s_m)$ are bounded additive functions of segments. This statement completes the proof of the auxiliary theorem.*

Since $u(M)$ is harmonic, the application of Poisson's integral yields the results:

$$F(r, s) = \int_{\sigma(r, s)} u(M) dM = \int_{\sigma(r, s)} \left[\frac{1}{A'_n r'_i} \int_{S_{r'_i}} \frac{r'^2 - r^2}{M M'^n} u(M') dM' \right] dM, \quad r'_i > r$$

$$= \lim_{i \rightarrow \infty} \int_{S_{r'_i}} p(M'; r'_i, r; s) u(M') dM'$$

where

$$p(M'; r'_i, r; s) = \frac{1}{A'_n r'_i} \int_{\sigma(r, s)} \frac{r'^2 - r^2}{M M'^n} dM$$

Hence we may write

$$\begin{aligned} F(r, s) &= \lim_{i \rightarrow \infty} \int_{S_{r'_i}} p(M'; r'_i, r; s) dF(r'_i, s'_m) \\ &= \int_S p(P; 1, r; s) d\overline{F(s'_p)} \end{aligned}$$

*This method of proof, suggested by Professor G. C. Evans, was developed for the circle by C. H. Dix and submitted as part of a thesis for the degree of Master of Arts at The Rice Institute.

The Stieltjes integrals are formed with respect to the net G of the theorem just proved. The last result is a consequence of the Helly-Bray theorem (see (22)), since $p(M'; r'_i, r; s)$ approaches $p(P; 1, r; s)$ uniformly. Hence

$$\begin{aligned} \lim_{r=1} F(r, s) &= \lim \int_S p(P; 1, r; s) d\overline{F(s'_p)} \\ &= \int_S q(P, s) d\overline{F(s'_p)} \end{aligned}$$

by the definition of the generalized Stieltjes integral, since the function $q(P, s)$ is the limit of the sequence of bounded continuous functions $p(P; 1, r; s)$. Therefore condition (a) is satisfied for any segment s and for any sequence $\{r\}$ which has 1 for a limit. Consequently the following theorem summarizes the results obtained:

Theorem: A necessary and sufficient condition that $u(M)$, harmonic within the unit hypersphere S be given by the formula

$$(I) \quad u(M) = \frac{1}{A'_n} \int_S \frac{1-r^2}{MP^n} dF(s'_p)$$

or by the formula

$$(II) \quad u(M) = \frac{1}{A'_n} \int_S \frac{1-r^2}{MP^n} dF(s_p)$$

in which $F(s)$ is a bounded additive function of segments and the integrals are evaluated on the basis of a net G of segments s and on all segments s , with the same system of coordinates, respectively, is that

$$(b) \quad T(r, S_r) = \int_{S_r} |u(M)| dM$$

remain bounded as r approaches 1 over a denumerable set of values,
or, in fact, in any way.

The function

$$F(r, s) = \int_{\sigma(r, s)} u(M) dM$$

has as a limit $F_1(s)$ as r approaches 1 for every segment s and

$$(26) \quad \lim_{r \rightarrow 1} F(r, s) = \lim_{r \rightarrow 1} \int_S p(P; 1, r; s) \\ = \int_S q(P, s) dF(s_P) = F_1(s).$$

is identical with $F(s)$ on all segments of continuity and has
regular discontinuities.

An obvious consequence of the last theorem is the following

Corollary: A necessary and sufficient condition for (I)
or (II) is that $u(M)$ be the difference of two functions har-
monic and non-negative within the unit hypersphere.

