RICE UNIVERSITY

SEQUENTIAL GRADIENT-RESTORATION ALGORITHM
FOR OPTIMAL CONTROL PROBLEMS
WITH GENERAL BOUNDARY CONDITIONS

by

SALVADOR GONZALEZ

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

Angelo Mice

Houston, Texas
June 1978

INFORMATION TO USERS

This was produced from a copy of a document sent to us for microfilming. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help you understand markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure you of complete continuity.
2. When an image on the film is obliterated with a round black mark it is an indication that the film inspector noticed either blurred copy because of movement during exposure, or duplicate copy. Unless we meant to delete copyrighted materials that should not have been filmed, you will find a good image of the page in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed the photographer has followed a definite method in "sectioning" the material. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For any illustrations that cannot be reproduced satisfactorily by xerography, photographic prints can be purchased at additional cost and tipped into your xerographic copy. Requests can be made to our Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases we have filmed the best available copy.

University
Microfilms
International

300 N. ZEEB ROAD, ANN ARBOR, MI 48106
18 BEDFORD ROW, LONDON WC1R 4EJ, ENGLAND

8003861

GONZALEZ, SALVADOR

SEQUENTIAL GRADIENT-RESTORATION ALGORITHM FOR OPTIMAL
CONTROL PROBLEMS WITH GENERAL BOUNDARY CONDITIONS

Rice University

PH.D.

1979

University
Microfilms
International

300 N. Zeeb Road, Ann Arbor, MI 48106

18 Bedford Row, London WC1R 4EJ, England

ABSTRACT

Sequential Gradient-Restoration Algorithm
for Optimal Control Problems
with General Boundary Conditions

by

Salvador Gonzalez

This thesis considers the numerical solution of two classes of optimal control problems, called Problem P1 and Problem P2 for easy identification.

Problem P1 involves a functional I subject to differential constraints and general boundary conditions. It consists of finding the state, the control, and the parameter so that the functional I is minimized while the constraints and the boundary conditions are satisfied to a predetermined accuracy. Problem P2 extends Problem P1 to include nondifferential constraints to be satisfied along the interval of integration. Algorithms are developed for both Problem P1 and Problem P2.

The approach taken is a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase involves one iteration and is designed to decrease the value of the functional, while the constraints are satisfied to first order. The restoration phase involves one or more iterations and is designed to force constraint

satisfaction to a predetermined accuracy while the norm squared of the variations of the control and the parameter is minimized.

The principal property of both algorithms is that they produce a sequence of feasible suboptimal solutions: the functions obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the values of the functional I corresponding to any two elements of the sequence are comparable.

The stepsize of the gradient phase is determined by a one-dimensional search on the augmented functional J , while the stepsize of the restoration phase is obtained by a one-dimensional search on the constraint error P . The gradient stepsize and the restoration stepsize are chosen so that the restoration phase preserves the descent property of the gradient phase. Therefore, the value of the functional I at the end of any complete gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

The algorithms presented in this thesis differ from those of Refs. 1 and 2, in that it is not required that the state vector be given at the initial point. Instead, the initial conditions can be absolutely general. In analogy with Refs. 1 and 2, the present algorithms are capable of handling general

final conditions; therefore, they are suited for the solution of optimal control problems with general boundary conditions. Their importance lies in the fact that many optimal control problems involve initial conditions of the type considered here.

Numerical examples are presented to illustrate the performance of the algorithms associated with Problem P1 and Problem P2. The numerical results show the feasibility as well as the convergence characteristics of these algorithms.

ACKNOWLEDGEMENTS

The author is specially indebted to Dr. A. Miele for the invaluable advice and help given throughout the years, not only at the university level but also on every aspect of his life. He would like to thank Messieurs A.K. Wu and C.T. Liu for helpful discussions and Miss J. Romero for doing a very precise and professional job in typing both the first draft and the final version of this thesis. This research was supported in part by the Office of Scientific Research, Office of Aerospace Research, United States Air Force, Grant No. AF-AFOSR-76-3075, and by CONACYT, Consejo Nacional de Ciencia y Tecnologia, Mexico City, Mexico.

DEDICATION

The author dedicates this thesis to his beloved wife, whose encouragement and support provided the magic ingredients to keep him moving uphill even during the most difficult moments.

TABLE OF CONTENTS

1.	Introduction	Page 1
2.	Statement of Problem P1	Page 4
3.	Algorithm for Problem P1	Page 8
4.	Summary of the Algorithm for Problem P1	Page 22
5.	Numerical Experiments for Problem P1	Page 26
6.	Statement of Problem P2	Page 39
7.	Algorithm for Problem P2	Page 43
8.	Summary of the Algorithm for Problem P2	Page 56
9.	Numerical Experiments for Problem P2	Page 60
10.	Discussion and Conclusions	Page 76
	References	Page 79

1. Introduction

Over the past several years, a successful family of algorithms for the solution of optimal control problems has been developed at Rice University by Professor Angelo Miele and his students and associates (see Refs. 1-8). They are known as sequential gradient-restoration algorithms and have been designed for the solution of different classes of optimal control problems. Some of these algorithms are of the ordinary-gradient type (Refs. 1-4), while the rest are of the conjugate-gradient type (Refs. 5-8). All of them have shown to be robust and reliable; however, they all require the state to be given at the initial point. Owing to the fact that optimal control problems exist, which require satisfaction of more general boundary conditions, we have undertaken the task of designing two new members of the aforementioned family of algorithms. These new members are of the ordinary-gradient type and have the following characteristics: (i) they retain the robustness, reliability, and convergence characteristics of the algorithms discussed in Refs. 1-4; (ii) they are able to handle all of the optimal control problems treated in Refs. 1-8; and (iii) they have the additional capability of handling optimal control problems with general boundary conditions.

Specifically, two types of optimal control problems are

considered here. Problem P1 consists of minimizing a functional I which depends on the n -vector state $x(t)$, the m -vector control $u(t)$, and the p -vector parameter π . The state and the parameter are required to satisfy r scalar relations at the initial point and q scalar relations at the final point, with $r + q \leq 2n + p$. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. Problem P2 differs from Problem P1 in that the state, the control, and the parameter are required to satisfy k additional scalar relations along the interval of integration.

The approach taken in the development of the new algorithms is a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase involves one iteration and is designed to decrease the value of the functional, while the constraints are satisfied to first order. The restoration phase involves one or more iterations, and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control and the parameter is minimized.

The principal property of the algorithms presented here is that a sequence of feasible suboptimal solutions is produced. In other words, at the end of each gradient-restoration cycle, the constraints are satisfied to a predetermined accuracy. Therefore, the values of the functional

I corresponding to any two elements of the sequence are comparable.

The stepsize of the gradient phase is determined by a one-dimensional search on the augmented functional J , while the stepsize of the restoration phase is obtained by a one-dimensional search on the constraints error P . The gradient stepsize and the restoration stepsize are chosen so that the restoration phase preserves the descent property of the gradient phase. As a consequence, the value of the functional I at the end of any complete gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

A time normalization is used in order to simplify the numerical computations. Specifically, the actual time θ is replaced by the normalized time $t = \theta/\tau$, which is defined in such a way that the initial time is $t = 0$ and final time is $t = 1$. The actual final time τ , if it is free, is regarded as a component of the vector parameter π to be optimized. In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time.

2. Statement of Problem P1

Notation. Let t denote the independent variable, and let $x(t)$, $u(t)$, π denote the dependent variables. The time t is a scalar, the state $x(t)$ is an n -vector, the control $u(t)$ is an m -vector, and the parameter π is a p -vector,¹

The state $x(t)$ is partitioned into vectors $y(t)$ and $z(t)$, defined as follows: $y(t)$ is an a -vector including those components of the state that are prescribed at the initial point, and $z(t)$ is a b -vector including those components of the state that are not prescribed at the initial point. Clearly, $a + b = n$.

Problem P1. With the above definitions, Problem P1 can be stated as follows. Minimize the functional

$$I = \int_0^1 f(x, u, \pi, t) dt + [h(z, \pi)]_0 + [g(x, \pi)]_1 \quad (1)$$

with respect to the state $x(t)$, the control $u(t)$, and the parameter π which satisfy the differential constraints

$$\dot{x} - \phi(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (2)$$

and the boundary conditions

$$y(0) = \text{given}, \quad (3)$$

$$[\omega(z, \pi)]_0 = 0, \quad (4)$$

$$[\psi(x, \pi)]_1 = 0. \quad (5)$$

¹All vectors are column vectors.

In the above equations, the quantities I, f, h, g are scalar, the function ϕ is an n -vector, the function ω is a c -vector, and the function ψ is a q -vector. The number of initial conditions $r = a + c$ and the number of final conditions q must satisfy the following relation:

$$r + q \leq 2n + p. \quad (6)$$

First-Order Conditions. From calculus of variations, it can be seen that the previous problem is one of the Bolza type, and it can be recast as that of minimizing the augmented functional

$$J = I + L, \quad (7)$$

subject to (2)-(5). The functional L is defined as

$$L = \int_0^1 [\lambda^T (\dot{x} - \phi)] dt + (\sigma^T \omega)_0 + (\mu^T \psi)_1, \quad (8)$$

where the n -vector $\lambda(t)$ is a variable Lagrange multiplier, the c -vector σ is a constant Lagrange multiplier, and the q -vector μ is a constant Lagrange multiplier. After integrating by parts the term $\lambda^T \dot{x}$, the functional (8) can be rewritten as

$$L = - \int_0^1 (\dot{\lambda}^T x + \lambda^T \phi) dt + (-\lambda^T x + \sigma^T \omega)_0 + (\lambda^T x + \mu^T \psi)_1. \quad (9)$$

The functions $x(t)$, $u(t)$, π and the multipliers $\lambda(t)$, σ ,

μ solving Problem P1 must satisfy the feasibility equations (2)-(5) and the following optimality conditions:

$$\dot{\lambda} - f_x + \phi_x \lambda = 0, \quad 0 \leq t \leq 1, \quad (10)$$

$$f_u - \phi_u \lambda = 0, \quad 0 \leq t \leq 1, \quad (11)$$

$$\int_0^1 (f_\pi - \phi_\pi \lambda) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 = 0, \quad (12)$$

$$(-\gamma + h_z + \omega_z \sigma)_0 = 0, \quad (13)$$

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (14)$$

Remark. Just as the state vector $x(t)$ is partitioned into an a-vector $y(t)$ and a b-vector $z(t)$, the multiplier vector $\lambda(t)$ is partitioned into an a-vector $\beta(t)$ and a b-vector $\gamma(t)$, having the following meaning: $\beta(t)$ is associated with $y(t)$ and $\gamma(t)$ is associated with $z(t)$. With reference to Eq. (13), $\gamma(0)$ denotes the portion of the initial Lagrange multiplier $\lambda(0)$ which is associated with $z(0)$, the portion of the initial state vector $x(0)$ which is not prescribed.

Approximate Methods. Since in general the differential system (2)-(5) and (10)-(14) is nonlinear, approximate methods are employed to find a solution iteratively. In this connection, let the norm squared of a vector v be defined by

$$N(v) = v^T v. \quad (15)$$

Then, the constraint error P can be written as

$$P = \int_0^1 N(\dot{x} - \phi) dt + N(\omega)_0 + N(\psi)_1, \quad (16)$$

and the error in the optimality conditions Q is given by

$$\begin{aligned} Q = & \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda) dt + \int_0^1 N(f_u - \phi_u \lambda) dt \\ & + N \left[\int_0^1 (f_\pi - \phi_\pi \lambda) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right] \\ & + N(-\gamma + h_z + \omega_z \sigma)_0 + N(\lambda + g_x + \psi_x \mu)_1. \end{aligned} \quad (17)$$

For the exact optimal solution, one must have

$$P = 0, \quad Q = 0. \quad (18)$$

For an approximation to the optimal solution, the following relations are to be satisfied:

$$P \leq \varepsilon_1, \quad Q \leq \varepsilon_2, \quad (19)$$

where ε_1 and ε_2 are small and positive preselected numbers.

3. Algorithm for Problem P1

Description of the Algorithm. The technique employed is characterized by a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase is started only when Ineq. (19-1) is satisfied; it involves one iteration and is designed to decrease the value of the functional I or the augmented functional J , while the constraints are satisfied to first order. The restoration phase is started only when Ineq. (19-1) is violated; it involves one or more iterations, each designed to decrease the constraint error P , while the norm squared of the variations of the control $u(t)$ and the parameters π and $z(0)$ is minimized. The restoration phase is terminated whenever the constraints are satisfied to a predetermined accuracy, that is, whenever Ineq. (19-1) is satisfied.

A complete gradient-restoration cycle is designed so that the value of the functional I decreases while the constraints are satisfied to the accuracy (19-1) both at the beginning and at the end of the cycle. Finally, the algorithm as a whole is terminated whenever Ineqs. (19) are satisfied simultaneously.

Notation. For any iteration of the gradient phase or the restoration phase, the following terminology is adopted: $x(t)$, $u(t)$, π denote the nominal functions; $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ denote the varied functions; and $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ denote the displacements leading from the nominal functions to the varied

functions. These quantities satisfy the relations

$$\tilde{x}(t) = x(t) + \Delta x(t), \quad \tilde{u}(t) = u(t) + \Delta u(t), \quad \tilde{\pi} = \pi + \Delta \pi. \quad (20)$$

Because the vector x is partitioned into y and z , Eq. (20-1) implies that

$$\tilde{y}(t) = y(t) + \Delta y(t), \quad \tilde{z}(t) = z(t) + \Delta z(t). \quad (21)$$

Let α be a positive number representing the stepsize (either the gradient stepsize or the restoration stepsize). Then, we define the displacements per unit of stepsize as follows:

$$A(t) = \Delta x(t)/\alpha, \quad B(t) = \Delta u(t)/\alpha, \quad C = \Delta \pi/\alpha. \quad (22)$$

The vector A is partitioned into vectors D and E associated with y and z , respectively. Therefore, Eq. (22-1) implies that

$$D(t) = \Delta y(t)/\alpha, \quad E(t) = \Delta z(t)/\alpha. \quad (23)$$

Upon combining (20)-(21) with (22)-(23), we see that

$$\tilde{x}(t) = x(t) + \alpha A(t), \quad \tilde{u}(t) = u(t) + \alpha B(t), \quad \tilde{\pi} = \pi + \alpha C, \quad (24)$$

and that

$$\tilde{y}(t) = y(t) + \alpha D(t), \quad \tilde{z}(t) = z(t) + \alpha E(t). \quad (25)$$

Desired Properties. The functions $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ must be determined so as to produce some desirable effect at every iteration, namely, the decrease of the functionals I , and/or J , and/or P . Thus, the following descent properties are required:

$$\tilde{I} < I, \quad \text{and/or} \quad \tilde{J} < J, \quad \text{and/or} \quad \tilde{P} < P, \quad (26)$$

where I, J, P are associated with the nominal functions and $\tilde{I}, \tilde{J}, \tilde{P}$ are associated with the varied functions. In turn, the functions $A(t)$, $B(t)$, C are chosen so that

$$\delta I < 0, \quad \text{and/or} \quad \delta J < 0, \quad \text{and/or} \quad \delta P < 0, \quad (27)$$

where the symbol $\delta(\dots)$ denotes the first variation. Then, by choosing the stepsize α sufficiently small, the satisfaction of relations (26) is guaranteed. Ineqs. (26-1), (26-2) and (27-1), (27-2) characterize the gradient phase, while Ineqs. (26-3) and (27-3) characterize the restoration phase.

First Variations. Next, we give the expressions for the first variations of the functionals I, J, P ; after simple manipulations, omitted for the sake of brevity, they take the form^{2,3}

²Implicit in Eqs. (28)-(30) is the assumption $D(0) = 0$.

³The first variation of the augmented functional J is computed by varying the functions $x(t)$, $u(t)$, π , while holding the multipliers $\lambda(t)$, σ , μ unchanged.

$$\delta I/\alpha = \int_0^1 (f_x^T A + f_u^T B + f_\pi^T C) dt + (h_\pi^T C + h_z^T E)_0 + (g_x^T A + g_\pi^T C)_1, \quad (28)$$

and

$$\begin{aligned} \delta J/\alpha = & -\int_0^1 (\dot{\lambda} - f_x + \phi_x \lambda)^T A dt + \int_0^1 (f_u - \phi_u \lambda)^T B dt \\ & + \left[\int_0^1 (f_\pi - \phi_\pi \lambda) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right]^T C \\ & + \left[(-\gamma + h_z + \omega_z \sigma)^T E \right]_0 + \left[(\lambda + g_x + \psi_x \mu)^T A \right]_1, \end{aligned} \quad (29)$$

and

$$\begin{aligned} \delta P/2\alpha = & \int_0^1 (\dot{x} - \phi)^T (\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C) dt \\ & + \left[\omega^T (\omega_\pi^T C + \omega_z^T E) \right]_0 + \left[\psi^T (\psi_x^T A + \psi_\pi^T C) \right]_1. \end{aligned} \quad (30)$$

For the purposes of this thesis, Eqs. (28)-(30) must be completed by the following relation:

$$K/\alpha^2 = \int_0^1 B^T B dt + C^T C + (E^T E)_0, \quad (31)$$

which constitutes a measure of the overall change of the control, the parameter, and the missing components of the initial state.

Gradient Phase. Suppose that nominal functions $x(t)$, $u(t)$, π are available, which satisfy (2)-(5). To first order, the perturbations per unit stepsize $A(t)$, $B(t)$, C must satisfy the linearized constraint equations

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C = 0, \quad 0 \leq t \leq 1, \quad (32)$$

$$D(0) = 0, \quad (33)$$

$$(\omega_z^T E + \omega_\pi^T C)_0 = 0, \quad (34)$$

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0. \quad (35)$$

Next, special variations are defined so that δJ , given by Eq. (29), can be made negative. This is achieved by employing the following variations per unit stepsize for the control, the parameter, and the missing components of the initial state:

$$B = -(f_u - \phi_u \lambda), \quad 0 \leq t \leq 1, \quad (36)$$

$$C = - \left[\int_0^1 (f_\pi - \phi_\pi \lambda) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right], \quad (37)$$

$$E(0) = -(-\gamma + h_z + \omega_z \sigma)_0. \quad (38)$$

The multipliers $\lambda(t)$, σ , μ appearing in (36)-(38) must be consistent with the differential equation

$$\dot{\lambda} - f_x + \phi_x \lambda = 0, \quad 0 \leq t \leq 1, \quad (39)$$

and the final condition

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (40)$$

When the variations defined by (32)-(40) are employed, the first variation of the augmented functional (29) becomes

$$\delta J = -\alpha Q, \quad (41)$$

where Q is the error in the optimality conditions (17), which reduces to

$$Q = \int_0^1 B^T B dt + C^T C + (E^T E)_0. \quad (42)$$

Since $Q > 0$, Eq. (41) shows that $\delta J < 0$. Hence, for α sufficiently small, it is guaranteed that the augmented functional J decreases. Furthermore, because of the fact that the constraints are satisfied to first order, the following relation is true:

$$\delta I = \delta J. \quad (43)$$

Hence, the descent property on the augmented functional J implies a descent property on the functional I .

In closing, an analytical justification can be given to the variations defined by (32)-(40). Let $K = \text{const}$ and $\alpha = \text{const}$ in (31); then, the variations defined by (32)-(40)

minimize the functional δI , given by (28), subject to the linearized constraints (32) - (35) and the quadratic isoperimetric constraint (31).

Linear, Two-Point Boundary-Value Problem for the Gradient Phase. The technique used to solve the LTP-BVP (32)-(40) is a backward-forward integration scheme in combination with the method of particular solutions (Ref. 9). The technique requires the execution of $q+1$ independent sweeps of the differential system (32)-(40), each characterized by a different value of the multiplier μ .

The generic sweep is started by assigning particular values to the components of μ ; then, the multiplier $\lambda(1)$ is obtained from (40). Next, Eq. (39) is integrated backward to obtain the function $\lambda(t)$, and Eq. (36) is employed to obtain $B(t)$. With $\lambda(0)$ known, $\gamma(0)$ is known. Therefore, Eqs. (34), (37), (38) constitute a system of $b+c+p$ linear relations in which the unknowns are the $b+c+p$ components of the vectors $E(0)$, σ , C . For this system to have a unique solution, the following disequation must hold:⁴

$$\det(\omega_z^T \omega_z + \omega_\pi^T \omega_\pi)_0 \neq 0. \quad (44)$$

With $E(0)$ known and because of (33), the vector $A(0)$ is known. Then, $A(t)$ is obtained by forward integration of (32). In this way, the sweep is completed: for the arbitrary value

assigned to μ , it leads to the satisfaction of all of the equations of the system (32)-(40), except Eq. (35).

In order to satisfy Eq. (35) and because the system (32)-(40) is nonhomogeneous, $q+1$ independent sweeps must be executed employing $q+1$ different multiplier vectors μ_i , $i=1, \dots, q+1$. The first q sweeps are performed by choosing the vectors $\mu_1, \mu_2, \dots, \mu_q$ to be the columns of the identity matrix of order q . The last sweep is executed by choosing μ_{q+1} to be the null vector. As a result, one generates the functions and multipliers

$$A_i(t), B_i(t), C_i, \lambda_i(t), \sigma_i, \mu_i, i=1, \dots, q+1. \quad (45)$$

Now, we introduce the $q+1$ undetermined, scalar constants k_i and form the linear combinations

$$A(t) = \sum k_i A_i(t), \quad B(t) = \sum k_i B_i(t), \quad C = \sum k_i C_i, \quad (46)$$

$$\lambda(t) = \sum k_i \lambda_i(t), \quad \sigma = \sum k_i \sigma_i, \quad \mu = \sum k_i \mu_i, \quad (47)$$

where the summations are taken over the index i . The $q+1$

⁴Disequation (44) is obtained from (34), (37), (38) after elimination of $E(0)$ and C . The resulting linear equation in σ admits a unique solution providing (44) is satisfied.

coefficients k_i are obtained by forcing the linear combinations (46-1) and (46-3) to satisfy Eq. (35), together with the normalization condition (Ref. 9)

$$\sum k_i = 1. \quad (48)$$

Once the constants k_i are known, the solution of the LTP-BVP (32)-(40) is given by (46)-(47).

Gradient Stepsize. With the functions $A(t)$, $B(t)$, C known, the one-parameter family of varied functions (24) can be formed. After substitution of Eqs. (24) into (1), (7), (16), the following functions of the stepsize are obtained:

$$\tilde{I} = \tilde{I}(\alpha), \quad \tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha). \quad (49)$$

Then, a one-dimensional search scheme is applied to (49-2), and a value of the stepsize α is selected for which the following relations are satisfied:

$$\tilde{J}(\alpha) \leq \tilde{J}(0), \quad \tilde{P}(\alpha) \leq P_*, \quad \tilde{\tau}(\alpha) \geq 0, \quad (50)$$

where τ is the final time and P_* is a preselected number, not necessarily small. Satisfaction of Ineq. (50-1) is possible because of the descent property of the gradient phase. Ineq. (50-2) is introduced to prevent excessive constraint violation. And Ineq. (50-3) is required for problems with free final time.