3. Behaviour of $u(M)$ as M approaches the boundary of S .

In 1.33 it was shown that the existence of a bounded additive function of segments on a net is sufficient for the definition of a bounded additive function $F_1(w)$ of segments w referred to an arbitrary system of rectangular axes. It will be useful to write at this point the formulae which give the salient properties of $F_1(w)$. By definition,

$$(27) \quad F_1(w) = \int_S q(P, w) dF(s'_p)$$

where the segments w are referred to an arbitrary system of axes and $q(P, w)$ is the metric density function. It is seen from this definition that the following relations are valid:

$$(28) \quad F_1(w) = \int_{S'} q(P, w) dF_1(w_p)$$

i.e., $F_1(w)$ has regular discontinuities; moreover,

$$(29) \quad F_1(s) = \int_S q(P, s) dF(s'_p) = \int_S q(P, s) dF_1(w_p)$$

If the positive variation function of $F_1(w)$ be denoted by $P_1(w)$, then

$$(30) \quad P_1(w) = \int_S q(P, w) dP(s'_p)$$

and corresponding expressions hold for $N_1(w)$ and $T_1(w)$ which are respectively the negative and total variation functions of $F_1(w)$.

The following theorem will now be stated:

Theorem: If $u(M)$ is given by $(I)_{\lambda}^{\delta r(I)}$, and $F_1(w)$ is the additive bounded function of plurisegments having regular discontinuities associated with $F(s)$ by the relation

$$(31) \quad F_1(w) = \int_S q(P, w) dF(s_P)$$

then

$$(32) \quad \lim_{r \rightarrow 1} (F(r, w) = \int_{\sigma(r, w)} u(M) dM) = F_1(w) = P_1(w) - N_1(w)$$

$$(33) \quad \lim_{r \rightarrow 1} (T(r, w) = \int_{\sigma(r, w)} |u(M)| dM) = T_1(w) = P_1(w) + N_1(w)$$

where $\sigma(r, w)$ is the projection of the region bounded by w on the hypersphere of radius r .

The proof is carried out as in the three dimensional case.

Definition: Let P be a regular point on S relative to a given system of axes. Then the proper segment

$$(34) \quad S(P, h): \begin{cases} \bar{\theta}_1 - h \leq \theta_1 \leq \bar{\theta}_1 + h \\ \bar{\theta}_2 - h \leq \theta_2 \leq \bar{\theta}_2 + h \\ \dots \dots \dots \\ \bar{\theta}_{n-2} - h \leq \theta_{n-2} \leq \bar{\theta}_{n-2} + h \\ \bar{\theta}_{n-1} - h \leq \theta_{n-1} \leq \bar{\theta}_{n-1} + h \end{cases}$$

where $\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_{n-1}, \bar{\theta}_n$ are the parametric coordinates of P , is said to be a proper segment regular relative to P . It is to be understood that h is sufficiently small that $S(P, h)$ contains no singular points.

Likewise, if P is a singular point of S having for a given system of coordinates the parameters $\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_{i-1}, \theta_i = 0 \text{ or } \pi, \theta_{i+1}, \dots, \theta_{n-1}$ arbitrary, the cap

$$(35) C_h(P): \begin{cases} \bar{\theta}_1 - h \leq \theta_1 \leq \bar{\theta}_1 + h \\ \bar{\theta}_2 - h \leq \theta_2 \leq \bar{\theta}_2 + h \\ \bar{\theta}_{i-1} - h \leq \theta_{i-1} \leq \bar{\theta}_{i-1} + h \\ 0 \leq \theta_i \leq h \quad \text{or} \quad \pi - h \leq \theta_i \leq \pi \\ 0 \leq \theta_{i+1} \leq \pi \\ 0 \leq \theta_{n-1} \leq \pi \\ 0 \leq \theta_n \leq 2\pi \end{cases}$$

will be said to be regular relative to P .

It will be recalled that a plurisegment on S , referred to a given system of axes, has been defined as a collection of a finite number or a denumerable infinity of proper segments and caps. Now let $\{p_i\}$ be a set of plurisegments, not ^{all} necessarily referred to the same system of axes, such that each member of the set has a point P either as an interior point or a boundary point. In particular, the $\{p_i\}$ might be caps or proper segments regular relative to P for a certain system of axes.