Prior to the satisfaction of (50), a scanning process is employed, leading to the bracketing of the minimum point for $\tilde{J}(\alpha)$. This operation is then followed by a Hermitian cubic interpolation process (Ref. 10), which is stopped whenever the following relation is satisfied:⁵

$$|\tilde{J}_\alpha(\alpha)| \leq \epsilon_3, \quad (51)$$

subject to an upper limit for the number of search steps. Once a stepsize α_0 has been selected consistently with either (51) or the prescribed upper limit for the number of search steps, Ineqs. (50) must be checked. If satisfaction occurs, then the stepsize α_0 is accepted. If any violation occurs, then the stepsize α_0 must be bisected progressively until satisfaction of (50) is finally achieved.

Restoration Phase. Let $x(t)$, $u(t)$, π denote nominal functions not satisfying (2)-(5). To first order, the perturbations per unit stepsize $A(t)$, $B(t)$, C must satisfy the linearized constraint equations⁶

⁵The symbol ϵ_3 denotes a small, preselected number.

⁶It is assumed that the nominal functions satisfy (3).

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1, \quad (52)$$

$$D(0) = 0, \quad (53)$$

$$(\omega_z^T E + \omega_\pi^T C + \omega)_0 = 0, \quad (54)$$

$$(\psi_x^T A + \psi_\pi^T C + \psi)_1 = 0. \quad (55)$$

The linearized equations (52)-(55) admit an infinite number of solutions, each of which is characterized by a descent property in the constraint error P . This descent property can be seen by combining (30) with (52)-(55): the first variation of P becomes

$$\delta P = -2\alpha P. \quad (56).$$

Since $P > 0$, Eq. (56) shows that $\delta P < 0$; hence, for α sufficiently small, a decrease in the constraint error P is guaranteed.

Among the infinite number of solutions of Eqs. (52)-(55), the one that minimizes the functional (31) is selected. Thus, we seek the minimum of the quadratic functional (31) with respect to the functions $A(t)$, $B(t)$, C which satisfy the linearized constraints (52)-(55).

By applying standard techniques of optimal control theory or calculus of variations, the following optimality conditions are obtained:

$$B = \phi_u \lambda, \quad 0 \leq t \leq 1, \quad (57)$$

$$C = \int_0^1 \phi_\pi \lambda dt - (\omega_\pi \sigma)_0 - (\psi_\pi \mu)_1, \quad (58)$$

$$E(0) = (\gamma - \omega_z \sigma)_0, \quad (59)$$

$$\dot{\lambda} + \phi_x \lambda = 0, \quad 0 \leq t \leq 1, \quad (60)$$

$$(\lambda + \psi_x \mu)_1 = 0. \quad (61)$$

Summarizing, we seek functions $A(t)$, $B(t)$, C and multipliers $\lambda(t)$, σ , μ which satisfy the linearized constraints (52)-(55) and the optimality conditions (57)-(61).

Linear, Two-Point Boundary-Value Problem for the Restoration Phase. The technique used to solve the LTP-BVP (52)-(55) and (57)-(61), associated with the restoration phase, is analogous to that described for the gradient phase; hence, it is not repeated, for the sake of brevity.

Restoration Stepsize. With the functions $A(t)$, $B(t)$, C known, the one-parameter family of varied functions (24) can be formed. For this one-parameter family, the constraint error (16) becomes a function of the form

$$\tilde{P} = \tilde{P}(\alpha). \quad (62)$$

Then, the stepsize α must be selected so that the following relations are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \quad \tilde{f}(\alpha) \geq 0. \quad (63)$$

Satisfaction of Ineq. (63-1) is possible because of the descent property of the restoration phase. Ineq. (63-2) is required for problems with free final time.

In order to achieve satisfaction of (63), a bisection process is applied to the restoration stepsize α , starting from the reference stepsize $\alpha_0 = 1$. This reference stepsize has the property of yielding one-step restoration for the case where the constraints (2)-(5) are linear.

Iterative Procedure for the Restoration Phase. The descent property (56) of the restoration phase guarantees satisfaction of Ineq.(63-1) at the end of any iteration, but not satisfaction of Ineq. (19-1). Therefore, the restoration algorithm must be employed iteratively until Ineq. (19-1) is satisfied. At this point, the restoration phase is terminated.

Descent Property of a Cycle. A descent property exists for a complete gradient-restoration cycle under the assumption of small stepsizes. Let α_g denote the gradient stepsize and α_r the restoration stepsize. Simple manipulations, omitted for the sake of brevity, show that the gradient corrections are of $O(\alpha_g)$, while the restoration corrections are of $O(\alpha_r \alpha_g^2)$. Hence, for α_g sufficiently small, the restoration corrections are negligible with respect to the gradient corrections. Therefore, the restoration phase preserves the descent property

of the gradient phase,

More specifically, let I , $\tilde{I}$, $\hat{I}$ denote the values of the functional (1) at the beginning of the gradient phase, at the end of the gradient phase, and at the end of the subsequent restoration phase. Note that I and $\tilde{I}$ are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, I and $\hat{I}$ are comparable, and the gradient stepsize α_g can be selected so that

$$\hat{I} < I. \quad (64)$$

This inequality constitutes the descent property of a complete gradient-restoration cycle. In order to enforce it, one proceeds as follows. At the end of the restoration phase, one must verify Ineq. (64). If it is satisfied, the next gradient phase is started; otherwise, the previous gradient stepsize is bisected as many times as needed until, after restoration, Ineq. (64) is satisfied.

4. Summary of the Algorithm for Problem P1

This algorithm includes cycles composed of a gradient phase and a restoration phase. The objective of each cycle is to decrease the functional I so that Ineq. (64) is satisfied, while the constraints are satisfied to a predetermined accuracy(19-1).

Gradient Phase. This phase involves a single iteration, and its objective is to decrease the augmented functional J , while the constraints are satisfied to first order. The gradient phase can be summarized as follows.

(a) Assume nominal functions $x(t)$, $u(t)$, π which satisfy the constraints(2)-(5) within the preselected accuracy (19-1).

(b) For the nominal functions, compute the vectors f_x , f_u , f_π and the matrices ϕ_x , ϕ_u , ϕ_π along the interval of integration. At the initial point, compute the vectors h_z , h_π and the matrices ω_z , ω_π . At the final point, compute the vectors g_x , g_π and the matrices ψ_x , ψ_π .

(c) Solve the LTP-BVP (32) - (40) using the method of particular solutions. In this way, obtain the functions $A(t)$, $B(t)$, C and the multipliers $\lambda(t)$, σ , μ .

(d) Using the functions in (c), compute the gradient stepsize by a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$ until satisfaction of Ineq. (51) occurs. Then, bisect the resulting stepsize (if necessary), until satisfaction of Ineqs. (50) occurs.

(e) Once the gradient stepsize is known, compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ with Eqs. (24).

Restoration Phase. This phase involves one or more iterations, and its objective is to reduce the constraint error P , until satisfaction of (19-1) occurs. Within a single iteration, the objective is to decrease the constraint error to a level compatible with Ineq. (26-3), while the norm squared of the variations of the control, the parameter, and the missing components of the initial state is minimized.

The nominal functions $x(t)$, $u(t)$, π are chosen as follows: for the first restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous gradient iteration; for any subsequent restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous restorative iteration. With this understanding, the restoration phase can be summarized as follows.

(a) Assume nominal functions $x(t)$, $u(t)$, π which satisfy condition (3), but not necessarily conditions (2), (4), (5).

(b) For the nominal functions, compute the vector $\dot{x} - \phi$ and the matrices ϕ_x , ϕ_u , ϕ_π along the interval of integration. At the initial point, compute the vector ω and the matrices ω_z , ω_π . At the final point, compute the vector ψ and the matrices ψ_x , ψ_π .

(c) Solve the LTP-BVP (52)-(55) and (57)-(61) using the method of particular solutions. In this way, obtain the functions $A(t)$, $B(t)$, C and the multipliers $\lambda(t)$, σ , μ .

(d) Using the functions in (c), compute the restoration stepsize by a one-dimensional search on the constraint error $\tilde{P}(\alpha)$. To this effect, perform a bisection process on α , starting from $\alpha = 1$, until Ineqs. (63) are satisfied.

(e) Once the restoration stepsize is known, compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ with Eqs. (24).

(f) Verify whether the varied functions in (e) satisfy Ineq. (19-1). If this is the case, the restoration phase is terminated. Otherwise, return to (a) and continue the process until satisfaction of (19-1) occurs.

Gradient-Restoration Cycle. After the restoration phase is completed, verify whether Ineq. (64) is satisfied. If this is the case, start the next cycle of the sequential gradient-restoration algorithm. If not, return to the previous gradient phase and reduce the gradient stepsize (using a bisection process) until, after restoration, Ineq. (64) is satisfied.

Computational Considerations. Here, special conditions relevant to the computer implementation of the sequential gradient-restoration algorithm are presented.

(a) Starting Condition. The present algorithm can be

started with nominal functions $x(t)$, $u(t)$, π satisfying condition (3) and violating none, some, or all of conditions (2), (4), (5). If the nominal functions are such that Ineq. (19-1) is violated, the algorithm starts with a restoration phase; hence, the first cycle is a half cycle, involving a restoration phase only. On the other hand, if the nominal functions are such that Ineq. (19-1) is satisfied, the algorithm starts with a gradient phase; hence, the first cycle is a complete gradient-restoration cycle.

(b) Bypassing Condition. At the end of the gradient phase of any cycle, the constraint error P must be computed. If Ineq. (19-1) is violated, a restoration phase is started. Otherwise, the restoration phase is bypassed, and the next gradient phase of the algorithm is started.

(c) Stopping Conditions. For the restoration phase taken individually, convergence is achieved whenever Ineq. (19-1) is satisfied. For the sequential gradient-restoration algorithm taken as a whole, convergence is achieved whenever Ineqs. (19) are satisfied simultaneously.

5. Numerical Experiments for Problem P1

Experimental Conditions. In order to evaluate the theory, several examples were solved. The sequential gradient-restoration algorithm associated with Problem P1 was programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic.

Computations were performed at Rice University using an IBM 370/155 computer. For each example, the interval of integration was divided into 100 steps. The differential equations were integrated using Hamming's modified predictor-corrector method with a special Runge-Kutta starting procedure (Ref. 11). The definite integrals I, J, P, Q were computed using a modified Simpson's rule. The method of particular solutions (Ref. 9) was used to solve the linear, two-point boundary value problems associated with both the gradient phase and the restoration phase.

(a) Convergence Conditions. The parameters $\epsilon_1, \epsilon_2, \epsilon_3$ appearing in Ineqs. (19) and (51) were set at the levels⁷

$$\epsilon_1 = E - 08, \quad \epsilon_2 = E - 04, \quad \epsilon_3 = E - 04, \quad (65)$$

⁷The symbol $E_{\pm ab}$ stands for $10^{\pm ab}$.

The tolerance level (65-1) characterizes the restoration phase; the tolerance levels (65-1) and (65-2), employed in combination, characterize the algorithm as a whole; and the tolerance level (65-3) characterizes the one-dimensional search for the gradient stepsize.

(b) Safeguard. For the gradient phase, the parameter P_* appearing in Ineq. (50-2) was set at the level

$$P_* = 10. \quad (66)$$

The tolerance level (66) limits the constraint violation which is permissible during the gradient phase.

(c) Nonconvergence Conditions. The sequential gradient-restoration algorithm was programmed to stop whenever violation of any of the following inequalities occurred:

$$N_c \leq 30, \quad N \leq 100, \quad N_r \leq 10, \quad (67)$$

$$N_{bg} \leq 10, \quad N_{br} \leq 10, \quad N_{bc} \leq 5, \quad (68)$$

$$M \leq 0.83 \text{ E} + 75. \quad (69)$$

Here, N_c is the number of cycles, N is the total number of iterations, N_r is the number of restorative iterations per cycle, N_{bg} is the number of bisections of the gradient stepsize required to satisfy Ineqs. (50), N_{br} is the number of bisections of the restoration stepsize required to satisfy

Ineqs. (63), N_{bc} is the number of bisections of the gradient stepsize required to satisfy Ineq. (64), and M is the modulus of any of the quantities employed in the algorithm.

Numerical Examples. In this section, five numerical examples are described employing scalar notation. In particular, the symbols $x_i(t)$, $i=1, \dots, n$, denote the components of the state; the symbols $u_i(t)$, $i=1, \dots, m$, denote the components of the control; and the symbols π_i , $i=1, \dots, p$, denote the components of the parameter.

For all of the examples, a time normalization is used in order to simplify the numerical computations. Specifically, the actual time θ is replaced by the normalized time

$$t = \theta/\tau, \quad (70)$$

which is defined in such a way that $t=0$ at the initial point and $t=1$ at the final point. The actual final time τ , if it is free, is regarded as a component of the vector parameter π to be optimized. In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time.

Example 5.1. This is a linear-quadratic problem with (i) initial state partially given and partially free and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 (x_1^2 + x_2^2 + u_1^2) dt, \quad (71)$$

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = u_1, \quad (72)$$

$$x_1(0) = 2, \quad x_2(0) = \text{free}, \quad (73)$$

$$x_1(1) = \text{free}, \quad x_2(1) = 1. \quad (74)$$

The assumed nominal functions are:

$$x_1(t) = 2, \quad x_2(t) = 1, \quad u_1(t) = -1. \quad (75)$$

The numerical results are given in Tables 1-2. Convergence to the desired stopping condition occurs in $N=5$ iterations, which include 1 restorative iteration and 4 gradient iterations.

Example 5.2. This is a linear-quadratic problem, with (i) a linear relation between the components of the initial state and (ii) fixed final time $\tau=1$:

$$I = \int_0^1 (x_1^2 + x_2^2 + u_1^2) dt, \quad (76)$$

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = u_1, \quad (77)$$

$$x_1(0) + x_2(0) = 3, \quad (78)$$

$$x_1(1) = \text{free}, \quad x_2(1) = 1. \quad (79)$$

The assumed nominal functions are:

$$x_1(t) = 2, \quad x_2(t) = 1, \quad u_1(t) = -1. \quad (80)$$

The numerical results are given in Tables 3-4. Convergence to the desired stopping condition occurs in $N=4$ iterations, which include 1 restorative iteration and 3 gradient iterations.

Example 5.3. This is a minimum time problem with (i) a component of the initial state given, (ii) a linear relation between the remaining components of the initial state, and (iii) free final time τ . After setting $\pi_1 = \tau$, the problem is as follows:

$$I = \pi_1, \quad (81)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (x_1^2 - u_1^2), \quad \dot{x}_3 = \pi_1 (u_1 - x_2^2 + x_1), \quad (82)$$

$$x_1(0) = 0, \quad x_2(0) + x_3(0) = 1, \quad (83)$$

$$x_1(1) = 1, \quad x_2(1) = 0, \quad x_3(1) = 2. \quad (84)$$

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = 0, \quad x_3(t) = 1+t, \quad u_1(t) = 1, \quad \pi_1 = 1. \quad (85)$$

The numerical results are given in Tables 5-6. Convergence to the desired stopping condition occurs in $N=9$ iterations, which include 7 restorative iterations and 2 gradient iterations.

Example 5.4. This is a minimum time problem with (i) a component of the initial state given, (ii) a nonlinear relation between the remaining components of the initial state, and (iii) free final time τ . After setting $\pi_1 = \tau$, the problem is as follows:

$$I = \pi_1, \quad (86)$$

$$\dot{x}_1 = \pi_1 x_3 \cos u_1, \quad \dot{x}_2 = \pi_1 x_3 \sin u_1, \quad \dot{x}_3 = \pi_1 \sin u_1, \quad (87)$$

$$x_1(0) = 0, \quad x_2(0)x_3(0) = 0, \quad (88)$$

$$x_1^2(1) + x_3(1) = 1, \quad x_2(1) = \text{free}. \quad (89)$$

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = 1, \quad x_3(t) = 0, \quad u_1(t) = 1, \quad \pi_1 = 1. \quad (90)$$

The numerical results are given in Tables 7-8. Convergence to the desired stopping condition occurs in $N = 24$ iterations, which include 13 restorative iterations and 11 gradient iterations.

Example 5.5. This is a minimum time problem with (i) a component of the initial state given, (ii) a nonlinear relation between the remaining components of the initial state, and (iii) free final time τ . After setting $\pi_1 = \tau$, the

problem is as follows:

$$I = \pi_1, \quad (91)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (x_1^2 - u_1^2), \quad \dot{x}_3 = \pi_1 (u_1 - x_2^2 + x_1), \quad (92)$$

$$x_1(0) = 0, \quad x_2^2(0) + x_3^2(0) = 1, \quad (93)$$

$$x_1(1)x_2(1) = 0, \quad x_3(1) = 2. \quad (94)$$

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = 0, \quad x_3(t) = 1+t, \quad u_1(t) = 1, \quad \pi_1 = 1. \quad (95)$$

The numerical results are given in Tables 9-10. Convergence to the desired stopping condition occurs in $N=9$ iterations, which include 7 restorative iterations and 2 gradient iterations.

Table 1. Convergence history, Example 5.1.

N	Phase	P	Q	I
0		0.10E+01		
1	Rest	0.94E-28	0.92E+01	7.72083
2	Grad	0.18E-26	0.15E+00	5.89315
3	Grad	0.10E-25	0.85E-02	5.84026
4	Grad	0.18E-25	0.19E-03	5.83854
5	Grad	0.36E-25	0.14E-04	5.83848

Table 2. Converged solution, Example 5.1.

t	x_1	x_2	u_1
0.0	2.0000	1.0207	-2.0305
0.1	1.9140	0.8420	-1.5505
0.2	1.8584	0.7092	-1.1118
0.3	1.8338	0.6186	-0.7064
0.4	1.8413	0.5671	-0.3271
0.5	1.8824	0.5525	0.0329
0.6	1.9591	0.5732	0.3805
0.7	2.0741	0.6284	0.7217
0.8	2.2303	0.7176	1.0628
0.9	2.4315	0.8411	1.4093
1.0	2.6821	1.0000	1.7689

$$\tau = 1.0000$$

Table 3. Convergence history, Example 5.2.

N	Phase	P	Q	I
0		0.10E+01		
1	Rest	0.54E-28	0.97E+01	5.99629
2	Grad	0.44E-27	0.76E-01	4.23041
3	Grad	0.54E-27	0.17E-02	4.20801
4	Grad	0.59E-27	0.28E-04	4.20768

Table 4. Converged solution, Example 5.2.

t	x_1	x_2	u_1
0.0	1.2109	1.7890	-2.4277
0.1	1.1561	1.5668	-2.0228
0.2	1.1197	1.3832	-1.6559
0.3	1.1017	1.2346	-1.3208
0.4	1.1027	1.1182	-1.0117
0.5	1.1233	1.0316	-0.7232
0.6	1.1647	0.9730	-0.4503
0.7	1.2284	0.9411	-0.1880
0.8	1.3160	0.9352	0.0683
0.9	1.4299	0.9548	0.3233
1.0	1.5726	1.0000	0.5814

$\tau = 1.0000$

Table 5. Convergence history, Example 5.3.

N	Phase	P	Q	I
0		0.86E+00		
1	Rest	0.22E-01		
2	Rest	0.20E-04		
3	Rest	0.10E-10	0.21E+00	1.14398
4	Grad	0.44E-01		
5	Rest	0.56E-03		
6	Rest	0.13E-07		
7	Rest	0.45E-17	0.86E-03	1.05689
8	Grad	0.20E-05		
9	Rest	0.68E-13	0.40E-05	1.05650

Table 6. Converged solution, Example 5.3.

t	x_1	x_2	x_3	u_1
0.0	0.0000	0.5866	0.4133	1.5686
0.1	0.1553	0.3586	0.5535	1.3827
0.2	0.2938	0.1825	0.7080	1.2445
0.3	0.4194	0.0469	0.8699	1.1358
0.4	0.5344	-0.0540	1.0353	1.0424
0.5	0.6398	-0.1227	1.2019	0.9532
0.6	0.7356	-0.1595	1.3683	0.8596
0.7	0.8211	-0.1645	1.5332	0.7553
0.8	0.8947	-0.1379	1.6951	0.6361
0.9	0.9549	-0.0818	1.8517	0.5006
1.0	1.0000	0.0000	2.0000	0.3500

$\tau = \pi_1 = 1.0565$

Table 7. Convergence history, Example 5.4.

N	Phase	P	Q	I
0		0.17E+01		
1	Rest	0.28E+00		
2	Rest	0.61E-02		
3	Rest	0.26E-05		
4	Rest	0.15E-13	0.45E-01	1.02941
5	Grad	0.39E-03		
6	Rest	0.36E-08	0.56E-02	1.00642
7	Grad	0.93E-06		
8	Rest	0.67E-14	0.30E-02	1.00393
9	Grad	0.24E-06		
10	Rest	0.16E-15	0.19E-02	1.00264
11	Grad	0.10E-06		
12	Rest	0.13E-16	0.12E-02	1.00184
13	Grad	0.51E-07		
14	Rest	0.20E-17	0.89E-03	1.00130
15	Grad	0.30E-07		
16	Rest	0.58E-18	0.62E-03	1.00092
17	Grad	0.15E-07		
18	Rest	0.12E-18	0.44E-03	1.00065
19	Grad	0.11E-07		
20	Rest	0.61E-19	0.30E-03	1.00044
21	Grad	0.53E-08	0.20E-03	1.00029
22	Grad	0.22E-07		
23	Rest	0.21E-18	0.12E-03	1.00019
24	Grad	0.22E-08	0.79E-04	1.00011

Table 8. Converged solution, Example 5.4.

t	x_1	x_2	x_3	u_1
0.0	0.0000	1.0000	0.0000	1.5707
0.1	0.0000	1.0050	0.1000	1.5661
0.2	0.0001	1.0200	0.2000	1.5615
0.3	0.0004	1.0450	0.3000	1.5569
0.4	0.0009	1.0800	0.4000	1.5522
0.5	0.0019	1.1250	0.5000	1.5476
0.6	0.0033	1.1799	0.5999	1.5430
0.7	0.0052	1.2449	0.6999	1.5384
0.8	0.0078	1.3199	0.7999	1.5338
0.9	0.0112	1.4048	0.8998	1.5291
1.0	0.0154	1.4997	0.9997	1.5245
$\tau = \pi_1 = 1.0001$				

Table 9. Convergence history, Example 5.5.