An arbitrary system of axes is now selected and P is referred to this system. If P is a non-singular point for this system, a set of proper segments regular relative to P is constructed, while if P is a singular point, a set of caps is taken about P . If h_1 is the smallest h such that every point of p_1 is either an interior point or a boundary point of $S(P, h_1)$ or $C_{h_1}(P)$, according as P is not or is a singular point for the system of axes to which $S(P, h_1)$ and $C_{h_1}(P)$ are referred, the following definition may be constructed:

Definition: If there exists a number $\alpha > 0$, independent of

P and 1, such that

$$(36) \frac{m(p_i)}{mS(P, h_i)} > \alpha, \quad \frac{m(p_i)}{mC_{h_i}(P)} > \alpha$$

the set $\{p_i\}$ is said to form a regular family relative to P.

Definition: If $F(s)$ is a bounded additive function of segments having regular discontinuities, the upper and lower symmetric derivatives of $F(s)$ at P are defined by the expressions

$$(37) \left\{ \begin{array}{ll} \overline{\lim}_{mS(P, h) \rightarrow 0} \frac{F[S(P, h)]}{mS(P, h)} & \underline{\lim}_{mS(P, h) \rightarrow 0} \frac{F[S(P, h)]}{mS(P, h)} \\ \overline{\lim}_{mC_h(P) \rightarrow 0} \frac{F[C_h(P)]}{mC_h(P)} & \underline{\lim}_{mC_h(P) \rightarrow 0} \frac{F[C_h(P)]}{mC_h(P)} \end{array} \right.$$

according as P is not or is a singular point for the system of axes to which $S(P, h)$, $C_h(P)$ are referred. If these limits have a unique value, $F(s)$ is said to have a unique symmetric derivative at P.

Definition: If $\{p_i\}$ is a regular family of plurisegments relative to P, the general derivative of $F(s)$, a bounded additive function of segments having regular discontinuities, at P is defined by

$$(38) \overline{DF}(P) = \overline{\lim}_{mp_i \rightarrow 0} \frac{F(p_i)}{mp_i} = \underline{DF}(P) = \underline{\lim}_{mp_i \rightarrow 0} \frac{F(p_i)}{mp_i}$$

If a unique limit does not exist, the upper and lower derivatives are defined by the expressions

$$(39) \quad \overline{DF(P)} = \overline{\lim_{m p_i \rightarrow 0} \frac{F(p_i)}{m p_i}} ; \quad \underline{DF(P)} = \underline{\lim_{m p_i \rightarrow 0} \frac{F(p_i)}{m p_i}}$$

The value of the derivative is independent of the regular family chosen, except on a set of points of zero measure on S . Moreover, it is to be remembered that if $F(s)$ is a non-negative, additive, bounded function of segments, it may be written in the form *

$$(40) \quad F(s) = \int_s F + \int_s DF(P) d\sigma_P$$

where the first term on the right is the discard function of $F(s)$ and the second term is an absolutely continuous function of plurisegments which may be treated as an absolutely continuous function of point sets.

The following theorem may now be stated and proved†:

Theorem: Let $F(s)$ be a bounded additive function of segments on the unit hypersphere S having regular discontinuities on S and a unique derivative at the point A . If $M_1, M_2, \dots$, is a sequence of points interior to S such that

$$(41) \quad \lim_{i \rightarrow \infty} M_i = A$$

then

$$(42) \quad \lim_{i \rightarrow \infty} v(M_i) = f(A)$$

where $f(A)$ denotes the derivative of $F(s)$ at A , if the ratio $\theta_i / (1-r_i)$ remains bounded, θ_i being the angle between OM_i and

* Maria, loc. cit., page 463.

† See Bray and Evans, loc. cit., page 173.

$\vec{OA}$, and r_1 the distance $\vec{OM}_1$.

It will be recalled that $u(M)$, harmonic at any point M within S , is defined by the formula

$$(43) \quad u(M) = \frac{1}{A'_n} \int_S \frac{1-r^2}{MP^n} dF(s_p)$$

where, by hypothesis, $F(s)$ is defined on all segments having a given system of rectangular axes.