N	Phase	P	Q	I
0		0.86E+00		
1	Rest	0.93E-02		
2	Rest	0.28E-04		
3	Rest	0.53E-10	0.31E+00	1.02610
4	Grad	0.15E+00		
5	Rest	0.96E-02		
6	Rest	0.10E-05		
7	Rest	0.24E-12	0.69E-03	0.86466
8	Grad	0.97E-06		
9	Rest	0.80E-13	0.33E-05	0.86430

Table 10. Converged solution, Example 5.5.

t	x_1	x_2	x_3	u_1
0.0	0.0000	0.6487	0.7609	1.6249
0.1	0.1312	0.4495	0.8721	1.4236
0.2	0.2475	0.2962	0.9929	1.2730
0.3	0.3522	0.1771	1.1187	1.1544
0.4	0.4476	0.0857	1.2473	1.0548
0.5	0.5348	0.0186	1.3767	0.9645
0.6	0.6144	-0.0260	1.5060	0.8767
0.7	0.6863	-0.0492	1.6341	0.7866
0.8	0.7502	-0.0518	1.7599	0.6907
0.9	0.8055	-0.0348	1.8823	0.5873
1.0	0.8515	0.0000	2.0000	0.4760

$$\tau = \pi_1 = 0.8643$$

6. Statement of Problem P2

Notation. It is the same as for Problem P1. Hence, it is not repeated here.

Problem P2. This problem consists of minimizing the functional

$$I = \int_0^1 f(x, u, \pi, t) dt + [h(z, \pi)]_0 + [g(x, \pi)]_1 \quad (96)$$

with respect to the state $x(t)$, the control $u(t)$, and the parameter π which satisfy the differential constraints

$$\dot{x} - \phi(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (97)$$

the nondifferential constraints

$$S(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (98)$$

and the boundary conditions

$$y(0) = \text{given}, \quad (99)$$

$$[\omega(z, \pi)]_0 = 0, \quad (100)$$

$$[\psi(x, \pi)]_1 = 0. \quad (101)$$

In the above equations, the quantities I, f, h, g are scalar, the function ϕ is an n -vector, the function S is a k -vector, the function ω is a c -vector, and the function ψ is a q -vector. The number of initial conditions $r = a + c$ and

the number of final conditions q must satisfy the following relation:

$$r + q \leq 2n + p. \quad (102)$$

First-Order Conditions. From calculus of variations, it can be seen that the previous problem is one of the Bolza type, and it can be recast as that of minimizing the augmented functional

$$J = I + L, \quad (103)$$

subject to (97)-(101). The functional L is defined as

$$L = \int_0^1 [\lambda^T (\dot{x} - \phi) + \rho^T S] dt + (\sigma^T \omega)_0 + (\mu^T \psi)_1, \quad (104)$$

where the n -vector $\lambda(t)$ is a variable Lagrange multiplier, the k -vector $\rho(t)$ is a variable Lagrange multiplier, the c -vector σ is a constant Lagrange multiplier, and the q -vector μ is a constant Lagrange multiplier. After integrating by parts the term $\lambda^T \dot{x}$, the functional (104) can be rewritten as

$$L = - \int_0^1 (\dot{\lambda}^T x + \lambda^T \phi - \rho^T S) dt + (-\lambda^T x + \sigma^T \omega)_0 + (\lambda^T x + \mu^T \psi)_1. \quad (105)$$

The functions $x(t)$, $u(t)$, π and the multipliers $\lambda(t)$, $\rho(t)$, σ , μ solving Problem P2 must satisfy the feasibility

equations (97)-(101) and the following optimality conditions:

$$\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (106)$$

$$f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \quad (107)$$

$$\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 = 0, \quad (108)$$

$$(-\gamma + h_z + \omega_z \sigma)_0 = 0, \quad (109)$$

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (110)$$

Approximate Methods. Since in general the differential system (97)-(101) and (106)-(110) is nonlinear, approximate methods are employed to find a solution iteratively. In this connection, let the norm squared of a vector v be defined by

$$N(v) = v^T v. \quad (111)$$

Then, the constraint error P can be written as

$$P = \int_0^1 N(\dot{x} - \phi) dt + \int_0^1 N(S) dt + N(\omega)_0 + N(\psi)_1, \quad (112)$$

and the error in the optimality conditions Q is given by

$$\begin{aligned}
Q = & \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho) dt + \int_0^1 N(f_u - \phi_u \lambda + S_u \rho) dt \\
& + N \left[\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right] \\
& + N(-\gamma + h_z + \omega_z \sigma)_0 + N(\lambda + g_x + \psi_x \mu)_1 .
\end{aligned} \tag{113}$$

For the exact optimal solution, one must have

$$P = 0, \quad Q = 0. \tag{114}$$

For an approximation to the optimal solution, the following relations are to be satisfied:

$$P \leq \varepsilon_1, \quad Q \leq \varepsilon_2, \tag{115}$$

where ε_1 and ε_2 are small and positive preselected numbers.

7. Algorithm for Problem P2

Description of the Algorithm. The technique employed is characterized by a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase is started only when Ineq. (115-1) is satisfied; it involves one iteration and is designed to decrease the value of the functional I or the augmented functional J , while the constraints are satisfied to first order. The restoration phase is started only when Ineq. (115-1) is violated; it involves one or more iterations, each designed to decrease the constraint error P , while the norm squared of the variations of the control $u(t)$ and the parameters π and $z(0)$ is minimized. The restoration phase is terminated whenever the constraints are satisfied to a predetermined accuracy, that is, whenever Ineq. (115-1) is satisfied.

A complete gradient-restoration cycle is designed so that the value of the functional I decreases while the constraints are satisfied to the accuracy (115-1) both at the beginning and at the end of the cycle. Finally, the algorithm as a whole is terminated whenever Ineqs. (115) are satisfied simultaneously.

Notation. For any iteration of the gradient phase or the restoration phase, the notation employed for Problem P2 is identical with that employed for Problem P1. Hence, Eqs. (20)-(25) are retained.

Desired Properties. For any iteration of the gradient phase or the restoration phase, the desired properties for Problem P2 are identical with those pertaining to Problem P1. Hence, Ineqs. (26)-(27) are retained.

First Variations. Next, we give the expressions for the first variations of the functionals I, J, P; after simple manipulations, omitted for the sake of brevity, they take the form^{8,9}

$$\delta I/\alpha = \int_0^1 (f_x^T A + f_u^T B + f_\pi^T C) dt + (h_\pi^T C + h_z^T E)_0 + (g_x^T A + g_\pi^T C)_1, \quad (116)$$

and

$$\begin{aligned} \delta J/\alpha = & - \int_0^1 (\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho)^T A dt + \int_0^1 (f_u - \phi_u \lambda + S_u \rho)^T B dt \\ & + \left[\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right]^T C \\ & + \left[(-\gamma + h_z + \omega_z \sigma)^T E \right]_0 + \left[(\lambda + g_x + \psi_x \mu)^T A \right]_1, \end{aligned} \quad (117)$$

⁸Implicit in Eqs. (116)-(118) is the assumption $D(0) = 0$.

⁹The first variation of the augmented functional J is computed by varying the functions $x(t)$, $u(t)$, π , while holding the multipliers $\lambda(t)$, $\rho(t)$, σ , μ unchanged.

and

$$\begin{aligned} \delta P/2\alpha = & \int_0^1 (\dot{x} - \phi)^T (\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C) dt + \int_0^1 s^T (s_x^T A + s_u^T B + s_\pi^T C) dt \\ & + \left[\omega^T (\omega_\pi^T C + \omega_z^T E) \right]_0 + \left[\psi^T (\psi_x^T A + \psi_\pi^T C) \right]_1 . \end{aligned} \quad (118)$$

For the purposes of this thesis, Eqs. (116)-(118) must be completed by the following relation:

$$K/\alpha^2 = \int_0^1 B^T B dt + C^T C + (E^T E)_0 , \quad (119)$$

which constitutes a measure of the overall change of the control, the parameter, and the missing components of the initial state.

Gradient Phase. Suppose that nominal functions $x(t)$, $u(t)$, π are available, which satisfy (97)-(101). To first order, the perturbations per unit stepsize $A(t)$, $B(t)$, C must satisfy the linearized constraint equations

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C = 0 , \quad 0 \leq t \leq 1 , \quad (120)$$

$$s_x^T A + s_u^T B + s_\pi^T C = 0 , \quad 0 \leq t \leq 1 , \quad (121)$$

$$D(0) = 0 , \quad (122)$$

$$(\omega_z^T E + \omega_\pi^T C)_0 = 0 , \quad (123)$$

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0 . \quad (124)$$

Next, special variations are defined so that δJ , given by Eq. (117), can be made negative. This is achieved by employing the following variations per unit stepsize for the control, the parameter, and the missing components of the initial state:

$$B = -(f_u - \phi_u \lambda + S_u \rho), \quad 0 \leq t \leq 1, \quad (125)$$

$$C = - \left[\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 \right], \quad (126)$$

$$E(0) = -(-\gamma + h_z + \omega_z \sigma)_0. \quad (127)$$

The multipliers $\lambda(t)$, $\rho(t)$, σ , μ appearing in (125)-(127) must be consistent with the differential equation

$$\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (128)$$

and the final condition

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (129)$$

When the variations defined by (120)-(129) are employed, the first variation of the augmented functional (117) becomes

$$\delta J = -\alpha Q, \quad (130)$$

where Q is the error in the optimality conditions (113), which reduces to

$$Q = \int_0^1 B^T B dt + C^T C + (E^T E)_0. \quad (131)$$

Since $Q > 0$, Eq. (130) shows that $\delta J < 0$. Hence, for α sufficiently small, it is guaranteed that the augmented functional J decreases. Furthermore, because of the fact that the constraints are satisfied to first order, the following relation is true:

$$\delta I = \delta J. \quad (132)$$

Hence, the descent property on the augmented functional J implies a descent property on the functional I .

In closing, an analytical justification can be given to the variations defined by (120)-(129). Let $K = \text{const}$ and $\alpha = \text{const}$ in (119); then, the variations defined by (120)-(129) minimize the functional δI , given by (116), subject to the linearized constraints (120)-(124) and the quadratic isoperimetric constraint (119).

Linear, Two-Point Boundary-Value Problem for the Gradient Phase. The technique used to solve the LTP-BVP (120)-(129) is a forward integration scheme in combination with the method of particular solutions (Ref. 9). The technique requires the execution of $n + p + 1$ independent sweeps of the differential system (120)-(129), each characterized by a different value of

the $(n+p)$ -vector w , whose components are the n components of the initial multiplier $\lambda(0)$ and the p components of the parameter C .

The generic sweep is started by assigning particular values to the components of w , that is, the components of the vectors $\lambda(0)$ and C . With $\lambda(0)$ known, $\gamma(0)$ is known, Therefore, Eqs. (123) and (127) constitute a system of $b+c$ linear relations in which the unknowns are the $b+c$ components of the vectors $E(0)$ and σ . For this system to have a unique solution, the following disequation must hold:

$$\det(\omega_z^T \omega_z)_0 \neq 0. \quad (133)$$

With $E(0)$ known and because of (122), the vector $A(0)$ is known, Then $A(t)$ and $\lambda(t)$ together with $B(t)$ and $\rho(t)$ are obtained by forward integration of (120) and (128), subject to (121) and (125). Note that, at each time station t , $0 \leq t \leq 1$, Eqs. (121) and (125) constitute a system of $m+k$ linear relations in which the unknowns are the $m+k$ components of the vectors $B(t)$ and $\rho(t)$. For this system to have a unique solution, the following disequation must hold:

$$\det(S_u^T S_u) \neq 0, \quad 0 \leq t \leq 1. \quad (134)$$

As a result of the procedure, the sweep is completed: for the arbitrary value assigned to w , it leads to the satisfaction

of all of the equations of the system (120)-(129) except Eqs. (124), (126), (129).

In order to satisfy Eqs. (124), (126), (129) and because the system (120)-(129) is nonhomogeneous, $n+p+1$ independent sweeps must be executed employing $n+p+1$ different vectors w_i , $i=1, \dots, n+p+1$. The first $n+p$ sweeps are performed by choosing the vectors $w_1, w_2, \dots, w_{n+p}$ to be the columns of the identity matrix of order $n+p$. The last sweep is executed by choosing w_{n+p+1} to be the null vector. As a result, one generates the functions and multipliers

$$A_i(t), B_i(t), C_i, \lambda_i(t), \rho_i(t), \sigma_i, \quad i=1, \dots, n+p+1. \quad (135)$$

Now, we introduce the $n+p+1$ undetermined, scalar constants k_i and form the linear combinations

$$A(t) = \sum k_i A_i(t), \quad B(t) = \sum k_i B_i(t), \quad C = \sum k_i C_i, \quad (136)$$

$$\lambda(t) = \sum k_i \lambda_i(t), \quad \rho(t) = \sum k_i \rho_i(t), \quad \sigma = \sum k_i \sigma_i, \quad (137)$$

where the summations are taken over the index i . The $n+p+1$ coefficients k_i and the q components of the multiplier μ are obtained by forcing the linear combinations (136)-(137) to satisfy Eqs. (124), (126), (129), together with the normalization condition (Ref. 9)

$$\sum k_i = 1. \quad (138)$$

Once the constants k_i are known, the solution of the LTP-BVP (120)-(129) is given by (136)-(137).

Gradient Stepsize. With the functions $A(t)$, $B(t)$, C known, the one-parameter family of varied functions (24) can be formed. After substitution of Eqs. (24) into (96), (103), (112), the following functions of the stepsize are obtained:

$$\tilde{I} = \tilde{I}(\alpha), \quad \tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha). \quad (139)$$

Then, a one-dimensional search scheme is applied to (139-2), and a value of the stepsize α is selected for which the following relations are satisfied:

$$\tilde{J}(\alpha) \leq \tilde{J}(0), \quad \tilde{P}(\alpha) \leq P_*, \quad \tilde{\tau}(\alpha) \geq 0, \quad (140)$$

where τ is the final time and P_* is a preselected number, not necessarily small. Satisfaction of Ineq. (140-1) is possible because of the descent property of the gradient phase. Ineq. (140-2) is introduced to prevent excessive constraint violation. And Ineq. (140-3) is required for problems with free final time.

Prior to the satisfaction of (140), a scanning process is employed, leading to the bracketing of the minimum point for $\tilde{J}(\alpha)$. This operation is then followed by a Hermitian cubic interpolation process (Ref. 10), which is stopped whenever the following relation is satisfied:¹⁰

$$|\tilde{J}_\alpha(\alpha)| \leq \varepsilon_3, \quad (141)$$

subject to an upper limit for the number of search steps. Once a stepsize α_0 has been selected consistently with either (141) or the prescribed upper limit for the number of search steps, Ineqs. (140) must be checked. If satisfaction occurs, then the stepsize α_0 is accepted. If any violation occurs, then the stepsize α_0 must be bisected progressively until satisfaction of (140) is finally achieved.

Restoration Phase. Let $x(t)$, $u(t)$, π denote nominal functions not satisfying (97)-(101). To first order, the perturbations per unit stepsize $A(t)$, $B(t)$, C must satisfy the linearized constraint equations¹¹

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1, \quad (142)$$

$$S_x^T A + S_u^T B + S_\pi^T C + S = 0, \quad 0 \leq t \leq 1, \quad (143)$$

$$D(0) = 0, \quad (144)$$

$$(\omega_z^T E + \omega_\pi^T C + \omega)_0 = 0, \quad (145)$$

$$(\psi_x^T A + \psi_\pi^T C + \psi)_1 = 0. \quad (146)$$

¹⁰The symbol ε_3 denotes a small, preselected number.

¹¹It is assumed that the nominal functions satisfy (99).

The linearized equations (142)-(146) admit an infinite number of solutions, each of which is characterized by a descent property in the constraint error P . This descent property can be seen by combining (118) with (142)-(146): the first variation of P becomes

$$\delta P = -2\alpha P. \quad (147)$$

Since $P > 0$, Eq. (147) shows that $\delta P < 0$; hence, for α sufficiently small, a decrease in the constraint error P is guaranteed.

Among the infinite number of solutions of Eqs. (142)-(146), the one that minimizes the functional (119) is selected. Thus, we seek the minimum of the quadratic functional (119) with respect to the functions $A(t)$, $B(t)$, C which satisfy the linearized constraints (142)-(146).

By applying standard techniques of optimal control theory or calculus of variations, the following optimality conditions are obtained:

$$B = \phi_u \lambda - S_u \rho, \quad 0 \leq t \leq 1, \quad (148)$$

$$C = \int_0^1 (\phi_\pi \lambda - S_\pi \rho) dt - (\omega_\pi \sigma)_0 - (\psi_\pi \mu)_1, \quad (149)$$

$$E(0) = (\gamma - \omega_z \sigma)_0, \quad (150)$$

$$\dot{\lambda} + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (151)$$

$$(\lambda + \psi_x \mu)_1 = 0. \quad (152)$$

Summarizing, we seek functions $A(t)$, $B(t)$, C and multipliers $\lambda(t)$, $\rho(t)$, σ , μ which satisfy the linearized constraints (142)-(146) and the optimality conditions (148)-(152).

Linear, Two-Point Boundary-Value Problem for the Restoration Phase. The technique used to solve the LTP-BVP (142)-(146) and (148)-(152), associated with the restoration phase, is analogous to that described for the gradient phase; hence, it is not repeated, for the sake of brevity.

Restoration Stepsize. With the functions $A(t)$, $B(t)$, C known, the one-parameter family of varied functions (24) can be formed. For this one-parameter family, the constraint error (112) becomes a function of the form

$$\tilde{P} = \tilde{P}(\alpha). \quad (153)$$

Then, the stepsize α must be selected so that the following relations are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \quad \tilde{\tau}(\alpha) \geq 0. \quad (154)$$

Satisfaction of Ineq. (154-1) is possible because of the descent property of the restoration phase. Ineq. (154-2) is required for problems with free final time.

In order to achieve satisfaction of (154), a bisection process is applied to the restoration stepsize α , starting from the reference stepsize $\alpha_0 = 1$. This reference stepsize has the property of yielding one-step restoration for the case where the constraints (97)-(101) are linear.

Iterative Procedure for the Restoration Phase. The descent property (147) of the restoration phase guarantees satisfaction of Ineq. (154-1) at the end of any iteration, but not satisfaction of Ineq. (115-1). Therefore, the restoration algorithm must be employed iteratively until Ineq. (115-1) is satisfied. At this point, the restoration phase is terminated.

Descent Property of a Cycle. A descent property exists for a complete gradient-restoration cycle under the assumption of small stepsizes. Let α_g denote the gradient stepsize and α_r the restoration stepsize. Simple manipulations, omitted for the sake of brevity, show that the gradient corrections are of $O(\alpha_g)$, while the restoration corrections are of $O(\alpha_r \alpha_g^2)$. Hence, for α_g sufficiently small, the restoration corrections are negligible with respect to the gradient corrections. Therefore, the restoration phase preserves the descent property of the gradient phase.

More specifically, let I , $\tilde{I}$, $\hat{I}$ denote the values of the functional (96) at the beginning of the gradient phase, at the end of the gradient phase, and at the end of the subsequent

restoration phase. Note that I and $\tilde{I}$ are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, I and $\hat{I}$ are comparable, and the gradient stepsize α_g can be selected so that

$$\hat{I} < I. \quad (155)$$

This inequality constitutes the descent property of a complete gradient-restoration cycle. In order to enforce it, one proceeds as follows. At the end of the restoration phase, one must verify Ineq. (155). If it is satisfied, the next gradient phase is started; otherwise, the previous gradient stepsize is bisected as many times as needed until, after restoration, Ineq. (155) is satisfied.

8. Summary of the Algorithm for Problem P2

This algorithm includes cycles composed of a gradient phase and a restoration phase. The objective of each cycle is to decrease the functional I so that Ineq. (155) is satisfied, while the constraints are satisfied to a predetermined accuracy (115-1).

Gradient Phase. This phase involves a single iteration, and its objective is to decrease the augmented functional J , while the constraints are satisfied to first order. The gradient phase can be summarized as follows.

(a) Assume nominal functions $x(t)$, $u(t)$, π which satisfy the constraints (97)-(101) within the preselected accuracy (115-1).

(b) For the nominal functions, compute the vectors f_x , f_u , f_π and the matrices ϕ_x , ϕ_u , ϕ_π , S_x , S_u , S_π along the interval of integration. At the initial point, compute the vectors h_z , h_π and the matrices ω_z , ω_π . At the final point, compute the vectors g_x , g_π and the matrices ψ_x , ψ_π .

(c) Solve the LTP-BVP (120)-(129) using the method of particular solutions. In this way, obtain the functions $A(t)$, $B(t)$, C and the multipliers $\lambda(t)$, $\rho(t)$, σ , μ .

(d) Using the functions in (c), compute the gradient stepsize by a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$ until satisfaction of Ineq. (141) occurs. Then,

bisect the resulting stepsize (if necessary), until satisfaction of Ineq. (140) occurs.

(e) Once the gradient stepsize is known, compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ with Eqs. (24).

Restoration Phase. This phase involves one or more iterations, and its objective is to reduce the constraint error P , until satisfaction of (115-1) occurs. Within a single iteration, the objective is to decrease the constraint error to a level compatible with Ineq. (26-3), while the norm squared of the variations of the control, the parameter, and the missing components of the initial state is minimized.

The nominal functions $x(t)$, $u(t)$, π are chosen as follows: for the first restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous gradient iteration; for any subsequent restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous restorative iteration. With this understanding, the restoration phase can be summarized as follows.

(a) Assume nominal functions $x(t)$, $u(t)$, π which satisfy condition (99), but not necessarily conditions (97)-(98) and (100)-(101).

(b) For the nominal functions, compute the vectors $(\dot{x} - \phi)$, S and the matrices ϕ_x , ϕ_u , ϕ_π , S_x , S_u , S_π along the

interval of integration. At the initial point, compute the vector ω and the matrices ω_z, ω_π . At the final point, compute the vector ψ and the matrices ψ_x, ϕ_π .

(c) Solve the LTP-BVP (142)-(146) and (148)-(152) using the method of particular solutions. In this way, obtain the functions $A(t), B(t), C$ and the multipliers $\lambda(t), \rho(t), \sigma, \mu$.

(d) Using the functions in (c), compute the restoration stepsize by a one-dimensional search on the constraint error $\tilde{P}(\alpha)$. To this effect, perform a bisection process on α , starting from $\alpha=1$, until Ineqs. (154) are satisfied.

(e) Once the restoration stepsize is known, compute the varied functions $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ with Eqs. (24).

(f) Verify whether the varied functions in (e) satisfy Ineq. (115-1). If this is the case, the restoration phase is terminated. Otherwise, return to (a) and continue the process until satisfaction of (115-1) occurs.

Gradient-Restoration Cycle. After the restoration phase is completed, verify whether Ineq. (155) is satisfied. If this is the case, start the next cycle of the sequential gradient-restoration algorithm. If not, return to the previous gradient phase and reduce the gradient stepsize (using a bisection process) until, after restoration, Ineq. (155) is satisfied.

Computational Considerations. Here, special conditions

relevant to the computer implementation of the sequential gradient-restoration algorithm are presented.