In the present discussion, $F(s)$ will be written in the form

$$(44) \quad F(s) = \sigma f(A) + g(s)$$

where $g(s)$ is a bounded additive function of segments having regular discontinuities and a zero derivative at A .

If a bounded additive function of segments has a zero derivative at A , the same is true of its positive and negative variation functions. In order to demonstrate this, consider a regular family $\{p_n\}$ of plurisegments at A . The function $F(s)$ may be written in the form

$$(45) \quad F(s) = P(s) - N(s)$$

Then, by definition of $P(s)$, there exists a finite plurisegment

$p'_n < p_n$ such that

$$F(p'_n) > P(p_n) - \epsilon, \quad \epsilon > 0, \text{ and arbitrary}$$

If p''_n is the complement of p'_n relative to p_n , $p'_n + p''_n = p_n$

$$F(p) = F(p') + F(p'') > P(s) - \epsilon + F(p'')$$

$$N(p) < -F'(p'') + \epsilon$$

Either $mp'_n \geq (1/2)mp_n$, or $mp''_n \geq (1/2)mp_n$. If those p'_n which satisfy the first inequality are collected into a family $\{p'_i\}$, and those p''_n which satisfy the second inequality into a family $\{p''_j\}$, the result is at least one regular family such that the parameter of regularity is not more than twice as great as the parameter of regularity of the family $\{p_n\}$. Accordingly, if $\{p_i\}$ is such a family

$$\lim_{i \rightarrow \infty} \frac{F'(p_i)}{mp_i} = 0$$

Hence, taking $\epsilon = (1/2)mp_1$,

$$\lim_{i \rightarrow \infty} \frac{P(p_i)}{mp_i} \leq \lim_{i \rightarrow \infty} \frac{F'(p_i)}{mp_i} \frac{mp_i}{mp_i} + \lim_{i \rightarrow \infty} \frac{1}{2^i} = 0$$

Consequently,

$$\lim_{i \rightarrow \infty} \frac{P(p_i)}{mp_i} = 0$$

since

$$\frac{P(p_i)}{mp_i} = 0$$

Likewise, if the family $\{p''_j\}$ exists,

$$\lim_{j \rightarrow \infty} \frac{N(p_j)}{mp_j} = 0$$

since

$$\frac{N(p_j)}{mp_j} = \frac{-F'(p_j) + P(p_j)}{mp_j}$$

Hence

$$\lim_{j \rightarrow \infty} \frac{P(p_j)}{m p_j} = \lim_{j \rightarrow \infty} \frac{F(p_j) + N(p_j)}{m p_j} = 0$$

and since $\{i\} + \{j\} = \{n\}$,

$$(46) \quad \lim_{n \rightarrow \infty} \frac{P(p_n)}{m p_n} = 0$$

and the statement is proved.

It is seen from this discussion that $g(s)$ may be taken to be positive in character.

Since the discontinuities of $F(s)$ are taken to be regular, the definitions of $F(s)$ and $g(s)$ may be extended to any system of segments defined with respect to a system of rectangular axes other than the original system of axes. If the segments and caps relative to this second system of axes be denoted by w , $u(M)$ may be written

$$(47) \quad u(M) = \frac{1}{A'_n} \int_S \frac{1 - r^2}{M P^n} dF(w_p)$$

As has been pointed out, $g(s)$ is non-negative and has a zero derivative at A with respect to a regular family $\{s\}$ of segments having A as an interior point and defined relative to the original system of axes. Furthermore, it is seen from the definition of the derivative that $g(s)$ has a zero derivative at A relative to any regular family of segments defined with respect to any system of axes.