(a) Starting condition. The present algorithm can be started with nominal functions $x(t)$, $u(t)$, π satisfying condition (99) and violating none, some, or all of conditions (97)-(98) and (100)-(101). If the nominal functions are such that Ineq. (115-1) is violated, the algorithm starts with a restoration phase; hence, the first cycle is a half cycle, involving a restoration phase only. On the other hand, if the nominal functions are such that Ineq. (115-1) is satisfied, the algorithm starts with a gradient phase; hence, the first cycle is a complete gradient-restoration cycle.

(b) Bypassing Condition. At the end of the gradient phase of any cycle, the constraint error P must be computed. If Ineq. (115-1) is violated, a restoration phase is started. Otherwise, the restoration phase is bypassed, and the next gradient phase of the algorithm is started.

(c) Stopping Conditions. For the restoration phase taken individually, convergence is achieved whenever Ineq. (115-1) is satisfied. For the sequential gradient-restoration algorithm taken as a whole, convergence is achieved whenever Ineqs. (115) are satisfied simultaneously.

9. Numerical Experiments for Problem P2

Experimental Conditions. The experimental conditions for Problem P2 are the same as for Problem P1. Hence, they are not repeated here.

Numerical Examples. In this section, five numerical examples are described employing scalar notation. The symbols used for Problem P2 are the same as those employed for Problem P1. For all of the examples, the time normalization (70) is employed.

Example 9.1. This is a problem with (i) a component of the initial state given, (ii) a component of the initial state free, (iii) a linear relation between the remaining components of the initial state, and (iv) fixed final time $\tau = \pi/2$:

$$I = \int_0^1 \tau (u_1^2 - x_1^2 + \tau t) dt, \quad (156)$$

$$\dot{x}_1 = \tau u_1, \quad \dot{x}_2 = \tau (2 - 4x_1^2), \quad \dot{x}_3 = \tau x_4, \quad \dot{x}_4 = \tau u_2, \quad (157)$$

$$4x_1 u_1 - x_3 u_2 - x_4^2 = 0, \quad (158)$$

$$x_1(0) = 0, \quad x_2(0) = \text{free}, \quad 2x_3(0) + x_4(0) = 1, \quad (159)$$

$$x_1(1) = 1, \quad x_2(1) = 0, \quad x_3(1) = \text{free}, \quad x_4(1) = \text{free}. \quad (160)$$

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = 4t(1-t), \quad x_3(t) = 1-2t, \quad (161)$$

$$x_4(t) = -1, \quad u_1(t) = 1/\tau, \quad u_2(t) = 0. \quad (162)$$

The numerical results are given in Tables 11-13. Convergence to the desired stopping condition occurs in $N = 36$ iterations, which include 16 restorative iterations and 20 gradient iterations.

Example 9.2. This is a problem with (i) a nonlinear relation between the components of the initial state and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 (x_1^2 + u_1^2) dt, \quad (163)$$

$$\dot{x}_1 = x_1^2 - u_1, \quad \dot{x}_2 = u_2, \quad (164)$$

$$x_1^2 - u_1 - 2x_2 u_2 = 0, \quad (165)$$

$$\sin^2[x_1(0)] + x_2^2(0) = 0.95, \quad (166)$$

$$x_1(1) = 1, \quad x_2(1) = \text{free}. \quad (167)$$

The assumed nominal functions are:

$$x_1(t) = 1, \quad x_2(t) = \sqrt{0.1}, \quad u_1(t) = 1, \quad u_2(t) = 1. \quad (168)$$

The numerical results are given in Tables 14-16. Convergence to the desired stopping conditions occurs in $N=19$ iterations, which include 14 restorative iterations and 5 gradient iterations.

Example 9.3. This is a problem with (i) a component of the initial state given, (ii) a nonlinear relation between the remaining components of the initial state, and (iii) fixed final time $\tau = 1$:^{12,13}

$$I = \int_0^1 \left[2x_2 u_1^3 / (1 + u_1^2) \right] dt + x_2^2(0), \quad (169)$$

$$\dot{x}_1 = x_2^2, \quad \dot{x}_2 = u_1, \quad (170)$$

$$u_1 - u_2^2 = 0, \quad (171)$$

$$x_1(0) = 0, \quad \left[x_2(0) - 0.35 \right] \left[1 - x_2(0) \right] - \pi_1^2 = 0, \quad (172)$$

$$x_1(1) = 1/2, \quad x_2(1) = \text{free}. \quad (173)$$

¹²The nondifferential constraint (171) implies that $u_1 \geq 0$.

¹³The initial condition (172-2) implies that $0.35 \leq x_2(0) \leq 1$.

The assumed nominal functions are:¹⁴

$$x_1(t) = (t^3 + 3t^2 + 3t)/14, \quad x_2(t) = C(t+1), \quad (174)$$

$$u_1(t) = C, \quad u_2(t) = \sqrt{C}, \quad \pi_1 = \sqrt{[(C - 0.35)(1 - C)]}. \quad (175)$$

The numerical results are given in Tables 17-19. Convergence to the desired stopping condition occurs in $N=10$ iterations, which include 7 restorative iterations and 3 gradient iterations.

Example 9.4. This is a minimum time problem with (i) a nonlinear relation between the components of the initial state and (ii) free final time τ . After setting $\pi_1 = \tau$, the problem is as follows:

$$I = \pi_1, \quad (176)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2 - 1/2), \quad (177)$$

$$u_1^2 - x_1^2 - u_2^2 = 0, \quad (178)$$

$$x_1(0) + x_2^2(0) = 0, \quad (179)$$

$$x_1(1)x_2(1) = -\pi/4. \quad (180)$$

¹⁴Here, $C = \sqrt{1/7}$. For this value of C , the constraints (170-2) and (171)-(173) are satisfied.

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = (\pi/4)t, \quad u_1(t) = 1, \quad u_2(t) = 1, \quad \pi_1 = 1. \quad (181)$$

The numerical results are given in Tables 20-22. Convergence to the desired stopping condition occurs in $N=19$ iterations, which include 12 restorative iterations and 7 gradient iterations.

Example 9.5. This is a problem with (i) a linear relation between the first two components of the initial state, (ii) a nonlinear relation between the remaining components of the initial state, and (iii) fixed final time $\tau = 1$:

$$I = \int_0^1 u_1^2 dt, \quad (182)$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u_2, \quad (183)$$

$$u_1 + 2x_3 u_2 + 2x_4^2 = 0, \quad (184)$$

$$x_1(0) + x_2(0) = 1, \quad -x_3^2(0) - 2x_3(0)x_4(0) = 0.85, \quad (185)$$

$$x_1(1) = 0, \quad x_2(1) = -1. \quad (186)$$

The assumed nominal functions are:

$$x_1(t) = 0, \quad x_2(t) = 1-2t, \quad x_3(t) = (1-2t)/(0.15), \quad (187)$$

$$x_4(t) = (2t - 1)/2\sqrt{0.15}, \quad u_1(t) = 1, \quad u_2(t) = 0. \quad (188)$$

The numerical results are given in Tables 23-25. Convergence to the desired stopping condition occurs in $N = 21$ iterations, which include 12 restorative iterations and 9 gradient iterations.

Table 11. Converged history , Example 9.1.

N	Phase	P	Q	I
0		0.23E+01		
1	Rest	0.79E-01		
2	Rest	0.10E-03		
3	Rest	0.49E-09	0.52E-01	1.27196
4	Grad	0.22E-03		
5	Rest	0.86E-09	0.27E-01	1.25174
6	Grad	0.60E-05		
7	Rest	0.13E-12	0.69E-02	1.24498
8	Grad	0.30E-05		
9	Rest	0.37E-13	0.85E-02	1.24163
10	Grad	0.45E-06		
11	Rest	0.37E-15	0.25E-02	1.23965
12	Grad	0.33E-06		
13	Rest	0.29E-15	0.41E-02	1.23832
14	Grad	0.92E-07		
15	Rest	0.99E-17	0.13E-02	1.23737
16	Grad	0.65E-07		
17	Rest	0.10E-16	0.23E-02	1.23667
18	Grad	0.27E-07		
19	Rest	0.67E-18	0.82E-03	1.23613
20	Grad	0.16E-07		
21	Rest	0.52E-18	0.14E-02	1.23572
22	Grad	0.10E-07		
23	Rest	0.70E-19	0.56E-03	1.23538
24	Grad	0.54E-08	0.87E-03	1.23513
25	Grad	0.18E-07		
26	Rest	0.89E-19	0.41E-03	1.23491
27	Grad	0.23E-08	0.58E-03	1.23473
28	Grad	0.80E-08	0.30E-03	1.23459
29	Grad	0.15E-07		
30	Rest	0.46E-19	0.44E-03	1.23446
31	Grad	0.83E-09	0.19E-03	1.23436
32	Grad	0.34E-08	0.40E-03	1.23425
33	Grad	0.62E-08	0.11E-03	1.23417
34	Grad	0.13E-07		
35	Rest	0.10E-18	0.53E-03	1.23404
36	Grad	0.42E-09	0.57E-04	1.23397

Table 12. Converged state variables, Example 9.1.

t	x_1	x_2	x_3	x_4
0.0	0.0000	-0.0549	0.9945	-0.9891
0.1	0.1541	0.2542	0.8277	-1.1312
0.2	0.3045	0.5340	0.6406	-1.2461
0.3	0.4476	0.7578	0.4377	-1.3317
0.4	0.5806	0.9043	0.2242	-1.3803
0.5	0.6983	0.9596	0.0060	-1.3930
0.6	0.7988	0.9197	-0.2120	-1.3792
0.7	0.8811	0.7885	-0.4257	-1.3366
0.8	0.9433	0.5775	-0.6303	-1.2626
0.9	0.9833	0.3062	-0.8208	-1.1575
1.0	1.0000	0.0000	-0.9924	-1.0238

Table 13. Converged control variables, Example 9.1.

t	u_1	u_2
0.0	0.9842	-0.9838
0.1	0.9734	-0.8209
0.2	0.9379	-0.6403
0.3	0.8825	-0.4412
0.4	0.8029	-0.1793
0.5	0.6946	0.0054
0.6	0.5836	0.1748
0.7	0.4622	0.3698
0.8	0.3269	0.5721
0.9	0.1813	0.7635
1.0	0.0301	0.9347

$$\tau = \pi/2$$

Table 14. Convergence history, Example 9.2.

N	Phase	P	Q	I
0		0.14E+01		
1	Rest	0.47E-01		
2	Rest	0.17E-02		
3	Rest	0.25E-06		
4	Rest	0.80E-14	0.10E+01	2.46749
5	Grad	0.15E+01		
6	Rest	0.64E-01		
7	Rest	0.76E-03		
8	Rest	0.21E-06		
9	Rest	0.16E-13	0.23E+01	1.57725
10	Grad	0.16E+01		
11	Rest	0.34E-01		
12	Rest	0.52E-04		
13	Rest	0.74E-10	0.28E+01	1.27436
14	Grad	0.81E-01		
15	Rest	0.51E-04		
16	Rest	0.83E-10	0.14E-01	0.45497
17	Grad	0.13E-05		
18	Rest	0.53E-13	0.32E-03	0.45019
19	Grad	0.10E-08	0.78E-05	0.45009

Table 15. Converged state variables, Example 9.2.

t	x_1	x_2
0.0	0.4087	0.8899
0.1	0.4286	0.9010
0.2	0.4542	0.9151
0.3	0.4862	0.9324
0.4	0.5253	0.9532
0.5	0.5726	0.9777
0.6	0.6294	1.0063
0.7	0.6974	1.0395
0.8	0.7792	1.0781
0.9	0.8783	1.1232
1.0	1.0000	1.1761

Table 16. Converged control variables, Example 9.2.

t	u_1	u_2
0.0	-0.0036	0.0959
0.1	-0.0427	0.1256
0.2	-0.0807	0.1568
0.3	-0.1178	0.1899
0.4	-0.1541	0.2256
0.5	-0.1896	0.2646
0.6	-0.2242	0.3082
0.7	-0.2579	0.3580
0.8	-0.2905	0.4163
0.9	-0.3218	0.4867
1.0	-0.3516	0.5746

$\tau = 1.0000$

Table 17. Convergence history, Example 9.3.

N	Phase	P	Q	I
0		0.31E-01		
1	Rest	0.12E-02		
2	Rest	0.49E-05		
3	Rest	0.28E-09	0.65E-01	0.31197
4	Grad	0.28E-03		
5	Rest	0.19E-07		
6	Rest	0.14E-15	0.17E-02	0.28910
7	Grad	0.76E-05		
8	Rest	0.24E-10	0.20E-03	0.28774
9	Grad	0.10E-07		
10	Rest	0.60E-16	0.22E-04	0.28764

Table 18. Converged state variables, Example 9.3.

t	x_1	x_2
0.0	0.0000	0.4148
0.1	0.0210	0.4993
0.2	0.0497	0.5685
0.3	0.0856	0.6274
0.4	0.1283	0.6787
0.5	0.1776	0.7235
0.6	0.2329	0.7621
0.7	0.2935	0.7947
0.8	0.3589	0.8212
0.9	0.4281	0.8409
1.0	0.5000	0.8527

Table 19. Converged control variables, Example 9.3.

t	u_1	u_2
0.0	0.9427	0.9709
0.1	0.7586	0.8710
0.2	0.6340	0.7962
0.3	0.5480	0.7403
0.4	0.4795	0.6925
0.5	0.4167	0.6455
0.6	0.3560	0.5966
0.7	0.2961	0.5441
0.8	0.2329	0.4826
0.9	0.1586	0.3983
1.0	0.0767	0.2770

$\tau = 1,$	$\pi_1 = 0.1947$
-------------	------------------

Table 20. Convergence history, Example 9.4.

N	Phase	P	Q	I
0		0.31E+01		
1	Rest	0.36E+00		
2	Rest	0.72E-01		
3	Rest	0.19E-02		
4	Rest	0.26E-06		
5	Rest	0.18E-13	0.97E-01	1.62388
6	Grad	0.27E-02		
7	Rest	0.83E-05		
8	Rest	0.85E-12	0.20E-01	1.58997
9	Grad	0.46E-04		
10	Rest	0.18E-08	0.39E-02	1.58424
11	Grad	0.34E-05		
12	Rest	0.29E-10	0.13E-02	1.58298
13	Grad	0.51E-06		
14	Rest	0.12E-11	0.54E-03	1.58253
15	Grad	0.93E-07		
16	Rest	0.51E-13	0.23E-03	1.58235
17	Grad	0.31E-07		
18	Rest	0.10E-13	0.12E-03	1.58227
19	Grad	0.72E-08	0.62E-04	1.58218

Table 21. Converged state variables, Example 9.4.

t	x_1	x_2
0.0	-0.0931	-0.3051
0.1	0.0367	-0.2779
0.2	0.1658	-0.2536
0.3	0.2908	-0.2424
0.4	0.4086	-0.2533
0.5	0.5164	-0.2931
0.6	0.6145	-0.3622
0.7	0.7200	-0.4412
0.8	0.8434	-0.5203
0.9	0.9880	-0.5994
1.0	1.1573	-0.6785

Table 22. Converged control variables, Example 9.4.

t	u_1	u_2
0.0	0.8169	0.8116
0.1	0.8218	0.8210
0.2	0.8062	0.7890
0.3	0.7705	0.7135
0.4	0.7157	0.5876
0.5	0.6449	0.3862
0.6	0.6194	0.0795
0.7	0.7200	0.0007
0.8	0.8434	0.0000
0.9	0.9880	0.0000
1.0	1.1573	0.0000

$\tau = \pi_1 = 1.5821$

Table 23. Convergence history, Example 9.5.

N	Phase	P	Q	I
0		0.22E+02		
1	Rest	0.26E+00		
2	Rest	0.19E+00		
3	Rest	0.43E-03		
4	Rest	0.17E-08	0.46E+00	3.32348
5	Grad	0.15E-01		
6	Rest	0.21E-05		
7	Rest	0.67E-13	0.55E-01	3.05708
8	Grad	0.70E-04		
9	Rest	0.93E-12	0.15E-01	3.01848
10	Grad	0.44E-04		
11	Rest	0.52E-11	0.65E-02	3.00785
12	Grad	0.19E-05		
13	Rest	0.29E-14	0.25E-02	3.00350
14	Grad	0.12E-05		
15	Rest	0.27E-14	0.13E-02	3.00161
16	Grad	0.89E-07		
17	Rest	0.89E-17	0.50E-03	3.00075
18	Grad	0.47E-07		
19	Rest	0.38E-17	0.30E-03	3.00035
20	Grad	0.44E-08	0.10E-03	3.00016
21	Grad	0.55E-08	0.66E-04	3.00007

Table 24. Converged state variables, Example 9.5.

t	x_1	x_2	x_3	x_4
0.0	0.5025	0.4974	0.5652	-1.0344
0.1	0.5378	0.2127	0.4658	-0.9496
0.2	0.5461	-0.0425	0.3762	-0.8365
0.3	0.5303	-0.2683	0.3002	-0.6723
0.4	0.4934	-0.4644	0.2446	-0.4245
0.5	0.4384	-0.6297	0.2181	-0.0969
0.6	0.3685	-0.7633	0.2242	0.2035
0.7	0.2867	-0.8670	0.2546	0.3827
0.8	0.1961	-0.9411	0.2971	0.4527
0.9	0.0995	-0.9854	0.3430	0.4567
1.0	0.0000	-1.0000	0.3872	0.4233

Table 25. Converged control variables, Example 9.5.

t	u_1	u_2
0.0	-2.9883	0.7498
0.1	-2.7013	0.9633
0.2	-2.4054	1.3367
0.3	-2.1091	2.0065
0.4	-1.8112	2.9656
0.5	-1.4929	3.3783
0.6	-1.1837	2.4541
0.7	-0.8894	1.1710
0.8	-0.5924	0.3071
0.9	-0.2940	-0.1795
1.0	0.0038	-0.4675

$\tau = 1.0000$

10. Discussion and Conclusions

In this thesis, two new members of the family of sequential gradient-restoration algorithms for the solution of optimal control problems are presented. These algorithms are of the ordinary-gradient type. One is associated with the solution of Problem P1, Eqs. (1)-(5), and the other is associated with the solution of Problem P2, Eqs. (96)-(101).

The algorithms presented in this thesis differ from those of Refs. 1-8, in that it is not required that the state vector be given at the initial point. Instead, the initial conditions can be absolutely general. In analogy with Refs. 1-8, the present algorithms are capable of handling general final conditions; therefore, they are suited for the solution of optimal control problems with general boundary conditions.

The importance of the present algorithms lies in that many optimal control problems either arise naturally in the format of Problems P1 and P2 or can be brought to such a format by means of suitable transformations (see Ref. 4). Therefore, a great variety of optimal control problems can be handled, as it is shown by the numerical examples presented.

The new algorithms include a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase involves one iteration and is designed to decrease the value of the functional I , while the constraints

are satisfied to first order. The restoration phase involves one or more iterations and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control and the parameter is minimized.

The principal property of the algorithms is that they produce a sequence of feasible suboptimal solutions, each satisfying the constraints to the same predetermined accuracy. Therefore, the values of the functional I corresponding to any two elements of the sequence are comparable.

The gradient phase is characterized by a descent property on the augmented functional J , which implies a descent property on the functional I . The restoration phase is characterized by a descent property on the constraint error P . The gradient stepsize and the restoration stepsize are chosen such that the restoration phase preserves the descent property of the gradient phase. Hence, the value of the functional I at the end of any complete gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

It must be noted that certain limitations exist. The algorithm associated with Problem P1 is applicable providing inequation (44) is satisfied at every iteration. By the same token, the algorithm associated with Problem P2 is applicable providing inequations (133)-(134) are satisfied at every iteration.

Numerical examples are presented to illustrate the performance of the algorithms associated with Problem P1 and Problem P2. The numerical results show the feasibility as well as the convergence characteristics of these algorithms.

In summary, the new members of the family of sequential gradient-restoration algorithms described here have the following properties: (i) they retain the robustness, reliability, and convergence characteristics of the algorithms discussed in Refs. 1-4; (ii) they are able to handle all of the optimal control problems treated in Refs. 1-8; and (iii) they have the additional capability of handling optimal control problems with general boundary conditions.

References

1. MIELE, A., PRITCHARD, R.E., and DAMOULAKIS, J.N., Sequential Gradient-Restoration Algorithm for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol. 5, No. 4, 1970.
2. MIELE, A., DAMOULAKIS, J.N., CLOUTIER, J.R., and TIETZE, J.L., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Journal of Optimization Theory and Applications, Vol. 13, No. 2, 1974.
3. HENNIG, G.R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State, Journal of Optimization Theory and Applications, Vol. 12, No. 1, 1973.
4. MIELE, A., Recent Advances in Gradient Algorithms for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol. 17, Nos. 5-6, 1975.
5. HEIDEMAN, J.C., and LEVY, A.V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 1, Theory, Journal of Optimization Theory and Applications, Vol. 15, No. 2, 1975.

6. HEIDEMAN, J.C., and LEVY, A.V., Sequential Conjugate Gradient Restoration Algorithm for Optimal Control Problems, Part 2, Examples, Journal of Optimization Theory and Applications, Vol. 15, No. 2, 1975.
7. MIELE, A., CLOUTIER, J.R., MOHANTY, B.P., and WU, A.K., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 1, International Journal of Control (to appear).
8. MIELE, A., CLOUTIER, J.R., MOHANTY, B.P., and WU, A.K., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, International Journal of Control (to appear).
9. MIELE, A., Method of Particular Solutions for Linear, Two-Point Boundary-Value Problems, Journal of Optimization Theory and Applications, Vol. 2, No. 4, 1968.
10. MIELE, A., BONARDO, F., and GONZALEZ, S., Modifications and Alternatives to the Cubic Interpolation Process for One-Dimensional Search, Rice University, Aero-Astronautics Report No. 135, 1976.
11. RALSTON, A., Numerical Integration Methods for the Solution of Ordinary Differential Equations, Mathematical Methods for Digital Computers, Vol. 1, Edited by A.

Ralston and H.S. Wilf, John Wiley and Sons, New York, New York, 1960.

Additional Bibliography

12. KNAPP, C.H., The Maximum Principle and the Method of Gradients, IEEE Transactions on Automatic Control, Vol. 11, No. 4, 1966.
13. LASTMAN, G.J., A Modified Newton's Method for Solving Trajectory Optimization Problems, AIAA Journal, Vol. 6, No. 5, 1968.
14. LASTMAN, G.J., and TAPLEY, B.D., Optimization of Nonlinear Systems with Inequality Constraints Explicitly Containing the Control, International Journal of Control, Vol. 12, No. 3, 1970.
15. HONTOIR, Y., and CRUZ, J.B., JR., A Manifold Imbedding Algorithm for Optimization Problems, Automatica, Vol. 8, No. 5, 1972.
16. PIRONNEAU, O., and POLAK, E., A Dual Method for Optimal Control Problems with Initial and Final Boundary Constraints, SIAM Journal on Control, Vol. 11, No. 3, 1973.