A second system of rectangular axes is now taken so that the direction $\overrightarrow{OA}$ determines the positive direction on the x_1 axis. The other axes are disposed of arbitrarily. A cap of the first class is defined relative to this system of axes:

$$(48) \quad C: \begin{cases} 0 \leq \varphi_1 \leq \alpha' \\ 0 \leq \varphi_2 \leq \pi \\ \text{-----} \\ 0 \leq \varphi_{n-2} \leq \pi \\ 0 \leq \varphi_{n-1} \leq 2\pi \end{cases}$$

A point M_1 is now selected so that

$$\theta_i < \alpha'/2$$

Then if A'_1 is the projection of M_1 on S , a system of rectangular axes is so selected that the positive direction on the x_1 axis is determined by the direction OA'_1 , while the other axes $x_2, \dots, x_n$ are disposed of arbitrarily. If $\varphi_1, \varphi_2, \dots, \varphi_{n-2}, \varphi_{n-1}$ denote the parameters of S with respect this system of axes, a cap of the first class may be taken about A'_1 as a pole:

$$(49) \quad C: \begin{cases} 0 \leq \varphi_1 \leq \alpha_i \\ 0 \leq \varphi_2 \leq \pi \\ \text{-----} \\ 0 \leq \varphi_{n-2} \leq \pi \\ 0 \leq \varphi_{n-1} \leq 2\pi \end{cases} \quad \alpha_i = \alpha' - \theta_i$$

This is valid, since the parameter φ_1 is additive on S , i.e., if two points P_1 and P_2 lie in the section of S by a two parametered spread containing the x_1 axis, and if P_2 has a greater φ_1 than P_1 , the displacement of P_2 relative to P_1 in the two parametered spread or plane is equal to $\varphi'_1 - \varphi_1$, where φ'_1 and φ_1 refer respectively to P_2 and P_1 .

If a series of caps of the type (48) are taken, and corresponding caps of the type (49) are formed, the caps (49)

constitute a regular family. For the measure of $(\mathcal{L})$ is given by

$$mC = \int_0^{\alpha'} \sin^{n-2} \varphi_1 d\varphi_1 \int_0^{\pi} \sin^{n-3} \varphi_2 d\varphi_2 \cdots \int_0^{\pi} \sin \varphi_{n-2} d\varphi_{n-2} \int_0^{2\pi} d\varphi_{n-1}$$

and measure of (H) is given by

$$mc = \int_0^{\alpha'} \sin^{n-2} \psi_1 d\psi_1 \int_0^{\pi} \sin^{n-3} \psi_2 d\psi_2 \cdots \int_0^{\pi} \sin \psi_{n-2} d\psi_{n-2} \int_0^{2\pi} d\psi_{n-1}$$

Hence

$$\begin{aligned} \frac{mC}{mc} &= \frac{\int_0^{\alpha'} \sin^{n-2} \varphi_1 d\varphi_1}{\int_0^{\alpha'} \sin^{n-2} \psi_1 d\psi_1} = 1 + \frac{\int_{\alpha'}^{\alpha'} \sin^{n-2} \varphi_1 d\varphi_1}{\int_0^{\alpha'} \sin^{n-2} \psi_1 d\psi_1} \\ &\leq 1 + \frac{\int_{\alpha'/2}^{\alpha'} \varphi_1^{n-2} d\varphi_1}{\int_0^{\alpha'/2} \psi_1^{n-2} d\psi_1} \leq 2^{n-1} \end{aligned}$$

All of the caps $(\mathcal{L})$, having A_1^i as a pole and containing A either as an interior point or as a boundary point, are members of a regular family, the regularity being defined as in the preceding paragraph. Let $\{c_i\}$ denote the set of such caps for a given A_1^i .

If α' is chosen sufficiently small, the function $F(c_i)$ may be written for the set $\{c_i\}$

$$(50) \quad F(c) = [f(A) + h(c)]\sigma$$

where σ , the measure of c , is defined by

$$(51) \quad \sigma = \int_0^{\varphi_i} \sin^{n-2} \varphi_1 d\varphi_1 \int_0^{\pi} \sin^{n-3} \varphi_2 d\varphi_2 \cdots \int_0^{\pi} \sin \varphi_{n-2} d\varphi_{n-2} \int_0^{2\pi} d\varphi_{n-1}$$

$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^{\varphi_i} \sin^{n-2} \varphi_1 d\varphi_1.$$

The quantity $h(c)$ is a function which becomes arbitrarily small as φ' approaches zero. Otherwise, there would be a regular family of caps on which the derivative of g would not be zero. In fact, so far as the present discussion is concerned, φ' will be chosen so small that

$$(52) \quad 0 \leq h(c) < \epsilon$$

where ϵ is an arbitrarily small positive quantity.

For a set of proper segments and caps taken with reference to the rectangular axes having OA'_1 as the x_1 axis, $u(M_1)$ is given by the formula

$$(53) \quad u(M_1) = \frac{1}{A'_n} \int \frac{1-r^2}{M_1 P^n} dF(w_p)$$

Since

$$F(w) = \sigma f(A) + g(w)$$

$$(54) \quad dF(w_p) = f(A) d\sigma_p + dg(w_p)$$

and $u(M_1)$ becomes

$$\begin{aligned} u(M_i) &= \frac{f(A)}{A'_h} \int_S d\sigma_p + \frac{1}{A'_h} \int_S \frac{1-r_i^2}{M_i \rho^n} dg(w_p) \\ &= f(A) + \frac{1}{A'_h} \int_S \frac{1-r_i^2}{M_i \rho^n} dg(w_p) \end{aligned}$$

Hence

$$(55) \quad |u(M_i) - f(A)| = \frac{1}{A'_h} \int_S \frac{1-r_i^2}{M_i \rho^n} dg(w_p) = I$$

Therefore, if it can be demonstrated that

$$(56) \quad \lim_{i \rightarrow \infty} I < \epsilon$$

the theorem will be proved, ϵ being an arbitrarily small positive number.

The integral I can be replaced by

$$(57) \quad I' = \frac{1}{A'_h} \int_S \phi(P, c_i) \frac{1-r_i^2}{M_i \rho^n} dg(w_p)$$

This is permissible, since the cap c_1 has the cap C as a limiting configuration as i approaches infinity. Hence in $S - c_1$ the denominator of the integrand has a lower bound greater than zero, as r_1 approaches 1. The function $\phi(P, c_i)$ is a function having the value zero for all points exterior to c_1 , and the value 1 for all points interior to and on the boundary of c_1 .

For all caps c_1 having A'_1 as a pole, $g(c_1)$ may be written

$$(58) \quad g(c_i) = h_i(\psi_i)$$

Hence $g(c_1)$ is a function of ψ_i , of limited variation, for all i . In fact, $h_i(\psi_i)$ is a non-negative, monotone function of ψ_i , decreasing as ψ_i decreases. Moreover

$$(59) \quad h_i(\psi_i) = n_i(\psi_i) \left[\int_0^{\psi_i} \sin^{n-2} \psi_i d\psi_i \right] \cdot \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \quad (39)$$

$$= n_i(\psi_i) \cdot \sigma(\psi_i)$$

where

$$\theta_i \leq \psi_i \leq \alpha_i$$

and

$$(60) \quad h_i(\psi_i) \leq n_i(\theta_i) \left[\int_0^{\theta_i} \sin^{n-2} \psi_i d\psi_i \right] \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$$

where

$$0 \leq \psi_i < \theta_i.$$

This distinction takes into consideration the fact that if $\psi_i \geq \theta_i$, the corresponding caps are members of a regular family about A, and

$$(61) \quad F(C) = [f(A) + n_i(C_i)] \sigma$$

i.e., $(62) \quad g(C_i) = n_i(C_i) \cdot \sigma$

whereas, if $\psi_i < \theta_i$, the corresponding caps having A'_1 as a pole are not members of a regular family about A. However, since $n_i(\psi_i)$ is non-negative and additive,

$$(63) \quad h_i(\psi_i) \leq h_i(\theta_i), \quad 0 \leq \psi_i < \theta_i.$$

The integral (57)

$$(64) \quad I' = \frac{1}{A'_n} \int_S \phi(P, C_i) \frac{1 - r_i^2}{M_i P^n} dg(w_P)$$

fulfills the inequality

$$(65) \quad I \leq \frac{1}{A'_h} \int_0^{\alpha_i} \frac{1-r_i^2}{M_i P^n} dh_i(\psi_i) \\ \leq \frac{1}{A'_h} \int_0^{\alpha_i} \frac{(1-r_i^2) dh_i(\psi_i)}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n}{2}}}$$

An application of the Stieltjes theorem on integration by parts yields

$$(66) \quad I \leq \frac{[1-r_i^2]}{A'_h} \left[\frac{h_i(\psi_i)}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n}{2}}} \right]_0^{\alpha_i} \\ + \frac{n[1-r_i^2]}{2 A'_h} \int_0^{\alpha_i} \frac{h_i(\psi_i) \cdot 2r_i \sin \psi_i d\psi_i}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n+2}{2}}}$$

For $\psi_i = \alpha_i$, the first term yields the quantity

$$\frac{[1-r_i^2]}{A'_h} \left[\frac{h_i(\alpha_i)}{[1+r_i^2-2r_i \cos \alpha_i]^{\frac{n}{2}}} \right]$$

Since the denominator has a fixed value greater than zero, this quantity vanishes as r_i approaches 1. For $\psi_i = 0$, more detailed consideration is necessary. For values of $\psi_i < \theta_i$,

$$h_i(\psi_i) \leq h_i(\theta_i) \cdot \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^{\theta_i} \sin^{n-2} \psi_i d\psi_i$$

$$0 \leq \left[\frac{(1-r_i^2) h_i(\psi_1)}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n}{2}}} \right]_0^{\theta_i} \leq \frac{h_i(\theta_i) [1+r_i^2] K \theta_i^{n-1}}{(1-r_i)^{n-1}}$$

By hypothesis,

$$\theta_i/(1-r_i) < B, \text{ a constant, for all } i.$$

Hence, since $h_i(\theta_i)$ approaches 0 with θ_i ,

$$(67) \quad \lim_{i \rightarrow \infty} \left[\frac{(1-r_i^2) h_i(\psi_1)}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n}{2}}} \right]_0^{\theta_i} = 0$$

The integral

$$\frac{n[1-r_i^2]}{2A'_n} \int_0^{\theta_i} \frac{2r_i h_i(\psi_1) \sin \psi_1 d\psi_1}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n+2}{2}}}$$

is divided into two parts:

$$(68) \quad \int_0^{\theta_i} + \int_{\theta_i}^{\theta_i} \quad \text{and (60)}$$

because of the inequalities (59). For convenience, these last integrals are replaced by the integrals

$$(69) \quad \int_0^{\theta_i} + \int_0^{\theta_i}$$

The first of these fulfills the following inequalities:

$$\begin{aligned} \frac{n}{A'_n} \int_0^{\theta_i} \frac{[1-r_i^2] h_i(\psi_1) \sin \psi_1 d\psi_1}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n+2}{2}}} &\leq \frac{n}{A'_n} \int_0^{\theta_i} \frac{(1-r_i^2) h_i(\theta_i) r_i \sin \psi_1 d\psi_1}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n+2}{2}}} \\ &\leq \frac{n[1-r_i^2] h_i(\theta_i)}{A'_n} \int_0^{\theta_i} \frac{r_i \sin \psi_1 d\psi_1}{[1+r_i^2-2r_i \cos \psi_1]^{\frac{n+2}{2}}} = \end{aligned}$$

$$\begin{aligned}
&= \frac{[1-r_i^2] h_i(\theta_i)}{A'_n} \left[\frac{1}{[1+r_i^2-2r_i \cos \theta_i]^{\frac{n}{2}}} \right]_0^{\theta_i} \\
&= \frac{[1-r_i^2] h_i(\theta_i)}{A'_n} \left[\frac{1}{(1-r_i)^n} - \frac{1}{[1+r_i^2-2r_i \cos \theta_i]^{\frac{n}{2}}} \right] \\
&\leq \frac{\eta_i(\theta_i) \cdot 2\pi^{\frac{n-1}{2}} \theta_i^{n-1} [1-r_i]}{A'_n \Gamma(\frac{n-1}{2})} \left[\frac{1}{(1-r_i)^n} - \frac{1}{[1+r_i^2-2r_i \cos \theta_i]^{\frac{n}{2}}} \right]
\end{aligned}$$

The term

$$(70) \quad \frac{\eta_i(\theta_i) \cdot 2\pi^{\frac{n-1}{2}} [1-r_i]}{A'_n \Gamma(\frac{n-1}{2})} \cdot \frac{\theta_i^{n-1}}{(1-r_i)^{n-1}}$$

has zero as a limit since

$$\frac{\theta_i}{1-r_i} < B, \text{ a positive constant,}$$

as i approaches infinity.

Since

$$1+r_i^2-2r_i \cos \theta_i \geq 1+r_i^2-2r_i,$$

the term

$$(71) \quad \frac{[1-r_i] \eta_i(\theta_i) \cdot 2\pi^{\frac{n-1}{2}} \theta_i^{n-1}}{A'_n \Gamma(\frac{n-1}{2})} \frac{1}{[1+r_i^2-2r_i \cos \theta_i]^{\frac{n}{2}}}$$

satisfies the inequality

$$(71) \leq \frac{\eta_i(\theta_i) \cdot 2\pi^{\frac{n-1}{2}} (1+r_i)}{A'_n \Gamma(\frac{n-1}{2})} \frac{\theta_i^{n-1}}{(1-r_i)^{n-1}}$$

Hence

$$\lim_{i \rightarrow \infty} () = 0$$

The integral

$$\frac{\eta [1-r_i^2]}{2 A'_n} \int_0^{d_i} \frac{2r_i h_i(\psi_i) \sin \psi_i d\psi_i}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n+2}{2}}}$$

is less than

$$(72) \frac{\eta}{2} \frac{[1-r_i^2] 2\pi^{\frac{n-1}{2}}}{A'_n \Gamma(\frac{n-1}{2})} \int_0^{d_i} \frac{2r_i \left[\int_0^{\psi_i} \sin \psi_i d\psi_i \right] \sin \psi_i d\psi_i}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n+2}{2}}}$$

where

$$h_i(\psi_i) < \epsilon \cdot \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^{\psi_i} \sin \psi_i d\psi_i$$

An integration by parts applied to the integral (72) yields

$$\frac{\epsilon [1-r_i^2] \pi^{\frac{n-1}{2}}}{A'_n \Gamma(\frac{n-1}{2})} \left[- \frac{\int_0^{\psi_i} \sin \psi_i d\psi_i}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n}{2}}} \right]_0^{d_i} + \frac{2\epsilon [1-r_i^2] \pi^{\frac{n-1}{2}}}{A'_n \Gamma(\frac{n-1}{2}) r_i} \int_0^{d_i} \frac{r_i \sin^n \psi_i d\psi_i}{[1+r_i^2-2r_i \cos \psi_i]^{\frac{n}{2}}}$$

For $\psi_i = d_i$, the first term vanishes as r_i approaches 1; for $\psi_i = 0$, it has the value 0 for all i . Hence this term vanishes as r_i

approaches 1.

A series of integrations by parts shows that the integral term in the last expression vanishes as r_1 approaches 1.

This completes the demonstration of the fact that

$$(73) \quad \overline{\lim}_{i=\infty} I < \epsilon, \epsilon > 0, \text{ arbitrarily small.}$$

This implies

$$(74) \quad \lim_{i=\infty} u(M_i) = f(A),$$

and the theorem is proved.

approaches 1.

A series of integrations by parts shows that the integral term in the last expression vanishes as r_1 approaches 1.

This completes the demonstration of the fact that

$$(73) \quad \overline{\lim}_{i \rightarrow \infty} I = 0$$

This implies

$$(74) \quad \lim_{i \rightarrow \infty} u(M_i) = f(A).$$

and the theorem is proved.